Moment Invariants for Object Recognition

Jan Flusser

I. Introduction

Analysis and interpretation of an image which was acquired by a real (i.e. non-ideal) imaging system is the key problem in many application areas such as remote sensing, astronomy and medicine, among others. Since real imaging systems as well as imaging conditions are usually imperfect, the observed image represents only a degraded version of the original scene. Various kinds of degradations (geometric as well as radiometric) are introduced into the image during the acquisition by such factors as imaging geometry, lens aberration, wrong focus, motion of the scene, systematic and random sensor errors, etc.

In the general case, the relation between the ideal image $f(x,y)$ and the observed image $g(x,y)$ is described as $g = \mathcal{D}(f)$, where $\mathcal{D}$ is a degradation operator. In the case of a linear shift-invariant imaging system, $\mathcal{D}$ has a form of

$$g(\tau(x,y)) = (f * h)(x,y) + n(x,y), \tag{1}$$

where $h(x,y)$ is the point-spread function (PSF) of the system, $n(x,y)$ is an additive random noise, τ is a transform of spatial coordinates due to projective imaging geometry and $*$ denotes a 2-D convolution. Knowing the image $g(x,y)$, our objective is to analyze the unknown scene $f(x,y)$.

By the term "scene analysis" we usually understand a complex process consisting of three basic stages. First, the image is segmented in order to extract objects of potential interest. Secondly, the extracted objects are "recognized", which means they are classified as elements of one class from the set of pre-defined object classes. Finally, spatial relations among the objects can be analyzed. In this tutorial, we focus on object recognition.

Recognition of objects and patterns that are deformed in various ways has been a goal of much recent research. There are basically three major approaches to this problem – brute force, image normalization, or invariant features. In brute force approach we search the space of all possible image degradations. That means the training set of each class should consist not only all class representatives but also all their rotated, scaled, blurred, and deformed versions. Clearly, this approach would lead to extreme time complexity and is practically inapplicable. In normalization approach, the objects are transformed into some standard position before they are classified. This could be very efficient in the classification stage but the object normalization usually requires to solve complex inverse problems which are often ill posed. The approach using invariant features appears to be the most promising. Its basic idea is to describe the objects by a set of features which are not sensitive to particular deformations and which provide enough discrimination power to distinguish among objects from different classes. From mathematical point of view, we have to find functional I defined on the space of all admissible image functions (let's imagine $L_1(R^2)$ space for instance) which are invariant with respect to degradation operator $\mathcal{D}$, i.e. which satisfies the condition $I(f) = I(\mathcal{D}(f))$ for any image function f.

In this article we present non-linear invariant functionals, which are composed of various projections of f into the space of polynomials. Such projections are known as *image moments* and the respective functionals are called *moment invariants*. We present several groups of moment invariants with respect to the most common degradations – image rotation and scaling, image affine transform, and image blurring (convolution with an unknown filter). We explain a general theory how to construct these functionals and show also a few of them in explicit forms. Then we briefly discuss numerical algorithms for efficient moment calculation.

J. Flusser is with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Prague 8, Czech Republic. Email: flusser@utia.cas.cz

II. Brief history

The history of moment invariants begun many years before the appearance of first computers, in the 19th century under the framework of the theory of algebraic invariants. The theory of algebraic invariants probably originates from famous German mathematician David Hilbert [1] and was thoroughly studied also in [2], [3].

Moment invariants were firstly introduced to the pattern recognition community in 1962 by Hu [4], who employed the results of the theory of algebraic invariants and derived his seven famous invariants to rotation of 2-D objects. Since that time, numerous works have been devoted to various improvements and generalizations of Hu's invariants and also to its use in many application areas.

Dudani [5] and Belkasim [6] described their application to aircraft silhouette recognition, Wong and Hall [7], Goshtasby [8] and Flusser and Suk [9] employed moment invariants in template matching and registration of satellite images, and many other authors used moment invariants for character recognition [6], [10]. Maitra [11] and Hupkens [12] made them invariant also to contrast changes, Wang [13] proposed illumination invariants particularly suitable for texture classification. Li [14] and Wong [15] presented the systems of invariants up to the orders nine and five, respectively. Unfortunately, no one of them paid attention to mutual dependence/independence of the invariants. The invariant sets presented in their papers are algebraically dependent. Most recently, Flusser [16], [17] has proposed a method how to derive independent sets of invariants of any orders.

There is also a group of papers [18], [19] that use *Zernike moments* to construct rotation invariants. Their motivation comes from the fact that Zernike polynomials are orthogonal on a unit circle. Thus, Zernike moments do not contain any redundant information and are more convenient for image reconstruction. However, Teague [18] showed that Zernike invariants of 2nd and 3rd orders are equivalent to Hu's ones when expressing them in terms of geometric moments. He presented the invariants up to eight order in explicit form but no general rule how to derive them is given. Wallin [19] described an algorithm for a formation of moment invariants of any order. Since Teague [18] as well as Wallin [19] were particularly interested in reconstruction abilities of the invariants, they didn't pay much attention to the question of independence.

Flusser and Suk [20] and Reiss [21] contributed significantly to the theory of moment invariants by correcting the Fundamental Theorem and deriving invariants to general affine transform.

Several papers studied recognitive and reconstruction aspects, noise tolerance and other numerical properties of various kinds of moment invariants and compared their performance experimentally [6], [22], [23], [24], [25], [26], [27]. Moment invariants were shown to be also a useful tool for geometric normalization of an image [28], [29]. Large amount of effort has been spent to find effective algorithms for moment calculation of binary [30] as well as gray-level images [31].

All the above mentioned invariants deal with geometric distortion of the objects. Much less attention has been paid to invariants with respect to changes of the image intensity function (we call them radiometric invariants) and to combined radiometric-geometric invariants. In fact, just the invariants both to radiometric and geometric image degradations are necessary to resolve practical object recognition tasks because usually both types of degradations are present in input images.

Van Gool et al. introduced so-called affine-photometric invariants of graylevel [32] and color [33] images. These features are invariant to the affine transform and to the change of contrast and brightness of the image simultaneously. A pioneer work on this field was done by Flusser and Suk [34] who derived invariants to convolution with an arbitrary centrosymmetric PSF. From the geometric point of view, their descriptors were invariant to translation only. Despite of this, the invariants have found successful applications in face recognition on out-of-focused photographs [35], in normalizing blurred images into the canonical forms [36], [37], in template-to-scene matching of satellite images [34], in blurred digit and character recognition [38], [13], in registration of images obtained by digital subtraction angiography [39] and in focus/defocus quantitative measurement [40]. Other sets of blur invariants (but still only shift-invariant) were proposed for some particular kinds of PSF -– axisymmetric blur invariants [41] and motion blur invariants [42], [43]. A significant improvement motivated by a problem of registration of blurred images was made by Flusser et al. They introduced so-called combined blur-rotation invariants [44] and combined blur-affine invariants [45] and reported their successful usage in satellite image registration [46] and in camera motion estimation [47].

As general references we recommend the books by Flusser et al. [48] and Papakostas [49]. The former is a monograph which can be used as a textbook at university courses, the latter is a collection of research papers showing the latest developments on the field.

III. Basic Terms

First we define basic terms which will be then used in the construction of the invariants.

Definition 1: By *image function* (or *image*) we understand any real function $f(x,y)$ having a bounded support and a finite nonzero integral.

Definition 2: *Geometric moment* m_{pq} of image $f(x,y)$, where p,q are non-negative integers and $(p+q)$ is called the *order* of the moment, is defined as

$$m_{pq} = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} x^p y^q f(x,y)\mathrm{d}x\mathrm{d}y. \tag{2}$$

Corresponding *central moment* μ_{pq} and *normalized moment* ν_{pq} are defined as

$$\mu_{pq} = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} (x-x_c)^p (y-y_c)^q f(x,y)dxdy, \tag{3}$$

$$\nu_{pq} = \frac{\mu_{pq}}{\mu_{00}^{\omega}}, \tag{4}$$

respectively, where $x_c = m_{10}/m_{00}$ and $y_c = m_{01}/m_{00}$ denote the coordinates of the centroid of $f(x,y)$, and $\omega = (p+q+2)/2$.

Definition 3: *Complex moment* c_{pq} of image $f(x,y)$ is defined as

$$c_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^p (x-iy)^q f(x,y)dxdy \tag{5}$$

where i denotes imaginary unit. Definitions of central and normalized complex moments are analogous to (3) and (4).

Geometric moments and complex moments carry the same amount of information. Each complex moment can be expressed in terms of geometric moments as

$$c_{pq} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^{q-j}\cdot i^{p+q-k-j}\cdot m_{k+j,p+q-k-j} \tag{6}$$

and vice versa:

$$m_{pq} = \frac{1}{2^{p+q}i^q}\sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}(-1)^{q-j}\cdot c_{k+j,p+q-k-j}. \tag{7}$$

The reason for introducing complex moments is in their favorable behavior under image rotation, as will be shown later.

IV. INVARIANTS TO ROTATION, TRANSLATION, AND SCALING

Invariants to similarity transformation group were the first invariants that appeared in pattern recognition literature. It was caused partly because of their simplicity, partly because of great demand for invariant features that could be used in position-independent object classification. In this problem formulation, degradation operator $\mathcal{D}$ is supposed to act solely in spatial domain and to have a form of similarity transform. Equation (1) then reduces to

$$g(\tau(x, y)) = f(x, y), \tag{8}$$

where $\tau(x, y)$ denotes arbitrary rotation, translation, and scaling.

Invariants to translation and scaling are trivial – central and normalized moments themselves can play this role. As early as in 1962, Hu [4] published seven rotation invariants, consisting of second and third order moments:

$$\begin{aligned}
\phi_1 &= \mu_{20} + \mu_{02}, \\
\phi_2 &= (\mu_{20} - \mu_{02})^2 + 4\mu_{11}^2, \\
\phi_3 &= (\mu_{30} - 3\mu_{12})^2 + (3\mu_{21} - \mu_{03})^2, \\
\phi_4 &= (\mu_{30} + \mu_{12})^2 + (\mu_{21} + \mu_{03})^2, \\
\phi_5 &= (\mu_{30} - 3\mu_{12})(\mu_{30} + \mu_{12})((\mu_{30} + \mu_{12})^2 - \\
&\quad 3(\mu_{21} + \mu_{03})^2) + (3\mu_{21} - \mu_{03}) \\
&\quad (\mu_{21} + \mu_{03})(3(\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2), \\
\phi_6 &= (\mu_{20} - \mu_{02})((\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2) + \\
&\quad 4\mu_{11}(\mu_{30} + \mu_{12})(\mu_{21} + \mu_{03}), \\
\phi_7 &= (3\mu_{21} - \mu_{03})(\mu_{30} + \mu_{12})((\mu_{30} + \mu_{12})^2 - \\
&\quad 3(\mu_{21} + \mu_{03})^2) - (\mu_{30} - 3\mu_{12})(\mu_{21} + \mu_{03}) \\
&\quad (3(\mu_{30} + \mu_{12})^2 - (\mu_{21} + \mu_{03})^2).
\end{aligned} \tag{9}$$

The Hu's invariants became classical and have found numerous successful applications in various areas, namely in solving simple tasks. Major weakness of the Hu's method is that its generalization is difficult. By means of it, we could not easily derive invariants from higher-order moments and invariants to more general transformations. Another drawback is that the Hu's invariants are mutually dependent and incomplete (see [16] for details). These limitations were overcome thirty years later.

After Hu, there have been published various approaches to the theoretical derivation of moment-based rotation invariants. Li [14] used Fourier-Mellin transform, Teague [18] and Wallin [19] proposed to use Zernike moments, Wong [15] used complex monomials which originate from the theory of algebraic invariants, and Mostafa and Psaltis [24] employed complex moments. Here, we present a scheme introduced by Flusser [16], [17], which is based on the complex moments.

In polar coordinates, (5) becomes the form

$$c_{pq} = \int_0^\infty \int_0^{2\pi} r^{p+q+1} e^{i(p-q)\theta} f(r, \theta) dr d\theta. \tag{10}$$

It follows from the definition that $c_{pq} = c_{qp}^*$ (the asterisk denotes complex conjugate). Furthermore, it follows immediately from (10) that the moment magnitude $|c_{pq}|$ is invariant to rotation of the image while the phase is shifted by $(p-q)\alpha$, where α is the angle of rotation. More precisely, it holds for the moment of the rotated image

$$c'_{pq} = e^{-i(p-q)\alpha} \cdot c_{pq}. \tag{11}$$

Any approach to the construction of rotation invariants is based on a proper kind of phase cancelation. The simplest method proposed by many authors is to use the moment magnitudes themselves as the invariants. However, they do not generate a complete set of invariants. In the following Theorem, phase cancelation is achieved by multiplication of appropriate moment powers.

Theorem 1: Let $n \geq 1$ and let k_i, p_i, and q_i $(i = 1, \cdots, n)$ be non-negative integers such that

$$\sum_{i=1}^{n} k_i (p_i - q_i) = 0.$$

Then

$$I = \prod_{i=1}^{n} c_{p_i q_i}^{k_i} \tag{12}$$

is invariant to rotation.

Theorem 1 allows us to construct an infinite number of the invariants for any order of moments, but only few of them are mutually independent. The knowledge of their basis is a crucial point because dependent features do not contribute to the discrimination power of the system at all and may even cause object misclassifications due to the "curse of dimensionality".

Fundamental theorem on how to construct an invariant basis for a given set of moments was firstly formulated and proven in [16] and later in more general form (which is shown below) in [17].

Theorem 2: Let us consider complex moments up to the order $r \geq 2$. Let a set of rotation invariants $\mathcal{B}$ be constructed as follows:

$$\mathcal{B} = \{\Phi(p, q) \equiv c_{pq} c_{q_0 p_0}^{p-q} | p \geq q \wedge p + q \leq r\},$$

where p_0 and q_0 are arbitrary indices such that $p_0 + q_0 \leq r$, $p_0 - q_0 = 1$ and $c_{p_0 q_0} \neq 0$ for all images involved. Then $\mathcal{B}$ is a basis of a set of all rotation invariants created from the moments up to the order r.

If we apply Theorem 2 with the choice of $r = 3$, $p_0 = 2$, and $q_0 = 1$, we obtain a complete an independent alternative to the Hu's invariants:

$$\begin{aligned}
\Phi(1, 1) &= c_{11}, \\
\Phi(2, 1) &= c_{21} c_{12}, \\
\Phi(2, 0) &= c_{20} c_{12}^2, \\
\Phi(3, 0) &= c_{30} c_{12}^3.
\end{aligned} \tag{13}$$

V. INVARIANTS TO AFFINE TRANSFORM

In practice we often face object deformations that are beyond the rotation-translation-scaling model. An exact model of photographing a planar scene by a pin-hole camera whose optical axis is not perpendicular to the scene is *projective transform* of spatial coordinates. Since the projective transform is not linear, its Jacobian is a function of spatial coordinates and projective moment invariants from a finite number of moments cannot exist [52], [53].

For small objects and large camera-to-scene distance is the perspective effect negligible and the projective transform can be well approximated by *affine transform*

$$\begin{aligned} x' &= a_0 + a_1 x + a_2 y, \\ y' &= b_0 + b_1 x + b_2 y. \end{aligned} \tag{14}$$

Thus, having powerful affine moment invariants for object description and recognition is in great demand.

A pioneer work on this field was done independently by Reiss [21] and Flusser and Suk [20], [51], who introduced affine moment invariants (AMI's) and proved their applicability in simple recognition tasks. They derived only few invariants in explicit forms and they did not study the problem of their mutual independence.

Here we present a new general method how to systematically derive arbitrary number of the AMI's of any weights and any orders, This method is based on representation of the AMI's by graphs (see [54] for more details).

Let us consider an image f and two arbitrary points (x_1, y_1), (x_2, y_2) from its support. Let us denote the "cross-product" of these points as T_{12}:

$$T_{12} = x_1 y_2 - x_2 y_1.$$

After an affine transform it holds $T'_{12} = J \cdot T_{12}$, where J is the Jacobian of the transform. The basic idea of the AMI's generating is the following. We consider various numbers of points and we integrate their cross-products (or some powers of their cross-products) on the support of f. These integrals can be expressed in terms of moments and, after eliminating the Jacobian by proper normalization, they yield affine invariants.

More precisely, having N points ($N \geq 2$) we define functional I depending on N and on non-negative integers n_{kj} as

$$I(f) = \int_{-\infty}^{\infty} \prod_{k,j=1}^{N} T_{kj}^{n_{kj}} \cdot \prod_{i=1}^{N} f(x_i, y_i) dx_i dy_i. \tag{15}$$

Note that it is meaningful to consider only $j > k$, because $T_{kj} = -T_{jk}$ and $T_{kk} = 0$. After an affine transform, I becomes

$$I' = J^w |J|^N \cdot I,$$

where $w = \sum_{k,j} n_{kj}$ is called the *weight* of the invariant and N is called the *degree* of the invariant.

If I is normalized by μ_{00}^{w+N} we get a desirable affine invariant

$$\left(\frac{I}{\mu_{00}^{w+N}}\right)' = \left(\frac{I}{\mu_{00}^{w+N}}\right)$$

(if w is odd and $J < 0$ there is an additional factor -1).

We illustrate the general formula (14) on two simple invariants. First, let $N = 2$ and $n_{12} = 2$. Then

$$\begin{aligned} I(f) &= \int_{-\infty}^{\infty} (x_1 y_2 - x_2 y_1)^2 f(x_1, y_1) f(x_2, y_2) dx_1 dy_1 dx_2 dy_2 \\ &= 2(m_{20} m_{02} - m_{11}^2). \end{aligned} \tag{16}$$

Similarly, for $N = 3$ and $n_{12} = 2, n_{13} = 2, n_{23} = 0$ we get

$$\begin{aligned} I(f) &= \int_{-\infty}^{\infty} (x_1 y_2 - x_2 y_1)^2 (x_1 y_3 - x_3 y_1)^2 f(x_1, y_1) \\ &\quad f(x_2, y_2) f(x_3, y_3) dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 \\ &= m_{20}^2 m_{04} - 4 m_{20} m_{11} m_{13} + 2 m_{20} m_{02} m_{22} \\ &\quad + 4 m_{11}^2 m_{22} - 4 m_{11} m_{02} m_{31} + m_{02}^2 m_{40}. \end{aligned} \tag{17}$$

The above idea has an analogy in graph theory. Each invariant generated by formula (14) can be represented by a graph, where each point (x_k, y_k) corresponds to one node and each cross-product T_{kj} corresponds to one edge of the graph. If $n_{kj} > 1$, the respective term $T_{kj}^{n_{kj}}$ corresponds to n_{kj} edges connecting k-th and j-th nodes. Thus, the number of nodes equals the degree of the invariant and the total number of the graph edges equals the weight w of the invariant. From the graph one can also learn about the orders of the moments the invariant is composed of and about its structure. The number of edges originating from each node equals the order of the moments involved. Each invariant of the form (14) is in fact a sum where each term is a product of certain number of moments. This number is constant for all terms of one invariant and is equal to the total number of the graph nodes. Particularly, for the invariants (15) and (16) the corresponding graphs are shown in Fig. 1.

Fig. 1. The graphs corresponding to the invariants (15) (left) and (16) (right)

Now one can see that the problem of derivation of the AMI's up to the given weight w is equivalent to generating all graphs with at least two nodes and at most w edges. This is a combinatorial task with exponential complexity but formally easy to implement. Unfortunately, most resulting graphs are useless because they generate invariants, which are dependent. Identifying and discarding them is very important but very complicated task.

There might be various kinds of dependencies in the set of all AMI's (i.e. in the set of all graphs). The invariant which equals to linear combinations of other invariants or of products of other invariants is called *reducible* invariant. Other invariants than reducible are called *irreducible* invariants.

Unfortunately, "irreducible" does not mean "independent" – there may be higher-order polynomial dependencies among irreducible invariants. Current methods [54] perfectly eliminate reducible invariants but identification of dependencies among irreducible invariants has not been resolved yet.

For illustration, let us consider AMI's up to the weight 10. Using the graph method we got, after discarding isomorphic graphs, 1519 AMI's in explicit forms. Then we applied the algorithms eliminating reducible invariants, which led to 362 irreducible invariants.

VI. Invariants to convolution

Two previous sections were devoted to the invariants with respect to transformation of spatial coordinates only. Now let us consider an imaging system with ideal geometry, i.e. $\tau(x, y) = (x, y)$, but suffering from non-ideal optical/radiometrical properties. Assuming the system is shift invariant, degradation operator $\mathcal{D}$ has a form of

$$g(x, y) = (f * h)(x, y), \tag{18}$$

where $h(x, y)$ is the point-spread function (PSF) of the system. This is a simple but realistic model of degradations introduced by out-of-focused camera ($h(x, y)$ has then a cylindrical shape), by camera and/or scene motion ($h(x, y)$ has a form of rectangular pulse), and by photographing through turbulent medium ($h(x, y)$ is then a Gaussian), to name a few. However, in real applications the PSF has more complex form because it use to be a composition of several degradation factors. Neither the shape nor the parameters of the PSF use to be known. This high-level uncertainty prevents us from solving (17) as an inverse problem. Although such attempts were published (see [55] or [56] for a basic survey), they did not yield satisfactory results.

In this section, we present functionals invariant to convolution with arbitrary centrosymmetric PSF (in image analysis literature they are often called "blur invariants" because common PSF's have a character of a low-pass filter). Blur invariants were firstly introduced by Flusser and Suk [34]. They have found successful applications in face recognition on out-of-focused photographs [35], in normalizing blurred images into the canonical forms [36], [37], in template-to-scene matching of satellite images [34], in blurred digit and character recognition [38], [13], in registration of images obtained by digital subtraction angiography [39] and in focus/defocus quantitative measurement [40].

The assumption of centrosymmetry is not a significant limitation of practical utilization of the method. Most real sensors and imaging systems, both optical and non-optical ones, have the PSF with certain degree of symmetry. In many cases they have even higher symmetry than the central one, such as axial or radial symmetry.

Principal theorem on convolution invariants is the following.

Theorem 3: Let functional $C : L_1(R^2) \times \mathbf{N_0} \times \mathbf{N_0} \to \mathbf{R}$ be defined as follows:
If $(p + q)$ is even then

$$C(p, q)^{(f)} = 0.$$

If $(p + q)$ is odd then

$$C(p, q)^{(f)} = \mu_{pq}^{(f)} - \frac{1}{\mu_{00}^{(f)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \binom{q}{m} C(p - n, q - m)^{(f)} \cdot \mu_{nm}^{(f)}.$$

Then

$$C(p, q)^{(f*h)} = C(p, q)^{(f)}$$

for any image function f, any non-negative integers p and q, and for any centrosymmetric PSF h.

Theorem 3 tells that blur invariants are recursively defined functionals consisting mainly from odd-order moments. Although they do not have straightforward "physical" interpretation, let us make a few notes to provide a better insight into their meaning. Any invariant (even different from those presented here) to convolution with a centrosymmetric PSF must give a constant response on centrosymmetric images. This is because any centrosymmetric image can be considered as a blurring PSF acting on delta-function. It can be proven that if f is centrosymmetric then $C(p, q)^{(f)} = 0$ for any p and q. The opposite implication is valid as well. Thus, what image properties are reflected by the $C(p, q)$'s? Let us consider a Fourier-based decomposition $f = f_c + f_a$ where f_c, f_a are centrosymmetric and antisymmetric components of f, respectively. Function f_a can be exactly recovered from odd-order moments of f (while even-order moments of f_a equal zero) and vice versa. A similar relation holds for the invariants $C(p, q)$. Thus, all $C(p, q)$'s reflect mainly properties of the antisymmetric component of the image, while all symmetric images are in their null-space.

The above theory was significantly generalized in [48] and [58], where invariants to N-fold symmetric PSF were introduced.

VII. Algorithms for moment computation

Since computing complexity of all moment invariants depends almost solely on the computing complexity of the moments themselves, we review efficient algorithms for moment calculation in a discrete space. Majority of the methods are focused on binary images because most of the applications work with binary images only but some fast algorithms for for graylevel images also exist.

In case of binary images, moment computation algorithms can be categorized into two groups referred as *decomposition methods* and *boundary-based* methods.

The idea behind all decomposition methods is the following. Having a binary object B, we decompose it into $K \geq 1$ blocks $B_1, B_2, \ldots, B_K$ such that $B_i \cap B_j = \emptyset$ for any $i \neq j$ and $B = \bigcup_{k=1}^{K} B_k$. Then

$$m_{pq}^{B} = \sum_{k=1}^{K} m_{pq}^{B_k}.$$

If we can calculate the moment of each block in $\mathcal{O}(1)$ time (as we can for rectangular blocks for instance) then the overall complexity of m_{pq}^{B} is $\mathcal{O}(K)$. If $K \ll MN$ the speed-up may be significant. The power of any decomposition method depends on our ability to decompose the object into a small number of blocks in a reasonable time. Individual decomposition methods differ from one another namely by the decomposition algorithms. Simple algorithms produce relatively high number of components but perform fast, while more sophisticated decomposition methods end up with small number of blocks but require more time. The choice of the method depends also on the number of the moments needed – for high number of moments the "overhead" time spent by the object decomposition becomes less significant while the number of blocks is an important parameter, and vice versa. A recent survey of decomposition methods along with their complexity comparison can be found in [30].

Boundary-based methods calculate the object moments just from the boundary, employing Green's theorem and/or boundary approximation by a polygon.

Among few methods for graylevel image moments, the *slicing technique* [59] is worth noting. The slice is a set of all pixels having the same intensity value, i.e. each slice is in fact a binary image. The original image can be considered to be a sum of the slices multiplied by their respective intensity. In this way we transform the problem of calculating graylevel moments to the previous task of calculating binary moments. The main problem is that the intensity slices are usually sparse or fragmented and cannot be effectively decomposed into blocks. The method can be modified such that the bit planes are used instead of the intensity slices and lower bit planes are replaced by a constant. This may lead to a significant speed-up with a negligible loss of accuracy.

In the discrete case, the integral in the moment definition must be replaced by a summation. The most common way (but not the only one) how to do that is to employ the rectangular (i.e. zero-order) method of numeric integration. Then (2) turns to the well-known form

$$m_{pq} = \sum_{x=1}^{N} \sum_{y=1}^{N} x^p y^q f_{ij}, \tag{19}$$

where N is the size of the image and f_{ij} are the grey levels of individual pixels.

VIII. Conclusion

This article presented a review of moment-based invariant functionals, their history, basic principles, and methods how to construct them. We demonstrated that invariant functionals can be used in image analysis as features for description and recognition of objects in degraded images.

Invariant-based approach is a significant step towards robust and reliable object recognition methods. It has a deep practical impact because many pattern recognition problems would not be solvable otherwise. In practice, image acquisition is always degraded by unrecoverable errors and the knowledge of invariants with respect to these errors is a crucial point.

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