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DEA models equivalent to general Nth order stochastic dominance efficiency tests

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ABSTRACT

We introduce data envelopment analysis (DEA) models equivalent to efficiency tests with respect to the Nth order stochastic dominance (NSD). In particular, we focus on strong and weak variants of convex NSD efficiency and NSD portfolio efficiency. The proposed DEA models are in relation with strong and weak Pareto–Koopmans efficiencies and employ Nth order lower and co-lower partial moments.

1. Introduction and notation

In this paper, we establish a link between data envelopment analysis (DEA) models [7] and Nth order stochastic dominance efficiency tests [13]. Such a link can be very useful because DEA literature provides many results on stability and sensitivity with respect to the input data, algorithmic issues, and methods on ranking efficient units (super-efficiency, cross-efficiency, etc., see [8] for a review) which can be applied to stochastic dominance theory. Moreover, the utility-based interpretation of stochastic dominance relations and efficiency (see [9,11,12,14]) can be used for the proposed DEA models. We generalize results on equivalence obtained for second-order stochastic dominance (SSD) by [5].

The proposed DEA models are derived from NSD efficiency tests introduced in [13]. We show how equivalent DEA models can be obtained using particular directional distance measures, cf. [4], which modify the well-known directional distance function, cf. [6]. The tests by [13] were proposed for the weak variants of the convex NSD efficiency and NSD portfolio efficiency only. Thus, we extend the analysis to the strong efficiencies and we show how these notations differ on a simple example. Note that the optimal solutions of the proposed DEA models are weakly or strongly Pareto–Koopmans efficient.

Let $x_{j,r}$ denote return of asset $j \in \{1, \dots, M\}$ taken with probability p_r, $r = 1, \dots, R$. We assume that the columns of $\{x_{j,r}\}$ are sorted in ascending order according to prospect $\tau = (\tau_1, \dots, \tau_M)$, i.e. if we set $x_r^* = \sum_{j=1}^M x_{j,r}\tau_j$, we have $x_1^* \le x_2^* \le \cdots \le x_R^*$. We denote by $y_1 < \cdots < y_S$ all sorted returns, where $S \le MR$. We use $q_{j,s} = \sum_{r=1}^R p_r \mathbb{I}_{(x_{j,r}=y_s)}$ for all $j \in \{1, \dots, M\}, s \in \{1, \dots, S\}$, where $\mathbb{I}_{(\cdot)}$ is equal to one if the condition $\cdot$ is fulfilled and to zero otherwise.

Portfolios are identified by (nonnegative) weights $\lambda = (\lambda_1, \dots, \lambda_M)$ such that $\sum_{j=1}^M \lambda_j = 1$. The set of all feasible portfolio weights is denoted by Λ. For $n = 0, 1, \dots,$ we define nth order lower partial moment of the ith asset by

$$\mathrm{LPM}_i^n(w) = \sum_{r=1}^R p_r [w - x_{i,r}]_+^n = \sum_{s=1}^S q_{i,s}[w - y_s]_+^n,$$

and nth order co-lower partial moment [1] by

$$\mathrm{coLPM}_{\tau,\lambda}^n(w) = \sum_{r=1}^R p_r \left(w - \sum_{j=1}^M x_{j,r}\lambda_j \right) \left[w - \sum_{j=1}^M x_{j,r}\tau_j \right]_+^n,$$

where $[y]_+^n = y^n \mathbb{I}_{(y \ge 0)}$. Note that its most important property is linearity with respect to the weights λ_j when portfolio weights τ_j and threshold w are given.

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2. DEA models

We employ DEA models based on directional distance measures. The choice of the (positive) direction is motivated by the paper [15]; see also [4] for a discussion in financial area. Employing a directional distance measure which expresses relative improvement necessary to reach the efficient frontier, see [4], a decision making unit (asset, portfolio) is classified as efficient if and only if the optimal value of the directional distance DEA model is equal to zero.

2.1. Weak Pareto–Koopmans efficiency

Paper [13] considered weak convex NSD efficiency and weak NSD portfolio efficiency.

2.1.1. Weak convex NSD efficiency

Let $U_N = \{u(x) : (-1)^n u^{(n)}(x) \le 0,\ \forall x,\ n = 1, \dots, N\}$ denote a subset of all utility functions, where $u^{(n)}$ is the nth derivative of u. Moreover, set $\bar{x}_j = \sum_{r=1}^{R} p_r x_{j,r},\ \forall j \in \{1, \dots, M\}$.

Definition 2.1 (*[13]*)**.** The ith asset is weakly convex NSD efficient (relative to the set of assets $\{1, \dots, M\}$), $N \ge 1$, if there exists a utility function $u \in U_N$ for which it is preferred to every asset:

$$\sum_{r=1}^{R} p_r u(x_{i,r}) \ge \sum_{r=1}^{R} p_r u(x_{j,r}) \quad \forall j = 1, \dots, M.$$

If such an admissible utility function does not exist the ith asset is called weakly convex NSD inefficient.

Proposition 2.1. *Consider the directional distance DEA model*

$$\max_{\lambda_j, \theta} \theta$$

$$\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \ge \bar{x}_i,$$

$$\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^n(y_S) \le \mathrm{LPM}_i^n(y_S), \quad n = 2, \dots, N-2,$$

$$\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^{N-1}(y_k) \le \mathrm{LPM}_i^{N-1}(y_k), \quad k = 1, \dots, S-1,$$

$$\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^{N-1}(y_S) \le \mathrm{LPM}_i^{N-1}(y_S) - \theta \cdot d,$$

$$\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \ge 0,\ j = 1, 2, \dots, M,$$

with the direction

$$d = \mathrm{LPM}_i^{N-1}(y_S) - \min_j \mathrm{LPM}_j^{N-1}(y_S).$$

If $d > 0$ then ith asset is weakly convex NSD efficient if and only if the optimal value of the directional distance DEA model is equal to zero, that is, the ith asset is DEA efficient. Moreover, if the direction is equal to zero, then the ith asset is weakly convex NSD efficient.

Proof. We employ the problem formulated in [13]; see Theorem 2. Since $y_S \ge x_{j,r},\ \forall j \in \{1, \dots, M\},\ \forall r \in \{1, \dots, R\}$ we obtain for $n = 1$ and for all $j \in \{1, \dots, M\}$:

$$\mathrm{LPM}_j^1(y_S) = \sum_{r=1}^{R} p_r [y_S - x_{j,r}]_+^1 = y_S - \bar{x}_j.$$

The first constraint then easily follows from [13, constraint (23.1)] and the rest of the proof for $d > 0$ is straightforward. If the direction is equal to zero, then the ith asset is weakly convex NSD efficient, because no improvement to the efficient frontier is possible. □

Proposition 2.1 shows that if expected return serves as the output and the lower partial moments given in the constraints as the inputs to the directional distance DEA model then an asset is classified as convex NSD efficient if and only if either it is DEA efficient or the direction is equal to zero. Our model uses a special case of general directional distance function [6], where we consider a directional vector with only one positive element d corresponding to input $\mathrm{LPM}_j^{N-1}(y_S)$.

2.1.2. Weak NSD portfolio efficiency

Weakly convex NSD efficiency generally do not allow for fully diversification across the assets. Therefore we consider also weakly NSD portfolio efficiency. The notation of the employed diversification-consistent DEA models was established by [10] and further investigated by [2,3] in relation with the Pareto–Koopmans efficiency.

Definition 2.2 (*[13]*)**.** The portfolio $\tau \in \Lambda$ is weakly NSD portfolio efficient (relative to Λ), $N \ge 2$, if there exists a utility function $u \in U_N$ for which the portfolio τ is preferred to all portfolios $\lambda \in \Lambda$:

$$\sum_{r=1}^{R} p_r u\left(\sum_{j=1}^{M} x_{j,r}\tau_j\right) \ge \sum_{r=1}^{R} p_r u\left(\sum_{j=1}^{M} x_{j,r}\lambda_j\right) \quad \forall \lambda \in \Lambda.$$

If such an admissible utility function does not exist the portfolio τ is called weakly NSD portfolio inefficient.

Proposition 2.2. *Consider diversification-consistent DEA model based on a directional distance measure*

$$\max_{\lambda_j, \theta} \theta$$

$$\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \ge \sum_{j=1}^{M} \tau_j \cdot \bar{x}_j,$$

$$\mathrm{coLPM}_{\tau,\lambda}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right),$$

$$n = 2, \dots, N-2,$$

$$\mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right),$$

$$k = 1, \dots, R-1,$$

$$\mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) - \theta \cdot d,$$

$$\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \ge 0,\ j = 1, 2, \dots, M,$$

with the direction

$$d = \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) - \min_{\lambda \in \Lambda} \mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right).$$

If $d > 0$ then portfolio τ is weakly NSD portfolio efficient if and only if the optimal value of the diversification-consistent DEA model is equal to zero, that is, portfolio τ is DEA efficient. Moreover, if the direction is equal to zero, then portfolio τ is weakly NSD portfolio efficient.

Proof. The result follows by considering the second model in the proof of Theorem 3 of [13]. There we replace υ by θd in the instance where it is a slack variable. And we replace it by θ in the instance where it appears in the objective function. Since d is constant, the optimal values of λ do not change.

For $w = \sum_{j=1}^{M} x_{j,R}\tau_j$ and $n = 1$

$$\begin{aligned}\mathrm{coLPM}^0_{\tau,\lambda}(w) &= \sum_{r=1}^{R} p_r \left(w - \sum_{j=1}^{M} x_{j,r}\lambda_j\right)\left[w - \sum_{j=1}^{M} x_{j,r}\tau_j\right]_+^0 \\ &= \sum_{r=1}^{R} p_r \left(w - \sum_{j=1}^{M} x_{j,r}\lambda_j\right) \\ &= w - \sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j.\end{aligned}$$

Similarly $\mathrm{coLPM}^0_{\tau,\tau}(w) = w - \sum_{j=1}^{M} \tau_j \cdot \bar{x}_j$. Thus, if we subtract w and multiply the constraint by -1, we obtain an equivalent constraint for $n = 1$:

$$\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \geq \sum_{j=1}^{M} \tau_j \cdot \bar{x}_j.$$

If the direction d is equal to zero, then portfolio τ is weakly NSD portfolio efficient, because no improvement to the efficient frontier is possible. □

The above theorem shows that to reach the equivalence with the NSD portfolio efficiency test [13, Theorem 3], we need to consider the diversification-consistent model where co-lower partial moments serve as the inputs and the expected return as the only output. Moreover, a positive direction is used to find an improvement in $\mathrm{coLPM}^{N-2}_{\tau,\cdot}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right)$.

2.2. Strong Pareto–Koopmans efficiency

In this section, the DEA models are generalized to be consistent with the strong Pareto–Koopmans NSD efficiency. Firstly, we slightly modify a set of considered utility functions as follows:

$$U_N^+ = \{u(x) : (-1)^n u^{(n)}(x) < 0,\ \forall x,\ n = 1, \ldots, N\}.$$

The strong counterparts of convex NSD and NSD portfolio efficiencies can be then obtained by a slight modification of Definitions 2.1 and 2.2, where we substitute U_N by U_N^+. Since set U_N^+ is not closed, we use an approximate set to construct the equivalent DEA models where the derivatives of utility functions are bounded by infinitesimal positive constants w_n, w'_k. Such infinitesimal constants are often used in DEA models.

We continue with a simple example where strongly and weakly SSD portfolio efficient assets do not match. Consider three assets $j \in \{1, 2, 3\}$ with three scenarios of returns $x_{j,r}$ listed in the following table

Asset/scenario	1	2	3
$x_{1,r}$	2	1	5
$x_{2,r}$	0	6	4
$x_{3,r}$	1	3	5

The first asset is both weakly and strongly SSD portfolio inefficient, whereas the second asset is strongly and weakly SSD efficient. The third asset is weakly SSD portfolio efficient but not strongly SSD efficient.

2.2.1. Strong convex NSD efficiency

We propose a DEA model with directional distance measure which is equivalent to the strong convex NSD efficiency test. We show that the model has the same structure of inputs and outputs as the one formulated in Proposition 2.1 for the weak convex NSD efficiency. However, the directional distance measure has to be extended to measure a possible improvement in each input and output.

Proposition 2.3. *Define the nonnegative directions*

$$\begin{aligned}d_1 &= \max_j \bar{x}_j - \bar{x}_i,\\ d_n &= \mathrm{LPM}_i^n(y_S) - \min_j \mathrm{LPM}_j^n(y_S), \quad n = 2, \ldots, N-2,\\ d'_k &= \mathrm{LPM}_i^{N-1}(y_k) - \min_j \mathrm{LPM}_j^{N-1}(y_k), \quad k = 1, \ldots, S,\end{aligned}$$

and consider the directional distance DEA model

$$\max_{\lambda_j,\theta_n,\theta'_k} \frac{1}{N+S-2}\left(\sum_{n=1}^{N-2}\theta_n + \sum_{k=1}^{S}\theta'_k\right)$$

$$\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \geq \bar{x}_i + \theta_1 \cdot d_1,$$

$$\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^n(y_S) \leq \mathrm{LPM}_i^n(y_S) - \theta_n \cdot d_n, \quad n = 2, \ldots, N-2,$$

$$\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^{N-1}(y_k) \leq \mathrm{LPM}_i^{N-1}(y_k) - \theta'_k \cdot d'_k, \quad k = 1, \ldots, S,$$

$$\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \geq 0,\ j = 1, 2, \ldots, M,$$

$$\begin{aligned}\theta_n &= 0, \quad \text{if } d_n = 0,\ n = 1, \ldots, N-2,\\ \theta_n &\geq 0, \quad \text{if } d_n > 0,\ n = 1, \ldots, N-2,\\ \theta'_k &= 0, \quad \text{if } d'_k = 0,\ k = 1, \ldots, S,\\ \theta'_k &\geq 0, \quad \text{if } d'_k > 0,\ k = 1, \ldots, S.\end{aligned}$$

If at least one direction is positive, then ith asset is strongly convex NSD efficient if and only if the optimal value of the directional distance DEA model is equal to zero, that is, the ith asset is DEA efficient. Moreover, if all directions are equal to zero, then the ith asset is strongly convex NSD efficient.

Proof. Following [9], we add conditions on positivity of derivatives, see Appendix A of [13] (proof of Theorem 2) where in model (25.1)–(25.3) decision variables b_n and c_k related to the derivatives of a utility function appear. In particular, if we add $b_n \geq w_n > 0$, $n = 1, \ldots, N-2$, and $c_k \geq w'_k > 0$, $k = 1, \ldots, S$, we obtain the dual LP formulation

$$\max_{\lambda_j,\upsilon_n,\upsilon'_k} \sum_{n=1}^{N-2} w_n\upsilon_n + \sum_{k=1}^{S} w'_k\upsilon'_k$$

$$\sum_{j=1}^{M}\sum_{s=1}^{S}(\lambda_j q_{j,s} - q_{i,s})(y_S - y_s)^n + \upsilon_n = 0, \quad n = 1, 2, \ldots, N-2,$$

$$\sum_{j=1}^{M}\sum_{s=1}^{k}(\lambda_j q_{j,s} - q_{i,s})(y_k - y_s)^{N-1} + \upsilon'_k = 0, \quad k = 1, 2, \ldots, S,$$

$$\sum_{j=1}^{M} \lambda_j = 1,\ \lambda_j \geq 0, \quad j = 1, 2, \ldots, M,$$

$$\begin{aligned}\upsilon_n &\geq 0, \quad n = 1, \ldots, N-2,\\ \upsilon'_k &\geq 0, \quad k = 1, \ldots, S.\end{aligned}$$

Using the definition of lower partial moments, the problem can be reformulated as

$$\max_{\lambda_j,\upsilon_n,\upsilon'_k} \sum_{n=1}^{N-2} w_n\upsilon_n + \sum_{k=1}^{S} w'_k\upsilon'_k$$

$$\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^n(y_S) \le \mathrm{LPM}_i^n(y_S) - \upsilon_n, \quad n = 1, \dots, N-2,$$

$$\sum_{j=1}^{M} \lambda_j \cdot \mathrm{LPM}_j^{N-1}(y_k) \le \mathrm{LPM}_i^{N-1}(y_k) - \upsilon'_k, \quad k = 1, \dots, S,$$

$$\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \ge 0,\ j = 1, 2, \dots, M,$$

$$\upsilon_n \ge 0, \quad n = 1, \dots, N-2,$$

$$\upsilon'_k \ge 0, \quad k = 1, \dots, S.$$

The constraint for $n = 1$ is transformed as in the proof of Proposition 2.1. Using the positive directions multiplied by decision variables θ_n, θ'_k instead of slack variables υ_n, υ'_k, i.e. we put $\upsilon_n = \theta_n d_n$, $\upsilon'_k = \theta'_k d'_k$, and by setting the positive weights in the objective function to

$$w_n = \frac{1}{(N+S-2)d_n}, \qquad w'_k = \frac{1}{(N+S-2)d'_k},$$

we obtain the final form of the DEA model. For zero directions, the corresponding decision variables θ_n, θ'_k are set to zero, because no improvement in these inputs and the output is possible. Moreover, if all directions are equal to zero, then no improvement to reach the efficient frontier is possible, thus the ith asset is strongly convex NSD efficient. □

Note that the objective function is usually normalized by the number of the decision variables in the objective function, in our case by $(N+S-2)$. However, this normalization has no influence on the efficiency classification.

2.2.2. *Strong NSD portfolio efficiency*

The DEA model equivalent to the strong NSD portfolio efficiency test employs the same inputs and outputs as the model introduced in Proposition 2.2 for the weak NSD portfolio efficiency, whereas the directional distance measure has to be again extended to investigate a possible improvement in any input and output.

Proposition 2.4. *Define the nonnegative directions*

$$d_1 = \max_j \bar{x}_j - \bar{x}_i,$$

$$d_n = \mathrm{coLPM}_{\tau,\tau}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) - \min_{\lambda\in\Lambda} \mathrm{coLPM}_{\tau,\lambda}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right),$$
$$n = 2, \dots, N-2,$$

$$d'_k = \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right) - \min_{\lambda\in\Lambda} \mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right),$$
$$k = 1, \dots, R,$$

and consider diversification-consistent DEA model based on directional distance measure

$$\max_{\lambda_j,\theta_n,\theta'_k} \frac{1}{N+R-2}\left(\sum_{n=1}^{N-2}\theta_n + \sum_{k=1}^{R}\theta'_k\right)$$

$$\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \ge \sum_{j=1}^{M} \tau_j \cdot \bar{x}_j + \theta_1 \cdot d_1,$$

$$\mathrm{coLPM}_{\tau,\lambda}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) - \theta_n \cdot d_n,$$
$$n = 2, \dots, N-2,$$

$$\mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right) - \theta'_k \cdot d'_k,$$
$$k = 1, \dots, R,$$

$$\sum_{j=1}^{M} \lambda_j = 1,\ \lambda_j \ge 0, \quad j = 1, 2, \dots, M,$$

$$\theta_n = 0, \quad \text{if } d_n = 0,\ n = 1, \dots, N-2,$$
$$\theta_n \ge 0, \quad \text{if } d_n > 0,\ n = 1, \dots, N-2,$$
$$\theta'_k = 0, \quad \text{if } d'_k = 0,\ k = 1, \dots, R,$$
$$\theta'_k \ge 0, \quad \text{if } d'_k > 0,\ k = 1, \dots, R.$$

If at least one direction is positive, then portfolio τ is strongly NSD portfolio efficient if and only if the optimal value of the diversification-consistent DEA model is equal to zero, that is, portfolio τ is DEA efficient. Moreover, if all directions are equal to zero, then portfolio τ is strongly NSD portfolio efficient.

Proof. Following [9], we add conditions on positivity of the derivatives, see Appendix of [13] (proof of Theorem 3). We obtain the dual formulation

$$\max_{\lambda_j,\upsilon_n,\upsilon'_k} \sum_{n=1}^{N-2} w_n\upsilon_n + \sum_{k=1}^{R} w'_k\upsilon'_k$$

$$\sum_{r=1}^{R} p_r\left(\sum_{j=1}^{M} x_{j,r}\tau_j - \sum_{j=1}^{M} x_{j,r}\lambda_j\right)\left(\sum_{j=1}^{M} x_{j,R}\tau_j - \sum_{j=1}^{M} x_{j,r}\tau_j\right)^{n-1} + \upsilon_n$$
$$= 0, \quad n = 1, 2, \dots, N-2,$$

$$\sum_{r=1}^{k} p_r\left(\sum_{j=1}^{M} x_{j,r}\tau_j - \sum_{j=1}^{M} x_{j,r}\lambda_j\right)\left(\sum_{j=1}^{M} x_{j,k}\tau_j - \sum_{j=1}^{M} x_{j,r}\tau_j\right)^{N-2} + \upsilon'_k$$
$$= 0, \quad k = 1, 2, \dots, R,$$

$$\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \ge 0,\ j = 1, 2, \dots, M,$$

$$\upsilon_n \ge 0, \quad n = 1, \dots, N-2, \qquad \upsilon'_k \ge 0, \quad k = 1, \dots, R.$$

We can rewrite the model using the co-lower partial moments which can serve to quantify the risk:

$$\max_{\lambda_j,\upsilon_n,\upsilon'_k} \sum_{n=1}^{N-2} w_n\upsilon_n + \sum_{k=1}^{R} w'_k\upsilon'_k$$

$$\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \ge \sum_{j=1}^{M} \tau_j \cdot \bar{x}_j + \upsilon_1,$$

$$\mathrm{coLPM}_{\tau,\lambda}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{n-1}\left(\sum_{j=1}^{M} x_{j,R}\tau_j\right) - \upsilon_n,$$
$$n = 2, \dots, N-2,$$

$$\mathrm{coLPM}_{\tau,\lambda}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right) \le \mathrm{coLPM}_{\tau,\tau}^{N-2}\left(\sum_{j=1}^{M} x_{j,k}\tau_j\right) - \upsilon'_k,$$
$$k = 1, 2, \dots, R,$$

$$\sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \ge 0,\ j = 1, 2, \dots, M,$$

$$\upsilon_n \ge 0, \quad n = 1, \dots, N-2,$$
$$\upsilon'_k \ge 0, \quad k = 1, \dots, R.$$

The constraint for $n = 1$ is transformed as in the proof of Proposition 2.2. Instead of the slack variables υ_n, υ'_k in the constraints and objective, we can substitute the positive directions multiplied by the new decision variables θ_n, θ'_k, i.e. we put $\upsilon_n = \theta_n d_n$ and $\upsilon'_k = \theta'_k d'_k$. These variables are then maximized in the objective function using weights

$$w_n = \frac{1}{(N+R-2)d_n}, \qquad w'_k = \frac{1}{(N+R-2)d'_k}.$$

For zero directions, the corresponding decision variables θ_n, θ'_k are set to zero, because no improvement in these inputs and the output is possible. Moreover, if all directions are equal to zero, then no improvement in all inputs and outputs is possible and the claim follows. □

Note that the computation of directions d_n, d'_k, in particular the minimization of co-lower partial moment $\text{coLPM}^n_{\tau,\lambda}(w)$ over weights λ_j, leads to a linear programming problem for arbitrary argument w:

$$\max_{\lambda_j} \sum_{j=1}^{M} \lambda_j \sum_{r=1}^{R} p_r x_{j,r} \left[w - \sum_{j=1}^{M} x_{j,r}\tau_j \right]_+^n$$

$$\text{s.t.} \sum_{j=1}^{M} \lambda_j = 1, \quad \lambda_j \geq 0,\ j = 1, 2, \ldots, M,$$

where the objective function is linear in decision variables λ_j.

2.2.3. Explicit linear programming formulation

All DEA models proposed in this paper can be solved as linear programming problems. Below, we provide a reformulation of the DEA model introduced in Proposition 2.4 for the strong NSD portfolio efficiency. Let us denote

$$a_{k,r} = \left[\sum_{j=1}^{M} x_{j,k}\tau_j - \sum_{j=1}^{M} x_{j,r}\tau_j \right]_+ .$$

Then we obtain the following linear program:

$$\max_{\lambda_j, \theta_n, \theta'_k} \frac{1}{N+R-2} \left(\sum_{n=1}^{N-2} \theta_n + \sum_{k=1}^{R} \theta'_k \right)$$

$$\sum_{j=1}^{M} \lambda_j \cdot \bar{x}_j \geq \sum_{j=1}^{M} \tau_j \cdot \bar{x}_j + \theta_1 \cdot d_1,$$

$$\sum_{j=1}^{M} \lambda_j \sum_{r=1}^{R} p_r x_{j,r} a_{R,r}^{n-1} \geq \sum_{j=1}^{M} \tau_j \sum_{r=1}^{R} p_r x_{j,r} a_{R,r}^{n-1} + \theta_n \cdot d_n, \quad n = 2, \ldots, N-2,$$

$$\sum_{j=1}^{M} \lambda_j \sum_{r=1}^{R} p_r x_{j,r} a_{k,r}^{N-2} \geq \sum_{j=1}^{M} \tau_j \sum_{r=1}^{R} p_r x_{j,r} a_{k,r}^{N-2} + \theta'_k \cdot d'_k, \quad k = 1, \ldots, R,$$

$$\sum_{j=1}^{M} \lambda_j = 1,\ \lambda_j \geq 0, \quad j = 1, 2, \ldots, M,$$

$$\theta_n = 0, \quad \text{if } d_n = 0,\ n = 1, \ldots, N-2,$$

$$\theta_n \geq 0, \quad \text{if } d_n > 0,\ n = 1, \ldots, N-2,$$

$$\theta'_k = 0, \quad \text{if } d'_k = 0,\ k = 1, \ldots, R,$$

$$\theta'_k \geq 0, \quad \text{if } d'_k > 0,\ k = 1, \ldots, R.$$

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