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Parabolic partial differential equations with discrete state-dependent delay: Classical solutions and solution manifold

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Abstract

Classical solutions to PDEs with discrete state-dependent delay are studied. We prove the well-posedness in a set X_F which is analogous to the solution manifold used for ordinary differential equations with state-dependent delay. We prove that the evolution operators are C^1-smooth on the solution manifold.

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1. Introduction

Differential equations play an important role in describing mathematical models of many real-world processes. For many years the models are successfully used to study a number of physical, biological, chemical, control and other problems. A particular interest is in differential equations

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with many variables such as partial differential equations (PDE) and/or integral differential equations (IDE) in the case when one of the variables is time. Such equations are frequently called *evolution equations*. They received much attention from researchers from different fields since such equations could (in one way or another) discover future states of a model. It is generally known that taking into account the *past states* of the model, in addition to the present one, makes the model more realistic. This leads to the so-called delay differential equations (DDE). Historically, the theory of DDE was first initiated for the simplest case of ordinary differential equations (ODE) with constant delay (see the monographs [2,7,4,13] and references therein). Recently many important results have been extended to the case of delay PDEs with constant delay (see e.g., [26,6,25,28]).

Investigating the models described by DDEs it is clear that the constancy of delays is an extra assumption which significantly simplifies the study mathematically but is rarely met in the underlying real-world processes. The value of the delays can be time or state-dependent. Recent results showed that the theory of state-dependent delay equations (SDDE) essentially differs from the ones of constant and time-dependent delays. The basic results on ODEs with state-dependent delay can be found in [5,10,11,17,12,16,27] and the review [8]. The starting point of many mathematical studies is the well-posedness of an initial-value problem for a differential equation. It is directly connected with the choice of the space of initial functions (phase space). For DDEs with constant delay the natural phase space is the space of continuous functions. However, SDDEs non-uniqueness of solutions with continuous initial function has been observed in [5] for ODE case. The example in [5] was designed by choosing a non-Lipschitz initial function $\varphi \in C[-h, 0]$ and a state-dependent delay such that the value $-r(\varphi) \in [-h, 0]$ (at the initial function) is a non-Lipschitz point of φ. In order to overcome this difficulty, i.e., to guarantee unique solvability of initial value problems it was necessary to restrict the set of initial functions (and solutions) to a set of smoother functions. This approach includes the restrictions to layers in the space of Lipschitz functions, C^1 functions or the so-called solution manifold (a subset of $C^1[-h, 0]$). As noted in [8, p. 465] "...typically, the IVP is uniquely solved for initial and other data which satisfy suitable Lipschitz conditions." The idea to investigate ODEs with state-dependent delays in the space of Lipschitz continuous functions is very fruitful, see e.g. [17,27]. In the present work we rely on the study of solution manifold for ODEs [14,16,27].

The study of PDEs with state-dependent delay is naturally more difficult and was initiated only recently [19–24]. In contrast to the ODEs with state-dependent delays, the possibility to exploit the space of Lipschitz continuous functions in the case of PDEs with state-dependent delays meets additional difficulties. One difficulty is that the solutions of PDEs usually do not belong to the space of Lipschitz continuous functions. Another difficulty is that the time-derivative of a solution belongs to a wider space comparing to the space to which the solution itself belongs. This fact makes the choice of the appropriate Lipschitz property more involved, and it depends on a particular model under consideration. It was already found (see [22] and [24]) that non-local operators could be very useful in such models and bring additional smoothness to the solutions. Further studies also show that approaches using C^1-spaces and solution manifolds (see [14,27] and [8] for ODE case) could also be used for PDE models, see [22,24]. In this work we combine the results for ODEs [8,16,27] and PDEs [22,24].

We also mention that a simple and natural additional property concerning the state-dependent delay which guarantees the uniqueness of solutions in the whole space of continuous functions was proposed in [21] and generalized in [23]. We will not develop this approach here.

Our goal in this paper is to investigate classical solutions to parabolic PDEs with discrete state-dependent delay. We find conditions for the well-posedness and prove the existence of a *solution manifold*. We prove that the evolution operators $G_t : X_F \to X_F$ are C^1-smooth for all $t \geq 0$. Our considerations rely on the result [27] and we try to be as close as possible to the line of the proof in [27] to clarify which parts of the proof need additional care in the PDE case. As in [22,24] it is shown that non-local (in space coordinates) operators are useful in our case. We notice that in [22,24] neither classical solutions nor C^1-smoothness of the evolution operators was discussed. In the final section we consider an example of a state-dependent delay which is defined by a threshold condition.

2. Preliminaries and the well-posedness

We are interested in the following parabolic partial differential equation with discrete state-dependent delay (SDD)

$$\frac{du(t)}{dt} + Au(t) = F\big(u_t\big), \quad t > 0 \tag{1}$$

with the initial condition

$$u_0 = u|_{[-h,0]} = \varphi \in C \equiv C([-h,0]; L^2(\Omega)). \tag{2}$$

As usual for delay equations [7], for any real $a \leq b$, $t \in [a,b]$ and any continuous function $u : [a-h,b] \to L^2(\Omega)$, we denote by u_t the element of C defined by the formula $u_t = u_t(\theta) \equiv u(t+\theta)$ for $\theta \in [-h,0]$.

We assume

(H1) *Operator A is the infinitesimal generator of a compact C_0-semigroup in $L^2(\Omega)$.*
(H2) *Nonlinear map F has the form*

$$F(\varphi) \equiv B(\varphi(-r(\varphi))), \qquad F : C \to L^2(\Omega), \tag{3}$$

where $B : L^2(\Omega) \to L^2(\Omega)$ is a bounded and Lipschitz operator. Here the state-dependent delay $r : C([-h,0]; L^2(\Omega)) \to [0,h]$ is a Lipschitz mapping.

In our study we use the standard (cf. [18, Def. 2.3, p. 106] and [18, Def. 2.1, p. 105])

Definition 1. A function $u \in C([-h,T]; L^2(\Omega))$ is called a `mild solution` on $[-h,T)$ of the initial value problem (1), (2) if it satisfies (2) and

$$u(t) = e^{-At}\varphi(0) + \int_0^t e^{-A(t-s)} F(u_s)\, ds, \quad t \in [0,T). \tag{4}$$

A function $u \in C([-h,T); L^2(\Omega)) \bigcap C^1((0,T); L^2(\Omega))$ is called a `classical solution` on $[-h,T)$ of the initial value problem (1), (2) if it satisfies (2), $u(t) \in D(A)$ for $0 < t < T$ and (1) is satisfied on $(0,T)$.

Theorem 1. *Assume (H1)–(H2) are satisfied. Then for any $\varphi \in C$ there is $t_\varphi > 0$ such that initial-value problem (1), (2) has a mild solution for $t \in [0, t_\varphi)$.*

The proof is standard since F is continuous (see [6]).

We notice that F is not a Lipschitz mapping from C to $L^2(\Omega)$, so we cannot, in general, guarantee the uniqueness of mild solutions (for ODE case see [5]).

Let us fix any mild solution u of (1), (2) and consider

$$g(t) \equiv F\big(u_t\big), \qquad t \in [0, t_\phi). \tag{5}$$

Mapping g is continuous (from $[0, t_\varphi)$ to $L^2(\Omega)$) since B, u and r are continuous. Choose $T \in (0, t_\varphi)$. We have $g \in C([0, T]; L^2(\Omega))$, hence $g \in L^2(0, T; L^2(\Omega))$. The initial value problem

$$\frac{dv(t)}{dt} + Av(t) = g(t), \quad v(0) = x \in L^2(\Omega) \tag{6}$$

has a unique mild solution, which is $v = u$ if we choose $x = u(0)$.

Now we assume that

(H3) *operator A is the infinitesimal generator of an* **analytic** *(compact) semigroup in* $L^2(\Omega)$.

Below we always assume that (H1)–(H3) are satisfied.

As usual, we denote the family of all Hölder continuous functions with exponent $\alpha \in (0, 1)$ in $I \subset R$ by $C^\alpha(I; L^2(\Omega))$. By [18, Theorem 3.1, p. 110] the solution v $(= u)$ of (6) is Hölder continuous with exponent $1/2$ on $[\varepsilon, T]$ for every $\varepsilon \in (0, T)$. If additionally $x \in D(A)$ then $v \in C^{\frac{1}{2}}([0, T]; L^2(\Omega))$.

Now we show that $g \in C^{\frac{1}{4}}([0, T]; L^2(\Omega))$ if $\varphi \in C^{\frac{1}{2}}([-h, 0]; L^2(\Omega)) \subset C$. Since for $u \in C^{\frac{1}{2}}([-h, T]; L^2(\Omega))$ and $t \in [0, T]$ one has $||u_t - u_s||_C \le H_u|t - s|^{\frac{1}{2}}$ and

$$\begin{aligned} ||g(t) - g(s)|| &\le L_B||u(t - r(u_t)) - u(s - r(u_s))|| \le L_B H_u |t - s + r(u_t) - r(u_s)|^{\frac{1}{2}} \\ &\le L_B H_u \left(|t - s| + L_r||u_t - u_s||_C\right)^{\frac{1}{2}}. \end{aligned} \tag{7}$$

Here H_u is the Hölder constant of u on $[-h, T]$, L_B and L_r are Lipschitz constants of B and r respectively. We get from (7) that

$$\begin{aligned} ||g(t) - g(s)|| &\le L_B H_u \left((T^{\frac{1}{2}} + L_r H_u)|t - s|^{\frac{1}{2}}\right)^{\frac{1}{2}} \\ &\le L_B H_u \left(T^{\frac{1}{2}} + L_r H_u\right)^{\frac{1}{2}} |t - s|^{\frac{1}{4}}, \quad s, t \in [0, T]. \end{aligned}$$

Here we used $|t - s| \le T^{\frac{1}{2}}|t - s|^{\frac{1}{2}}$. We have shown that $g \in C^{\frac{1}{4}}([0, T]; L^2(\Omega))$. It gives, by [18, Corollary 3.3, p. 113], that our mild solution u is *classical* (under assumptions $\varphi \in C^{\frac{1}{2}}([-h, 0]; L^2(\Omega)) \subset C$ and $u(0) \in D(A)$).

Set

$$X \equiv \left\{\varphi \in C^1([-h, 0]; L^2(\Omega)),\ \varphi(0) \in D(A)\right\}, \tag{8}$$

$$||\varphi||_X \equiv \max_{\theta\in[-h,0]} ||\varphi(\theta)|| + \max_{\theta\in[-h,0]} ||\dot\varphi(\theta)|| + ||A\varphi(0)||. \tag{9}$$

Clearly, X is a Banach space since A is closed. We show that problem (1), (2) has a *unique* solution for any $\varphi \in X$.

As mentioned before, F is not Lipschitz on C, but if φ is Lipschitz (with Lipschitz constant L_φ), then one easily gets the following estimate (see (3))

$$||F(\varphi) - F(\psi)|| \le L_B ||\varphi(-r(\varphi)) - \psi(-r(\psi))||$$
$$\le L_B(L_\varphi |r(\varphi) - r(\psi)| + ||\varphi - \psi||_C) \le L_B(L_\varphi L_r + 1)||\varphi - \psi||_C. \tag{10}$$

Here L_B and L_r are Lipschitz constants of maps B and r.

By [18, Theorem 3.5, p. 114] (item (ii)), Au and du/dt are continuous on $[0, T]$, so u is Lipschitz from $[-h, T]$ to $L^2(\Omega)$. This property together with (10) implies the uniqueness of solution to (1), (2).

The above proves the following

Theorem 2. *Assume (H1)–(H3) are satisfied. Then for any $\varphi \in X$ there is $t_\varphi > 0$ such that initial value problem (1), (2) has a unique classical solution on $[-h, t_\phi)$.*

3. Solution manifold

Let $U \subset$ be an open subset of X. We need the following assumption.

(S) *The map $F : U \to L^2(\Omega)$ is continuously differentiable, and for every $\varphi \in U$ the derivative $DF(\phi) \in L_c(X; L^2(\Omega))$ has an extension $D_e F(\phi)$ which is an element of the space of bounded linear operators $L_c(X_0; L^2(\Omega))$, where $X_0 = \{\varphi \in C([-h, 0]; L^2(\Omega)), \varphi(0) \in D(A)\}$ is a Banach space with the norm $||\varphi||_{X_0} = \max_{\theta \in [-h,0]} ||\varphi(\theta)|| + ||A\varphi(0)||$.*

Condition (S) is analogous to that of [8, p. 467].

Let us consider the subset

$$X_F = \{\varphi \in C^1([-h, 0]; L^2(\Omega)), \varphi(0) \in D(A), \dot{\varphi}(0) + A\varphi(0) = F(\varphi)\} \tag{11}$$

of X. X_F will be called *solution manifold* according to the terminology of [27]. The equation in (11) is understood as equation in $L^2(\Omega)$. We have the following analogue to [27, Proposition 1].

Lemma 1. *If condition (S) holds and $X_F \neq \emptyset$ then X_F is a C^1 submanifold of X.*

Proof of Lemma 1. Consider any $\bar{\varphi} \in X_F \subset X$ (see (11) and also (8)). Choose $b > 0$ so large that

$$||D_e F(\bar{\varphi})||_{L_c(X_0; L^2(\Omega))} < b.$$

Define $a : [-h, 0] \ni s \mapsto se^{bs} \in \mathbb{R}$. Then

$$a(0) = 0, \quad a'(0) = 1, \quad |a(s)| \le \frac{1}{eb} \quad (-h \le s \le 0).$$

Define the closed subspaces Y and Z of X as follows:

$$Y = \{a(\cdot)y^0 : y^0 \in L^2(\Omega)\} \subset X$$

and

$$Z = \{\varphi \in X : \dot{\varphi}(0) = 0\} \subset X.$$

Clearly $Y \cap Z = \{0\}$, and $X = Y \oplus Z$.

We can define the projections

$$P_Y \phi = a(\cdot)\dot{\phi}(0), \qquad P_Z \phi = \phi - a(\cdot)\dot{\phi}(0).$$

Use $\phi = y + z = P_Y \phi + P_Z \phi$.

We define

$$G : X = Y \oplus Z \ni \phi \mapsto \dot{\phi}(0) + A\phi(0) - F(\phi) \in L^2(\Omega).$$

Clearly $\phi \in X_F \iff G(\phi) = 0$. For the bounded linear map $D_Y G(\bar{\varphi}) \in L_c(Y; L^2(\Omega))$ we have

$$D_Y G(\bar{\varphi})y = \dot{y}(0) + Ay(0) - DF(\bar{\varphi})y = y^0 - DF(\bar{\varphi})a(\cdot)y^0 = y^0 - D_e F(\bar{\varphi})a(\cdot)y^0$$

since $y = a(\cdot)y^0$ for some $y^0 \in L^2(\Omega)$, $\dot{y}(0) = y^0$, $y(0) = 0$.

Using the choices of a and $b \in R$ we obtain

$$||D_Y G(\bar{\varphi})y||_{L^2(\Omega)} \geq ||y^0||_{L^2(\Omega)} \left(1 - \frac{||D_e F(\bar{\varphi})||}{eb}\right) \geq \frac{1}{2}||y^0||_{L^2(\Omega)}.$$

Then $D_Y G(\bar{\varphi}) : Y \to L^2(\Omega)$ is a linear isomorphism. The Implicit function theorem can be applied to complete the proof of lemma. □

For the convenience of the reader we remind some properties of the semigroup $\{e^{-At}\}_{t\geq 0}$.

Lemma 2. *(See [9, Theorem 1.4.3, p. 26] or [18, Theorem 2.6.13, p. 74].) Let A be a sectorial operator in the Banach space Y and $Re\,\sigma(A) > \delta > 0$. Then*

(i) *for $\alpha \geq 0$ there exists $C_\alpha < \infty$ such that*

$$||A^\alpha e^{-At}|| \leq C_\alpha t^{-\alpha} e^{-\delta t} \text{ for } t > 0; \tag{12}$$

(ii) *if $0 < \alpha \leq 1$, $x \in D(A^\alpha)$,*

$$||(e^{-At} - I)x|| \leq \frac{1}{\alpha} C_{1-\alpha} t^\alpha ||A^\alpha x|| \text{ for } t > 0. \tag{13}$$

Also C_α is bounded for α in any compact interval of $(0, \infty)$ and also bounded as $\alpha \to 0+$.

Remark 1. It is important to notice that we can write $||(e^{-At} - I)A\varphi(0)|| \le ||e^{-At} - I|| \cdot ||A\varphi(0)||$, but $||e^{-At} - I|| \not\to 0$ as $t \to 0+$ because e^{-At} is not a uniformly continuous semigroup since A is *unbounded* (see [18, Theorem 1.2, p. 2]).

Remark 2. We also notice that the (linear) mapping $D(A) \ni \xi \longmapsto (e^{-At} - I)\xi \in C^1([0,T]; L^2(\Omega))$ is continuous, while $L^2(\Omega) \ni \xi \longmapsto (e^{-At} - I)\xi \in C^1((0,T]; L^2(\Omega))$ is not.

We need the following

Lemma 3. *Let A be a sectorial operator in the Banach space Y and $f : (0,T) \to Y$ be locally Hölder continuous with $\int_0^\rho ||f(s)||\, ds < \infty$ for some $\rho > 0$. For $0 \le t < T$, define*

$$I_T(f)(t) = \mathcal{F}(t) \equiv \int_0^t e^{-A(t-s)} f(s)\, ds. \tag{14}$$

Then

(i) *$\mathcal{F}(\cdot)$ is continuous on $[0,T)$;*
(ii) *$\mathcal{F}(\cdot)$ continuously differentiable on $(0,T)$, with $\mathcal{F}(t) \in D(A)$ for $0 < t < T$, and $d\mathcal{F}(t)/dt + A\mathcal{F}(t) = f(t)$ on $0 < t < T$, $\mathcal{F}(t) \to 0$ in X as $t \to 0+$.*
(iii) *If additionally $f : (0,T) \to Y$ satisfies*

$$||f(t) - f(s)|| \le K(s)(t-s)^\gamma \quad \text{for} \quad 0 < s < t < T < \infty,$$

where $K : (0,T) \to \mathbb{R}$ is continuous with $\int_0^T K(s)\, ds < \infty$, then for every $\beta \in [0,\gamma)$ the function $\mathcal{F}(t)$ is continuously differentiable $\mathcal{F} : (0,T) \to Y^\beta \equiv D(A^\beta)$ with

$$\left\| \frac{d\mathcal{F}(t)}{dt} \right\|_\beta \le Mt^{-\beta}||f(t)|| + M \int_0^t (t-s)^{\gamma-\beta-1} K(s)\, ds \tag{15}$$

for $0 < t < T$. Here M is a constant independent of γ, β, $f(\cdot)$.
Further, if $\int_0^h K(s)\, ds = O(h^\delta)$ as $h \to 0+$, for some $\delta > 0$, then $t \to d\mathcal{F}(t)/dt$ is locally Hölder continuous from $(0,T)$ into Y^β.
(iv) *If $f : [0,T] \to Y$ is Hölder continuous (on the compact $[0,T]$ the local and global Hölder properties coincide), then $\mathcal{F} \in C^1([0,T]; Y)$.*

Proof of Lemma 3. Items (i) and (ii) are proved in [9, Lemma 3.2.1, p. 50]. Item (iii) is proved in [9, Lemma 3.5.1, p. 70]. The proof of (iv) is contained in the proof of [18, Theorem 3.5, item (ii), p. 114]. We briefly outline the main steps. Using properties (ii) (i.e. $d\mathcal{F}(t)/dt + A\mathcal{F}(t) = f(t)$ on $0 < t < T$) and $f \in C([0,T]; Y)$ it is enough to show that $A\mathcal{F}$ is continuous at $t = 0$. We write $\mathcal{F}(t) = \int_0^t e^{-A(t-s)}[f(s) - f(t)]\, ds + \int_0^t e^{-A(t-s)} f(t)\, ds = v_1(t) + v_2(t)$. The property $Av_1 \in C^\gamma([0,T]; Y)$ is proved in [18, Lemma 3.4, p. 113]. To show that $Av_2 \in C([0,T]; Y)$ one uses

$$Av_2(t) = \int_0^t Ae^{-A(t-s)} f(t)\, ds = \int_0^t Ae^{-A\tau} f(t)\, d\tau = \int_0^t \left\{ -\frac{d}{d\tau} e^{-A\tau} f(t) \right\} d\tau$$
$$= f(0) - e^{-At} f(t) = f(0) - e^{-At} f(0) + e^{-At}(f(0) - f(t)).$$

Hence $\|Av_2(t)\| \le \|f(0) - e^{-At} f(0)\| + \|e^{-At}\| \|f(0) - f(t)\| \le \|f(0) - e^{-At} f(0)\| + M\|f(0) - f(t)\| \to 0$ as $t \to 0+$ due to the continuity of e^{-At} and $f(t)$. It completes the proof of Lemma 3. □

To simplify the calculations we assume the following Lipschitz property holds

$$\exists \alpha \in (0,1),\ \exists L_{B,\alpha} \ge 0 \text{ so that}$$
$$||A^\alpha (B(u) - B(v))|| \le L_{B,\alpha} ||u - v|| \text{ holds for all } u, v \in L^2(\Omega). \tag{16}$$

Remark 3. It is easy to see that (16) implies similar property with $\alpha = 0$, i.e.,

$$\exists\, L_{B,0} \ge 0 \text{ so that } ||B(u) - B(v)|| \le L_{B,0} ||u - v|| \text{ holds for all } u, v \in L^2(\Omega). \tag{17}$$

Example 1. Let us consider $B(u) = \int_\Omega f(x-y) b(u(y))\, dy$ which is a convolution of a function $f \in H^1(\Omega)$ and composition $b \circ u$ with $b : R \to R$ Lipschitz. We use the properties of a convolution (see e.g. [3, pp. 104, 108]) $(f \star g)(x) = \int_\Omega f(x-y) g(y)\, dy$, namely $||f \star g||_{L^p} \le ||f||_{L^1} ||g||_{L^p}$ for any $f \in L^1$ and $g \in L^p$, $1 \le p \le \infty$ and also $D^\beta (f \star g) = (D^\beta f) \star g$, particularly, $\nabla(f \star g) = (\nabla f) \star g$ (for details see e.g. [3, Proposition 4.20, p. 107]).

If we consider Laplace operator with Dirichlet boundary conditions $A \sim (-\Delta)_D$, then $||A^{1/2} \cdot ||$ is equivalent to $|| \cdot ||_{H^1}$, so $||A^{1/2}(B(u) - B(v))|| \le C_1^2 ||B(u) - B(v)||^2 + C_1^2 ||\nabla(B(u) - B(v))||^2 \le C_1^2 ||f||_{L^1}^2 ||b(u) - b(v)||^2 + C_1^2 ||\nabla f||_{L^1}^2 ||b(u) - b(v)||^2$. Using the Lipschitz property of b, we get (16) with $\alpha = 1/2$ and $L_{B,\alpha} = C_1 L_b (||f||_{L^1}^2 + ||\nabla f||_{L^1}^2)^{1/2}$.

Using (16) and (3) one easily gets the Lipschitz property for F. Namely, for Lipschitz ψ and Lipschitz SDD r

$$||A^\alpha (F(\psi) - F(\chi))|| \le ||A^\alpha (B(\psi(-r(\psi))) - B(\chi(-r(\chi))))||$$
$$\le L_{B,\alpha} ||\psi(-r(\psi))) - \chi(-r(\chi)))||$$
$$\le L_{B,\alpha} L_\psi L_r ||\psi - \chi||_C + L_{B,\alpha} ||\psi - \chi||_C = L_{F,\alpha} ||\psi - \chi||_C, \quad L_{F,\alpha} = L_{B,\alpha}(L_\psi L_r + 1). \tag{18}$$

Using (17), similarly to (18), one gets

$$||F(\psi) - F(\chi)|| \le L_{F,0} ||\psi - \chi||_C, \quad L_{F,0} = L_{B,0}(L_\psi L_r + 1). \tag{19}$$

We use all notations of [27], changing $\mathbb{R}^n$ for $L^2(\Omega)$ when necessary. For example, we use the notation E_T (see [27, p. 50])

$$E_T : C^1([-h,0]) \to C^1([-h,T]), \qquad (E_T \varphi)(t) \equiv \begin{cases} \varphi(t), & \text{for } t \in [-h,0), \\ \varphi(0) + t\dot\varphi(0) & \text{for } t \in [0,T]. \end{cases} \tag{20}$$

Let B be a real Banach space. The open ball in B with radius $\mu > 0$ and centre 0 is denoted by B_μ. For $m \in M$ and $\mu > 0$ we denote $M_{m,\mu} = M \cap (m + B_\mu)$.

On the other hand, some notations should be changed. For example, for any $\psi \in X_F$ and $\mu > 0$ we set (remind that $||\cdot||_X$ is not just C^1-norm, see (8), (9), (11))

$$X_{\psi,\mu} \equiv X_F \bigcap \left\{ \psi + (C^1([-h,0]; L^2(\Omega)))_{X,\mu} \right\} = \{ \psi \in X_F : ||\varphi - \psi||_X < \mu \}. \tag{21}$$

For $T > 0$ (to be chosen below), we split a map $x \in C^1([-h,T]) \equiv C^1([-h,T]; L^2(\Omega))$ with $x_0 = \varphi \in X_F$ given, as $x = y + \hat{\varphi}$, where for short $\hat{\varphi}(t) = (E_T \varphi)(t)$ is defined in (20).

We look for a fixed point of the following map (φ is the parameter)

$$R_{T\mu}(\varphi, y) \equiv \begin{bmatrix} e^{-At}\varphi(0) - \varphi(0) - t\dot{\varphi}(0) + \int_0^t e^{-A(t-\tau)} F(y_\tau + \hat{\varphi}_\tau)\, d\tau, & t \in [0,T], \\ 0 & t \in [-h, 0), \end{bmatrix} \tag{22}$$

where $R_{T\mu} : X_{\psi,\mu} \times (C_0^1([-h,T]; L^2(\Omega)))_\varepsilon \to C_0^1([-h,T]; L^2(\Omega))$, and $X_{\psi,\mu}$ defined in (21). Here for $T > 0$ we denote by $C_0^1([-h,T]; L^2(\Omega))$ the closed subspace of maps $z \in C^1([-h,T]; L^2(\Omega))$ which vanish on $[-h, 0]$.

Proposition 1. $R_{T\mu} : X_{\psi,\mu} \times (C_0^1([-h,T]; L^2(\Omega))) \to C_0^1([-h,T]; L^2(\Omega))$.

To prove that the image of $R_{T\mu}(\varphi, y) = z$ belongs to $C_0^1([-h,T]; L^2(\Omega))$, we notice that $y \in C^1([-h,T]; L^2(\Omega))$ implies $y + \hat{\varphi} \in Lip([-h,T]; L^2(\Omega))$, which together with (10) gives that $F(y_\tau + \hat{\varphi}_\tau)$, $\tau \in [0,T]$ is Lipschitz, so [9, Lemma 3.2.1, p. 50] can be applied to the integral term in $R_{T\mu}$ (see (22)). This gives $z \in C^1(0,T; L^2(\Omega))$.

The property $||z(t)|| \to 0$ as $t \to 0+$ is simple. The last step is to show that $||\dot{z}(t)|| \to 0$ as $t \to 0+$. Using [9, Lemma 3.2.1, p. 50] and property $\varphi \in X_F$, we have

$$\dot{z}(t) = -Ae^{-At}\varphi(0) - \dot{\varphi}(0) - A \int_0^t e^{-A(t-\tau)} F(y_\tau + \hat{\varphi}_\tau)\, d\tau + F(y_t + \hat{\varphi}_t)$$

$$= -Ae^{-At}\varphi(0) + A\varphi(0) - F(\varphi) - A \int_0^t e^{-A(t-\tau)} F(y_\tau + \hat{\varphi}_\tau)\, d\tau + F(y_t + \hat{\varphi}_t).$$

Hence

$$||\dot{z}(t)|| \le ||(e^{-At} - I)A\varphi(0)|| + ||F(y_t + \hat{\varphi}_t) - F(\varphi)|| + \left\| A \int_0^t e^{-A(t-\tau)} F(y_\tau + \hat{\varphi}_\tau)\, d\tau \right\|. \tag{23}$$

The first two terms in (23) tend to zero as $t \to 0+$ since $\varphi(0) \in D(A)$, e^{-At} is strongly continuous, F is continuous and $||y_t + \hat{\varphi}_t - \varphi||_C \to 0$ as $t \to 0+$. To estimate the last term in (23) we use (18) for $\psi = 0$ and the property $||A^\alpha e^{-At}|| \le C_\alpha t^{-\alpha} e^{-\delta t}$, $\alpha \ge 0$ (remind that e^{-At} is analytic and see Lemma 2 and [9, Theorem 1.4.3, p. 26], [18, Theorem 2.6.13, p. 74]). So

$$\left\| A \int_0^t e^{-A(t-\tau)} F(y_\tau + \hat{\varphi}_\tau)\, d\tau \right\| = \left\| \int_0^t A^{1-\alpha} e^{-A(t-\tau)} A^\alpha F(y_\tau + \hat{\varphi}_\tau)\, d\tau \right\|$$

$$\leq \int_0^t C_{1-\alpha} (t-\tau)^{\alpha-1} e^{-\delta(t-\tau)} L_{B,\alpha} ||y_\tau + \hat{\varphi}_\tau||_C \, d\tau$$

$$\leq L_{B,\alpha} C_{1-\alpha} \cdot \max_{s\in[0,T]} ||y_s + \hat{\varphi}_s||_C \int_0^t (t-\tau)^{\alpha-1} e^{-\delta(t-\tau)}\, d\tau \to 0$$

as $t \to 0+$ since the last integral is convergent for $\alpha > 0$. It completes the proof of Proposition 1. □

Remark 4. It is important in the proof of Proposition 1 to have the property (16) with $\alpha > 0$ for the convergence of the last integral.

As in [27, p. 56] we will use local charts of the manifold X_F and a version of Banach's fixed point theorem with parameters (see e.g., Proposition 1.1 of Appendix VI in [4, p. 497]).

Remark 5. More precisely, we look for a fixed point of $R_{T\mu}(\varphi, y)$ as a function of y where parameter is the image of φ under a local chart map instead of $\varphi \in X_{\psi,\mu}$. The reason is that the parameter should belong to an open subset of a Banach space, but $X_{\psi,\mu}$ is not even linear (it is a subset of the manifold X_F).

We remind that for short we denoted by $\hat{\varphi} \equiv E_T\varphi$, where $E_T\varphi$ is defined in (20).

Proposition 2. *(See [27, Prop. 2].) For every $\varepsilon > 0$ there exist $T = T(\varepsilon) > 0$ and $\mu = \mu(\varepsilon)$ such that for all $\varphi \in \psi + (C^1([-h, 0]; L^2(\Omega)))_\mu$ and all $t \in [0, T]$,*

$$\hat{\varphi}_t \in \psi + (C^1([-h, 0]; L^2(\Omega)))_\varepsilon$$

The proof is unchanged as in [27, Proposition 2], so we omit it here.

Let us denote $M_T > 0$ a constant satisfying $||e^{-As}|| \leq M_T$ for all $s \in [0, T]$. Now we prove an analogue to [27, Proposition 3].

Proposition 3. *For all $\varphi \in X_{\psi,\mu}$ and $y, w \in (C_0^1([-h, T]; L^2(\Omega)))_\varepsilon$ one has*

$$||R_{T\mu}(\varphi, y) - R_{T\mu}(\varphi, w)||_{C^1([-h,T];L^2(\Omega))} \leq L_{R_{T\mu}} ||y - w||_{C^1([-h,T];L^2(\Omega))}, \tag{24}$$

where we denoted for short the Lipschitz constant

$$L_{R_{T\mu}} \equiv T L_{F,0,\varepsilon}(M_T + 1) + T^\alpha C_{1-\alpha} M_T \alpha^{-1} L_{F,\alpha,\varepsilon} \tag{25}$$

with $L_{F,\alpha,\varepsilon} = L_{B,\alpha}(\varepsilon L_r + 1)$ and $L_{F,0,\varepsilon} = L_{B,0}(\varepsilon L_r + 1)$ (cf. (18), (19)). The constant α is defined in (16).

Proof of Proposition 3. Using (19), we have for all $||\psi||_{C^1} \le \varepsilon$

$$||F(\psi) - F(\chi)|| \le L_{F,0,\varepsilon}||\psi - \chi||_C, \quad L_{F,0,\varepsilon} = L_{B,0}(\varepsilon L_r + 1).$$

Let $z = R_{T\mu}(\varphi, y)$, $v = R_{T\mu}(\varphi, w)$ for $y, w \in (C_0^1([-h, T]; L^2(\Omega)))_\varepsilon$. For all $t \in [0, T]$, one gets

$$\begin{aligned} ||z(t) - v(t)|| &\le || \int_0^t e^{-A(t-\tau)}(F(y_\tau + \hat{\varphi}_\tau) - F(w_\tau + \hat{\varphi}_\tau))\, d\tau|| \\ &\le T M_T L_{F,0,\varepsilon}||y - w||_{C^1([-h,T];(L^2(\Omega))}. \end{aligned} \tag{26}$$

Next $\|\dot{z}(t) - \dot{v}(t)\| \le \|F(y_t + \hat{\varphi}_t) - F(w_t + \hat{\varphi}_t)\| + \|A \int_0^t e^{-A(t-\tau)}(F(y_\tau + \hat{\varphi}_\tau) - F(w_\tau + \hat{\varphi}_\tau))\, d\tau\| \le L_{F,0,\varepsilon}\|y_t - w_t\|_C + \int_0^t \|A^{1-\alpha}e^{-A(t-\tau)}\|\|A^\alpha(F(y_\tau + \hat{\varphi}_\tau) - F(w_\tau + \hat{\varphi}_\tau))\|\, d\tau$. To estimate the first term we write

$$\begin{aligned} ||y_t - w_t||_C &= \max_{s\in[-h,0]} || \int_0^{t+s} (\dot{y}(\tau) - \dot{w}(\tau))\, d\tau|| \le \int_0^T ||\dot{y}(\tau) - \dot{w}(\tau)||\, d\tau \\ &\le T||y - w||_{C^1([-h,T];L^2(\Omega))}. \end{aligned}$$

For the second term, as in Proposition 1, we use the property $||A^\alpha e^{-At}|| \le C_\alpha t^{-\alpha} e^{-\delta t}$, $\alpha \ge 0$ (see [9, Theorem 1.4.3, p. 26] or [18, Theorem 2.6.13, p. 74]), the Lipschitz property (18) and calculations $\int_0^t (t - \tau)^{\alpha-1}\, d\tau = t^\alpha/\alpha$ to get

$$\begin{aligned} &\int_0^t \|A^{1-\alpha}e^{-A(t-\tau)}\|\|A^\alpha(F(y_\tau + \hat{\varphi}_\tau) - F(w_\tau + \hat{\varphi}_\tau))\|\, d\tau \\ &\le C_{1-\alpha}T^\alpha \alpha^{-1} M_T L_{F,\alpha,\varepsilon}\|y - w\|_{C^1([-h,T];L^2(\Omega))}. \end{aligned}$$

Hence

$$||\dot{z}(t) - \dot{v}(t)|| \le \left\{ T L_{F,0,\varepsilon} + T^\alpha C_{1-\alpha} M_T \alpha^{-1} L_{F,\alpha,\varepsilon} \right\} ||y - w||_{C^1([-h,T];L^2(\Omega))}.$$

The last estimate and (26) combined give (24). □

The following statement is an analogue to [27, Proposition 4 and Corollary 1].

Proposition 4. *For given $\delta > 0$ there exist $T = T(\delta) > 0$, $\mu = \mu(\delta) > 0$, such that for all $\varphi \in X_{\psi,\mu}$ ($||\psi - \varphi||_X \le \mu$) one has*

$$||R_{T\mu}(\varphi, 0)||_{C^1([-h,T];L^2(\Omega))} < \delta.$$

Moreover, for a positive ε there exist $\delta > 0$ (and $T = T(\delta) > 0$, $\mu = \mu(\delta) > 0$ as above) and $\lambda \in (0, 1)$, such that $R_{T\mu}$ (defined in (22)) maps the subset $X_{\psi,\mu} \times (C^1_0([-h, T]; L^2(\Omega)))_\varepsilon$ into the closed ball $Cl\,(C^1_0([-h, T]; L^2(\Omega)))_{\lambda\varepsilon} \subset (C^1_0([-h, T]; L^2(\Omega)))_\varepsilon$.

Proof of Proposition 4. Consider $z \equiv R_{T\mu}(\varphi, 0)$. For $t \in [0, T]$, we write

$$z(t) = e^{-At}\varphi(0) - \varphi(0) - t\dot{\varphi}(0) + \int_0^t e^{-A(t-\tau)} F(\hat{\varphi}_\tau)\, d\tau$$
$$= (e^{-At} - I)(\varphi(0) - \psi(0)) + (e^{-At} - I)\psi(0) - t \cdot (\dot{\varphi}(0) - \dot{\psi}(0)) - t\dot{\psi}(0)$$
$$+ \int_0^t e^{-A(t-\tau)} \left\{ F(\hat{\varphi}_\tau) - F(\hat{\psi}_\tau) \right\} d\tau + \int_0^t e^{-A(t-\tau)} F(\hat{\psi}_\tau)\, d\tau. \tag{27}$$

We estimate different parts of (27) in the following ten steps.

1. Using the property $||(e^{-At} - I)x|| \le \frac{1}{\alpha} C_{1-\alpha} t^\alpha ||A^\alpha x||$ (see [9, Theorem 1.4.3]) one gets

$$||(e^{-At} - I)(\varphi(0) - \psi(0))|| \le C_{\frac{1}{2}} t^{\frac{1}{2}} ||A^{\frac{1}{2}}(\varphi(0) - \psi(0))|| \le \hat{C} t^{\frac{1}{2}} ||A(\varphi(0) - \psi(0))||$$
$$\le \hat{C} t^{\frac{1}{2}} ||\varphi - \psi||_X.$$

2. $||t \cdot (\dot{\varphi}(0) - \dot{\psi}(0))|| \le t \cdot ||\varphi - \psi||_X$.
3. $|| \int_0^t e^{-A(t-\tau)} \left\{ F(\hat{\varphi}_\tau) - F(\hat{\psi}_\tau) \right\} d\tau|| \le M_T t L_{F,0} \max_{\tau\in[0,t]} ||\hat{\varphi}_\tau - \hat{\psi}_\tau||_C \le M_T t L_{F,0}(1 + T)||\varphi - \psi||_X$.
4. $|| \int_0^t e^{-A(t-\tau)} F(\hat{\psi}_\tau)\, d\tau|| \le M_T t L_{B,0} \max_{\tau\in[0,t]} ||\hat{\psi}_\tau||_C \le M_T t L_{B,0}(1 + T)||\psi||_X$.
   Now we proceed to estimate the time derivative of $z(t)$

$$\dot{z}(t) = -Ae^{-At}\varphi(0) - \dot{\varphi}(0) + F(\hat{\varphi}_t) - A \int_0^t e^{-A(t-\tau)} F(\hat{\varphi}_\tau)\, d\tau$$
$$= -Ae^{-At}\varphi(0) + A\varphi(0) + F(\varphi) + F(\hat{\varphi}_t) - A \int_0^t e^{-A(t-\tau)} F(\hat{\varphi}_\tau)\, d\tau$$
$$= (e^{-At} - I)A(\psi(0) - \varphi(0)) - (e^{-At} - I)A\psi(0)$$
$$+ [F(\hat{\varphi}_t) - F(\hat{\psi}_t)] + [F(\hat{\psi}_t) - F(\psi)] + [F(\psi) - F(\varphi)]$$
$$- \int_0^t Ae^{-A(t-\tau)} \{F(\hat{\varphi}_\tau) - F(\hat{\psi}_\tau)\} d\tau - \int_0^t Ae^{-A(t-\tau)} F(\hat{\psi}_\tau)\, d\tau. \tag{28}$$

   We use the following
5. $||(e^{-At} - I)A(\psi(0) - \varphi(0))|| \le (M_T + 1)||\varphi - \psi||_X$.
   We remind (19) for steps 6 and 7.

6. $||F(\hat{\varphi}_t) - F(\hat{\psi}_t)|| \le L_{F,0} \max_{\tau\in[0,t]} ||\hat{\varphi}_\tau - \hat{\psi}_\tau||_C \le L_{F,0}(1+T)||\varphi - \psi||_X$.
7. $||F(\varphi) - F(\psi)|| \le L_{F,0}||\varphi - \psi||_X$.
8. $||F(\hat{\psi}_t) - F(\psi)|| \to 0$ as $t \to 0+$ since $\hat{\psi}$ is continuous from $[-h, T]$ to $L^2(\Omega)$.
9. $|| \int_0^t Ae^{-A(t-\tau)}\{F(\hat{\varphi}_\tau) - F(\hat{\psi}_\tau)\}\, d\tau|| = || \int_0^t A^{1-\alpha}e^{-A(t-\tau)}A^\alpha\{F(\hat{\varphi}_\tau) - F(\hat{\psi}_\tau)\}\, d\tau||$

$$\le \int_0^t C_{1-\alpha}(t-\tau)^{\alpha-1}e^{-\delta(t-\tau)}L_{F,\alpha}||\hat{\varphi}_\tau - \hat{\psi}_\tau||_C\, d\tau \le C_{1-\alpha}L_{F,\alpha}D_{\alpha,T}||\varphi - \psi||_X,$$

   where $D_{\alpha,T} \equiv \int_0^T (T-\tau)^{\alpha-1}e^{-\delta(T-\tau)}\, d\tau$, $\alpha > 0$.
10. Similar to the previous case ($L_{B,\alpha}$ instead of $L_{F,\alpha}$)

$$|| \int_0^t Ae^{-A(t-\tau)}F(\hat{\psi}_\tau)\, d\tau|| \le C_{1-\alpha}L_{B,\alpha}D_{\alpha,T}||\psi||_X.$$

Now we can apply estimates 1–10 (combined) to (27), (28). It gives the possibility to choose small enough $T = T(\delta) > 0$, $r = r(\delta) > 0$ such that

$$||z||_{C^1([-h,T];L^2(\Omega))} \equiv ||R_{T\mu}(\varphi, 0)||_{C^1([-h,T];L^2(\Omega))} < \delta. \tag{29}$$

Remark 6. Small μ is used in 5–7 only. For all the other terms it is enough (to be small) to have a small T.

Now we prove the second part of Proposition 4. We have

$$\begin{aligned} ||R_{T\mu}(\varphi, y)||_{C^1([-h,T];L^2(\Omega))} &\le ||R_{T\mu}(\varphi, y) - R_{T\mu}(\varphi, 0)||_{C^1([-h,T];L^2(\Omega))} \\ &\quad + ||R_{T\mu}(\varphi, 0)||_{C^1([-h,T];L^2(\Omega))}. \end{aligned} \tag{30}$$

The first term in (30) is controlled by Proposition 3 (see (24)), while the second one by (29).

More precisely, we proceed as follows. First choose $\varepsilon > 0$, then choose small $T(\varepsilon) > 0$ to have the Lipschitz constant $L_{R_{T\mu}} < 1$ (see (24), (25)). Next we set $\delta \equiv \frac{\varepsilon}{2}(1 - L_{R_{T\mu}}) > 0$ and the corresponding $T = T(\delta) \in (0, T(\varepsilon)]$, $\mu = \mu(\delta) > 0$ as in the first part of Proposition 4, see (29). Finally, we set $\lambda \equiv \frac{1}{2}(1 + L_{R_{T\mu}}) \in (0, 1)$. Now estimates (30), (24) and (29) show that for any $y \in (C_0^1([-h, T]; L^2(\Omega)))_\varepsilon$ we have

$$\begin{aligned} ||R_{T\mu}(\varphi, y)||_{C^1([-h,T];L^2(\Omega))} &\le L_{R_{T\mu}}||y||_{C^1([-h,T];L^2(\Omega))} + \delta \le L_{R_{T\mu}}\varepsilon + \delta \\ &= L_{R_{T\mu}}\varepsilon + \frac{\varepsilon}{2}(1 - L_{R_{T\mu}}) = \varepsilon\frac{1}{2}(1 + L_{R_{T\mu}}) = \varepsilon\lambda < \varepsilon. \end{aligned}$$

It completes the proof of Proposition 4. □

We assume

(H4) *Nonlinear operators* $B : L^2(\Omega) \to D(A^\alpha)$ *for some* $\alpha > 0$ *and* $r : C([-h, 0]; L^2(\Omega)) \to [0, h]$ *are* C^1*-smooth.*

Remark 7. Assumption (H4) implies that the restriction $r : C([-h,0]; L^2(\Omega)) \supset C^1([-h,0]; L^2(\Omega)) \to [0,h]$ is also C^1-smooth. In addition, it is easy to see that (H4) implies condition (S).

Proposition 5. *Assume (H1)–(H4) are satisfied. Then $R_{T\mu}$ is C^1-smooth.*

The *proof of Proposition 5* follows the one of [27, Prop. 5]. The main essential difference is the following. The C^1-smoothness of $B : L^2(\Omega) \to D(A^\alpha)$ implies the C^1-smoothness of $\widetilde{F} : X_{\psi,\mu} \times C^1([-h,0]; L^2(\Omega)) \to D(A^\alpha)$ defined as $\widetilde{F}(\varphi, y) \equiv B(\varphi(-r(\varphi+y)) + y(-r(\varphi+y)))$.

We also use evident additional property of the C^1-smoothness of the map $X \ni \varphi \mapsto e^{-At}\varphi(0) \in C([0,T]; L^2(\Omega))$ (remind the definition of X in (8)). Here we use $I_T : C^1([0,T]; L^2(\Omega)) \to C^1([0,T]; L^2(\Omega))$ given by $I_T(y)(t) \equiv \int_0^t e^{-A(t-\tau)} y(\tau)\, d\tau$ instead of I_T used in [27, p. 50]. We rely on [9, Lemma 3.2.1, p. 50] (see Lemma 3, item (iv) above). □

As in [27, p. 56] we are ready to use local charts of the submanifold X_F and a version of Banach's fixed point theorem with parameters (see e.g., [4, Proposition 1.1 of Appendix VI]). Namely, Propositions 3–5 allow us to apply Banach's fixed point theorem to get for any $\varphi \in X_{\psi,\mu}$ the unique fixed point $y = y^\varphi \in (C_0^1([-h,T]; L^2(\Omega)))_\varepsilon$ of the map $R_{T\mu}$. We denote this correspondence by $Y_{T\mu} : X_{\psi,\mu} \to (C_0^1([-h,T]; L^2(\Omega)))_\varepsilon$ and it is C^1-smooth.

It also gives that the map

$$S_{T\mu} : X_{\psi,\mu} \to C^1([-h,T]; L^2(\Omega)), \tag{31}$$

defined by $S_{T\mu}\varphi = x^\varphi \equiv y^\varphi + \hat{\varphi} \equiv Y_{T\mu}(\varphi) + E_T\varphi$ is C^1-smooth. Here $E_T\varphi$ is defined in (20).

The local semiflow

$$F_{T\mu} : [0,T] \times X_{\psi,\mu} \to X_F \subset X$$

is given by

$$F_{T\mu}(t,\varphi) = x_t^\varphi = ev_t(S_{T\mu}(\varphi)). \tag{32}$$

Here we denoted the evaluation map

$$ev_t : C^1([-h,T]; L^2(\Omega)) \to C^1([-h,0]; L^2(\Omega)), \qquad ev_t x \equiv x_t \quad \text{for all} \quad t \in [0,T]. \tag{33}$$

Proposition 6. *Assume (H1)–(H4) are satisfied. Then $F_{T\mu}$ is continuous, and each solution map $F_{T\mu}(t,\cdot) : X_{\psi,\mu} \ni \phi \mapsto x_t^{(\phi)} \in X_F$, $t \in [0,T]$, is C^1-smooth. For all $t \in [0,T]$, all $\phi \in X_{\psi,\mu}$, and all $\chi \in T_\phi X_F$, one has $T_{F_{T\mu}(t,\phi)} \ni D_2 F_{T\mu}(t,\phi)\chi = v_t^{(\phi,\chi)}$, where the function $v \equiv v^{(\phi,\chi)} \in C^1([-h,T]; L^2(\Omega)) \cap C([0,T]; D(A))$ is the solution of the initial value problem*

$$\dot{v}(t) = Av(t) + DF(x_t^{(\phi)})v_t \quad \textit{for all} \quad t \in [0,T],\ v_0 = \chi. \tag{34}$$

Here $T_\phi X_F$ is the tangent space to the manifold X_F at point $\phi \in X_F$.

Proof of Proposition 6. We denote for short $G \equiv F_{T\mu}$ and $S \equiv S_{T\mu}$. Now we discuss the continuity of F (remind the definition of X in (8) and the norm $||\cdot||_X$ in (9)).

$$
\begin{aligned}
||G(s,\chi) - G(t,\varphi)||_X &= ||x_s^\chi - x_t^\varphi||_{C^1[-h,0]} + ||A(x^\chi(s) - x^\varphi(t))|| \\
\leq ||x_s^\chi - x_s^\varphi||_{C^1[-h,0]} &+ ||x_s^\varphi - x_t^\varphi||_{C^1[-h,0]} + ||A(x^\chi(s) - x^\varphi(s))|| + ||A(x^\varphi(s) - x^\varphi(t))|| \\
&\leq ||S(\chi) - S(\varphi)||_{C^1[-h,T]} + ||x_s^\varphi - x_t^\varphi||_{C^1[-h,0]} + ||A(x^\chi(s) - x^\varphi(s))|| \\
&\quad + ||A(x^\varphi(s) - x^\varphi(t))||.
\end{aligned}
\tag{35}
$$

Consider the third term in (35).

$$
\begin{aligned}
||A(x^\chi(s) - x^\varphi(s))|| &\leq ||e^{-As}A(\chi(0) - \varphi(0))|| \\
&+ \int_0^s ||e^{-A(s-\tau)}A^{1-\alpha}A^\alpha(F(x_\tau^\chi) - F(x_\tau^\varphi))||\,d\tau \\
&\leq ||\chi - \varphi||_X + C_{1-\alpha}T^\alpha\alpha^{-1}M_T L_{B,\alpha}(L_{x^\varphi}L_r + 1)||x^\chi - x^\varphi||_{C[-h,T]} \\
&\leq ||\chi - \varphi||_X + C_{1-\alpha}T^\alpha\alpha^{-1}M_T L_{B,\alpha}(L_{x^\varphi}L_r + 1)||S(\chi) - S(\varphi)||_{C[-h,T]}.
\end{aligned}
$$

We see that due to the continuity of $S \equiv S_{T\mu}$ (see (31)) the first and the third terms in (35) tend to zero when $||\chi - \varphi||_X \to 0$. The second term in (35) tends to zero as $|s-t| \to 0$ since $x \in C^1([-h,T]; L^2(\Omega))$. The last term in (35) vanishes due to [18, Theorem 3.5, item (ii), p. 114] (remind that $x^\varphi(0) \equiv \varphi(0) \in D(A)$). We proved the continuity of F. To verify the differential equation for v (see (34)), we follow the line of arguments presented in [27, p. 58]. More precisely, we first verify the integral equation (4) i.e. show that v is a mild solution to (34). The only difference in our case is the presence of the operator A which is linear. Hence it does not add any difficulties in the differentiability of $S \equiv S_{T\mu}$ when we define for fixed $\phi \in X_{\psi,\mu}$, and $\chi \in T_\phi X_F$ the function $v \equiv DS(\phi)\chi \in C^1([-h,T]; L^2(\Omega))$. Here DS is understood as the differential of a map between manifolds (see (31) for the definition of S and [1] for basic theory of manifolds). One can see [27, p. 58] that $v_0 = ev_0 DS(\phi)\chi = D(ev_0 \circ S)(\phi)\chi = \chi$. Here the evaluation map ev_t is defined in (33). Also for $t \in [0,T]$ and all $\varphi \in X_{\psi,\mu}$ one has $ev_t(S(\varphi)) = ev_t x^{(\varphi)} = x_t^{(\varphi)} = F(t,\varphi)$, which implies (see (32))

$$
v_t = ev_t DS(\phi)\chi = D(ev_t \circ S)(\phi)\chi = D_2F(t,\chi).
$$

To show that v satisfies the integral variant of equation (34) i.e., it is a mild solution to (34), we first remind (31) and notation $\hat{\varphi}(t) = (E_T\varphi)(t)$ (20). For $t > 0$ we have

$$
\begin{aligned}
S(\varphi)(t) &= x^{(\varphi)}(t) = y^{(\varphi)} + E_T\varphi \equiv Y_{T\mu}(\varphi) + E_T\varphi \\
&= e^{-At}\varphi(0) - \varphi(0) - t\dot{\varphi}(0) + \int_0^t e^{-A(t-\tau)}F(y_\tau + \hat{\varphi}_\tau)\,d\tau + \varphi(0) + t\dot{\varphi}(0) \\
&= e^{-At}\varphi(0) + \int_0^t e^{-A(t-\tau)}F(y_\tau + \hat{\varphi}_\tau)\,d\tau.
\end{aligned}
$$

Hence

$$S(\phi)(t) = e^{-At}\phi(0) + \int_0^t e^{-A(t-\tau)} F(x_\tau^{(\phi)})\, d\tau, \qquad t > 0, \tag{36}$$

and the definition $v \equiv DS(\phi)\chi \in C^1([-h,T]; L^2(\Omega))$ gives for $t > 0$

$$v(t) = (DS(\phi)\chi)(t) = \chi(0) + \int_0^t e^{-A(t-\tau)} DF(x_\tau^{(\phi)})\, v_\tau\, d\tau.$$

For more details see [27, p. 58]. So v is a mild solution to (34).

Remark 8. To differentiate the nonlinear term in (36) we apply the same result on the smoothness of the substitution operator as in [27, p. 51]. More precisely, we consider an open set $U \subset C^1([-h,0]; L^2(\Omega))$ and the open set

$$U_T \equiv \{\eta \in C([0,T]; C^1([-h,0]; L^2(\Omega))) : \eta(t) \in U \text{ for all } \quad t \in [0,T]\}.$$

It is proved in [4, Appendix IV, p. 490] that the substitution operator $F_T : U_T \ni \eta \mapsto F \circ \eta \in C([-h,0]; L^2(\Omega))$ is C^1-smooth, with $(DF_T(\eta)\chi)(t) = DF(\eta(t))\chi(t)$ for all $\eta \in U_T$, $\chi \in C([0,T]; C^1([-h,0]; L^2(\Omega)))$, $t \in [0,T]$.

To show that v is classical solution we remind first that Assumption (H4) gives the (local) Lipschitz property for the Frechet derivative $DF : X \supset U \to L^2(\Omega)$ here $U \subset X$ is an open set. We remind (see e.g. [8, p. 466]) the form of DF using the restricted evaluation map (not to be confused with the evaluation map ev_t defined in (33))

$$Ev : C^1([-h,0]; L^2(\Omega)) \times [-h,0] \ni (\phi, s) \mapsto \phi(s) \in L^2(\Omega)$$

which is continuously differentiable, with $D_1 Ev(\phi,s)\chi = Ev(\chi,s)$ and $D_2 Ev(\phi,s)1 = \varphi'(s)$. Hence we write our delay term F as the composition $F \equiv B \circ Ev \circ (id \times (-r))$ (see (3)) which is continuously differentiable from U to $L^2(\Omega)$, with

$$\begin{aligned} DF(\phi)\chi &= DB(\phi(-r(\phi)))[D_1 Ev(\phi, -r(\phi))\chi - D_2 Ev(\phi, -r(\phi)) Dr(\phi)\chi] \\ &= DB(\phi(-r(\phi)))[\chi(-r(\phi)) - \phi'(-r(\phi)) Dr(\phi)\chi] \end{aligned} \tag{37}$$

for $\phi \in U$ and $\chi \in C^1([-h,0]; L^2(\Omega))$.

Mappings B and r satisfy (H4) and we remind (see Remark 7) that our F satisfies the condition similar to (S) in [8, p. 467]. For an example of a delay term see below.

The (local) Lipschitz property for the Frechet derivative $DF : X \to L^2(\Omega)$ and the additional smoothness of the initial function $\chi \in T_\phi X_F \subset X$ gives the possibility to apply Theorem 2 to show that v is a classical solution to (34). □

Define the set $\Upsilon = \bigcup_{\phi \in X} [0, t(\phi)) \times \{\phi\} \subset [0, \infty) \times X$ and the map $G : \Upsilon \to X$ given by the formula $G(t, \phi) = x_t^{\phi}$. Propositions 1–6 combined lead to the following

Theorem 3. *Assume (H1)–(H4) are satisfied. Then G is continuous, and for every $t \geq 0$ such that $\Upsilon_t \neq \emptyset$ the map G_t is C^1-smooth. For every $(t, \phi) \in \Upsilon$ and for all $\chi \in T_\phi X$, one has $DG_t(\phi)\chi = v_t$ with $v : [-h, t(\phi)) \to L^2(\Omega)$ is C^1-smooth and satisfies $\dot{v}(t) = Av(t) + DF(G(t, \phi))v_t$, for $t \in [0, t(\phi))$, $v_0 = \chi$.*

4. Example of a state-dependent delay

Consider the following example of the delay term used, for example, in population dynamics [15, p. 191]. It is the so-called, threshold condition.

The state-dependent delay $r : C([-h, 0]; L^2(\Omega)) \to [0, h]$ is given implicitly by the following equation

$$R(r; \varphi) = 1, \tag{38}$$

where

$$R(r; \varphi) \equiv \int_{-r}^{0} \left(\frac{C_1}{C_2 + \int_\Omega \varphi^2(s)(x)\, dx} + C_3 \right) ds, \quad C_i > 0. \tag{39}$$

Since

$$D_1 R(r(\varphi); \varphi) \cdot Dr(\varphi)\psi + D_2 R(r(\varphi); \varphi)\psi = 0$$

and

$$D_1 R(r(\varphi); \varphi) \cdot 1 = \left(\frac{C_1}{C_2 + \int_\Omega \varphi^2(-r)(x)\, dx} + C_3 \right) \cdot 1 \neq 0, \quad C_i > 0,$$

$$D_2 R(r(\varphi); \varphi)\psi = -\int_{-r}^{0} \left\{ \frac{C_1}{[C_2 + \int_\Omega \varphi^2(s)(x)\, dx]^2} \cdot 2 \cdot \int_\Omega \varphi(s)(x) \cdot \psi(s)(x)\, dx \right\} ds,$$

we have

$$Dr(\varphi)\psi = \left(\frac{C_1}{C_2 + \int_\Omega \varphi^2(-r)(x)\, dx} + C_3 \right)^{-1}$$
$$\times \int_{-r(\varphi)}^{0} \left\{ \frac{C_1}{[C_2 + \int_\Omega \varphi^2(s)(x)\, dx]^2} \cdot 2 \cdot \int_\Omega \varphi(s)(x) \cdot \psi(s)(x)\, dx \right\} ds. \tag{40}$$

Now, we substitute the above form of $Dr(\varphi)\psi$ into (37) and arrive at

$$DF(\varphi)\psi = DB(\varphi(-r(\varphi)))\Bigg[\psi(-r(\varphi)) - \varphi'(-r(\varphi)) \times \left(\frac{C_1}{C_2 + \int_\Omega \varphi^2(-r)(x)\,dx} + C_3\right)^{-1} \times \int_{-r(\varphi)}^{0} \left\{ \frac{C_1}{[C_2 + \int_\Omega \varphi^2(s)(x)\,dx]^2} \cdot 2 \cdot \int_\Omega \varphi(s)(x) \cdot \psi(s)(x)\,dx \right\} ds \Bigg]. \tag{41}$$

We see that mapping r satisfies (H4). We also remind (see Remark 7) that in this example F satisfies the condition similar to (S) in [8, p. 467], provided operator $B : L^2(\Omega) \to D(A^\alpha)$ (for some $\alpha > 0$) is C^1-smooth.

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