B. Lagrange's Equations of Robot motion

Let us analyze robot-manipulator with n DOF. The kinetic energy may be described as a quadratic positive definite form

$$E_k = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} \tag{2}$$

and potential energy can be written as

$$E_p = -\sum_{j=1}^{n} m_j \mathbf{G}^T \mathbf{T}_0^j \mathbf{R}_{c,j} \tag{3}$$

If we use the equation (1), then equations of robot motion can be derived in the final form

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\,\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} \tag{4}$$

where $\mathbf{M}$ is an inertia matrix of the type $n \times n$, $\mathbf{q}$ is a vector of the type $n \times 1$, $\mathbf{C}$ is $n \times n$ matrix, which represents Coriolis and centrifugal forces, $\mathbf{g}$ is a vector of gravity influences, see [5], [6] and [7].

III. Hamiltonian Formalism

Physicists developed analytical mechanics in the form that can be used in all branches of physics. Hamilton's equations have a special meaning in quantum mechanics. Forces, velocities and accelerations are not as so important for study of elementary particles as energies and momentums. So, let us study the meaning of Hamiltonian formalism for a control purpose of robot-manipulators.

Physicists formulated Hamilton function and other notions with using components of positions and momentums. For aim of this paper, let a vector-matrix description be used. For example, generalized momentum p_j are defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, 2, \cdots, n \tag{5}$$

Let all vectors be defined classically as $\mathbf{p} = (p_1, \cdots, p_n)^{\mathrm{T}}$, $\mathbf{q} = (q_1, \cdots, q_n)^{\mathrm{T}}$, etc. Then, the relation (5) can be rewritten in the following form

$$\mathbf{p} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T \tag{6}$$

Form (6) is more suitable for our description of mathematical formulae. Similarly, the definition of Hamilton function is

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \quad \Rightarrow \quad H = \mathbf{p}^T \dot{\mathbf{q}} - L \tag{7}$$

Although the Lagrange function L is a function of vectors $\mathbf{q}$ ant its time derivation $\dot{\mathbf{q}}$, the Hamilton function is a function of $\mathbf{q}$ and $\mathbf{p}$. So, we can generally write

$$L = L(\mathbf{q}, \dot{\mathbf{q}}, t)\,, \qquad H = H(\mathbf{q}, \mathbf{p}, t) \tag{8}$$

Then the equations (1) can be rewritten

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}}\right)^T = \mathbf{F}\,. \tag{9}$$

The partial derivation of H from (7) with using (9) gives

$$\frac{\partial H}{\partial \mathbf{q}} = -\frac{\partial L}{\partial \mathbf{q}} \tag{10}$$

and hence from (6) and (9), we obtain the equation

$$\left(\frac{\partial H}{\partial \mathbf{q}}\right)^T = \mathbf{F} - \dot{\mathbf{p}} \tag{11}$$

Similarly the partial derivation of (7) with respect to vector $\mathbf{p}$ and using (9) and (6) follows to

$$\left(\frac{\partial H}{\partial \mathbf{p}}\right)^T = \dot{\mathbf{q}} \tag{12}$$

Both equations (11) and (12) may be rewritten in the form

$$\dot{\mathbf{q}} = \left(\frac{\partial H}{\partial \mathbf{p}}\right)^T, \qquad \dot{\mathbf{p}} = \mathbf{F} - \left(\frac{\partial H}{\partial \mathbf{q}}\right)^T \tag{13}$$

Set of equations (13) is a vector representation of well-known Hamilton's equations. These equations are usually written as components of the vectors defined in (13).

A. Main Idea

The momentums and moments of movements are very different in arbitrary configurations of robots. The classical methods of robot control use information on position and velocity. It predetermines, that control methods based on feedback of positions and generalized momentums, will be different in results. In robotics the generalized momentum is really momentum or moment of movement, respectively. Hence, Hamiltonian formalism may be better for aims of robot control than the Lagrangian one. In the following part we develop analogical differential equations of robot dynamics with using Hamilton's equations.

B. Differential Equations of Robot Dynamics

Arbitrary robot may be considered as the time invariant system. Then, it is well known that the Hamiltonian (7) is full energy that is the sum of kinetic and potential energies. Because the Lagrangian L depends on positions and velocities and Hamiltonian depends on positions and generalized momentums, we can rewrite the relations (8) as

$$L(\mathbf{q},\dot{\mathbf{q}}) = E_k(\mathbf{q},\dot{\mathbf{q}}) - E_p(\mathbf{q}) \tag{14}$$

$$H(\mathbf{q},\mathbf{p}) = E_k(\mathbf{q},\mathbf{p}) + E_p(\mathbf{q}) \tag{15}$$

If (10) is used, then we can derive a very interesting result

$$\frac{\partial E_k(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}} = -\frac{\partial E_k(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}} \tag{16}$$

The partial derivative of (14) yields

$$\frac{\partial L}{\partial \mathbf{q}} = \frac{\partial E_k(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}} - \mathbf{g}^T(\mathbf{q}) \tag{17}$$

where the derivative of the potential energy is

$$\frac{\partial E_p}{\partial \mathbf{q}} = \mathbf{g}^T(\mathbf{q}) \tag{18}$$

If we use (16), (17) and (10), then the vector $\dot{\mathbf{p}}$ in (13) yields the result

$$\dot{\mathbf{p}} = \mathbf{F} - \mathbf{g}(\mathbf{q},t) - \left(\frac{\partial E_k(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}}\right)^T \tag{19}$$

The relations (2) and (6) yield

$$\mathbf{p}^T = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial E_k}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}^T \mathbf{M} \tag{20}$$

Now we obtain from (20) the relation

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{21}$$

If (21) is substituted into (2), then we obtain the expression of the kinetic energy in the space ($\mathbf{q}$, $\mathbf{p}$)

$$E_k(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}^T\mathbf{M}^{-1}(\mathbf{q})\,\mathbf{p} \tag{22}$$

Expression (22) is right for Hamiltonian H. Equation (13) is then the same as (21). Now, equations (19) and (21) fully describe the robot dynamics in the Hamiltonian formalism.

C. Reduction for Robot Control

Let us define a skew symmetric matrix $\mathbf{S}$ [7] as follows

$$S_{ij} = \frac{1}{2}\sum_{k=1}^{n}\dot{q}_k\left(\frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right) \tag{23}$$

It can be proved that then holds

$$\mathbf{S}\,\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{M}}\,\dot{\mathbf{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}^T\mathbf{M}\,\dot{\mathbf{q}}\right)\right) \tag{24}$$

If we use (2) and (16), then the robot dynamic equations may be rewritten into the following compact form

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{25}$$

$$\dot{\mathbf{p}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{p} - \mathbf{g} + \mathbf{u} \tag{26}$$

where, the vector $\mathbf{F}$ was replaced by $\mathbf{u}$. The vector $\mathbf{u}$ will be a control vector similarly as in (4). The equations (25) and (26) are the final equations that describe the dynamics of robot motion.

IV. Position Control

This chapter considers conventional problems of robot control. For the simplicity, all following methods will be demonstrated in the join space that is for joint space control. The task of space control or motion and force control will be omitted. Since the space represented by coordinates ($\mathbf{q}$, $\mathbf{p}$) is Hamiltonian phase space [2], [3], we will call the control in this space simply *control in Hamilton space*. On the other hand, the control in Lagrangian phase space [4], that is represented by coordinates $(\mathbf{q},\dot{\mathbf{q}})$ [1] - [4], will be simply called *control in Lagrange space*.

A. Tracking Control in Hamiltonian Space

Let the following transformation vectors be defined as

$$\mathbf{e} = \mathbf{q}_d - \mathbf{q}\,, \quad \mathbf{z} = \mathbf{M}\,(\dot{\mathbf{e}} + \mathbf{A}\mathbf{e})\,, \quad \mathbf{y} = \mathbf{p} - \mathbf{z} \tag{38}$$

The control system described by (25) and (26) let be controlled by the control law

$$\mathbf{u} = \dot{\mathbf{y}} - \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{y} + \mathbf{g} - \mathbf{B}\,\mathbf{z} \tag{39}$$

where matrices $\mathbf{A}$ and $\mathbf{B}$ are suitably chosen tuning matrices. From (26) and (39), feedback equation can be obtained as

$$\dot{\mathbf{z}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{z} - \mathbf{B}\,\mathbf{z} \tag{40}$$

Let us define a quadratic form

$$W = \frac{1}{2}\mathbf{z}^T\mathbf{M}^{-}\ \mathbf{z} \geq 0 \tag{41}$$

where its time derivative along trajectory of (40) leads to

$$\dot{W} = -\mathbf{z}^T\mathbf{M}^{-}\ \mathbf{B}\,\mathbf{z} \leq\ 0 \tag{42}$$

The multiplication of matrices in the quadratic form (42) are positive definite. Now we can proceed as in [16] and derive exponential stability for robot control, where vectors **z**, **q** and its appropriate time derivatives are exponential stable considering positive definite matrices **A** and **B** as follows

$$\mathbf{A} = diag\,(a_1, \cdots, a_n), \ \ \mathbf{B} = diag\,(b_1, \cdots, b_n) \tag{43}$$

B. Tracking control in Lagrangian space

Like in the previous part let us define new vectors **y**, **z**

$$\mathbf{e} = \mathbf{q}_d - \mathbf{q}\,, \quad \mathbf{z} = \dot{\mathbf{e}} + \mathbf{A}\mathbf{e}\,, \quad \dot{\mathbf{y}} = \dot{\mathbf{q}} - \mathbf{z} \tag{44}$$

The control law for controlled system is defined by

$$\mathbf{u} = \mathbf{M}\,\ddot{\mathbf{y}} + \mathbf{C}\,\dot{\mathbf{y}} + \mathbf{g} - \mathbf{B}\,\mathbf{z} \tag{45}$$

If this control law is substituted into (4), then feedback is:

$$\dot{\mathbf{z}} = -\,\mathbf{M}^{-1}\mathbf{C}\,\mathbf{z} - \mathbf{M}^{-1}\mathbf{B}\,\mathbf{z} \tag{46}$$

Note, that asymptotical stability of the control process can be proved as in the previous part.

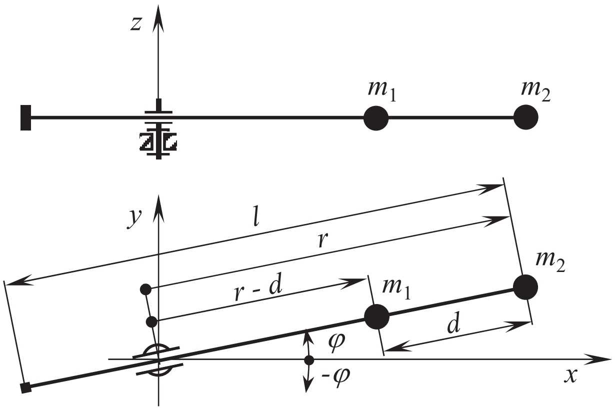

Fig. 1. Scheme of two-mass robot arm system.

VI. Solved Examples

Let us consider a two-mass robot-arm system in Fig. 1. It consists of the robot arm of length l with negligible mass in comparison with two masses m_1 and m_2 outlying for a distance d. The system has 2 DOF with corresponding two generalized coordinates φ and r and two momentums p_1 and p_2. The arm is led through a prismatic joint connected to the basic frame by a rotational joint. The system can be described in Hamiltonian formalism by (25) and (26) with parameters:

$$\begin{gathered}
\mathbf{q} = \begin{bmatrix}\varphi \\ r\end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix}p_1 \\ p_2\end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix}0 \\ 0\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix}u_1 \\ u_2\end{bmatrix} \\
\mathbf{M} = \begin{bmatrix}(m_1(r-d)^2 + m_2 r^2)^{-1} & 0 \\ 0 & (m_1 + m_2)^{-1}\end{bmatrix} \\
\left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right) = \begin{bmatrix}(m_1 + m_2)r\dot{r} - m_1 d\dot{r}\ , & -\dot{\varphi}((m_1 + m_2)r - m_1 d) \\ \dot{\varphi}((m_1 + m_2)r - m_1 d), & 0\end{bmatrix}
\end{gathered} \tag{47}$$

which were used for the control design both PD Control and model-based exponentially stable control. The testing trajectory (Fig. 2, ‘quatrefoil’) of mass point m_2 is composed according to [18] for maximum tangential velocity $v_t = 5\text{ms}^{-1}$.

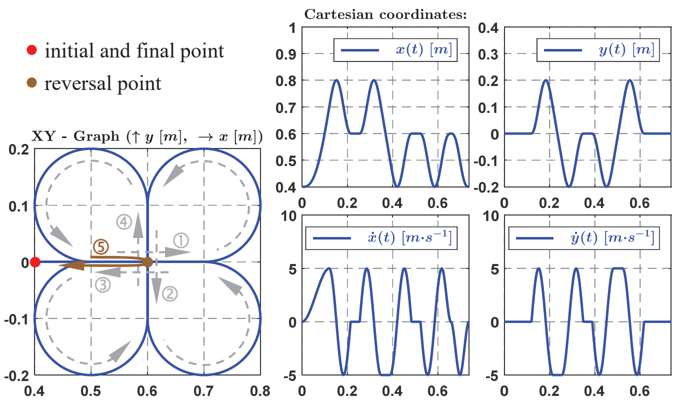

Fig. 2. Testing trajectory of mass point m_2 for tracking control.

Illustrative comparative examples of the tracking control applied to the robot arm system (Fig. 1), given by parameters $l = 1\text{m}$, $d = 0.2\text{m}$, $m_1 = m_2 = 10\text{kg}$, are shown in joint Fig. 3.

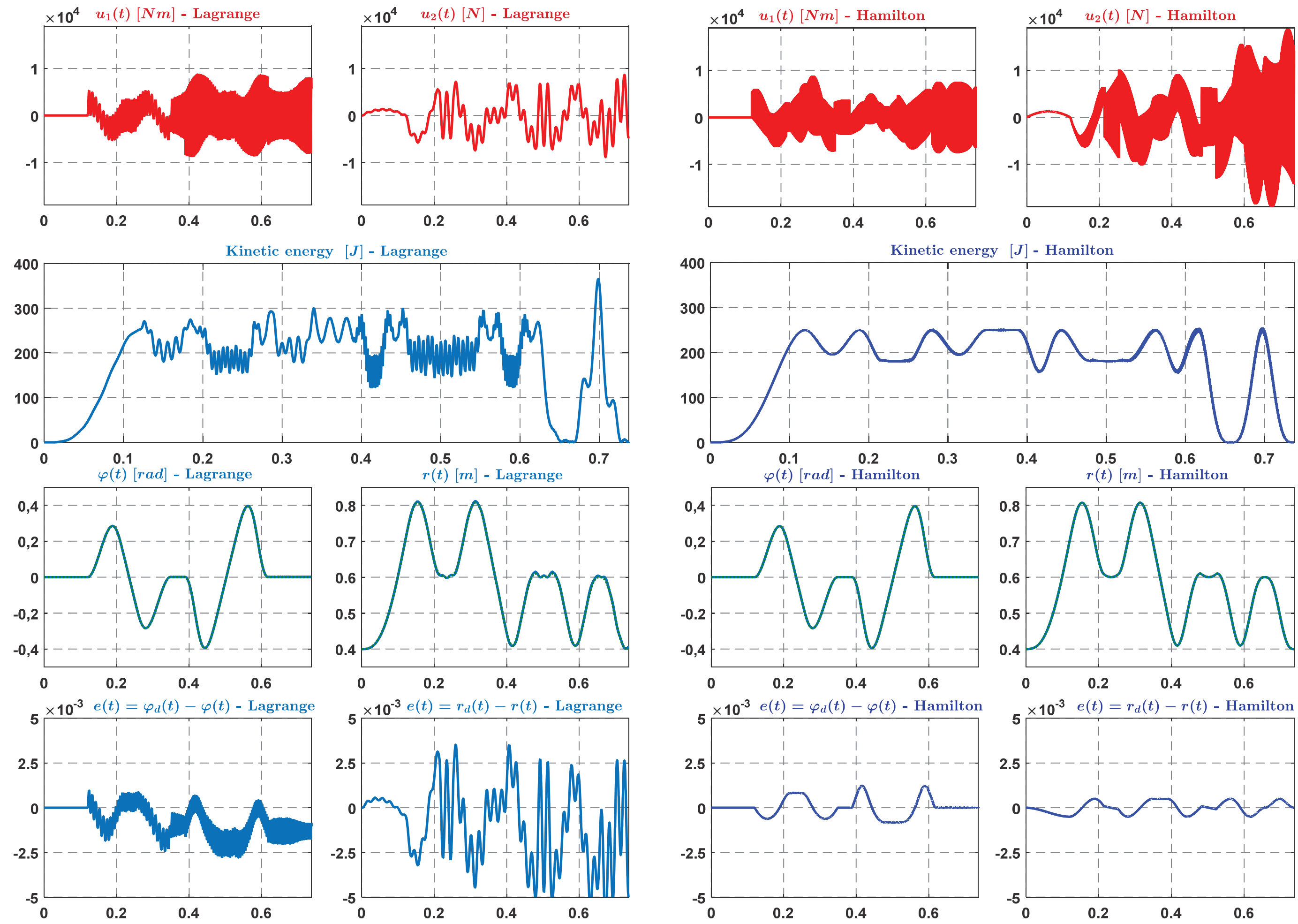

Fig. 3. Tracking control of system (Fig. 1) along trajectory (Fig. 2): time histories [s] of control actions, kinetic energy, generalized coordinates and their errors.

Fig. 3 shows the comparison of the tracking control in both Lagrandian and Hamiltonian formalisms for the identical system. Even though tracking of desired values looks similar, time histories of control errors (four bottom subfigures) and kinetic energies demonstrate better behavior of the control process in Hamiltonian space using different descriptive parameters.

Conclusion

The paper shows strong features of Hamiltonian formalism useful for efficient robot control. It is obvious that Hamiltonian formalism considers dynamics and internal energy distribution in robotic systems more naturally by means of specific quantities – momentums. It is a significant finding for future work.

References

[1] H. Golstein, Classical Mechanics, Addison-Wesley, Camb., 1950.

[2] R. Abraham and J. E. Marsden, Foundations of Mechanics, Benjamin Cummings, Reading, MA, 1978.

[3] A. Fasano and S. Marmi, Analytical Mechanics. Oxford Press, 2002.

[4] D. T. Greenwood, Advanced Dynamics, Cambridge Univ. Press, 2003.

[5] B. Siciliano, O. Khatib (Eds.). Handbook of Robotícs. Springer, 2008.

[6] C. Samson and M. Le Borgne and B. Espiau, Robot Control, The task Function Approach, Clarendorf Press - Oxford, 1991.

[7] S. Arimoto, Control Theory of Non-linear Mech. Systems, Oxford, 1996.

[8] V. Záda, Robotics, Mathematical Aspects of Analysis and Control. TU of Liberec, 2012. ISBN 978/80/7372/882/3. In Czech.

[9] I. D. Landau, R. Horowitz, “Synthesis of adaptive controllers for robot manipulators using a passive feedback system approach,” Proc. IEEE Int. Conf. on Robotics and Automation, 1988, USA, pp. 1028-1033.

[10] H. Berghuis, “Model-based robot control: from theory to practice,” Ph.D. Thesis, University of Twente, Enschende, The Netherlands, 1993.

[11] M Takegaki, S Arimoto, “A new feedback method for dynamic control of manipulators,” Trans. of ASME, J. of Dynamic Sytems, Measurement and control, Vol.102, pp. 119-125, 1981.

[12] J. T. Wen, D. Bayard, “New class of control laws dor robotic manipulators,” Int. J. Control, Vol.47(5), pp.1361-1385, 1988.

[13] J. P. LaSalle, “Some extension of Lyapunov’s second method,” IRE Trabsactions on Circuit Theory, 7, p. 520-527, 1960.

[14] J. Böhm, K. Belda, and M. Valášek, “Study of control of planar redundant paralel robot,” Proc. IASTED Int. Conf. MIC, 2001, pp. 694-699.

[15] C. Samson, “Robust control of a class of non/linear systems and applications to robotics”, Int. J. of Adaptive Control and Signal Processing, Vol. 1, 1987, pp. 49-68.

[16] V. Záda, “Exponential stabilization of open kinematic structures,” in: 1st Int. Conf. APLIMAT 2002, Slovak University of Technology, Bratislava, Slovak Republic, February, 2002 pp. 445-448.

[17] V. Záda, “Exponential stabilization of robot arms,” RAAD-Robotics in Alphe-Adria-Danube Region, Hungaria, 2006. pp. 218-222.

[18] K. Belda, P. Novotný, “Path simulator for machine tools and robots,” Proc. of the 17th IEEE Int. Conf. MMAR, 2012, pp. 373-378.