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Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making

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ABSTRACT

In this work we introduce a method for constructing linear orders between pairs of intervals by using aggregation functions. We adapt this method to the case of interval-valued Atanassov intuitionistic fuzzy sets and we apply these sets and the considered orders to a decision making problem.

1. Introduction

In decision making problems it may happen that, after the exploitation phase, the best alternatives are equally ranked and it is not possible to decide which one is the best. It has been noticed [1] that these troubles often appear when the entries of the considered fuzzy preference matrix are close to 0.5, that is, when the experts have doubts about their preferences of some alternatives over the others. In this situation, the systematic use of extensions of fuzzy sets has been shown to be a really useful tool [2]. Among those fuzzy sets, interval-valued fuzzy sets (IVFSs) [3–5] or, equivalently, Atanassov intuitionistic fuzzy sets (AIFs) [6] play indeed a crucial role.

In some special cases, despite the fact of using IVFSs and AIFs, still remain problems that are similar to those encountered in the previous ones. For these new last situations we may use the interval-valued Atanassov intuitionistic fuzzy sets (IVAIFSs) [7]. Besides, the use of intervals to represent membership and non-membership has, from our point of view, a double advantage:

1. If we want to model environments where there exist non-comparable elements, it will be enough to use classical partial orders between intervals. This is not the case in this work.
2. If we must represent ignorance [8] associated to the datum given by an expert, we can understand the length of the intervals as a representation of such ignorance. If, in these cases, we need to be able to compare any two data, then we can use any of the linear orders we consider here.

Once the decision of using IVAIFSs to deal with a decision making problem has been reached, we should choose, accordingly, a linear order between pairs of intervals. In this way, we will select as the best option the alternative which is associated to the largest pair of intervals, with respect to the considered linear order.

Moreover, in decision making problems we must also aggregate the information furnished by the experts by means of aggregation functions [9–11].

All these considerations have led us to aim the following objectives:

(1) To use aggregation functions for building linear orders for pairs of intervals whose end-points belong to the unit interval.
(2) To study methods for constructing linear orders on the set of IVAIFSs.

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(3) To deal with the exploitation phase of decision making problems through IVAIFSs, by using the previously built linear orders.

The structure of this paper is the following. In Section 2 we introduce the notation and recall some well-known notions. In Sections 3,4, we construct two classes of linear orders between pairs of intervals. Section 5 contains an application of our theoretical results to group decision making. In particular, we provide two algorithms. Some concluding remarks as well as suggestions for further research close the paper.

2. Previous concepts and results

We start by recalling some well-known concepts that will be useful for subsequent developments throughout the paper.

2.1. On orders and partially ordered sets

Given a partially ordered set (poset) $(P, \preceq)$, we say that

(a) 1_P is the top of the poset if for all $x \in P$ it holds $x \preceq 1_P$.
(b) 0_P is the bottom of the poset if for all $x \in P$ it holds $0_P \preceq x$.

In case they exist, 1_P and 0_P are unique.

Let $K([0,1]) \subset \mathbb{R}^2$ be given by

$$K([0,1]) = \{(\underline{x}, \overline{x}) \in [0,1] \times [0,1] | \underline{x} \leqslant \overline{x}\},$$

and let $L([0,1])$ be the set of all closed subintervals of the unit interval, that is

$$L([0,1]) = \{\mathbf{x} | \mathbf{x} = [\underline{x}, \overline{x}] \text{ such that } 0 \leqslant \underline{x} \leqslant \overline{x} \leqslant 1\}.$$

There is a straightforward bijection $i : K([0,1]) \to L([0,1])$ given by $i((\underline{x}, \overline{x})) = [\underline{x}, \overline{x}] = \mathbf{x}$. Through this bijection, the partial order on $\mathbb{R}^2$, $(a,b) \leq_2 (c,d)$ if and only if $a \leqslant c$ and $b \leqslant d$ induces an equivalent partial order on $L([0,1])$, namely,

$$\mathbf{x} \preceq_2 \mathbf{y} \text{ iff } \underline{x} \leqslant \underline{y} \text{ and } \overline{x} \leqslant \overline{y}. \quad (1)$$

In this way, $(L([0,1]), \preceq_2)$ is a poset whose bottom and top are, respectively, $\mathbf{0} = [0,0]$ and $\mathbf{1} = [1,1]$. In fact, the bijection above is a lattice isomorphism.[1]

We refer as $(L([0,1]))^2$, to the universe of pairs of intervals, that is,

$$(L([0,1]))^2 = \{(\mathbf{x}, \mathbf{y}) = ([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) \text{ with } \underline{x}, \overline{x}, \underline{y}, \overline{y} \in [0,1]\}.$$

Similarly to what happens in the case of $\mathbb{R}^2$ and $L([0,1])$, the partial order on $\mathbb{R}^4$, given by $(a_1, b_1, c_1, d_1) \leq_4 (a_2, b_2, c_2, d_2)$ if and only if $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$ and $c_1 \leqslant c_2$ and $d_1 \leqslant d_2$, also induces an equivalent partial order $\preceq_4$ on $(L([0,1]))^2$, given by

$$(\mathbf{x_1}, \mathbf{y_1}) \preceq_4 (\mathbf{x_2}, \mathbf{y_2}) \text{ if and only if } \underline{x}_1 \leqslant \underline{x}_2 \text{ and } \overline{x}_1 \leqslant \overline{x}_2 \text{ and } \underline{y}_1 \leqslant \underline{y}_2 \text{ and } \overline{y}_1 \leqslant \overline{y}_2. \quad (2)$$

In this way, $((L([0,1]))^2, \preceq_4)$ becomes a poset whose bottom and top are, respectively, $(\mathbf{0}, \mathbf{0}) = ([0,0],[0,0])$ and $(\mathbf{1}, \mathbf{1}) = ([1,1],[1,1])$.

Definition 2.1 [18]. An order $\preceq$ on $L([0,1])$ is said to be admissible if it is linear and refines the order $\preceq_2$, i.e., it is a linear order satisfying that for all $\mathbf{x}, \mathbf{y} \in L([0,1])$ such that $\mathbf{x} \preceq_2 \mathbf{y}$ it holds $\mathbf{x} \preceq \mathbf{y}$.

Example 2.1. The lexicographic orders on $L([0,1])$, given by

- $\mathbf{x} \preceq_{\text{lex1}} \mathbf{y}$ if and only if $(\underline{x} < \underline{y})$ or $(\underline{x} = \underline{y}$ and $\overline{x} \leqslant \overline{y})$ (lexicographic-1 order), and
- $\mathbf{x} \preceq_{\text{lex2}} \mathbf{y}$ if and only if $(\overline{x} < \overline{y})$ or $(\overline{x} = \overline{y}$ and $\underline{x} \leqslant \underline{y})$ (lexicographic-2 order), are admissible.

2.2. Extensions of fuzzy sets

Definition 2.2 [6]. Let U be a nonempty set usually called a universe. An Atanassov's Intuitionistic Fuzzy Set (AIFS) F over U is given by

$$F = \{\langle u, \mu_F(u), \nu_F(u)\rangle | u \in U\}$$

where $\mu_F : U \to [0,1]$ defines the membership degree of the element $u \in U$ to F and $\nu_F : U \to [0,1]$ defines its nonmembership degree to the same set F. Besides, the functions μ_F and ν_F satisfy that, for all $u \in U$, $\mu_F(u) + \nu_F(u) \leqslant 1$.

The pair $(\mu_F(u), \nu_F(u))$ is called an intuitionistic pair, $\mathcal{L}([0,1])$ being the set of all possible intuitionistic pairs, i.e.,

$$\mathcal{L}([0,1]) = \{\mathbf{a} | \mathbf{a} = (a_1, a_2), a_1, a_2 \in [0,1] \text{ and } a_1 + a_2 \leqslant 1\}.$$

In [6], Atanassov introduced a partial order in the universe of AIFSs.

Definition 2.3. Let F_1, F_2 be two AIFSs. According to the order given by Atanassov in [6]

$$F_1 \leqslant F_2 \text{ if and only if for all } u \in U, \mu_{F_1}(u) \leqslant \mu_{F_2}(u) \text{ and } \nu_{F_1}(u) \geqslant \nu_{F_2}(u).$$

Definition 2.4 [7]. Let U be a universe. An Interval-Valued Atanassov Intuitionistic Fuzzy Set (IVAIFS) G over U is given by

$$G = \{\langle u, m_G(u), n_G(u)\rangle | u \in U\}$$

where $m_G : U \to L([0,1])$ defines the membership degree of the element $u \in U$ to F and $n_G : U \to L([0,1])$ defines its nonmembership degree to the same universe U. Moreover, for all $u \in U$, the sum of the upper boundary values of $m_G(u)$ and $n_G(u)$ must be lower than or equal to 1.

The pair $(m_G(u), n_G(u))$ is called an interval-valued intuitionistic pair, being $\mathcal{L}_{IV}([0,1])$ the set of all possible interval-valued intuitionistic pairs, i.e.,

$$\mathcal{L}_{IV}([0,1]) = \{(\mathbf{x}, \mathbf{y}), \text{ with } \mathbf{x}, \mathbf{y} \in L([0,1]) \text{ and } \overline{x} + \overline{y} \leqslant 1\}.$$

Remark 1. Note that $\mathcal{L}_{IV}([0,1])$ consists of special types of intervals, while $(\mathcal{L}([0,1]))^2$ is a set of all possible intuitionistic pairs.

Definition 2.5. Let G_1, G_2 be two IVAIFSs. According to the order given by Atanassov in [7], $G_1 \preceq G_2$ if and only if, for all $u \in U$,

$$m_{G1}(u) \preceq_2 m_{G2}(u) \text{ and } n_{G2}(u) \preceq_2 n_{G1}(u),$$

where $\preceq_2$ is the partial order on $L([0,1])$ given in Eq. (1).

[1] This kind of sets, namely $K([0,1])$ and $L([0,1])$ have already been used, suitably equipped with some order and latticial structure [12,13], to construct some universal codomain where it was possible to represent different kinds of orderings as, e.g., total preorders, interval-orders and semiorders by means of a single function that preserves the ordinal structure. The bijection $i : K([0,1]) \to L([0,1])$ has also been considered in those approaches, and some other similar bijections and/or latticial isomorphism as well as order isotonies have also been introduced accordingly. By the way, another universal codomain to represent different kinds of orderings, which is essentially equivalent to $K([0,1])$, consists of triangular and symmetric fuzzy numbers. For further information see [14–17].

2.3. Aggregation functions

Definition 2.6. Given a poset $(P, \preceq_P)$ with bottom 0_P and top 1_P, an aggregation function M on P w.r.t the order $\preceq_P$ (also known as an $\preceq_P$-aggregation function) is a mapping $M : P^n \to P$ satisfying

- $M(0_P, \ldots, 0_P) = 0_P$, $M(1_P, \ldots, 1_P) = 1_P$, and
- $M(x_1, \ldots, x_n) \preceq_P M(y_1, \ldots, y_n)$ for $(x_1, \ldots, x_n) \preceq_P (y_1, \ldots, y_n)$

where $(x_1, \ldots, x_n) \preceq_P (y_1, \ldots, y_n)$ holds if and only if $x_i \preceq_P y_i$ for all $i \in \{1, \ldots, n\}$.

This definition extends the usual one for the unit interval $[0, 1]$. For further information see [19].

Proposition 2.1 [18]. *Let $B_1, B_2 : [0,1]^2 \to [0,1]$ be two continuous aggregation functions, such that for all $(p_1, p_2), (q_1, q_2) \in K([0,1])$, the equalities $B_1(p_1, p_2) = B_1(q_1, q_2)$ and $B_2(p_1, p_2) = B_2(q_1, q_2)$ only hold provided that $(p_1, p_2) = (q_1, q_2)$.*

The order $\preceq_{B_1,B_2}$ on $L([0,1])$, given by

$$\mathbf{x} \preceq_{B_1,B_2} \mathbf{y} \text{ if and only if } B_1(\underline{x}, \overline{x}) < B_1(\underline{y}, \overline{y}) \text{ or else } (B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y}) \text{ and } B_2(\underline{x}, \overline{x}) \leqslant B_2(\underline{y}, \overline{y})),$$

is an admissible order on $L([0,1])$.

The following results can be found in [9,11,20,21].

Definition 2.7. A function $T : [0,1]^2 \to [0,1]$ is called a t-norm if it is symmetric, associative, increasing with respect to the order $\leqslant$ and $T(x, 1) = x$ for all $x \in [0,1]$.

Definition 2.8. A function $S : [0,1]^2 \to [0,1]$ is called a t-conorm if it is symmetric, associative, increasing with respect to the order $\leqslant$ and $S(x, 0) = x$ for all $x \in [0,1]$.

A strictly decreasing and continuous function $n : [0,1] \to [0,1]$ such that $n(0) = 1$ and $n(1) = 0$ is called a strict negation. If, in addition, it is involutive (that is, $n(n(x)) = x$ for all $x \in [0,1]$), then n is said to be a strong negation. A t-norm T is dual to a t-conorm S (and vice versa) with respect to a strong negation n if $T(x, y) = n(S(n(x), n(y)))$ for all $x, y \in [0,1]$.

3. Admissible orders on $(L([0,1]))^2$

Although a partial order is enough to define aggregation functions, some special classes of aggregations actually require to have at hand a linear order. Examples of such classes are Choquet integrals and Sugeno integrals. The order given by Atanassov for IVAIFSs is partial, which is a undeniable handicap in the adaptation of such classes of aggregation operators to the IVAI setup. In this section we define the admissible linear orders on $(L([0,1]))^2$, generalizing the concept of admissible orders on $L([0,1])$.

Definition 3.1. An order $\preceq$ on $(L([0,1]))^2$ is said to be admissible if it is a linear and refines the order $\preceq_4$ in Eq. (2), i.e., it is linear order satisfying that for all $(\mathbf{x_1}, \mathbf{y_1}), (\mathbf{x_2}, \mathbf{y_2}) \in (L([0,1]))^2$, $(\mathbf{x_1}, \mathbf{y_1}) \preceq_4 (\mathbf{x_2}, \mathbf{y_2})$ implies $(\mathbf{x_1}, \mathbf{y_1}) \preceq (\mathbf{x_2}, \mathbf{y_2})$.

The elements $z_i = (\mathbf{x_i}, \mathbf{y_i}) \in (L([0,1]))^2$ can be visualized in a straightforward manner. Since $\mathbf{x_i}, \mathbf{y_i} \in L([0,1])$, each pair of intervals can be drawn as a rectangle for which the first interval lies in the horizontal axis and the second interval lies in the vertical one. In such a representation, the following statements hold true:

- The wider the first interval, the wider the rectangle.
- The wider the second interval, the higher the rectangle.

As a consequence, the area of the rectangle will be directly proportional to the width of the intervals. Furthermore, for any $z_1, z_2 \in (L([0,1]))^2, z_1 \preceq_4 z_2$ if and only if each corner of the rectangle of z_2 is located above and on the right side of its corresponding corner in the rectangle z_1.

Example 3.1. Let $z_1 = ([0.3, 0.6], [0.2, 0.7])$, $z_2 = ([0.5, 0.8], [0.55, 0.9])$, $z_3 = ([0.4, 0.5], [0.3, 0.35])$, $z_4 = ([0.1, 0.4], [0.4, 0.6])$. The intervals can be represented in the unit square $[0,1]^2$ as in Fig. 1. In that figure some visual interpretations can be drawn. For example, we have that the intervals of z_1 are wider than those of any other z_i, since its area is significantly greater. Alternatively, we have that $z_i \preceq_4 z_2$ for $i \in \{1, 3, 4\}$, since the corners of z_2 are located above and on the right side w.r.t the other rectangles. Similarly, we can deduce that z_1, z_3 and z_4 are incomparable in terms of $\preceq_4$.

In [18], Bustince et al. introduced a construction method of admissible orders on $L([0,1])$ by using two aggregation functions. Such method can also be generalized to handle elements in $(L([0,1]))^2$.

Proposition 3.1. *Let $A = < A_1, A_2, A_3, A_4 >$ be four aggregation functions,*[2] *$A_i : [0,1]^4 \to [0,1]$ such that for all $(\mathbf{p}, \mathbf{q}), (\mathbf{r}, \mathbf{s}) \in (L([0,1]))^2$ the equalities $A_i(\underline{p}, \overline{p}, \underline{q}, \overline{q}) = A_i(\underline{r}, \overline{r}, \underline{s}, \overline{s})$ for all $i = \{1, \ldots 4\}$ only hold if $(\mathbf{p}, \mathbf{q}) = (\mathbf{r}, \mathbf{s})$.*

An admissible order $\preceq_A$ on $(L([0,1]))^2$ can be defined as follows $(\mathbf{x_1}, \mathbf{y_1}) \preceq_A (\mathbf{x_2}, \mathbf{y_2})$ if and only if one of the (mutually exclusive) following conditions is satisfied.

(i) $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) < A_1(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$;
(ii) $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ *and* $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) < A_2(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$;
(iii) $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ *and* $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ *and* $A_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) < A_3(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$;
(iv) $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ *and* $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ *and* $A_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = A_3(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$ *and* $A_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) \leqslant A_4(\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$.

Proof. The order $\preceq_A$ refines $\leq_4$ since every A_i is an aggregation function. Moreover, the linearity is assured since the four equalities of A_i only hold simultaneously if $(\mathbf{x_1}, \mathbf{y_1}) = (\mathbf{x_2}, \mathbf{y_2})$. The transitivity follows from the transitivity of the standard order on $[0,1]$. □

Remark 2. Notice that any permutation of the aggregation functions A_i also produces an admissible order different from the former one.

Remark 3. In [18] it was proven that an admissible order on $K([0,1])$ cannot be induced by a single function. Clearly, this result also holds true since we are working in a larger space.

Henceforward, we use the order generated by four aggregation functions (in Proposition 3.1). Thus, all the ideas to be introduced till the end of this section refer to such family of admissible orders named 4-admissible.

Example 3.2. The lexicographic orders can be constructed from the four projections.

[2] *Warning: notice that here the order of appearance of the A_i's counts. See also Remark 2.*

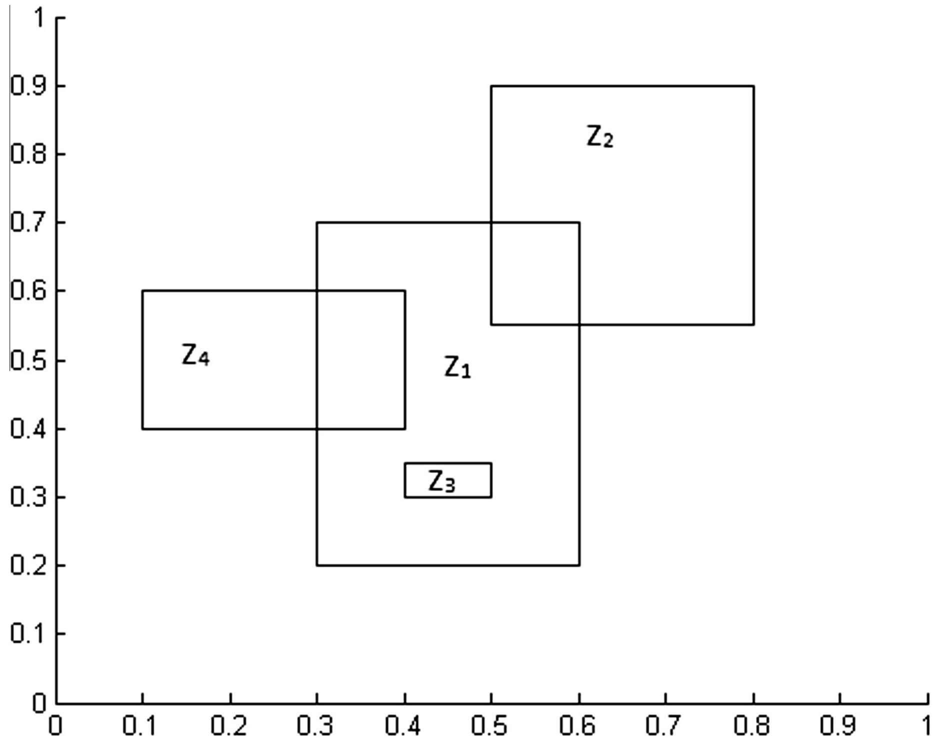

Fig. 1. Pairs of intervals.

1. The standard lexicographic order: let A_i be the aggregation function that maps to the ith component (i.e. the ith projection). In that case, $(\mathbf{x_1}, \mathbf{y_1}) \preceq_A (\mathbf{x_2}, \mathbf{y_2})$ if and only if
   - $(\underline{x}_1 < \underline{x}_2)$, or
   - $(\underline{x}_1 = \underline{x}_2$ and $\overline{x}_1 < \overline{x}_2)$, or
   - $(\underline{x}_1 = \underline{x}_2, \overline{x}_1 = \overline{x}_2$ and $\underline{y}_1 < \overline{y}_2)$, or
   - $(\underline{x}_1 = \underline{x}_2, \overline{x}_1 = \overline{x}_2, \underline{y}_1 = \underline{y}_2$ and $\overline{y}_1 \leqslant \overline{y}_2)$.

2. The reversed lexicographic order: let A_i be the aggregation function that maps to the $(5-i)$th component (i.e. the $(5-i)$th projection). In that case, $(\mathbf{x_1}, \mathbf{y_1}) \preceq_A (\mathbf{x_2}, \mathbf{y_2})$ if and only if
   - $(\overline{y}_1 < \overline{y}_2)$, or
   - $(\overline{y}_1 = \overline{y}_2$ and $\underline{y}_1 < \underline{y}_2)$, or
   - $(\overline{y}_1 = \overline{y}_2, \underline{y}_1 = \underline{y}_2$ and $\overline{x}_1 < \overline{x}_2)$, or
   - $(\overline{y}_1 = \overline{y}_2, \underline{y}_1 = \underline{y}_2, \overline{x}_1 = \overline{x}_2$ and $\underline{x}_1 \leqslant \underline{x}_2)$.

3. Any other permutation of the projections gives rise to an admissible order where we compare the components in a predetermined order.

Proposition 3.2. *Let $A =< A_1, A_2, A_3, A_4 >$ be four aggregation functions given by*

$$A_i(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = a_i\underline{x}_1 + b_i\overline{x}_1 + c_i\underline{y}_1 + d_i\overline{y}_1,$$

with $a_i, b_i, c_i, d_i \in [0, 1], a_i + b_i + c_i + d_i = 1$ and

$$|D| = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \neq 0.$$

Then (and only then), the order generated by the aggregation functions A_i is a 4-admissible order.

Proof. The functions A_i are weighted arithmetic means. Let $([\underline{x}_1, \overline{x}_1], [\underline{y}_1, \overline{y}_1]), ([\underline{x}_2, \overline{x}_2], [\underline{y}_2, \overline{y}_2]) \in (L([0, 1]))^2$, such that

$$a_i\underline{x}_1 + b_i\overline{x}_1 + c_i\underline{y}_1 + d_i\overline{y}_1 = a_i\underline{x}_2 + b_i\overline{x}_2 + c_i\underline{y}_2 + d_i\overline{y}_2,$$

for $i \in \{1, \ldots, 4\}$. Because of the regularity of D, both linear systems have a unique and common solution, i.e., $(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = (\underline{x}_2, \overline{x}_2, \underline{y}_2, \overline{y}_2)$. The result now follows from Proposition 3.1. □

Example 3.3. Let A contain the following aggregation functions:

- $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{3}{8}\underline{x}_1 + \frac{3}{8}\overline{x}_1 + \frac{1}{8}\underline{y}_1 + \frac{1}{8}\overline{y}_1$;
- $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{10}{20}\underline{x}_1 + \frac{5}{20}\overline{x}_1 + \frac{3}{20}\underline{y}_1 + \frac{2}{20}\overline{y}_1$;
- $A_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{20}\underline{x}_1 + \frac{10}{20}\overline{x}_1 + \frac{8}{20}\underline{y}_1 + \frac{1}{20}\overline{y}_1$;
- $A_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{4}\underline{x}_1 + \frac{1}{4}\overline{x}_1 + \frac{1}{4}\underline{y}_1 + \frac{1}{4}\overline{y}_1$.

Since $|D| = -0.0069$, the order generated by A, as in Proposition 3.1, is a 4-admissible order.

Remark 4. Notice that the value of the determinant is close to 0 but this is due to the fact that all the elements of the matrix are smaller than 1.

The construction of admissible orders through a 4-tuple of weighted arithmetic means has an interesting geometrical interpretation. If we consider A in the form of the corresponding four weighting vectors which generate $A_1, \ldots, A_4$, i.e.,

$$A \approx R = \{< a_1, b_1, c_1, d_1 >, < a_2, b_2, c_2, d_2 >, < a_3, b_3, c_3, d_3 >, < a_4, b_4, c_4, d_4 >\}$$

the condition in Proposition 3.2 means that R is a basis of the vector space $\mathbb{R}^4$. Hence, to any basis R of $\mathbb{R}^4$ which consists of weighting vectors there is a unique admissible order $\preceq_A$ constructed by means of the corresponding weighted means.

Finally, after changing the basis, the values of interval-valued intuitionistic pairs in the new basis, (which are now in $[0,1]^4$), are ordered through the standard lexicographic order.

Proposition 3.3. *Let a tuple $A = \langle A_1, \ldots, A_4 \rangle$ of aggregation functions generate an admissible order $\preceq_A$. Let $B_i : [0,1]^2 \to [0,1], i \in \{1,\ldots,4\}$ be four aggregation functions such that*

- $A_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_i(\underline{x}, \overline{x})$ *for* $i \in \{1,2\}$, *and*
- $A_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_j(\underline{y}, \overline{y})$ *for* $j \in \{3,4\}$.

Then, $(\mathbf{x_1}, \mathbf{y_1}) \preceq_A (\mathbf{x_2}, \mathbf{y_2})$ *if and only if*

(i) $(\mathbf{x_1} \prec_{B_1,B_2} \mathbf{x_2})$*, or*
(ii) $(\mathbf{x_1} = \mathbf{x_2}$ *and* $\mathbf{y}_1 \preceq_{B_3,B_4} \mathbf{y_2})$*,*

where $\preceq_{B_i,B_j}$ *is the order on* $L([0,1])$ *generated in Proposition 2.1.*

Proof. It is straightforward.

Notice that, if we use $B_1 = B_3$ and $B_2 = B_4$, the result is a 4-admissible order where we combine the standard lexicographic order with the order $\preceq_{B_1,B_2}$. The resulting order acts as follows: first we compare the intervals using $\preceq_{B_1,B_2}$ and, only if they are equal, we compare the second interval with that same order ($\preceq_{B_1,B_2}$). For instance, the standard lexicographic order can be seen as the composition of the lexicographic-1 order between intervals combined with itself.

Alternatively, notice that, if $A_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_i(\underline{y}, \overline{y})$ for $i \in \{1,2\}$, and $A_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_j(\underline{x}, \overline{x})$ for $j \in \{3,4\}$, then the resulting order is also 4-admissible.

A well-known class of binary aggregation functions is that of Atanassov's operators $\mathbb{K}_\alpha$ given by $\mathbb{K}_\alpha(a,b) = a + \alpha(b-a)$ with $\alpha \in [0,1]$.

In our particular case, the inputs being intervals, an Atanassov's operator acting on the endpoints of the intervals yields a point inside the corresponding intervals. □

Example 3.4. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$, with $\alpha_1 \neq \alpha_2$ and $\alpha_3 \neq \alpha_4$. Let $A =< A_1, \ldots, A_4 >$ be four aggregation functions given by

- $A_i(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \mathbb{K}_{\alpha_i}(\underline{x}_1, \overline{x}_1)$, for $i \in \{1,2\}$, and
- $A_j(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \mathbb{K}_{\alpha_j}(\underline{y}_1, \overline{y}_1)$, for $j \in \{3,4\}$.

The tuple A generates a 4-admissible order that renders in $(\mathbf{x_1}, \mathbf{y_1}) \preceq_A (\mathbf{x_2}, \mathbf{y_2})$ if and only if

- $(\mathbf{x_1} \prec_{\mathbb{K}_{\alpha_1},\mathbb{K}_{\alpha_2}} \mathbf{x_2})$, or
- $(\mathbf{x_1} =_{\mathbb{K}_{\alpha_1},\mathbb{K}_{\alpha_2}} \mathbf{x_2}$ and $\mathbf{y_1} \preceq_{\mathbb{K}_{\alpha_3},\mathbb{K}_{\alpha_4}} \mathbf{y_2})$.

From the construction in Example 3.4, we can retrieve some well-known orders. For example, if $\{\alpha_1, \alpha_2\} = \{0,1\}$ and $\{\alpha_3, \alpha_4\} = \{0,1\}$, we obtain lexicographic orders. Moreover, all these 4-admissible orders are particular examples of the construction in Proposition 3.2, with $c = d = 0$ for A_1, A_2 and $a = b = 0$ for A_3 and A_4.

In [18] it was proven that given an $\alpha \in [0,1)$ then all admissible orders $\preceq_{\alpha,\beta}$ on $L([0,1])$ with $\beta > \alpha$ coincide. Then, different aggregation functions could generate the same admissible order. This also affects to admissible orders generated as in Proposition 3.2. For instance,

$$|D_1| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{vmatrix} \neq 0, \quad |D_2| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{vmatrix} \neq 0$$

generate the same order.

4. IVAIF-admissible order on $\mathcal{L}_{IV}([0,1])$

The admissible orders defined in Section 3 refine the partial order $\preceq_4$. However, any of them could also refine the partial order given by Atanassov for IVAIFS [7]. In this section, we define a new family of linear orders with a crucial additional feature, namely, they refine Atanassov's partial order.

We remind the reader that in Atanassov's partial order, given two elements $(\mathbf{x_1}, \mathbf{y_1}), (\mathbf{x_2}, \mathbf{y_2}) \in \mathcal{L}_{IV}([0,1])$,

$$(\mathbf{x_1}, \mathbf{y_1}) \preceq (\mathbf{x_2}, \mathbf{y_2}) \text{ if and only if } \underline{x}_1 \leqslant \underline{x}_2,\ \overline{x}_1 \leqslant \overline{x}_2,\ \underline{y}_1 \geqslant \underline{y}_2, \text{ and } \overline{y}_1 \geqslant \overline{y}_2. \tag{3}$$

Definition 4.1. An order $\preceq$ on $\mathcal{L}_{IV}([0,1])$ is said to be an IVAIF-admissible order if it is a linear order and refines the partial order given by Atanassov for IVAIFS (Eq. (3)).

Notice that, if we have an IVAIF-admissible order on $\mathcal{L}_{IV}([0,1])$, as in Definition 4.1, then the bottom of $(\mathcal{L}_{IV}([0,1]), \preceq)$ is $(\mathbf{0}, \mathbf{1})$ and the top is $(\mathbf{1}, \mathbf{0})$.

As in Section 3, we can generate a visualization of the elements in $\mathcal{L}_{IV}([0,1]) \subset (L([0,1]))^2$ that captures the behaviour of the admissible orders in Definition 4.1. Following the visualization rules in Fig. 1 we have that, for any two elements z_1, z_2 in $\mathcal{L}_{IV}([0,1]), z_1 \preceq z_2$ if and only if the corners of z_2 are individually located below and to the right of those of z_1. For example, in Fig. 2, we have given $z_1 = ([0.1,0.4],[0.1,0.6]), z_2 = ([0.3,0.55],[0.05,0.25]), z_3 = ([0.05,0.2],[0.15,0.25]) \in \mathcal{L}_{IV}([0,1])$. Visually, it is evident that $z_1 \preceq z_2$ and $z_3 \preceq z_2$, but also that z_1 and z_3 are not comparable with the partial order in Definition 4.1. Notice that, in this visualization, no element is allowed to be in the grey zone of the rectangle in Fig. 2 due to the restrictions in the definitions of the membership and nonmembership degrees in an interval-valued intuitionistic pair.

In the sequel, two different constructions of IVAIF-admissible orders are introduced.

Proposition 4.1. *Let $B =< B_1, B_2, B_3, B_4 >$ be four aggregation functions $B_i : [0,1]^4 \to [0,1]$ which generate the orders $\preceq_{B_1,B_2}$ and $\preceq_{B_3,B_4}$ on $L([0,1])$ as in Proposition 2.1. Then the order relation $\preceq_B^{IV}$ given by*

$$(\mathbf{x}_1, \mathbf{y}_1) \preceq_B^{IV} (\mathbf{x}_2, \mathbf{y}_2) \text{ if and only if } \mathbf{x}_1 \prec_{B_1,B_2} \mathbf{x}_2 \text{ or } (\mathbf{x}_1 = \mathbf{x}_2 \text{ and } \mathbf{y}_2 \preceq_{B_3,B_4} \mathbf{y}_1),$$

is an IVAIF-admissible order.

Proof. The linearity of $\preceq_B^{IV}$ is straight as $\mathcal{L}_{IV}([0,1]) \subset (L([0,1]))^2$. In addition, it refines the partial order given by Atanassov due to the fact that the order relation, $\preceq_{B_3,B_4}$, has been reversed.

In particular, if $B_1 = B_3$ and $B_2 = B_4$, then $\preceq_{B_1,B_2} = \preceq_{B_3,B_4}$ and, consequently, the same order is used to compare both intervals although in the second one the order is reversed. □

Proposition 4.2. *Let $A =< A_1, A_2, A_3, A_4 >$ be four aggregation functions, $A_i : [0,1]^4 \to [0,1]$ such that for all (p_1, p_2, p_3, p_4),*

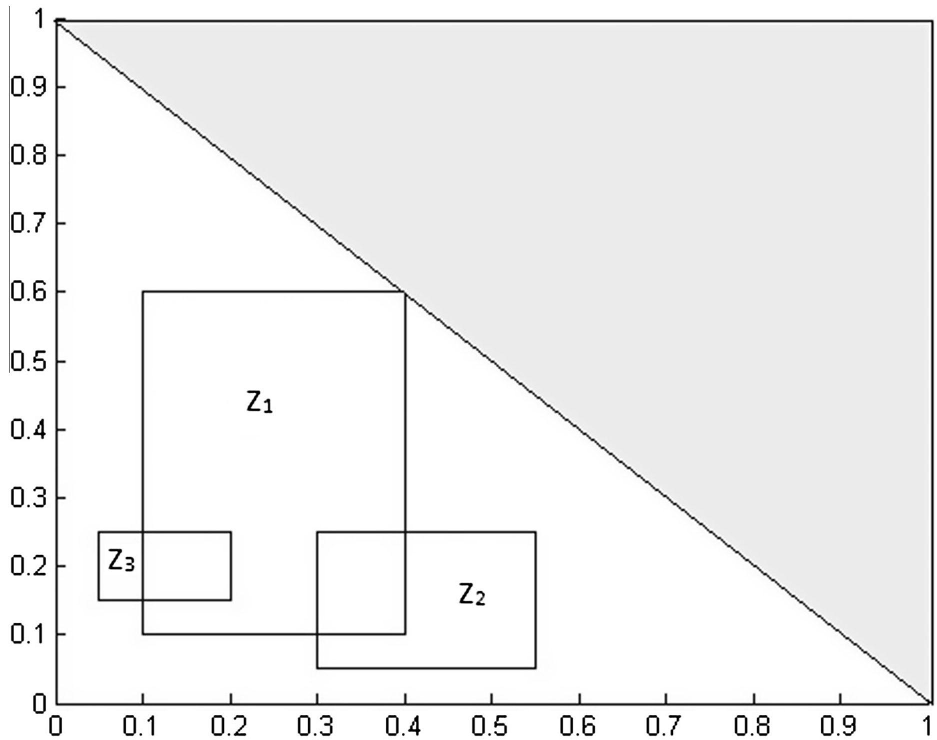

Fig. 2. Visual representation of interval-valued intuitionistic pairs. The white zone represents the subset of $\mathcal{L}_{IV}([0,1])$ in which such pairs are allowed.

$(q_1, q_2, q_3, q_4) \in [0,1]^4$ *the equalities* $A_i(p_1, p_2, p_3, p_4) = A_i(q_1, q_2, q_3, q_4)$ *for all* $i \in \{1, \ldots, 4\}$ *only hold if* $(p_1, p_2, p_3, p_4) = (q_1, q_2, q_3, q_4)$.

An IVAIF-*admissible order* $\preceq_A^{IV}$ *on* $\mathcal{L}_{IV}([0,1])$, *is defined as follows:* $(\mathbf{x}_1, \mathbf{y}_1) \preceq_A^{IV} (\mathbf{x}_2, \mathbf{y}_2)$ *if and only if one of the following (mutually exclusive) conditions is satisfied.*

(i) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) < A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
(ii) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$ *and* $A_2(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) < A_2(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
(iii) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$, $A_2(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$, *and* $A_3(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) < A_3(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$,
(iv) $A_1(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_1(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$, $A_2(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_2(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$, $A_3(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = A_3(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$, *and* $A_4(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) \leqslant A_4(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$.

Proof. The linearity is warranted because the equalities only hold if $(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) = (\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$.

To check the second condition (that of refining the partial order) in the statement of Definition 4.1, notice that if

$$\underline{x}_1 \leqslant \underline{x}_2,\ \overline{x}_1 \leqslant \overline{x}_2,\ \underline{y}_1 \geqslant \underline{y}_2,\ \text{and } \overline{y}_1 \geqslant \overline{y}_2.$$

then

$$\underline{x}_1 \leqslant \underline{x}_2, \overline{x}_1 \leqslant \overline{x}_2, 1-\underline{y}_1 \leqslant 1-\underline{y}_2,\ \text{and } 1-\overline{y}_1 \leqslant 1-\overline{y}_2,$$

so consequently $A_i(\underline{x}_1, \overline{x}_1, 1-\underline{y}_1, 1-\overline{y}_1) \leqslant A_i(\underline{x}_2, \overline{x}_2, 1-\underline{y}_2, 1-\overline{y}_2)$ for all $i \in \{1, \ldots, 4\}$.

From now on we name the order generated by four aggregation functions (as in Proposition 4.2) 4-IVAIF-admissible order. □

Remark 5. Given $\mathbf{y} \in L([0,1])$, it follows that $(1-\underline{y}, 1-\overline{y}) \in L^*([0,1])$, where

$$L^*([0,1]) = \{\mathbf{s} | \mathbf{s} = (s_1, s_2) \text{ such that } 0 \leqslant s_2 \leqslant s_1 \leqslant 1\}.$$

Then in Proposition 4.2 it is enough that to see, given $(\mathbf{p_1}, \mathbf{q_1}) = ([\underline{p}_1, \overline{p}_1], (\underline{q}_1, \overline{q}_1)), (\mathbf{p_2}, \mathbf{q_2}) = ([\underline{p}_2, \overline{p}_2], (\underline{q}_2, \overline{q}_2)) \in L([0,1]) \times L^*([0,1])$,the equalities $A_i(\underline{p}_1, \overline{p}_1, \underline{q}_1, \overline{q}_1) = A_i(\underline{p}_2, \overline{p}_2, \underline{q}_2, \overline{q}_2)$ hold if and only if $(\mathbf{p_1}, \mathbf{q_1}) = (\mathbf{p_2}, \mathbf{q_2})$.

However, in order to simplify notation we have imposed a slightly stronger restriction. Anyway, all the given examples in Section 3 satisfy it.

Let a tuple $A = \langle A_1, \ldots, A_4 \rangle$ of aggregation functions generate an admissible order. Let $B_i : [0,1]^2 \to [0,1]$ be four aggregations such that

- $A_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_i(\underline{x}, \overline{x})$ for $i \in \{1, 2\}$, and
- $A_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = B_j(\underline{y}, \overline{y})$ for $j \in \{3, 4\}$,

Then the orders $\preceq_A^{IV}$ and $\preceq_B^{IV}$ may be different. To guarantee that they are actually different it is enough that $B_3(\underline{y}_1, \overline{y}_1) < B_3(\underline{y}_2, \overline{y}_2)$ and simultaneously $B_3(1-\underline{y}_1, 1-\overline{y}_1) > B_3(1-\underline{y}_2, 1-\overline{y}_2)$ hold true for some $\mathbf{y}_1, \mathbf{y}_2 \in L([0,1])$.

For instance, let $B_3(\underline{y}, \overline{y}) = \underline{y}\overline{y}$. Here, we have that for $\mathbf{y}_1 = [0.2, 0.2]$ and $\mathbf{y}_2 = [0.1, 0.9]$

$$B_3(0.2, 0.2) = 0.04 < 0.09 = B_3(0.1, 0.9)$$
$$B_3(0.8, 0.8) = 0.64 > 0.09 = B_3(0.9, 0.1).$$

Proposition 4.3. *Let* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$, *with* $\alpha_1 \neq \alpha_2$ *and* $\alpha_3 \neq \alpha_4$. *If*

- $A_i(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = K_{\alpha_i}(\underline{x}, \overline{x})$ *for* $i \in \{1, 2\}$, *and*
- $A_j(\underline{x}, \overline{x}, \underline{y}, \overline{y}) = K_{\alpha_j}(\underline{y}, \overline{y})$ *for* $j \in \{3, 4\}$,

then the tuple $A = \langle A_1, \ldots, A_4 \rangle$ *generates a 4-IVAIF admissible order that is equal to* $\preceq_B^{IV}$ *being* $B = \langle \mathbb{K}_{\alpha_1}, \mathbb{K}_{\alpha_2}, \mathbb{K}_{\alpha_3}, \mathbb{K}_{\alpha_4} \rangle$.

Proof. The fact that the aggregation functions satisfy the conditions to generate a 4-IVAIF order is a simple calculation. To prove the equality between the two orders notice that in this case the conditions *i*) and *ii*) of the order $\preceq_A^{IV}$ are exactly equal to $\mathbf{x}_1 \preceq_{\mathbb{K}_{\alpha_1},\mathbb{K}_{\alpha_2}} \mathbf{x}_2$. Then, it is enough to prove that for all γ,

$$\mathbb{K}_\gamma(1-a_1, 1-b_1) < \mathbb{K}_\gamma(1-a_2, 1-b_2)$$

is equivalent to $\mathbb{K}_\gamma(a_1, b_1) > \mathbb{K}_\gamma(a_2, b_2)$.

But

$$\begin{aligned}
\mathbb{K}_\gamma(1-a_1,1-b_1) &< \mathbb{K}_\gamma(1-a_2,1-b_2)\\
&\iff 1-a_1+\gamma(1-b_1-(1-a_1)) < 1-a_2+\gamma(1-b_2-(1-a_2))\\
&\iff 1-a_1+\gamma(a_1-b_1) < 1-a_2+\gamma(a_2-b_2)\\
&\iff a_2-\gamma(a_2-b_2) < a_1-\gamma(a_1-b_1)\\
&\iff a_2+\gamma(b_2-a_2) < a_1+\gamma(b_1-a_1)\\
&\iff \mathbb{K}_\gamma(a_2,b_2) < \mathbb{K}_\gamma(a_1,b_1),
\end{aligned}$$

so the proof is complete. □

Example 4.1. Let $\preceq_A^{IV}$ be the order generated by $A = <\mathbb{K}_{0.25}, \mathbb{K}_{0.75}, \mathbb{K}_{0.25}, \mathbb{K}_{0.75}>$. Consider the elements $z_1 = ([0.15, 0.35], [0.2, 0.5])$ and $z_2 = ([0.15, 0.35], [0.1, 0.9]) \in \mathcal{L}_{IV}([0,1])$. Since their membership degrees are identical we only need to compare their nonmembership degrees.

In fact,

$$\begin{aligned}
\mathbb{K}_{0.25}(0.2, 0.5) &= 0.2 + 0.25\cdot(0.5-0.2) = 0.275 < 0.3\\
&= 0.1 + 0.25\cdot(0.9-0.1) = \mathbb{K}_{0.25}(0.1, 0.9)
\end{aligned}$$

and $([0.15, 0.35], [0.1, 0.9]) \preceq_A^{IV} ([0.15, 0.35], [0.2, 0.5])$.

5. Application to decision making

Decision making problems may be summarized as follows. We have a set of p alternatives:

$$Z = \{z_1, \cdots, z_p\}$$

and a set of $n > 2$ experts:

$$E = \{e_1, \cdots, e_n\}.$$

Each of the latter provides her/his preferences on the former set of alternatives by means of a preference relation in the following way:

$$r_{el} = \begin{pmatrix} - & r_{(el)12} & \cdots & r_{(el)1p}\\ r_{(el)21} & - & \cdots & r_{(el)2p}\\ \cdots & \cdots & - & \cdots\\ r_{(el)p1} & \cdots & \cdots & - \end{pmatrix}. \tag{4}$$

Here $r_{(el)ij}$, with $i \neq j$, expresses to what extent the expert l (with $l \in \{1, \cdots, n\}$) prefers the alternative z_i over the alternative z_j.

We must reach a decision of selecting either an alternative or a set of alternatives, which is (are) optimal as regards the experts assessments.

In [20], it is stated that the resolution of a group decision making problem consists of two steps:

(1) Uniform representation of information. In this phase, the heterogeneous information for the problem (the information can be represented by means of preference orderings or utility functions or fuzzy preference relations) is translated into homogeneous information by means of different transformation functions (see [22]).

(2) Application of a selection procedure. This procedure consists of two phases:

(2.1) Aggregation phase. A collective preference structure is built from the set of individual homogeneous preference structures.

(2.2) Exploitation phase. A given method is applied to the collective preference structure to obtain a selection of alternatives.

We use the theoretical developments in previous sections in the exploitation phase of the group decision making problem considered by Nayagam [23]. In particular, we consider the adaptation of this problem done by Zhang et al. [24]. In this adptation, authors consider that *there exists a panel with four possible alternatives for investment:*

(1) z_1 *is a car company,*
(2) z_2 *is a food company,*
(3) z_3 *is a computer company,*
(4) z_4 *is an arms company.*

It is necessary to choose the best company for investment.

Let the data in [24] be our collective preference matrix. In the exploitation phase we use the voting method which consists in aggregating the values in each row of the collective matrix R_c in such a way that, at the end, we have as many values (pairs of intervals) as rows. Since these latter values are not comparable through the partial order, we will select the alternative associated to the largest pair, according to a considered linear order.

$$R_c = \begin{pmatrix} - & ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.1,0.3],[0.5,0.6])\\ ([0.6,0.7],[0.2,0.3]) & - & ([0.6,0.7],[0.2,0.3]) & ([0.4,0.8],[0.1,0.2])\\ ([0.3,0.6],[0.3,0.4]) & ([0.5,0.6],[0.3,0.4]) & - & ([0.4,0.5],[0.1,0.3])\\ ([0.7,0.8],[0.1,0.2]) & ([0.6,0.7],[0.1,0.3]) & ([0.3,0.4],[0.1,0.2]) & - \end{pmatrix}.$$

To aggregate the values of each row of R_c we use the concept of interval-valued intuitionistic t-norms.

Definition 5.1. A mapping $\mathbb{T} : (\mathcal{L}_{IV}([0,1]))^2 \to \mathcal{L}_{IV}([0,1])$ is an interval-valued intuitionistic t-norm if it is symmetric, associative, increasing with respect to the partial order $\preceq$ given by Atanassov (also called monotone) and $\mathbb{T}((\mathbf{x},\mathbf{y}),(\mathbf{1},\mathbf{0})) = (\mathbf{x},\mathbf{y})$.

It is easy to see that, if we take the classical product t-norm, $T_P(x,y) = x\cdot y$, and its dual t-conorm with respect to the standard negation, $S_P(x,y) = x+y-x\cdot y$, the following expression is an interval-valued intuitionistic t-norm: $\mathbb{T}((\mathbf{x},\mathbf{y}),(\mathbf{z},\mathbf{t})) = ([\underline{x}\cdot\underline{z}, \overline{x}\cdot\overline{z}], [\underline{y}+\underline{t}-\underline{y}\cdot\underline{t}, \overline{y}+\overline{t}-\overline{y}\cdot\overline{t}])$.

Applying $\mathbb{T}$ to each row of R_c we get a new matrix, say Rg, given by:

$$Rg = \begin{pmatrix} z_1 = ([0.016, 0.090], [0.720, 0.856])\\ z_2 = ([0.144, 0.392], [0.424, 0.608])\\ z_3 = ([0.060, 0.180], [0.559, 0.748])\\ z_4 = ([0.126, 0.224], [0.271, 0.552]) \end{pmatrix}.$$

In this setting, as regards the partial order $\preceq$, it follows

$$z_1 \preceq z_3 \preceq z_2 \text{ and } z_1 \preceq z_3 \preceq z_4,$$

but z_2 and z_4 are not comparable.

For this reason we consider the 4-IVAIF-admissible order $\preceq_A^{IV}$ defined through the following aggregation functions.

- $A_1(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{2}{20}\underline{x}_1 + \frac{2}{20}\overline{x}_1 + \frac{8}{20}\underline{y}_1 + \frac{8}{20}\overline{y}_1$
- $A_2(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{10}{20}\underline{x}_1 + \frac{5}{20}\overline{x}_1 + \frac{3}{20}\underline{y}_1 + \frac{2}{20}\overline{y}_1$
- $A_3(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{20}\underline{x}_1 + \frac{10}{20}\overline{x}_1 + \frac{8}{20}\underline{y}_1 + \frac{1}{20}\overline{y}_1$
- $A_4(\underline{x}_1, \overline{x}_1, \underline{y}_1, \overline{y}_1) = \frac{1}{4}\underline{x}_1 + \frac{1}{4}\overline{x}_1 + \frac{1}{4}\underline{y}_1 + \frac{1}{4}\overline{y}_1$.

With this order, we have $z_1 \preceq_A^{IV} z_3 \preceq_A^{IV} z_2 \preceq_A^{IV} z_4$ and the selected firm is arms company.

However, as it happened in [18] different 4-IVAIFS-admissible orders can lead to different rankings and hence the selection of the best alternative for a given decision making problem can be forced. For instance, in our case if we take $\preceq_A^{IV}$ with $A =< \mathbb{K}_{0.25}, \mathbb{K}_{0.75}, \mathbb{K}_{0.25}, \mathbb{K}_{0.75} >$ it comes out that the best alternative is the second one, since $z_1 \preceq_A^{IV} z_3 \preceq_A^{IV} z_4 \preceq_A^{IV} z_2$. Nevertheless, for the order $([\underline{x}_1, \overline{x}_1], [\underline{y}_1, \overline{y}_1]) \preceq_A^{IV} ([\underline{x}_2, \overline{x}_2], [\underline{y}_2, \overline{y}_2])$ if and only if

- $(\overline{x}_1 < \overline{x}_2)$, or
- $(\overline{x}_1 = \overline{x}_2$ and $\underline{x}_1 < \underline{x}_2)$, or
- $(\overline{x}_1 = \overline{x}_2, \underline{x}_1 = \underline{x}_2$ and $\overline{y}_1 > \overline{y}_2)$, or else
- $(\overline{x}_1 = \overline{x}_2, \underline{x}_1 = \underline{x}_2, \overline{y}_1 = \overline{y}_2$ and $\underline{y}_1 \geqslant \underline{y}_2)$

we have that $z_1 \preceq_A^{IV} z_3 \preceq_A^{IV} z_4 \preceq_A^{IV} z_2$ and the best alternative is the second one.

To cope with this situation the following algorithm takes different 4-IVAIFS-admissible orders into account simultaneously.

(1) To select several linear orders built with the methods developed in the previous sections.
(2) For each order, to apply in the exploitation phase the voting method with the same aggregations. For instance, $\mathbb{T} = (T_P, S_P)$.
(3) To select the alternative which appears as the best placed in the majority of all the so-obtained rankings.

In our considered problem, the chosen alternative through this algorithm is the second one. That is, we must invest our money in a food company. Clearly, the nature of the problem will impose the number of linear orders to be considered and/or the conditions that will force us to use alternative methods.

6. Conclusions

In this work we have studied how to construct linear orders between pairs of intervals on $L([0, 1])$ that can be used to construct linear orders in Atanassov interval-valued intuitionistic fuzzy sets. We have applied this operator to group decision making problems giving two algorithms, the first one for a particular linear order and the second one which mixes different linear orders.

As a possible development for future research, somewhat related to the main ideas introduced throughout the present manuscript, we point out the introduction of different orderings on families of intervals of the real line could be also analyzed from the point of view of extensions of the canonical ordering of the real line to a superset (namely, $L([0, 1])$) following a suitable set of criteria established a priori. The real line can be immediately embedded into $L([0, 1])$ by just considering each real number x as the degenerate interval $[x, x]$.

A similar typical problem corresponds to the extension of linear orders from a finite set to its power set. Indeed, although it is always possible to extend a linear order from a given finite set U to its power set, a typical question that gave rise to some classical papers from the 1970s on (see e.g. [25–28]), is whether or not it is possible to perform an extension that follows a list of criteria imposed a priori. Sometimes, the extension is not possible because the criteria used are, so-to-say, contradictory. But, in addition, there are other situations in which the extension is not possible because of a combinatorial explosion which, due to the bigger cardinality of the power set of U, does not leave room to rank all the terms of the power set in an extended linear order, accomplishing all the criteria. Perhaps the most famous result in this direction is the so-called Kannai–Peleg impossibility theorem (see [26]).

However, when the extension does not affect to the whole power set, but to some suitable superset (smaller than the power set), perhaps it may still happen that an extension accomplishing aprioristic criteria is possible, after all. As far as we know, an analysis of this kind where we start with the canonical order of the real line (instead of a linear order on a finite set), and try to extend it to the set of closed intervals of real numbers, following some list of criteria that have been set beforehand, is an open problem.

We leave for future works the interpretation of the length of the intervals in a given decision making problem and its relation with ignorance functions and possibility theory.

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