

# Achievable Structures at Infinity of Linear Systems Decoupled by Non-regular Static State Feedback

Petr Zagalak\* and Vladimír Kučera\*\*

\* *Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, P.O.Box 18, 18208 Prague, Czech Republic (Tel: (420)266052367; e-mail: zagalak@utia.cas.cz).*

\*\* *Faculty of Electrical Engineering, Czech Technical University in Prague, Technick 2, 16627 Prague, Czech Republic (e-mail:kucera@fel.cvut.cz).*

**Abstract:** Morgan's problem, or the row-by-row decoupling of linear systems by state feedback, has attracted control theorists for fifty years. In spite of that, the problem has not been completely solved yet. This paper considers a simple case of Morgan's problem in which the system has already been decoupled and has an integrator dynamics. The objective is to characterize all achievable sets of the decouplability indices (infinite zero orders of the decoupled system).

Keywords: Linear systems; state feedback; decoupling.

## 1. INTRODUCTION

Morgan's problem, see (MorJr., 1964), is as follows: Given a linear, finite-dimensional, time-invariant system  $(A, B, C, D)$  governed by the equations

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D \in \mathbb{R}^{p \times m}$ . It is assumed that the pair  $(A, B)$  is controllable and the transfer function of the system,

$$T(s) := C(sI_n - A)^{-1}B + D$$

has rank  $p$ . Determine a state feedback control law  $(F, G)$  given by

$$u = Fx + Gv,$$

where  $F \in \mathbb{R}^{m \times n}$ ,  $G \in \mathbb{R}^{m \times p}$  with  $\text{rank } G = p$ , and a list of non-negative integers  $r_1 \geq r_2 \geq \dots \geq r_p$  such that the transfer function of the closed-loop system is diagonal,

$$T_{F,G}(s) := C(sI_n - A - BF)^{-1}BG = \Lambda_r^{-1}(s) \quad (1)$$

where  $\Lambda_r(s) := \text{diag}\{s^{r_i}\}_{i=1}^p$ . The integers  $r_1, r_2, \dots, r_p$  are called the decouplability indices of the system and, in fact, are equal to the orders of the zero of the decoupled system at  $s = \infty$ .

This formulation of Morgan's problem is specific in that (1) has only integrator dynamics. No matter what the decoupled dynamics is, however, it is the lists of the decouplability indices that classify all solutions to the problem.

If  $m = p$ , the matrix  $G$  is square and non-singular and the corresponding feedback is called regular. Such feedback has no potential to change the infinite zero orders of the system. If  $m > p$ , the matrix  $G$  is no longer square and the resulting feedback is called non-regular. Such a feedback law can alter the infinite zero orders of the system (Loiseau, 1988) and will thus be studied in this paper.

Without any loss of generality, suppose that the system has been put to the Morse canonical form (Morse, 1973; Kitapci and Silverman, 1984)

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{p_1} \end{bmatrix}$$

where

$$A_1 \in \mathbb{R}^{n_1 \times n_1}, \quad A_2 \in \mathbb{R}^{n_2 \times n_2}, \quad B_1 \in \mathbb{R}^{n_1 \times p_1},$$

$$B_2 \in \mathbb{R}^{n_2 \times (m-p)}, \quad C_1 \in \mathbb{R}^{p_1 \times n_1},$$

$n = n_1 + n_2$ ,  $n_2 > 0$  is the dimension of unobservable subspace of the system and  $p_1 + p_2 = p$  with  $p_2 = \text{rank } D$ .

The unobservability assumption allows for a non-trivial output-nulling controllability subspace that can be exploited to alter the decouplability indices of the system.

Denote  $\rho := \{\rho_i\}_{i=1}^p$ ,  $\rho_1 \geq \rho_2 \geq \dots \geq \rho_p$  the list of the controllability indices of  $(A_1, B_1)$ . Note that  $\rho$  yields the infinite zero structure of  $(A, B, C, D)$ .

Denote  $\sigma := \{\sigma_i\}_{i=1}^{m-p}$ ,  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p$  the list of the controllability indices of  $(A_2, B_2)$  and suppose, sacrificing no generality, that  $(A_2, B_2)$  is in the Brunovsky canonical form (Brunovský, 1970).

We now make the essential assumption that the system has already been decoupled. Thus,

$$T(s) = \begin{bmatrix} C_1(I_{n_1} - A_1)^{-1}B_1 & 0 \\ 0 & I_{p_2} \end{bmatrix} = \begin{bmatrix} \Lambda_\rho^{-1}(s) & 0 \\ 0 & I_{p_2} \end{bmatrix}$$

where  $\Lambda_\rho^{-1}(s) := \text{diag}\{s^{-\rho_i}\}_{i=1}^{p_1}$ . It follows that Morgan's problem for the system  $(A, B, C, D)$  reduces to Morgan's problem for the system  $(A, B, C, 0)$ , given by

$$\bar{A} = A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \bar{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \bar{C} = [C_1, 0]$$

whose transfer function is

$$\bar{T}(s) = C_1(I_{n_1} - A_1)^{-1}B_1 = \Lambda_\rho^{-1}(s)$$

The list  $\rho := \{\rho_i\}_{i=1}^p$ ,  $\rho_1 \geq \rho_2 \geq \dots \geq \rho_p$  provides the minimal decouplability indices, and the infinite zero orders, of  $(\bar{A}, \bar{B}, \bar{C}, 0)$ .

The objective of the paper is to characterize all achievable sets of decouplability indices  $r_1 \geq r_2 \geq \dots \geq r_p$  for which (1) holds when a non-regular state feedback control law  $(F, G)$  is applied to  $(A, B, C, D)$  in solving Morgan's problem. This amounts to characterizing all lists

$$r := \{r_i\}_{i=1}^p, r_1 \geq r_2 \geq \dots \geq r_p$$

of the decouplability indices of  $(\bar{A}, \bar{B}, \bar{C}, 0)$  for which

$$T_{\bar{F}, \bar{G}}(s) := \bar{C}(sI_{n_1} - \bar{A} - \bar{B}\bar{F})^{-1}\bar{B}\bar{G} = \Lambda_r^{-1}(s) \quad (2)$$

holds when state feedback  $(\bar{F}, \bar{G})$  is applied. The remaining decouplability indices  $r_{p_1} + 1 = \dots = r_p$  cannot be affected by state feedback and are zero.

The literature on the Morgan problem is rich. A complete solution is not available but many results exist that concern some special cases (Falb and Wolovich, 1967; Morse and Wonham, 1971; Descusse et al., 1988). In this paper, we shall follow the polynomial approach introduced in (Zagalak et al., 1993) and elaborate on the preliminary result (Eldem et al., 1997).

## 2. FINITE LISTS OF INTEGERS

Let  $\mathcal{L}_p$  denote the set of finite lists  $\xi := \{\xi_i\}_{i=1}^p$ ,  $\xi_1 \geq \xi_2 \geq \dots \geq \xi_p$  of non-negative integers such that  $\xi_1 + \xi_2 + \dots + \xi_p \leq p$ . Let  $l(\xi)$  denote the highest index  $i$  for which  $\xi_i > 0$ . The set  $\mathcal{L}_p$  is an ordered set with respect to the order relation  $\ll$ , which is called dominance, and is defined in (Loiseau et al., 1995) by

$$\alpha \ll \beta \iff \sum_{j=i}^p \alpha_j \leq \sum_{j=i}^p \beta_j$$

for  $i = 1, 2, \dots, p$  and  $\alpha, \beta \in \mathcal{L}_p$

Given a list  $\xi \in \mathcal{L}_p$  one can construct the so-called conjugated list  $\xi^*$ , which also lies in  $\mathcal{L}_p$  and which is defined by

$$\xi_i^* := \text{card}\{j \mid \xi_j \geq i\}$$

for  $i = 1, 2, \dots, p$ .

It can easily be observed that  $(\xi^*)^* = \xi$ , which means that there exists a one-to-one correspondence between  $\xi$  and  $\xi^*$ . Another order on  $\mathcal{L}_p$ , which is called *the conjugated majorization* and denoted by  $<$ , is defined as follows.

$$\alpha < \beta \iff \sum_{j=1}^i \alpha_j \leq \sum_{j=1}^i \beta_j$$

for  $i = 1, 2, \dots, p$  and  $\alpha, \beta \in \mathcal{L}_p$ . It has been shown in (Loiseau, 1988) that

$$\alpha \ll \beta \iff \beta^* > \alpha^*.$$

When dealing with the lists whose sums are equal, it is convenient to define the subset  $\mathcal{L}_p^*$  of the set  $\mathcal{L}_p$ :

$$\alpha, \beta \in \mathcal{L}_p^* \iff \sum_{j=1}^p \alpha_j = \sum_{j=1}^p \beta_j.$$

For such lists, which have been extensively studied in (Hardy et al., 1967; Brylawski, 1973), for instance, the following holds.

$$\begin{aligned} \alpha \ll \beta &\iff \beta^* \ll \alpha^* \\ \alpha < \beta &\iff \beta^* < \alpha^* \\ \alpha \ll \beta &\iff \alpha > \beta \end{aligned} \quad (3)$$

Denote  $\mathbf{p} := \{1, 2, \dots, p\}$  and define for  $q$ ,  $1 \leq q < p$ , the subset  $\mathbf{j}$  of  $\mathbf{p}$ ,

$$\mathbf{j} := \{0 < j_1 < j_2 < \dots < j_q \leq p\}.$$

The pair  $(\alpha, \beta)$ ,  $\alpha > \beta$ ,  $\alpha, \beta \in \mathcal{L}_p^*$  is called *reducible* if there exist a set of indexes  $\mathbf{j}$  for some  $q < p$  such that

$$\begin{aligned} \alpha^{(1)} &:= \{\alpha_{j_i}\}_{i=1}^q > \beta^{(1)} := \{\beta_{j_i}\}_{i=1}^q, \\ \alpha^{(2)} &:= \{\alpha_i\}_{i \in \mathbf{p}-\mathbf{j}} > \beta^{(2)} := \{\beta_i\}_{i \in \mathbf{p}-\mathbf{j}}; \end{aligned}$$

otherwise the lists are called *irreducible*. For every pair  $(\alpha, \beta)$  with  $\alpha, \beta \in \mathcal{L}_p^*$ ,  $\alpha > \beta$ , there exists  $k \geq 1$  such that  $(\alpha, \beta)$  can be decomposed as

$$\begin{aligned} \alpha &= \alpha^{(1)} \cup \alpha^{(2)} \cup \dots \cup \alpha^{(k)}, \\ \beta &= \beta^{(1)} \cup \beta^{(2)} \cup \dots \cup \beta^{(k)}, \end{aligned}$$

where  $\alpha^{(i)} \cap \alpha^{(j)} = \beta^{(i)} \cap \beta^{(j)} = \emptyset$ ,  $i, j \in \mathbf{k}$ ,  $i \neq j$  and the pairs  $(\alpha^{(i)}, \beta^{(i)})$ ,  $\alpha^{(i)} > \beta^{(i)}$ ,  $i \in \mathbf{k}$  are irreducible.

## 3. POLYNOMIAL AND RATIONAL MATRICES

Let  $\mathbb{R}^{p \times m}[s]$  denote the set of  $p \times m$  matrices of polynomials in the indeterminate  $s$  over the field of real numbers  $\mathbb{R}$ . A  $p \times p$  square matrix  $U(s) \in \mathbb{R}[s]$  is said to be unimodular if it is invertible and the inverse is polynomial. Let  $\mathbb{R}^{p \times m}(s)$  denote the set of  $p \times m$  matrices of rational functions in  $s$  over  $\mathbb{R}$ . In the sequel we denote by  $\Lambda_\alpha(s)$  the diagonal matrix  $\text{diag}\{s^{\alpha_i}\}_{i=1}^p$  where  $\alpha := \{\alpha_i\}_{i=1}^p$ . A matrix  $P(s) \in \mathbb{R}^{p \times m}(s)$  is proper rational if  $\lim_{s \rightarrow \infty} P(s)$  is a real  $p \times m$  matrix. A square proper rational matrix  $B(s) \in \mathbb{R}^{p \times m}(s)$  is said to be biproper if it is invertible and the inverse is proper rational. A non-square  $p \times m$  proper rational matrix  $B(s) \in \mathbb{R}^{p \times m}(s)$  with  $p < m$  is said to be row biproper if it can be completed to a biproper  $m \times m$  matrix.

The following two results can be found in (Rosenbrock, 1970; Kailath, 1980; Zagalak et al., 1993).

*Lemma 1.* Let  $P(s) \in \mathbb{R}^{m \times m}[s]$  with invariant factors  $\psi_1(s) \triangleright \psi_2(s) \triangleright \dots \triangleright \psi_m(s)$  where  $\psi_i(s) \triangleright \psi_{i+1}(s)$  means that  $\psi_{i+1}(s)$  divides (without remainder)  $\psi_i(s)$ . Let  $c_i := \text{deg}_{c_i} P(s)$  denote the  $i$ -th column degree of  $P(s)$  and let  $c_1 \geq c_2 \geq \dots \geq c_m$ . Then

$$\{c_i\} \gg \{\text{deg } \psi_i(s)\}.$$

In (Zagalak et al., 1993), implicit necessary and sufficient conditions for the existence of a solution to Morgan's problem were established. These conditions are adapted here to the case under consideration as follows.

*Proposition 2.* Let a system  $(\bar{A}, \bar{B}, \bar{C}, 0)$  and a list  $r := \{r_i\}_{i=1}^p$  of non-negative integers  $r_1 \geq r_2 \geq \dots \geq r_p$ ,  $r_i \geq \rho_i$ ,  $i = 1, 2, \dots, p_1$  be given and let  $\nu := \{\nu_i\}_{i=1}^p$  be the list of non-increasingly ordered differences  $r_i - \rho_i$ ,  $i = 1, 2, \dots, p_1$ .

Then there exists a state feedback  $(\bar{F}, \bar{G})$  such that (1) holds if and only if there exist a row biproper matrix  $B(s) \in \mathbb{R}^{(m-p) \times p_1}(s)$  and a biproper matrix  $V(s) \in \mathbb{R}^{(m-p_2) \times (m-p_2)}(s)$  such that

(i)

$$V(s) \begin{bmatrix} \Lambda_\nu^{-1}(s) \\ B(s) \end{bmatrix} = Z, \quad (4)$$

where  $Z \in \mathbb{R}^{(m-p_2) \times p_1}$  is a full column rank matrix, and

(ii)

$$V(s)\Lambda_{\rho\sigma}(s) \in \mathbb{R}^{m \times m}[s], \quad (5)$$

with  $\Lambda_{\rho\sigma}(s) := \text{block diag} \{\Lambda_\rho(s), \Lambda_\sigma(s)\}$

The state feedback  $(\bar{F}, \bar{G})$  is then given by the relationships

$$\bar{F} = -X^{-1}Y \quad (6)$$

and

$$\bar{G} = X^{-1} \begin{bmatrix} I_p \\ 0 \end{bmatrix} \quad (7)$$

where the matrices  $X \in \mathbb{R}^{m \times m}$  and  $Y \in \mathbb{R}^{m \times n}$  with  $X$  non-singular form a constant solution to the equation

$$V(s)\Lambda_{\rho\sigma}(s) = XD(s) + YN(s), \quad (8)$$

and the matrices  $N(s)$  and  $D(s)$  are given by

$$(sI_n - \bar{A})^{-1}\bar{B} = N(s)D^{-1}(s),$$

$$N(s) = \begin{bmatrix} N_1(s) & 0 \\ 0 & N_2(s) \end{bmatrix}, \quad D(s) = \begin{bmatrix} D_1(s) & 0 \\ 0 & D_2(s) \end{bmatrix},$$

$$N_1(s) = \text{block diag} \left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\rho_j-1} \end{bmatrix} \right\}_{j=1}^{p_1}, \quad D_1(s) = \Lambda_\rho(s)$$

$$N_2(s) = \text{block diag} \left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\sigma_j-1} \end{bmatrix} \right\}_{j=1}^{m-p}, \quad D_2(s) = \Lambda_\sigma(s)$$

The next lemma represents a key technical result in our development.

*Lemma 3.* Let  $\alpha, \beta \in \mathcal{L}_p^*$ ,  $\alpha < \beta$  be irreducible lists with  $q := l(\alpha) = l(\beta)$  and denote  $\epsilon_i := \alpha_i - \beta_i$ ,  $i = 1, 2, \dots, q$ . Let further  $p > q$  be an integer and define the  $p \times q$  matrix

$$Q(s) := \begin{bmatrix} \Lambda_\beta(s) \\ 0 \end{bmatrix}$$

Then there exist unimodular matrices  $U(s) \in \mathbb{R}^{p \times p}[s]$  and  $V(s) \in \mathbb{R}^{q \times q}[s]$  such that the following holds.

- (a) The matrix  $M(s) := U(s)Q(s)V(s)$  is column reduced with column degrees  $\deg_{ci} M(s) = \alpha_i$  for  $i = 1, 2, \dots, q$ ;
- (b)  $\deg_{ci} U(s) = \kappa_i$ , where  $\kappa_i = 0$  if  $\epsilon_i < 0$  and  $\kappa_i = \epsilon_i$  for  $\epsilon_i > 0$ ,  $i = 1, 2, \dots, q$ ; the remaining  $p - q$  columns of  $U(s)$  are arbitrary but such that  $U(s)$  is unimodular;
- (c)  $\deg_{ci} V(s) < \beta_i$  for  $i = 1, 2, \dots, q$ .

#### 4. MAIN RESULTS

We extend the list  $\sigma$  by appending  $2p_1 - (m - p_2)$  zeros so that it belongs to  $\mathcal{L}_p$ , namely

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{m-p} \geq \sigma_{m-p+1} = \dots = \sigma_{p_1}$$

and assume, for the sake of simplicity, that the lists  $\nu$  and  $\sigma$  are of equal sum, i.e.  $\nu, \sigma \in \mathcal{L}_p^*$ .

The proof of the following theorem can be simplified if the conformal mapping  $s = d^{-1}$  is used. In particular, if this

mapping is applied to a  $p \times m$  proper rational matrix  $P(s)$ , then the matrix  $P(d)$  is polynomial in  $d$  (d-polynomial, we say) if and only if all the poles of  $P(s)$  are located at  $s = 0$ .

*Theorem 4.* Given a system  $(\bar{A}, \bar{B}, \bar{C}, 0)$  and a list  $r := \{r_i\}$  of non-negative integers  $r_1 \geq r_2 \geq \dots \geq r_{p_1}$  such that  $r_i > \rho_i$  for  $i = 1, 2, \dots, p_1$ . Let  $\nu$  be the list defined in Proposition 2. Then there exists a state feedback  $(\bar{F}, \bar{G})$  such that (1) holds if and only if

$$\nu_i = 0, \quad i = m - p + 1, \dots, p_1, \quad (9)$$

$$\nu > \sigma, \quad (10)$$

$$r_i \geq \sigma_i, \quad i = 1, 2, \dots, p_1. \quad (11)$$

**Proof.** (Necessity). Without any loss of generality we can assume that the matrix  $X$  in (8) equals the identity matrix. Then by (8)

$$\begin{bmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{bmatrix} := V(s) = \begin{bmatrix} I_p - F_{11}N_1(s)D_1^{-1} & -F_{12}N_2(s)D_2^{-1} \\ -F_{21}N_1(s)D_1^{-1} & I_p - F_{22}N_2(s)D_2^{-1} \end{bmatrix} \quad (12)$$

where the matrices  $V(s)$ ,  $N(s)$ ,  $D(s)$ , and  $F$  are partitioned compatibly with  $A$  and  $B$ . The relationship (12) implies that the matrices  $V(s)$ ,  $V_{11}(s)$  and  $V_{22}(s)$  are biproper while  $V_{12}(s)$  and  $V_{21}(s)$  are strictly proper. Moreover, as they have all their poles at  $s = 0$ , it follows that they are d-polynomial.

The relationship (4) further reveals that the matrices  $\Lambda_\nu(s)$  and  $B(s)$  are right coprime matrices over the ring  $\mathbb{R}_p(s)$ . And since  $\Lambda_\nu^{-1}(s)$  is a proper matrix, the matrix  $B(s)$  has to be row biproper.

Taking the limit of both sides of (4) as  $s \rightarrow \infty$ , we obtain

$$Z = \begin{bmatrix} \Lambda_\nu^{-1}(\infty) \\ B(\infty) \end{bmatrix} \quad (13)$$

which is a matrix of rank  $p_1$ . Since  $\text{rank } B(\infty) = m - p$ , we conclude that  $\text{rank } \Lambda_\nu^{-1}(\infty) = p_1 - (m - p)$ . Thus,  $\nu_i = 0$  for  $i = m - p + 1, \dots, p_1$ , which is (9).

Consider the submatrix  $\bar{Z}$  of  $Z$  that consists of the first  $p_1$  rows and the first  $m - p$  columns of  $Z$ . It follows from (13) that  $\bar{Z} = 0$ . Denote  $\bar{\Lambda}_\nu^{-1}(s)$  the first  $m - p$  columns of  $\Lambda_\nu^{-1}(s)$  and  $\bar{B}(s)$  the first  $m - p$  columns of  $B(s)$ . It follows from (9) that  $\bar{B}(s)$  is biproper. Then the corresponding part of equation (4) yields

$$V_{11}(s)\bar{\Lambda}_\nu^{-1}(s) = -V_{12}(s)\bar{B}(s)$$

As both  $V_{11}(s)$  and  $B(s)$  are biproper, the matrices  $\bar{\Lambda}_\nu^{-1}(s)$  and  $V_{12}(s)$  are equivalent over  $\mathbb{R}_p(s)$ ; namely, they possess the same invariant factors  $s^{-\nu_i}$ ,  $i = 1, 2, \dots, m - p$ .

Now recall that  $V_{12}(d)$  is d-polynomial. Let  $c_i := \deg_{ci} V_{12}(d)$ ,  $i = 1, 2, \dots, m - p$  and let  $\psi_1(d) \triangleright \psi_2(d) \triangleright \dots \triangleright \psi_{m-p}(d)$  be the invariant factors of  $V_{12}(d)$ . Then

$$\psi_i(d) = d^{\nu_i} \chi_i(d), \quad i = 1, 2, \dots, m - p,$$

where the polynomials  $\chi_i(d)$  have no roots at  $d = 0$ . Further, as  $c_i \leq \sigma_i$ ,  $i = 1, 2, \dots, m - p$ , we have by Lemma 1 that

$$\{\nu_i\} \ll \{\deg \psi_i\} \ll \{c_i\} \ll \{\sigma_i\}$$

Taking into account that  $\nu, \sigma \in \mathcal{L}_p^*$ , the order relationship (3) can be used to prove condition (10). To establish

condition (11), postmultiply (4) by  $\Lambda_r(s)$  and write the result in the form

$$Z\Lambda_r(s) - \begin{bmatrix} V_{11}(s) \\ V_{21}(s) \end{bmatrix} \Lambda_\rho(s) = \begin{bmatrix} V_{12}(s) \\ V_{22}(s) \end{bmatrix} B(s)\Lambda_r(s)$$

The term on the left hand side is s-polynomial and thus the term on the right hand side is s-polynomial, too.

Now, as the denominator of  $\begin{bmatrix} V_{12}(s) \\ V_{22}(s) \end{bmatrix}$  has the highest power of  $s$  equal to  $\sigma$  in column  $i$  and this power can further be increased when this matrix is postmultiplied by  $B(s)$ , it follows that  $\sigma_i \leq r_i$ ,  $i = 1, 2, \dots, m-p$ .

(Sufficiency). Suppose that the conditions (9), (10), and (11) are satisfied. The sufficiency part of the proof consists in constructing a pair of matrices  $(\bar{F}, \bar{G})$  that define the desired state feedback. In fact, the matrices  $V(s)$  and  $B(s)$  that occur in Proposition 2 will be constructed. The appropriate feedback can then be found using (6), (7), and (8).

We start by applying the conformal mapping  $s = d^{-1}$  to the matrix  $\bar{\Lambda}_\nu^{-1}(s)$  introduced in the necessity part of the proof. Thus,  $\bar{\Lambda}_\nu(d) := \bar{\Lambda}_\nu(s^{-1})$  is a d-polynomial matrix.

As (10) holds, Lemma 3 can be applied to the  $p_1 \times (m-p)$  matrix  $\Lambda_\nu(d)$  and to the lists  $\{\sigma_i\}_{i=1}^{m-p}$  and  $\{\nu_i\}_{i=1}^{m-p}$ .

These lists are supposed to be irreducible. If not, decompose them first as a union of irreducible constituents and then apply Lemma 3 to each of them. Thus, there exist a d-unimodular  $p_1 \times p_1$  matrix  $V_{11}(d)$  and a d-unimodular  $(m-p) \times (m-p)$  matrix  $\bar{B}(d)$  such that

$$V_{12}(d) = -V_{11}(d)\bar{\Lambda}_\nu(d)\bar{B}^{-1}(d)$$

is a column reduced polynomial matrix with column degrees  $\deg_{ci} V_{12}(d) = \sigma_i$ ,  $i = 1, 2, \dots, m-p$ . Moreover,  $V_{11}(d)$  has column degrees  $\deg_{ci} V_{11}(d) = \kappa_i$ , where  $\kappa_i = 0$  if  $\sigma_i - \nu_i < 0$  and  $\kappa_i = \sigma_i - \nu_i$  if  $\sigma_i - \nu_i > 0$ ,  $i = 1, 2, \dots, m-p$ . The remaining  $p_1 - (m-p)$  columns of  $V_{11}(d)$  will be taken to be  $\begin{bmatrix} V_{12}(s) \\ V_{22}(s) \end{bmatrix}$ .

To construct the d-polynomial matrices  $V_{21}(d)$  and  $V_{22}(d)$ , consider the equation

$$V_{21}(d)\bar{\Lambda}_\nu(d) + V_{22}(d)\bar{B}(d) = I_{m-p} \quad (14)$$

By Lemma 3,  $\bar{B}(d)$  is d-unimodular so that we can take  $V_{21}(d) = 0$  and  $V_{22}(d) = \bar{B}^{-1}(d)$ . Lemma 2 also implies that  $\deg_{ci} \bar{B}^{-1}(d) < \sigma_i$ ,  $i = 1, 2, \dots, m-p$ .

Now define  $V(d)$  by

$$V(d) := \begin{bmatrix} V_{11}(d) & V_{12}(d) \\ V_{21}(d) & V_{22}(d) \end{bmatrix} \quad (15)$$

Expanding matrices  $\bar{\Lambda}_\nu(d)$  and  $\bar{B}(d)$  as follows,

$$\bar{\Lambda}_\nu(d) := \begin{bmatrix} \bar{\Lambda}_\nu(d) & 0 \\ 0 & I_{p_1 - (m-p)} \end{bmatrix}, B(d) := [\bar{B}(d), 0]$$

relationship (4) is satisfied for

$$Z := \begin{bmatrix} 0 & 0 \\ 0 & I_{p_1 - (m-p)} \\ I_{m-p} & 0 \end{bmatrix},$$

The column degrees of the matrix  $V(d)$  in (15) satisfy

$$\deg_{ci} \begin{bmatrix} V_{11}(d) \\ V_{21}(d) \end{bmatrix} = \kappa_i, \quad i = 1, 2, \dots, m-p$$

$$\deg_{ci} \begin{bmatrix} V_{11}(d) \\ V_{21}(d) \end{bmatrix} = 0, \quad i = m-p+1, \dots, p_1$$

and

$$\deg_{ci} \begin{bmatrix} V_{12}(d) \\ V_{22}(d) \end{bmatrix} = \sigma_i, \quad i = 1, 2, \dots, m-p$$

Since (11) holds,  $\kappa_i = \sigma_i - \nu_i \leq r_i - \nu_i = \rho_i$  so that  $\kappa_i \leq \rho_i$  for  $i = 1, 2, \dots, m-p$ . Consequently,  $V_{11}(d)$  has column degrees  $\deg_{ci} V_{11}(d) \leq \rho_i$ ,  $i = 1, 2, \dots, p_1$ .

Having constructed matrices  $V(d)$  and  $B(d)$ , we apply the inverse mapping  $d = s^{-1}$  to obtain  $V(s) := V(d^{-1})$  and  $B(s) := B(d^{-1})$ . These matrices clearly satisfy both conditions (i) and (ii) of Proposition 1. The appropriate feedback  $(\bar{F}, \bar{G})$  can then be found using (6), (7), and (8).

## ACKNOWLEDGEMENTS

P. Zagalak and V. Kučera were supported by Czech Science Foundation under projects 16-06678S and 16-17398S.

## REFERENCES

- Brunovský, P.(1970). A classification of controllable linear systems. *Kybernetika* 3, 173-188.
- Brylawski, T.(1973). The lattice of integer partitions. *Discrete Mathematics* 6, pp.201-219.
- Descusse J., J.F. Lafay, M. Malabre (1988). Solution to Morgan's Problem. *IEEE Trans. Automat. Control*, AC-33, No.8, pp.732-739.
- Eldem, V., K. Özçaldıran, and P. Zagalak(1997). A Note on Diagonal Decoupling. *Proc.36th IEEE Conference on Decision and Control, IEEE Contr.Sys.Soc., San Diego, California USA*, pp.1425-1430.
- Falb, P. L., and W. A. Wolovich (1967). Decoupling in the design of multivariable control systems. *IEEE Trans.Automat. Control*, AC-12, No.6, pp.651-669.
- Hardy, G.H., J. E. Littlewood, and G. Polya(1967). Inequalities. *Cambridge University Press*, Cambridge, UK.
- Kailath, T.(1980). Linear Systems. *Prentice Hall*, Englewood Cliffs, N.J.
- Kitapci, A. and L.M. Silverman(1984). Determination of Morse's canonical form using the structure algorithm. In: *Proc. 23rd Conf. Decision and Control*, Las Vegas, Nevada, USA, pp.1752-1757. .
- Loiseau, J.J.(1988). Sur la modification de la structure à l'infini par un retour d'état statique. *SIAM J. Control and Optimization*, Vol.26, No.2, pp.251-273.
- Loiseau, J.J., P. Zagalak, and V. Kučera (1995). Pole Structure Assignment via Non-regular Static State Feedback. *Proc. of the IFAC Conference on System Structure and Control*, Ecole Centrale, Nantes, pp.595-600.
- Morgan, B. Jr. (1964). Synthesis of linear multivariable systems by state feedback. *Proc. Joint Automatic Control Conference*, Standford University, California, USA, pp.468-472.
- Morse, A. S.(1973). Structural invariants of linear multivariable systems. *SIAM J. Control and Optimization*, Vol. 11, No., pp.446-465.
- Morse, A. S. and W.M. Wonham(1971). Status of Noninteracting Control. *IEEE Trans. Automat. Control*, AC-16, No.6, pp.568-581.

- Rosenbrock, H. H.(1970). *State-space and Multivariable Theory*. Wiley, New York.
- Zagalak, P., V.Kučera, and J.J.Loiseau(1993). Eigenstructure Assignment in Linear Systems by State Feedback. *Polynomial Methods in Optimal Control and Filtering*, ed.K.J.Hunt, Peter Peregrinus, London, pp.210–248.
- Zagalak, P., J.F.Lafay, and A.N.Herrera-Hernandez(1993). The Row-by-row Decoupling via State Feedback: A Polynomial Approach. *Automatica*, Vol.29, No.6, pp.1491–1499.