Mathematical Modeling of Industrial Robots Based on Hamiltonian Mechanics

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***Abstract*—The paper deals with an advanced mathematical modeling for robot motion control. The explanation focuses on a composition of suitable mathematical models of robot dynamics intended for control design. Instead of usual Lagrangian formulation of dynamics, this paper presents robot dynamics by Hamiltonian formulation that leads to different physical descriptive quantities considered for control design. In the case of Hamiltonian formulation, a momentum is such physical quantity. In the paper, as representative control approaches, PD control with gravity compensation and model-oriented Lyapunov-based control are considered. The control approaches considering Hamiltonian formulation are demonstrated for simplicity on two-mass robot-arm system. However, the presented results are generally applicable e.g. to usual articulated multi-purpose industrial robots-manipulators.**

Keywords—robot-manipulator; Hamiltonian formalism; mathematical modeling; PD control; model-oriented motion control

I. Introduction

In practice of mechanical engineers, conventional Newton's mechanics is predominantly used. For such formalism, a vector oriented notion of forces is typical. The vector description is not only one possibility that moreover sometimes is insufficient and unsuitable. Force interactions in any system can be also described by scalar functions in more general configuration space. Let us recall Lagrangian or Hamiltonian functions [1] - [3] corresponding to equations of motion: Lagrange's equations or Hamilton's equations, respectively.

The majority of engineers and scientists use Lagrange's equations for expression of robot dynamics [4], [5]. These equations are then used for realization of robot control. In general, it is assumed that there are given some limits for positions, velocities and accelerations, respectively. Usually, there are given some limits for control torques, too. The limits of velocities are usually constant for all position configurations of robots without respecting of the fact that inertia moments are very different for arbitrary configurations. Controllers for such robot control approach have usually to manage complicated control structures, to control strong nonlinear systems with inconvenient limits [5], [6] and [7].

The Lagrangian formalism is based on kinetic and potential energies and on a phase space formed by positions and velocities. Momentums are not respected there in this formalism. However, in robot-manipulator dynamics, all momentums change very quickly, usually in the rate 1/10 [7]. Hence, study of control methods is interesting not only on Lagrangian formalism [8], but also on Hamiltonian one. It was investigated as the approach that uses the property of passivity of the robot [9], [10]. Such approach can modify the natural energy of the robot so that it can satisfy the desired objectives (position or tracking control). Hamiltonian formalism with using a modified Hamiltonian [11] was used as new function there. Various choices are possible for the desired potential energy function [12]. An alternative approach for potential function is in [11].

Here, in this contribution, we would like to find an answer for the following question: *What are key features of Lagrangian and Hamiltonian formalism for robot control?* Hence, we shall omit such changes as [11], but shall compare almost the same algorithms on the same problems of robot controls defined in Lagrangian or Hamiltonian configuration spaces.

The paper is organized as follows. Section II and III summary dynamic models based on Lagrangian and Hamiltonian formalisms. Section IV explains PD control with gravity compensation. Section V explores model-oriented Lyapunov-based control. Finally, Section VI shows solved examples for two-mass robot-arm system.

II. Lagrangian Formalism

A. Lagrange's Equations

Lagrange's equations of classical mechanics [1] - [3] are often used for description of non-trivial mechanical systems. These equations are usually described by the following form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = F_j, \quad j = 1, 2, \cdots, n \tag{1}$$

where n is a number of degrees of freedom (DOF); a scalar function $L = E_k - E_p$ is Lagrange's function, E_k is kinetic energy and E_p is potential energy; F_j are generalized forces and q_j generalized coordinates. For technical applications [4], [8], the generalized forces F_j represent only a sum of non-conservative forces and complementarily conservative forces are represented by the potential energy E_p.

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B. Lagrange's Equations of Robot motion

Let us analyze robot-manipulator with n DOF. The kinetic energy may be described as a quadratic positive definite form

$$E_k = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} \tag{2}$$

and potential energy can be written as

$$E_p = -\sum_{j=1}^{n} m_j \mathbf{G}^T \mathbf{T}_0^j \mathbf{R}_{c,j} \tag{3}$$

If we use the equation (1), then equations of robot motion can be derived in the final form

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\,\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{u} \tag{4}$$

where $\mathbf{M}$ is an inertia matrix of the type $n \times n$, $\mathbf{q}$ is a vector of the type $n \times 1$, $\mathbf{C}$ is $n \times n$ matrix, which represents Coriolis and centrifugal forces, $\mathbf{g}$ is a vector of gravity influences, see [5], [6] and [7].

III. Hamiltonian Formalism

Physicists developed analytical mechanics in the form that can be used in all branches of physics. Hamilton's equations have a special meaning in quantum mechanics. Forces, velocities and accelerations are not as so important for study of elementary particles as energies and momentums. So, let us study the meaning of Hamiltonian formalism for a control purpose of robot-manipulators.

Physicists formulated Hamilton function and other notions with using components of positions and momentums. For aim of this paper, let a vector-matrix description be used. For example, generalized momentum p_j are defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, 2, \cdots, n \tag{5}$$

Let all vectors be defined classically as $\mathbf{p} = (p_1, \cdots, p_n)^T$, $\mathbf{q} = (q_1, \cdots, q_n)^T$, etc. Then, the relation (5) can be rewritten in the following form

$$\mathbf{p} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T \tag{6}$$

Form (6) is more suitable for our description of mathematical formulae. Similarly, the definition of Hamilton function is

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \quad \Rightarrow \quad H = \mathbf{p}^T \dot{\mathbf{q}} - L \tag{7}$$

Although the Lagrange function L is a function of vectors $\mathbf{q}$ ant its time derivation $\dot{\mathbf{q}}$, the Hamilton function is a function of $\mathbf{q}$ and $\mathbf{p}$. So, we can generally write

$$L = L(\mathbf{q}, \dot{\mathbf{q}}, t)\,, \qquad H = H(\mathbf{q}, \mathbf{p}, t) \tag{8}$$

Then the equations (1) can be rewritten

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}}\right)^T = \mathbf{F}\,. \tag{9}$$

The partial derivation of H from (7) with using (9) gives

$$\frac{\partial H}{\partial \mathbf{q}} = -\frac{\partial L}{\partial \mathbf{q}} \tag{10}$$

and hence from (6) and (9), we obtain the equation

$$\left(\frac{\partial H}{\partial \mathbf{q}}\right)^T = \mathbf{F} - \dot{\mathbf{p}} \tag{11}$$

Similarly the partial derivation of (7) with respect to vector $\mathbf{p}$ and using (9) and (6) follows to

$$\left(\frac{\partial H}{\partial \mathbf{p}}\right)^T = \dot{\mathbf{q}} \tag{12}$$

Both equations (11) and (12) may be rewritten in the form

$$\dot{\mathbf{q}} = \left(\frac{\partial H}{\partial \mathbf{p}}\right)^T, \qquad \dot{\mathbf{p}} = \mathbf{F} - \left(\frac{\partial H}{\partial \mathbf{q}}\right)^T \tag{13}$$

Set of equations (13) is a vector representation of well-known Hamilton's equations. These equations are usually written as components of the vectors defined in (13).

A. Main Idea

The momentums and moments of movements are very different in arbitrary configurations of robots. The classical methods of robot control use information on position and velocity. It predetermines, that control methods based on feedback of positions and generalized momentums, will be different in results. In robotics the generalized momentum is really momentum or moment of movement, respectively. Hence, Hamiltonian formalism may be better for aims of robot control than the Lagrangian one. In the following part we develop analogical differential equations of robot dynamics with using Hamilton's equations.

B. Differential Equations of Robot Dynamics

Arbitrary robot may be considered as the time invariant system. Then, it is well known that the Hamiltonian (7) is full energy that is the sum of kinetic and potential energies. Because the Lagrangian L depends on positions and velocities and Hamiltonian depends on positions and generalized momentums, we can rewrite the relations (8) as

$$L(\mathbf{q}, \dot{\mathbf{q}}) = E_k(\mathbf{q}, \dot{\mathbf{q}}) - E_p(\mathbf{q}) \tag{14}$$

$$H(\mathbf{q}, \mathbf{p}) = E_k(\mathbf{q}, \mathbf{p}) + E_p(\mathbf{q}) \tag{15}$$

If (10) is used, then we can derive a very interesting result

$$\frac{\partial E_k(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}} = -\frac{\partial E_k(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} \tag{16}$$

The partial derivative of (14) yields

$$\frac{\partial L}{\partial \mathbf{q}} = \frac{\partial E_k(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} - \mathbf{g}^T(\mathbf{q}) \tag{17}$$

where the derivative of the potential energy is

$$\frac{\partial E_p}{\partial \mathbf{q}} = \mathbf{g}^T(\mathbf{q}) \tag{18}$$

If we use (16), (17) and (10), then the vector $\dot{\mathbf{p}}$ in (13) yields the result

$$\dot{\mathbf{p}} = \mathbf{F} - \mathbf{g}(\mathbf{q}, t) - \left(\frac{\partial E_k(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}}\right)^T \tag{19}$$

The relations (2) and (6) yield

$$\mathbf{p}^T = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial E_k}{\partial \dot{\mathbf{q}}} = \dot{\mathbf{q}}^T \mathbf{M} \tag{20}$$

Now we obtain from (20) the relation

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{21}$$

If (21) is substituted into (2), then we obtain the expression of the kinetic energy in the space ($\mathbf{q}$, $\mathbf{p}$)

$$E_k(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^T \mathbf{M}^{-1}(\mathbf{q})\,\mathbf{p} \tag{22}$$

Expression (22) is right for Hamiltonian H. Equation (13) is then the same as (21). Now, equations (19) and (21) fully describe the robot dynamics in the Hamiltonian formalism.

C. Reduction for Robot Control

Let us define a skew symmetric matrix $\mathbf{S}$ [7] as follows

$$S_{ij} = \frac{1}{2}\sum_{k=1}^{n} \dot{q}_k \left(\frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right) \tag{23}$$

It can be proved that then holds

$$\mathbf{S}\,\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{M}}\,\dot{\mathbf{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}^T \mathbf{M}\,\dot{\mathbf{q}}\right)\right) \tag{24}$$

If we use (2) and (16), then the robot dynamic equations may be rewritten into the following compact form

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{p} \tag{25}$$

$$\dot{\mathbf{p}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{p} - \mathbf{g} + \mathbf{u} \tag{26}$$

where, the vector $\mathbf{F}$ was replaced by $\mathbf{u}$. The vector $\mathbf{u}$ will be a control vector similarly as in (4). The equations (25) and (26) are the final equations that describe the dynamics of robot motion.

IV. Position Control

This chapter considers conventional problems of robot control. For the simplicity, all following methods will be demonstrated in the join space that is for joint space control. The task of space control or motion and force control will be omitted. Since the space represented by coordinates ($\mathbf{q}$, $\mathbf{p}$) is Hamiltonian phase space [2], [3], we will call the control in this space simply *control in Hamilton space*. On the other hand, the control in Lagrangian phase space [4], that is represented by coordinates $(\mathbf{q}, \dot{\mathbf{q}})$ [1] - [4], will be simply called *control in Lagrange space*.

A. Position Control in Hamiltonian Space

Let us consider the simplest problem of position control. The controlled system (robot) is described by equations (25) and (26). Let the controller (regulator) be described as

$$\mathbf{u} = \mathbf{g} + \mathbf{A}\mathbf{e} + \mathbf{B}\,(\mathbf{0} - \mathbf{p}) \tag{27}$$

where $\mathbf{A}$ and $\mathbf{B}$ are positive definite diagonal matrices, $\mathbf{g}$ is gravity compensation for the robot and $\mathbf{e} = \mathbf{q}_d - \mathbf{q}$. This approach may be called *PD control with full gravity compensation*. The analogical versions for robot control described by Lagrange's equations are in [5], [7] etc.

The target position $\mathbf{q}_d$ in terms of joint coordinates is fixed. Consider a set point control problem, in which the posture of the robot arm is allowed to asymptotically approach the target state ($\mathbf{q}_d$, $\mathbf{p}$) = ($\mathbf{q}_d$, $\mathbf{0}$). Substitution the control law (27) into (26) yields

$$\dot{\mathbf{p}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{p} + \mathbf{A}\,\mathbf{e} + \mathbf{B}\,(\mathbf{0} - \mathbf{p}) \tag{28}$$

Let us consider a quadratic form

$$W = \frac{1}{2}\mathbf{p}^T\mathbf{M}^{-1}\mathbf{p} + \frac{1}{2}\mathbf{e}^T\mathbf{A}\,\mathbf{e} \;\geq\; 0 \tag{29}$$

The time derivation of (29) along the trajectory given by (28), and with using that $\mathbf{S}$ is skew symmetric, yields

$$\dot{W} = -\,\mathbf{p}^T\mathbf{M}^{-1}\mathbf{B}\,\mathbf{p} \;\leq\; 0 \tag{30}$$

Since $\mathbf{B}$ is a diagonal positive definite matrix and the inverse of matrix $\mathbf{M}$ is positive definite, then the multiplication of these matrices in (30) is positive definite, so the quadratic form (30) is negative semi-definite. Then by Lyapunov theory of stability the control process is stable. However, we would like to obtain a better result.

We have now to prove that as $\mathbf{p} = \mathbf{0}$, the robot does not reach a configuration $\mathbf{q} \neq \mathbf{q}_d$. This can be done by the La Salle invariant set theorem [13]. The set of points in the neighborhood of the equilibrium that satisfies

$$\dot{W} = 0 \tag{31}$$

is such that $\mathbf{p} = \mathbf{0}$ and $\dot{\mathbf{p}} = \mathbf{0}$. From (28), it follows that $\mathbf{e} = \mathbf{0}$. Hence, the equilibrium point given by $\mathbf{e} = \mathbf{0}$, $\mathbf{p} = \mathbf{0}$ is the only possible equilibrium for the controlled system and is the largest invariant set in that set of points. Hence, the equilibrium point is asymptotically stable.

B. Comparison with the Control in Lagrangian Space

Consider the equation (4), where

$$\mathbf{C} = \left(\frac{1}{2}\dot{\mathbf{M}} + \mathbf{S}\right) \tag{32}$$

The PD control law is given by [5] - [7] etc.

$$\mathbf{u} = \mathbf{g} + \mathbf{A}\,\mathbf{e} + \mathbf{B}\,(\mathbf{0} - \dot{\mathbf{q}}) \tag{33}$$

Let us consider a quadratic form

$$V = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\,\dot{\mathbf{q}} + \frac{1}{2}\mathbf{e}^T\mathbf{A}\,\mathbf{e} \;\geq\; 0 \tag{34}$$

Similarly to (30) we obtain the time derivation of V

$$\dot{V} = -\,\dot{\mathbf{q}}^T\mathbf{B}\,\dot{\mathbf{q}} \;\leq\; 0 \tag{35}$$

Let us consider the case, where at time $t = 0$ is $W = V$. Using equation (26) the equation (30) can be rewritten

$$\dot{W} = -\,\dot{\mathbf{q}}^T\mathbf{B}\,\mathbf{M}\,\dot{\mathbf{q}} \;\leq\; 0 \tag{36}$$

and hence the relations (35) and (36) follow to

$$\dot{W} \leq \dot{V} \;\Leftrightarrow\; \dot{\mathbf{q}}^T\mathbf{B}\,(\mathbf{M} - \mathbf{E})\,\dot{\mathbf{q}} \;\geq\; 0 \tag{37}$$

$\mathbf{B}$ is positive definite, hence $\dot{W} \leq \dot{V}$ if and only if the matrix $\mathbf{M} - \mathbf{E}$ is positive semidefinite, too. If this matrix is positive definite the trajectory of W is under the trajectory of V in time.

Note, that W and V are positive definite. If W and V represent the quality of control, then the control given by (27) in Hamilton phase space is better than the control (33) in Lagrange space for the same setting of the parameters in matrices $\mathbf{A}$ and $\mathbf{B}$.

V. Tracking Control

The tracking control problem in the joint space consists of a given time-varying trajectory $\mathbf{q}_d(t)$ and its derivatives. Several schemes for performing these objectives do exist.

The well-known is *inverse dynamic control* [14] and *computed torque control* [15] Any others are the *passivity based control* and *the Lyapunov-based control* [12]. Here, there is introduced one of Lyapunov-based control [16] that is model-based control with state transformation [17] just lending to exponentially stable control behavior in view of Lyapunov theory of stability.

A. Tracking Control in Hamiltonian Space

Let the following transformation vectors be defined as

$$\mathbf{e} = \mathbf{q}_d - \mathbf{q}\,, \quad \mathbf{z} = \mathbf{M}\,(\dot{\mathbf{e}} + \mathbf{A}\mathbf{e})\,, \quad \mathbf{y} = \mathbf{p} - \mathbf{z} \tag{38}$$

The control system described by (25) and (26) let be controlled by the control law

$$\mathbf{u} = \dot{\mathbf{y}} - \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{y} + \mathbf{g} - \mathbf{B}\,\mathbf{z} \tag{39}$$

where matrices **A** and **B** are suitably chosen tuning matrices. From (26) and (39), feedback equation can be obtained as

$$\dot{\mathbf{z}} = \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right)\mathbf{M}^{-1}\mathbf{z} - \mathbf{B}\,\mathbf{z} \tag{40}$$

Let us define a quadratic form

$$W = \frac{1}{2}\mathbf{z}^T\mathbf{M}^{-1}\,\mathbf{z} \geq 0 \tag{41}$$

where its time derivative along trajectory of (40) leads to

$$\dot{W} = -\mathbf{z}^T\mathbf{M}^{-1}\,\mathbf{B}\,\mathbf{z} \leq 0 \tag{42}$$

The multiplication of matrices in the quadratic form (42) are positive definite. Now we can proceed as in [16] and derive exponential stability for robot control, where vectors **z**, **q** and its appropriate time derivatives are exponential stable considering positive definite matrices **A** and **B** as follows

$$\mathbf{A} = diag\,(a_1, \cdots, a_n), \;\; \mathbf{B} = diag\,(b_1, \cdots, b_n) \tag{43}$$

B. Tracking control in Lagrangian space

Like in the previous part let us define new vectors **y**, **z**

$$\mathbf{e} = \mathbf{q}_d - \mathbf{q}\,, \quad \mathbf{z} = \dot{\mathbf{e}} + \mathbf{A}\mathbf{e}\,, \quad \dot{\mathbf{y}} = \dot{\mathbf{q}} - \mathbf{z} \tag{44}$$

The control law for controlled system is defined by

$$\mathbf{u} = \mathbf{M}\,\ddot{\mathbf{y}} + \mathbf{C}\,\dot{\mathbf{y}} + \mathbf{g} - \mathbf{B}\,\mathbf{z} \tag{45}$$

If this control law is substituted into (4), then feedback is:

$$\dot{\mathbf{z}} = -\,\mathbf{M}^{-1}\mathbf{C}\,\mathbf{z} - \mathbf{M}^{-1}\mathbf{B}\,\mathbf{z} \tag{46}$$

Note, that asymptotical stability of the control process can be proved as in the previous part.

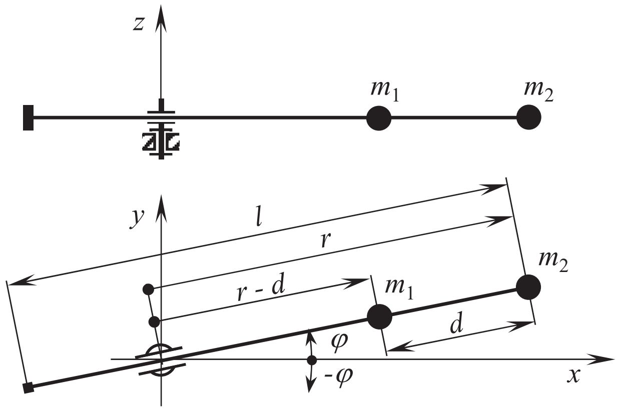

Fig. 1. Scheme of two-mass robot arm system.

VI. Solved Examples

Let us consider a two-mass robot-arm system in Fig. 1. It consists of the robot arm of length l with negligible mass in comparison with two masses m_1 and m_2 outlying for a distance d. The system has 2 DOF with corresponding two generalized coordinates φ and r and two momentums p_1 and p_2. The arm is led through a prismatic joint connected to the basic frame by a rotational joint. The system can be described in Hamiltonian formalism by (25) and (26) with parameters:

$$\begin{gathered}\mathbf{q} = \begin{bmatrix}\varphi \\ r\end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix}p_1 \\ p_2\end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix}0 \\ 0\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix}u_1 \\ u_2\end{bmatrix} \\ \mathbf{M} = \begin{bmatrix}(m_1(r-d)^2 + m_2 r^2)^{-1} & 0 \\ 0 & (m_1 + m_2)^{-1}\end{bmatrix} \\ \left(\frac{1}{2}\dot{\mathbf{M}} - \mathbf{S}\right) = \begin{bmatrix}(m_1+m_2)r\dot{r} - m_1 d\,\dot{r} & ,-\dot{\varphi}((m_1+m_2)r - m_1 d) \\ \dot{\varphi}((m_1+m_2)r - m_1 d), & 0\end{bmatrix}\end{gathered} \tag{47}$$

which were used for the control design both PD Control and model-based exponentially stable control. The testing trajectory (Fig. 2, 'quatrefoil') of mass point m_2 is composed according to [18] for maximum tangential velocity $v_t = 5\text{ms}^{-1}$.

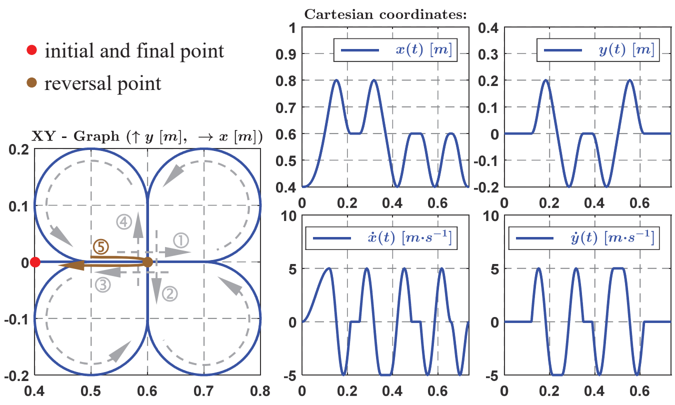

Fig. 2. Testing trajectory of mass point m_2 for tracking control.

Illustrative comparative examples of the tracking control applied to the robot arm system (Fig. 1), given by parameters $l = 1\text{m}$, $d = 0.2\text{m}$, $m_1 = m_2 = 10\text{kg}$, are shown in joint Fig. 3.

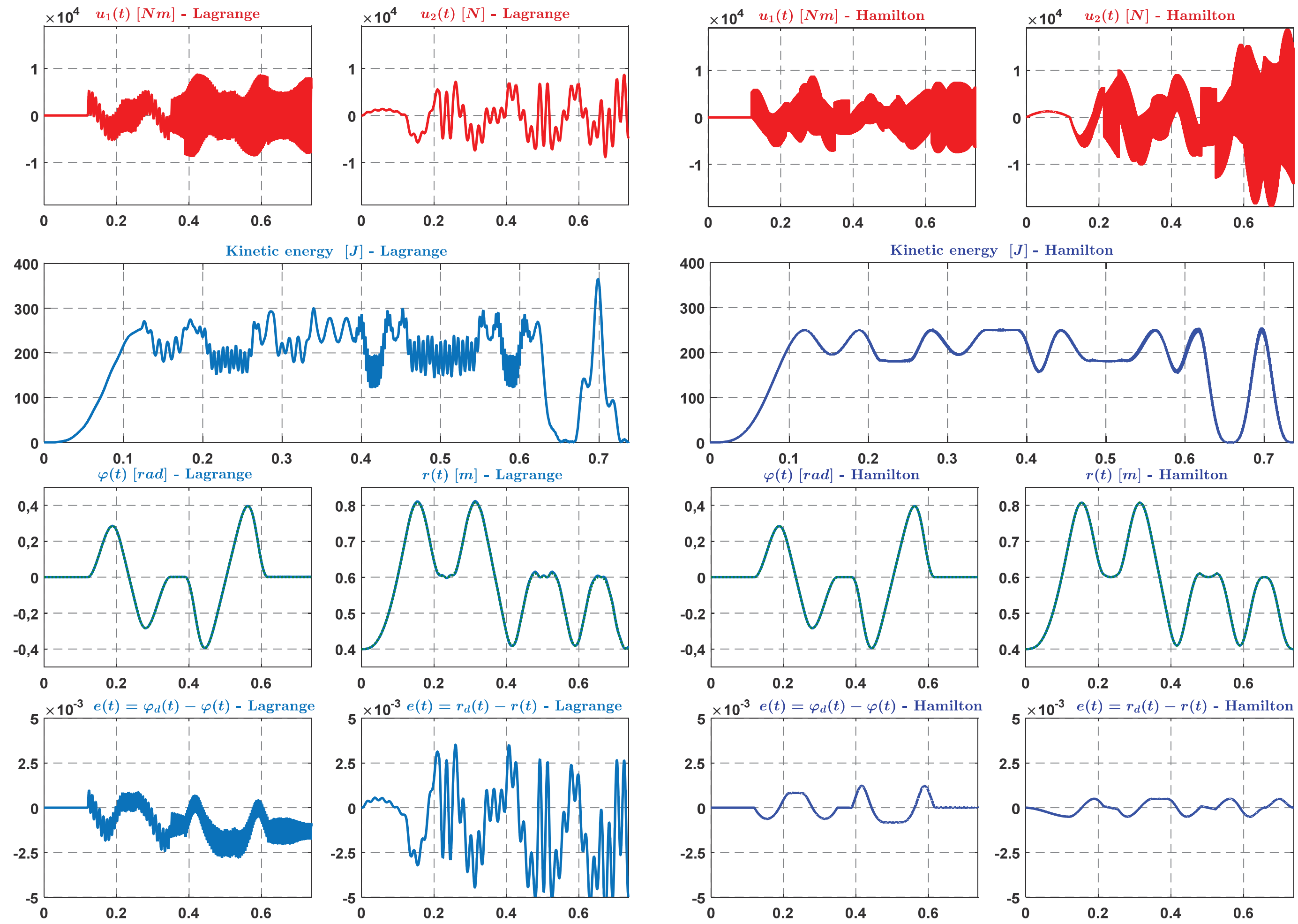

Fig. 3. Tracking control of system (Fig. 1) along trajectory (Fig. 2): time histories [s] of control actions, kinetic energy, generalized coordinates and their errors.

Fig. 3 shows the comparison of the tracking control in both Lagrandian and Hamiltonian formalisms for the identical system. Even though tracking of desired values looks similar, time histories of control errors (four bottom subfigures) and kinetic energies demonstrate better behavior of the control process in Hamiltonian space using different descriptive parameters.

CONCLUSION

The paper shows strong features of Hamiltonian formalism useful for efficient robot control. It is obvious that Hamiltonian formalism considers dynamics and internal energy distribution in robotic systems more naturally by means of specific quantities – momentums. It is a significant finding for future work.

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