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STOLARSKY'S INEQUALITY FOR CHOQUET-LIKE EXPECTATION

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ABSTRACT. Expectation is the fundamental concept in statistics and probability. As two generalizations of expectation, Choquet and Choquet-like expectations are commonly used tools in generalized probability theory. This paper considers the Stolarsky inequality for two classes of Choquet-like integrals. The first class generalizes the Choquet expectation and the second class is an extension of the Sugeno integral. Moreover, a new Minkowski's inequality without the comonotonicity condition for two classes of Choquet-like integrals is introduced. Our results significantly generalize the previous results in this field. Some examples are given to illustrate the results.

1. Introduction

Choquet expectation [7] plays an important role in analyzing many problems in statistics and probability, especially when dealing with non-additive data [26]. The classical integral inequalities are used in numerous applications in information theory, engineering, economics and statistics. Stolarsky's inequality is one of the most fundamental inequalities in analysis and its applications. The history and the development of this inequality are described in [12,21]. Stolarsky's inequality is a useful tool in several theoretical and applied fields. For instance, it plays a major role for the gamma function [21]. A weighted version of Stolarsky's inequality was presented in [1,12]. The mixing of Choquet expectation and integral inequalities can be applied to find solutions of many uncertain problems. Recently, Agahi *et al.* [2] proved the following Stolarsky type inequality for Choquet expectation.

Theorem 1.1. ([2: Theorem 4.1]) *Let $\mathcal{B}([0,1])$ be the Borel σ-algebra over $[0,1]$ and $([0,1],\mathcal{B}([0,1]),\mu)$ be a monotone probability space such that μ is a lower-semicontinuous monotone probability absolutely continuous with respect to the Lebesgue measure λ, i.e., if $\lambda(E)=0$ for some $E\in\mathcal{B}([0,1])$ then also $\mu(E)=0$. Then the Stolarsky inequality holds for the corresponding Choquet expectation, i.e., for any nonincreasing function $X\colon[0,1]\to[0,1]$ it holds*

$$\mathbb{E}_C^{\mu}\left[X(\omega^{\frac{1}{a+b}})\right] \geq \mathbb{E}_C^{\mu}[X(\omega^{\frac{1}{a}})]\mathbb{E}_C^{\mu}[X(\omega^{\frac{1}{b}})], \tag{1.1}$$

where $a,b\in(0,\infty)$.

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Sugeno integral was firstly introduced by Sugeno [23] and then was exploited by many authors [16, 17, 25]. The idea of mixing of Sugeno integral and integral inequalities was presented by Román-Flores et al. [18] and then followed by the authors [3,9,14]. For example, in [9], Stolarsky's inequality for a Lebesgue measure-based Sugeno integral and a continuous and strictly monotone function was obtained which has been extended by Agahi et al. [2].

Theorem 1.2. ([2: Theorem 3.5]) *Let* $X\colon [0,1] \to [0,1]$ *be a nondecreasing function and* $([0,1], \mathcal{B}[0,1], \mu)$ *be a monotone probability space. Let* β, γ *be automorphisms on* $[0,1]$ *(i.e.,* $\beta, \gamma\colon [0,1] \to [0,1]$ *are increasing bijections) and* $\alpha = (\beta^{-1} \star \gamma^{-1})^{-1}$*. If* $\star\colon [0,1]^2 \to [0,1]$ *be any continuous aggregation function which is jointly strictly increasing and bounded from above by* min*, then*

$$\mathrm{Su}_\mu[X(\alpha)] \geq (\mathrm{Su}_\mu[X(\beta)]) \star (\mathrm{Su}_\mu[X(\gamma)]).$$

In 1995, Mesiar [13] introduced two classes of Choquet-like integrals as generalizations of Choquet expectation and Sugeno integral. The first class is called "Choquet-like expectation" which generalizes the Choquet expectation (see Definition 2.7) and the second class is an extension of the Sugeno integral (see Definition 2.9). This motivates us to propose the following problem.

Problem. Under what conditions does the Stolarsky inequality hold for two classes of Choquet-like integrals?

We will give the answer to the above problem in Section 3 which will imply a generalization of [2, 9]. Our results expand the applicability of Stolarsky type inequality for Choquet expectation by combining the properties of pseudo-analysis.

Recently, there were obtained some of the classical integral inequalities for integrals with respect to non-additive measures based on the concept of comonotonicity [4, 6, 15, 28]. For example, Minkowski's inequality for Sugeno integral has studied in [4]. In [4], the authors showed that in general, the Minkowski inequality is not valid without the comonotonicity condition (See [4: Example 3.6]). However, the major question that arises in mind is the following problem.

Problem. How can Minkowski's inequality be explained without comonotonicity condition for Sugeno integral?

In this paper, we also give a new Minkowski's inequality without the comonotonicity condition for two classes of Choquet-like integrals. In special cases, our results give a new version of Minkowski's inequality for Sugeno integral without the comonotonicity condition.

The paper is organized as follows: Section 2 recalls the concepts of Choquet-like integrals while Sections 3, 4 present our main results. Finally, some concluding remarks are added.

2. Preliminaries

To prove our results, we shall first recall some basic definitions and previous results. For details, we refer to [13] (see also [19]).

Definition 2.1. ([24]) An operation $\oplus\colon [0,\infty]^2 \to [0,\infty]$ is called a pseudo-addition if the following properties are satisfied:

(P1) $a \oplus 0 = 0 \oplus a = a$ (neutral element);

(P2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);

(P3) $a \leq c$ and $b \leq d$ imply that $a \oplus b \leq c \oplus d$ (monotonicity);

(P4) $a_n \to a$ and $b_n \to b$ imply that $a_n \oplus b_n \to a \oplus b$ (continuity).

Definition 2.2. ([13, 24]) Let $\oplus$ be a given pseudo-addition on $[0, \infty]$. Another binary operation $\otimes$ on $[0, \infty]$ is said to be a pseudo-multiplication corresponding to $\oplus$ if the following properties are satisfied:

(M1) $a \otimes (x \oplus y) = (a \otimes x) \oplus (a \otimes y)$ (left distributivity);

(M2) $a \leq b$ implies $(a \otimes x) \leq (b \otimes x)$ and $(x \otimes a) \leq (x \otimes b)$ (monotonicity);

(M3) $a \otimes x = 0 \Leftrightarrow a = 0$ or $x = 0$ (absorbing element and no zero divizors);

(M4) exists $e \in (0, \infty]$ (i.e., there exist the neutral element e) such that $e \otimes x = x \otimes e = x$ for any $x \in [0, \infty]$ (neutral element);

(M5) $a_n \to a \in (0, \infty)$ and $x_n \to x$ imply$(a_n \otimes x_n) \to (a \otimes x)$ and $\infty \otimes x = \lim_{a \to \infty} (a \otimes x)$ (continuity);

(M6) $a \otimes x = x \otimes a$ (commutativity);

(M7) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ (associativity).

Theorem 2.3. ([13]) *Let $\otimes$ be a pseudo-multiplication corresponding to a given pseudo-addition $\oplus$ fulfilling axioms* (M1)–(M7).

(I) *If its identity element e is not an idempotent of $\oplus$, then there is a unique continuous strictly increasing function $g\colon [0, \infty] \to [0, \infty]$ with $g(0) = 0$ and $g(\infty) = \infty$, such that $g(e) = 1$ and*

$$a \oplus b = g^{-1}(g(a) + g(b)) \quad \oplus \text{ is called a } g\text{-addition},$$

$$a \otimes b = g^{-1}(g(a) \cdot g(b)) \quad \otimes \text{ is called a } g\text{-multiplication}.$$

(II) *If the identity element e of the pseudo-multiplication is also an idempotent of $\oplus$ (i.e., $e \oplus e = e$), then $\oplus = \vee$ (= sup, i.e., the logical addition).*

For $x \in [0, \infty]$ and $p \in (0, \infty)$, we will introduce the pseudo-power $x_{\otimes}^{(p)}$ as follows: If $p = n$ is a natural number, then $x_{\otimes}^{(n)} = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n-\text{times}}$. If p is not a natural number, then the corresponding power is defined by $x_{\otimes}^{(p)} = \sup \left\{ y_{\otimes}^{(m)} \mid y_{\otimes}^{(n)} \leqslant x, \text{ where } m, n \text{ are natural numbers such that } \frac{m}{n} \leqslant p \right\}$. Evidently, if $x \otimes y = g^{-1}(g(x) \cdot g(y))$, then $x_{\otimes}^{(p)} = g^{-1}(g^p(x))$.

Definition 2.4 ([11]). A monotone measure μ on a measurable space $(\Omega, \mathcal{F})$ is a set function $\mu\colon \mathcal{F} \to [0, \infty)$ satisfying

(i) $\mu(\emptyset) = 0$;

(ii) $\mu(\Omega) > 0$;

(iii) $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$;

moreover, μ is called finite if $\|\mu\| = \mu(\Omega) < \infty$. The triple $(\Omega, \mathcal{F}, \mu)$ is also called a *monotone measure space* if μ is a monotone measure on $\mathcal{F}$.

We call μ a *monotone probability*, if $\|\mu\| = 1$. When μ is a monotone probability, the triple $(\Omega, \mathcal{F}, \mu)$ is called a *monotone probability space*.

Definition 2.5. Let $(\Omega, \mathcal{F})$ be a measurable space. For each number $a \in (0, \infty]$, $\mathcal{M}_a^{(\Omega,\mathcal{F})}$ denotes the set of all monotone measures (in the sense of Definition 2.4) satisfying $\|\mu\| = a$.

Definition 2.6. Let $(\Omega, \mathcal{F}, \mu)$ be a monotone measure space and $X \colon \Omega \to [0, \infty)$ be an $\mathcal{F}$-measurable function. The Choquet expectation of X over $A \in \mathcal{F}$ w.r.t. the monotone measure μ is defined as

$$\mathbb{E}_C^{\mu}[X\mathbb{I}_A] = \int_0^{\infty} \mu(A \cap \{X \geq t\})\,\mathrm{d}t. \tag{2.1}$$

where the integral on the right-hand side is the (improper) Riemann integral. In particular, if $A = \Omega$, then

$$\mathbb{E}_C^{\mu}[Xt] = \int_0^{\infty} \mu(\{X \geq t\})\,\mathrm{d}t.$$

Mesiar [13] has shown that there are two classes of Choquet-like integral: the Choquet-like integral (denoted by $\mathbb{E}_{Cl,g}^{\mu}$) based on a g-addition and a g-multiplication and the Choquet-like integral based on $\vee$ and a corresponding pseudo-multiplication $\otimes$.

Definition 2.7. ([13]) Let $(\Omega, \mathcal{F})$ be a measurable space and $\mu \colon \mathcal{F} \to [0, \infty]$ be a monotone measure. Let $\oplus$ and $\otimes$ be generated by a generator g. The Choquet-like expectation of a non-negative measurable function X over $A \in \mathcal{F}$ w.r.t. the monotone measure μ can be represented as

$$\mathbb{E}_{Cl,g}^{\mu}[X\mathbb{I}_A] = g^{-1}(\mathbb{E}_C^{g(\mu)}[g(X)\mathbb{I}_A]) = g^{-1}\Bigl(\int_0^{\infty} g\mu(A \cap \{g(X) \geq t\})\,\mathrm{d}t\Bigr).$$

In particular, if $A = \Omega$, then

$$\mathbb{E}_{Cl,g}^{\mu}[X] = g^{-1}\Bigl(\mathbb{E}_C^{g(\mu)}[g(X)]\Bigr). \tag{2.2}$$

Remark 2.8. Notice that we sometimes call this kind of Choquet-like integral a g-Choquet integral (g-C-integral for short). It is plain that the g-C-integral is the original Choquet integral (expectation) whenever $g = i$ (the identity mapping).

Definition 2.9. ([13]) Let $\otimes$ be a pseudo-multiplication corresponding to $\vee$ and fulfilling (M1)–(M7). Then the Choquet-like integral (so-called $\mathbb{S}_{\mu}^{\otimes}$ integral) of a measurable function $X \colon \Omega \to [0, \infty)$ w.r.t. a finite monotone measure μ can be represented as

$$\mathbb{S}_{\mu}^{\otimes}[Xt] = \sup_{a \in [0,\infty]} (a \otimes \mu(\{X \geq a\})). \tag{2.3}$$

It is plain that the $\mathbb{S}_{\mu}^{\otimes}$ integral is the Sugeno integral (denoted by $\mathrm{Su}_{\mu}[\cdot]$) whenever $\otimes = \wedge$ [25]. During this paper, we always consider the existence of all $\mathbb{S}_{\mu}^{\otimes}[\cdot]$.

Remark 2.10. Notice that when working on $[0, 1]$, we mostly deal with $e = 1$, then $\otimes = \circledast$ is a semicopula (t-seminorm), i.e., a binary operation $\circledast \colon [0, 1]^2 \to [0, 1]$ which is non-decreasing in both components and has 1 as neutral element. Then $\otimes = \circledast$ satisfies $a \otimes b \leq \min(a, b)$ for all $(a, b) \in [0, 1]^2$, see [8].

Definition 2.11. The $\mathbb{S}_\mu^{\circledast}$ integral on the $[0,1]$ scale related to the semicopula $\circledast$ is given by

$$\mathbb{S}_\mu^{\circledast}[X] = \sup_{a\in[0,1]} (a \circledast \mu(\{X \geq a\}).$$

This type of integral was called seminormed integral in [22].

Remark 2.12. For a fixed strict t-norm T, the corresponding $\mathbb{S}_\mu^T$ integral is the so-called Sugeno-Weber integral [27]. If $\circledast$ is the standard product, then the Shilkret integral [20] (denoted by $\mathrm{Sh}_\mu[\cdot]$) can be recognized. Notice that the original Sugeno integral (denoted by $\mathrm{Su}_\mu[\cdot]$) which was introduced by Sugeno [23] in 1974 is a special seminormed integral when $\circledast = \min$.

[5: Theorem 2.13] helps us to reach the main results.

Theorem 2.13. *Let $X, Y \colon \Omega \to [0,\infty)$ be two nondecreasing functions and $(\Omega, \mathcal{F}, \mu)$ a monotone measure space and $\star\colon [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and $\varphi_i \colon [0,\infty) \to [0,\infty)$, $i = 0,1,2$ be continuous and strictly increasing functions. If $\otimes$ is a pseudo-multiplication (with neutral element $e = \|\mu\|$) corresponding to $\vee$ satisfying*

$$\varphi_0^{-1}(\varphi_0(p_1 \star p_2) \otimes c) \geq [\varphi_1^{-1}[(\varphi_1(p_1)) \otimes c] \star p_2] \vee [p_1 \star \varphi_2^{-1}[\varphi_2(p_2) \otimes c]]$$

then for any monotone measure $\mu \in \mathcal{M}_e^{(\Omega,\mathcal{F})}$ such that $\mathbb{S}_\mu^{\otimes}[\cdot] < \infty$, the inequality

$$\varphi_0^{-1}(\mathbb{S}_\mu^{\otimes}[\varphi_0(X \star Y)]) \geq \varphi_1^{-1}(\mathbb{S}_\mu^{\otimes}[\varphi_1(X)]) \star \varphi_2^{-1}(\mathbb{S}_\mu^{\otimes}[\varphi_2(Y)])$$

holds.

3. Stolarsky's inequality

The purpose of this section is to prove the Stolarsky inequality for two classes of Choquet-like integrals. Theorem 3.1 gives us the Stolarsky's inequality for the first class Choquet-like integrals, i.e., for Choquet-like expectation. Afterwards, we will obtain the Stolarsky's inequality for the second class in Theorem 3.3.

Theorem 3.1. *Let $([0,1], \mathcal{B}([0,1]), \mu)$ be a monotone probability space and let the pseudo-operations be generated by a generator $g\colon [0,\infty] \to [0,\infty]$, $g(1) = 1$. Let $a, b > 0$. Then the Stolarsky inequality holds for the corresponding Choquet-like expectation, i.e., for any nonincreasing real-valued function $X\colon [0,1] \to [0,1]$ it holds*

$$\mathbb{E}_{Cl,g}^{\mu}\big[X(\omega^{\frac{1}{a+b}})\big] \geq \big(\mathbb{E}_{Cl,g}^{\mu}\big[X(\omega^{\frac{1}{a}})\big]\big) \otimes \big(\mathbb{E}_{Cl,g}^{\mu}\big[X(\omega^{\frac{1}{b}})\big]\big), \tag{3.1}$$

where $g(\mu)$ is a lower-semicontinuous monotone probability absolutely continuous with respect to the Lebesgue measure λ, i.e., if $\lambda(E) = 0$ for some $E \in \mathcal{B}([0,1])$ then also $g(\mu(E)) = 0$.

Proof. Observe that

$$\mathbb{E}_{Cl,g}^{\mu}[X(\omega^{\frac{1}{a+b}})] = g^{-1}\big(\mathbb{E}_{C}^{g(\mu)}\big[g\big(X(\omega^{\frac{1}{a+b}})\big)\big]\big). \tag{3.2}$$

From (3.2) and using the Stolarsky inequality for Choquet expectation (1.1), we have

$$\begin{aligned}\mathbb{E}^{\mu}_{Cl,g}[X(\omega^{\frac{1}{a+b}})] &\geq g^{-1}\big(\mathbb{E}^{g(\mu)}_{C}\big[g\big(X(\omega^{\frac{1}{a}})\big)\big]\cdot\mathbb{E}^{g(\mu)}_{C}\big[g\big(X(\omega^{\frac{1}{b}})\big)\big]\big)\\ &= g^{-1}\big[g\big(g^{-1}\big(\mathbb{E}^{g\left(\mu\right)}_{C}\big[g\big(X(\omega^{\frac{1}{a}})\big)\big]\big)\big)\cdot g\big(g^{-1}\big(\mathbb{E}^{g\left(\mu\right)}_{C}\big[g\big(X(\omega^{\frac{1}{b}})\big)\big]\big)\big)\big]\\ &= g^{-1}\big(g\big(\mathbb{E}^{\mu}_{Cl,g}\big[X(\omega^{\frac{1}{a}})\big]\big)\cdot g\big(\mathbb{E}^{\mu}_{Cl,g}\big[X(\omega^{\frac{1}{b}})\big]\big)\big)\\ &= \big(\mathbb{E}^{\mu}_{Cl,g}\big[X(\omega^{\frac{1}{a}})\big]\big)\otimes\big(\mathbb{E}^{\mu}_{Cl,g}\big[X(\omega^{\frac{1}{b}})\big]\big).\end{aligned}$$

This completes the proof. □

***Example* 3.2.**

(i) Let $g(x) = x^{\alpha}$, $\alpha > 0$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^{\alpha}+y^{\alpha}}$ and $x \otimes y = xy$. Then (3.1) reduces on the following inequality

$$\sqrt[\alpha]{\left(\mathbb{E}^{\mu^{\alpha}}_{C}\left[(X(\omega^{\frac{1}{a+b}}))^{\alpha}\right]\right)} \geqslant \sqrt[\alpha]{\left(\mathbb{E}^{\mu^{\alpha}}_{C}\left[(X(\omega^{\frac{1}{a}}))^{\alpha}\right]\right)}\cdot\sqrt[\alpha]{\left(\mathbb{E}^{\mu^{\alpha}}_{C}\left[(X(\omega^{\frac{1}{b}}))^{\alpha}\right]\right)}.$$

(ii) Let $g(x) = 2^{x} - 1$. The corresponding pseudo-operations are $x \oplus y = \frac{1}{\ln 2}\ln(2^{x}+2^{y}-1)$ and $x \otimes y = \frac{1}{\ln 2}\ln\left((2^{x}-1)(2^{y}-1)+1\right)$. Then (3.1) reduces on the following inequality

$$\mathbb{E}^{2^{\mu}-1}_{C}\big[2^{X(\omega^{\frac{1}{a+b}})}-1\big] \geqslant \mathbb{E}^{2^{\mu}-1}_{C}\big[2^{X(\omega^{\frac{1}{a}})}-1\big]\cdot\mathbb{E}^{2^{\mu}-1}_{C}\big[2^{X(\omega^{\frac{1}{b}})}-1\big].$$

Now we consider the second class of the Choquet-like integral where is based on $\vee$ and a corresponding pseudo-multiplication $\otimes$ with neutral element $e = \|\mu\|$.

Theorem 3.3. *Let $X\colon [0,\infty) \to [0,\infty)$ be a nondecreasing function, $([0,\infty),\mathcal{B}[0,\infty),\mu)$ be a monotone measure space, $\star\colon [0,\infty)^{2} \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by* min *and $\varphi_i\colon [0,\infty) \to [0,\infty)$, $i = 0,1,2$ be continuous and strictly increasing functions. Let $\alpha,\beta,\gamma\colon [0,\infty) \to [0,\infty)$ be increasing bijections such that $\alpha \geqslant \max\{\beta,\gamma\}$. If $\otimes$ is a pseudo-multiplication (with neutral element $e = \|\mu\|$) corresponding to $\vee$ satisfying*

$$\varphi_0^{-1}(\varphi_0(p_1\star p_2)\otimes c) \geq \left[\varphi_1^{-1}[(\varphi_1(p_1))\otimes c]\star p_2\right]\vee\left[p_1\star\varphi_2^{-1}[\varphi_2(p_2)\otimes c]\right],$$

then for any monotone measure $\mu \in \mathcal{M}_e^{([0,\infty),\mathcal{B}[0,\infty))}$ such that $\mathbb{S}^{\otimes}_{\mu}[\cdot] < \infty$, the inequality

$$\varphi_0^{-1}\big(\mathbb{S}^{\otimes}_{\mu}\big[\varphi_0(X(\alpha))\big]\big) \geq \varphi_1^{-1}\big(\mathbb{S}^{\otimes}_{\mu}\big[\varphi_1\big(X(\beta)\big)\big]\big)\star\varphi_2^{-1}\big(\mathbb{S}^{\otimes}_{\mu}\big[\varphi_2(X(\gamma))\big]\big)$$

holds.

Proof. Since $\alpha \geq \beta$, $\alpha \geq \gamma$ and X is nondecreasing, we have $X(\alpha) \geq X(\beta)\wedge X(\gamma) \geq X(\beta)\star X(\gamma)$. Note that all the three functions $X(\alpha), X(\beta)$ and $X(\gamma)$ are nondecreasing. Now applying Theorem 2.13, by the monotonicity of $\mathbb{S}^{\otimes}_{\mu}$ integral there holds

$$\begin{aligned}\varphi_0^{-1}(\mathbb{S}^{\otimes}_{\mu}\left[\varphi_0\left(X\left(\alpha\right)\right)\right]) &\geq \varphi_0^{-1}(\mathbb{S}^{\otimes}_{\mu}\left[\varphi_0\left(X(\beta)\star X(\gamma)\right)\right])\\ &\geq \varphi_1^{-1}\left(\mathbb{S}^{\otimes}_{\mu}\left[\varphi_1\left(X\left(\beta\right)\right)\right]\right)\star\varphi_2^{-1}\left(\mathbb{S}^{\otimes}_{\mu}\left[\varphi_2\left(X\left(\gamma\right)\right)\right]\right).\end{aligned}$$

□

Since we work on the $[0,1]$ scale in Theorem 3.3 then $\otimes = \circledast$ is semicopula (t-seminorm) and the following result holds.

Corollary 3.4. *Let $X\colon [0,1] \to [0,1]$ be a nondecreasing function, $([0,1], \mathcal{B}[0,1], \mu)$ be a monotone probability space, $*\colon [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by* min *and $\varphi_i\colon [0,1] \to [0,1]$, $i = 0,1,2$ be continuous and strictly increasing functions. Let $\alpha, \beta, \gamma\colon [0,1] \to [0,1]$ be increasing bijections such that $\alpha \geqslant \max\{\beta, \gamma\}$. If semicopula $\circledast$ satisfying*

$$\varphi_0^{-1}\left(\varphi_0\left(p_1 \star p_2\right) \circledast c\right) \geq \left[\varphi_1^{-1}\left[\left(\varphi_1\left(p_1\right)\right) \circledast c\right] \star p_2\right] \vee \left[p_1 \star \varphi_2^{-1}\left[\varphi_2\left(p_2\right) \circledast c\right]\right],$$

then the inequality

$$\varphi_0^{-1}(\mathbb{S}_\mu^\circledast\left[\varphi_0\left(X(\alpha)\right)\right]) \geq \varphi_1^{-1}\left(\mathbb{S}_\mu^\circledast\left[\varphi_1\left(X\left(\beta\right)\right)\right]\right) \star \varphi_2^{-1}\left(\mathbb{S}_\mu^\circledast\left[\varphi_2\left(X\left(\gamma\right)\right)\right]\right)$$

holds.

Let $\varphi_0(x) = x^r$, $0 < r < \infty$, and $\varphi_1(x) = \varphi_2(x) = x^s$, $0 < s < \infty$, then for all $0 < s < \infty$, then we get the following result.

Corollary 3.5. *Let $X\colon [0,1] \to [0,1]$ be a nondecreasing function, $([0,1], \mathcal{B}[0,1], \mu)$ be a monotone probability space and $\star\colon [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by min. Let $\alpha, \beta, \gamma\colon [0,1] \to [0,1]$ be increasing bijections such that $\alpha \geqslant \max\{\beta, \gamma\}$. If semicopula $\circledast$ satisfying*

$$\left((a \star b)^r \circledast c)\right)^{\frac{1}{r}} \geq \left[\,(a^s \circledast c)^{\frac{1}{s}} \star b\right] \vee \left[a \star (b^s \circledast c)^{\frac{1}{s}}\right],$$

then the inequality

$$\left(\mathbb{S}_\mu^\circledast\left[(X(\alpha))^r\right]\right)^{\frac{1}{r}} \geq \left(\mathbb{S}_\mu^\circledast\left[(X\left(\beta\right))^s\right]\right)^{\frac{1}{s}} \star \left(\mathbb{S}_\mu^\circledast\left[(X\left(\gamma\right))^s\right]\right)^{\frac{1}{s}}$$

holds for all $0 < s < \infty$ and $0 < r < \infty$.

Specially, when $s = 1$, we have the following result.

Corollary 3.6. *Let $X\colon [0,1] \to [0,1]$ be a nondecreasing function, $([0,1], \mathcal{B}[0,1], \mu)$ be a monotone probability space and $\star\colon [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by min. Let $\alpha, \beta, \gamma\colon [0,1] \to [0,1]$ be increasing bijections such that $\alpha \geqslant \max\{\beta, \gamma\}$. If semicopula $\circledast$ satisfying*

$$\left((a \star b)^r \circledast c\right)^{\frac{1}{r}} \geq \left[(a \circledast c) \star b\right] \vee \left[a \star (b \circledast c)\right], \tag{3.3}$$

then the inequality

$$\left(\mathbb{S}_\mu^\circledast[(X(\alpha))^r]\right)^{\frac{1}{r}} \geq \left(\mathbb{S}_\mu^\circledast[X(\beta)]\right) \star \left(\mathbb{S}_\mu^\circledast\left[X\left(\gamma\right)\right]\right) \tag{3.4}$$

holds for all $0 < r < \infty$.

***Example* 3.7.** Let nondecreasing function X be defined as $X(\omega) = \sqrt{\omega}$. Let $\circledast$ and $\star$ be the standard product. Then for $r = 1$, (3.3) holds readily. If μ is the Lebesgue measure, then

$$\mathrm{Sh}_\mu\left[X\left(\omega\right)\right] = \frac{2}{9}\sqrt{3}, \qquad \mathrm{Sh}_\mu\left[X\left(\omega^2\right)\right] = \frac{1}{4} \qquad \text{and} \qquad \mathrm{Sh}_\mu\left[X\left(\omega^4\right)\right] = \frac{4}{27}.$$

Therefore,

$$\begin{aligned}\mathbb{S}_\mu^\circledast\left[X(\omega)\right] = \mathrm{Sh}_\mu\left[X(\omega)\right] &= \frac{2}{9}\sqrt{3} \\ &\geqslant \frac{1}{27} = \mathrm{Sh}_\mu\left[X\left(\omega^2\right)\right] \cdot \mathrm{Sh}_\mu\left[X\left(\omega^4\right)\right] = \left(\mathbb{S}_\mu^\circledast\left[X\left(\omega^2\right)\right]\right) \star \left(\mathbb{S}_\mu^\circledast\left[X\left(\omega^4\right)\right]\right).\end{aligned}$$

***Example* 3.8.** Let $\star(x,y) = (x+y-1)\vee 0$ in Example 3.7. Then a straightforward calculus shows that

$$\begin{aligned}\mathbb{S}^{\circledast}_{\mu}\left[X(\omega)\right] = \frac{2}{9}\sqrt{3} \geqslant \left(\frac{1}{4}+\frac{4}{27}-1\right)\vee 0 &= \left(\mathrm{Sh}_{\mu}\left[X\left(\omega^{2}\right)\right] + \mathrm{Sh}_{\mu}\left[X\left(\omega^{4}\right)\right]-1\right)\vee 0\\ &= \left(\mathbb{S}^{\circledast}_{\mu}\left[X(\omega^{2})\right]\right)\star\left(\mathbb{S}^{\circledast}_{\mu}\left[X\left(\omega^{4}\right)\right]\right).\end{aligned}$$

Let $\circledast = \min$ in Corollary 3.6. Then we get the following result.

Corollary 3.9. ([2: Theorem 3.5]) *Let $X\colon [0,1]\to[0,1]$ be a nondecreasing function and $([0,1],\mathcal{B}[0,1],\mu)$ a monotone probability space. Let β, γ be automorphisms on $[0,1]$ (i.e., $\beta,\gamma\colon [0,1]\to[0,1]$ are increasing bijections) and $\alpha = (\beta^{-1}\star\gamma^{-1})^{-1}$. If $\star\colon [0,1]^2\to[0,1]$ be any continuous aggregation function which is jointly strictly increasing and bounded from above by* min*, then*

$$\mathrm{Su}_{\mu}[X(\alpha)] \geq (\mathrm{Su}_{\mu}[X(\beta)])\star(\mathrm{Su}_{\mu}\left[X\left(\gamma\right)\right]). \tag{3.5}$$

Proof. As $\star$ is bounded from above by min, $\alpha^{-1} = \beta^{-1}\star\gamma^{-1} \leq \beta^{-1}\wedge\gamma^{-1}$ and thus $\alpha\geq\beta$, $\alpha\geq\gamma$. Since X is nondecreasing, we have $X(\alpha)\geq X(\beta)\wedge X(\gamma)\geq X(\beta)\star X(\gamma)$. Notice that if $\star$ is bounded from above by min, then for $r=1$, (3.3) holds readily. Applying the Corollary 3.6, the proof is completed. $\square$

4. A new inequality

This section intends to present a new Minkowski's inequality without the comonotonicity condition for two classes of Choquet-like integrals. Theorems 4.1 and 4.3 are the main results of this section. During this section, we always consider the existence of all Choquet expectation.

Theorem 4.1. *Let X,Y be two non-negative measurable functions and let the pseudo-operations be generated by a generator g. If*

$$mg\left(Y(\omega)\right) \leqslant g(X(\omega)) \leqslant Mg\left(Y(\omega)\right) \qquad \text{for all} \quad \omega\in\Omega, \tag{4.1}$$

where $0<m\leqslant M$, then for the Choquet-like expectation given by (2.2), the following inequalities

$$\left(\mathbb{E}^{\mu}_{Cl,g}\left[\left(X\oplus Y\right)^{(s)}\right]\right)^{\left(\frac{1}{s}\right)}_{\otimes} \geqslant K_1\otimes\left[\left(\mathbb{E}^{\mu}_{Cl,g}\left[X^{(s)}_{\otimes}\right]\right)^{\left(\frac{1}{s}\right)}_{\otimes}\oplus\left(\mathbb{E}^{\mu}_{Cl,g}\left[Y^{(s)}_{\otimes}\right]\right)^{\left(\frac{1}{s}\right)}_{\otimes}\right], \tag{4.2}$$

$$\left(\mathbb{E}^{\mu}_{Cl,g}\left[(X\oplus Y)^{(s)}\right]\right)^{\left(\frac{1}{s}\right)}_{\otimes} \leqslant K_2\otimes\left[\left(\mathbb{E}^{\mu}_{Cl,g}\left[X^{(s)}_{\otimes}\right]\right)^{\left(\frac{1}{s}\right)}_{\otimes}\oplus\left(\mathbb{E}^{\mu}_{Cl,g}\left[Y^{(s)}_{\otimes}\right]\right)^{\left(\frac{1}{s}\right)}_{\otimes}\right], \tag{4.3}$$

hold where $s\geqslant 1$ and the constants $K_1 = g^{-1}\left[\left(\frac{M(m+1)+(M+1)}{(M+1)(m+1)}\right)^{-1}\right]$, $K_2 = g^{-1}\left[\left(\frac{m(M+1)+(m+1)}{(m+1)(M+1)}\right)^{-1}\right]$ are independent of μ.

Proof. We will prove inequality (4.2) and the other case is similar. By (4.1), we have

$$g\left(X(\omega)\right) \leqslant M\left(g\left(X(\omega)\right)+g\left(Y(\omega)\right)\right) - Mg\left(X(\omega)\right).$$

So

$$\left(M+1\right)^{s}\left(g(X(\omega))\right)^{s} \leqslant M^{s}\left(g(X(\omega))+g(Y(\omega))\right)^{s},$$

and then

$$\left(\mathbb{E}_C^{g(\mu)}\left[g\left(X^s\right)\right]\right)^{\frac{1}{s}} \leqslant \frac{M}{M+1}\left(\mathbb{E}_C^{g(\mu)}\left[(g(X)+g(Y))^s\right]\right)^{\frac{1}{s}}. \tag{4.4}$$

Also, since $mg\left(Y(\omega)\right) \leqslant g\left(X(\omega)\right)$, we have $g\left(Y(\omega)\right) \leqslant \frac{1}{m}\left(g(X(\omega))+g\left(Y(\omega)\right)\right)-\frac{1}{m}g\left(Y\left(\omega\right)\right)$. So

$$\left(\frac{1}{m}+1\right)^s\left(g\left(Y(\omega)\right)\right)^s \leqslant \left(\frac{1}{m}\right)^s\left(g\left(X(\omega)\right)+g\left(Y(\omega)\right)\right)^s,$$

and then,

$$\left(\mathbb{E}_C^{g(\mu)}[g(Y^s)]\right)^{\frac{1}{s}} \leqslant \frac{1}{m+1}\left(\mathbb{E}_C^{g(\mu)}\left[(g(X)+g(Y))^s\right]\right)^{\frac{1}{s}}. \tag{4.5}$$

Therefore, (4.4) and (4.5) give us the following result:

$$g(K_1)\left[\left(\mathbb{E}_C^{g(\mu)}[g(X^s)]\right)^{\frac{1}{s}}+\left(\mathbb{E}_C^{g(\mu)}[g(Y^s)]\right)^{\frac{1}{s}}\right] \leqslant \left(\mathbb{E}_C^{g(\mu)}\left[(g(X)+g(Y))^s\right]\right)^{\frac{1}{s}}. \tag{4.6}$$

Now, observe that

$$\begin{aligned}
\left(\mathbb{E}_{Cl,g}^{\mu}\left[(X\oplus Y)^{(s)}\right]\right)_{\otimes}^{\left(\frac{1}{s}\right)} &= g^{-1}\Big(\left(\mathbb{E}_C^{g(\mu)}\left[g\left((X\oplus Y)_{\otimes}^{(s)}\right)\right]\right)^{\frac{1}{s}}\Big)\\
&= g^{-1}\Big(\left(\mathbb{E}_C^{g(\mu)} g\left(g^{-1}\left((g(X\oplus Y))^s\right)\right)\right)^{\frac{1}{s}}\Big)\\
&= g^{-1}\Big(\left(\mathbb{E}_C^{g(\mu)}\left[((g\circ X)+(g\circ Y))^s\right]\right)^{\frac{1}{s}}\Big).
\end{aligned}$$

By using (4.6), we have

$$\begin{aligned}
& g^{-1}\Big(\left(\mathbb{E}_C^{g(\mu)}\left[((g\circ X)+(g\circ Y))^s\right]\right)^{\frac{1}{s}}\Big)\\
&\geqslant g^{-1}\Big(g(K_1)\Big[\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ X)^s\right]\right)^{\frac{1}{s}}+\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ Y)^s\right]\right)^{\frac{1}{s}}\Big]\Big)\\
&= g^{-1}\Big(g(K_1)\Big[g\left(g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ X)^s\right]\right)^{\frac{1}{s}}\right)\right)+g\left(g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ Y)^s\right]\right)^{\frac{1}{s}}\right)\right)\Big]\Big)\\
&= g^{-1}\left(g(K_1)\left[g\left(g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ X)^s\right]\right)^{\frac{1}{s}}\right)\right)+g\left(g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ Y)^s\right]\right)^{\frac{1}{s}}\right)\right)\right]\right)\\
&= K_1\otimes g^{-1}\left(\left[g\left(g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ X)^s\right]\right)^{\frac{1}{s}}\right)\right)+g\left(g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[(g\circ Y)^s\right]\right)^{\frac{1}{s}}\right)\right)\right]\right)\\
&= K_1\otimes\left[g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[g\left(g^{-1}\left((g\circ X)^s\right)\right)\right]\right)^{\frac{1}{s}}\right)\oplus g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[g\left(g^{-1}\left((g\circ Y)^s\right)\right)\right]\right)^{\frac{1}{s}}\right)\right]\\
&= K_1\otimes\left[g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[g(X_{\otimes}^{(s)})\right]\right)^{\frac{1}{s}}\right)\oplus g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\left[g(Y_{\otimes}^{(s)})\right]\right)^{\frac{1}{s}}\right)\right]\\
&= K_1\otimes\left[g^{-1}\Big(\Big(g\big(g^{-1}\big(\mathbb{E}_C^{g(\mu)}\left[g(X_{\otimes}^{(s)})\right]\big)\big)\Big)^{\frac{1}{s}}\Big)\oplus g^{-1}\Big(\Big(g\big(g^{-1}\big(\mathbb{E}_C^{g(\mu)}\left[g(Y_{\otimes}^{(s)})\right]\big)\big)\Big)^{\frac{1}{s}}\Big)\right]\\
&= K_1\otimes\left[g^{-1}\left(\left(g\left(\mathbb{E}_{Cl,g}^{\mu}\left[X_{\otimes}^{(s)}\right]\right)\right)^{\frac{1}{s}}\right)\oplus g^{-1}\left(\left(g\left(\mathbb{E}_{Cl,g}^{\mu}\left[Y_{\otimes}^{(s)}\right]\right)\right)^{\frac{1}{s}}\right)\right]\\
&= K_1\otimes\left[\left(\mathbb{E}_{Cl,g}^{\mu}\left[X_{\otimes}^{(s)}\right]\right)_{\otimes}^{\left(\frac{1}{s}\right)}\oplus\left(\mathbb{E}_{Cl,g}^{\mu}\left[Y_{\otimes}^{(s)}\right]\right)_{\otimes}^{\left(\frac{1}{s}\right)}\right].
\end{aligned}$$

This completes the proof. □

***Example* 4.2.**

(i) Let $g(x)=\mathrm{e}^x$. The corresponding pseudo-operations are $x\oplus y=\ln(\mathrm{e}^x+\mathrm{e}^y)$ and $x\otimes y=x+y$. Then (4.2) and (4.3) reduce on the following inequalities

$$\ln\left(\mathbb{E}_C^{\mathrm{e}^{\mu}}\left[\left(\mathrm{e}^X+\mathrm{e}^Y\right)^s\right]\right)^{\frac{1}{s}}\geqslant K_1+\ln\left(\left(\mathbb{E}_C^{\mathrm{e}^{\mu}}\left[\mathrm{e}^{sX}\right]\right)^{\frac{1}{s}}+\left(\mathbb{E}_C^{\mathrm{e}^{\mu}}\left[\mathrm{e}^{sY}\right]\right)^{\frac{1}{s}}\right),$$

$$\ln\left(\mathbb{E}_C^{\mathrm{e}^{\mu}}\left[\left(\mathrm{e}^X+\mathrm{e}^Y\right)^s\right]\right)^{\frac{1}{s}}\leqslant K_2+\ln\left(\left(\mathbb{E}_C^{\mathrm{e}^{\mu}}\left[\mathrm{e}^{sX}\right]\right)^{\frac{1}{s}}+\left(\mathbb{E}_C^{\mathrm{e}^{\mu}}\left[\mathrm{e}^{sY}\right]\right)^{\frac{1}{s}}\right),$$

where $K_1=-\ln\left(\frac{M(m+1)+(M+1)}{(M+1)(m+1)}\right)$, $K_2=-\ln\left(\frac{m(M+1)+(m+1)}{(m+1)(M+1)}\right)$, $s\geqslant 1$.

(ii) Let $g(x)=x^{\alpha}$, $\alpha>0$. The corresponding pseudo-operations are $x\oplus y=\sqrt[\alpha]{x^{\alpha}+y^{\alpha}}$ and $x\otimes y=xy$. Then (4.2) and (4.3) reduce on the following inequalities

$$\sqrt[\alpha s]{\mathbb{E}_C^{\mu^{\alpha}}\left[(X^{\alpha}+Y^{\alpha})^s\right]}\geqslant K_1\sqrt[\alpha]{\left[\left(\mathbb{E}_C^{\mu^{\alpha}}\left[X^{\alpha s}\right]\right)^{\frac{1}{s}}+\left(\mathbb{E}_C^{\mu^{\alpha}}\left[Y^{\alpha s}\right]\right)^{\frac{1}{s}}\right]},$$

$$\sqrt[\alpha s]{\mathbb{E}_C^{\mu^{\alpha}}\left[(X^{\alpha}+Y^{\alpha})^s\right]}\leqslant K_2\sqrt[\alpha]{\left[\left(\mathbb{E}_C^{\mu^{\alpha}}\left[X^{\alpha s}\right]\right)^{\frac{1}{s}}+\left(\mathbb{E}_C^{\mu^{\alpha}}\left[Y^{\alpha s}\right]\right)^{\frac{1}{s}}\right]},$$

where $K_1=\left(\frac{M(m+1)+(M+1)}{(M+1)(m+1)}\right)^{-\frac{1}{\alpha}}$, $K_2=\left(\frac{m(M+1)+(m+1)}{(m+1)(M+1)}\right)^{-\frac{1}{\alpha}}$, $s\geqslant 1$.

Now, we focus on the second class of Choquet-like integrals.

THEOREM 4.3. *Let X,Y be two non-negative measurable functions such that*

$$0<m\leqslant\frac{X(\omega)}{Y(\omega)}\leqslant M\qquad\text{for all}\quad\omega\in\Omega.$$

Let $s>0$. If $\otimes$ is a pseudo-multiplication satisfying

$$(a\otimes b)\leqslant\min\left\{\frac{(m+1)^s}{m^s}\Big(\frac{m^s a}{(m+1)^s}\otimes b\Big),(M+1)^s\Big(\frac{a}{(M+1)^s}\otimes b\Big)\right\}\tag{4.7}$$

then, for any monotone measure μ, the $\mathbb{S}_{\mu}^{\otimes}$ integral (2.3) satisfies the inequality

$$\left(\mathbb{S}_{\mu}^{\otimes}\left[X^s\right]\right)^{\frac{1}{s}}+\left(\mathbb{S}_{\mu}^{\otimes}\left[Y^s\right]\right)^{\frac{1}{s}}\geqslant K\left(\mathbb{S}_{\mu}^{\otimes}\left[(X+Y)^s\right]\right)^{\frac{1}{s}}$$

where $K=\frac{m(M+1)+(m+1)}{(m+1)(M+1)}$.

Proof. Since $X(\omega)\geqslant m\left(X(\omega)+Y(\omega)\right)-mX(\omega)$, we have

$$(m+1)^s\left(X(\omega)\right)^s\geqslant m^s\left(X(\omega)+Y(\omega)\right)^s.$$

The monotonicity of $\mathbb{S}_{\mu}^{\otimes}$ integral and (4.7) imply that

$$\left(\mathbb{S}_{\mu}^{\otimes}\left[X^s\right]\right)^{\frac{1}{s}}\geqslant\left(\mathbb{S}_{\mu}^{\otimes}\left[\Big(\frac{m}{m+1}\Big)^s(X+Y)^s\right]\right)^{\frac{1}{s}}\geqslant\frac{m}{m+1}\left(\mathbb{S}_{\mu}^{\otimes}\left[(X+Y)^s\right]\right)^{\frac{1}{s}}.\tag{4.8}$$

Also, since $M\geqslant\frac{X(\omega)}{Y(\omega)}$ we have $Y(\omega)\geqslant\frac{1}{M}\left(X(\omega)+Y(\omega)\right)-\frac{1}{M}Y(\omega)$. So

$$\Big(\frac{1}{M}+1\Big)^s\left(Y(\omega)\right)^s\geqslant\Big(\frac{1}{M}\Big)^s\left(X(\omega)+Y(\omega)\right)^s,$$

and then,

$$\left(\mathbb{S}_{\mu}^{\otimes}\left[Y^s\right]\right)^{\frac{1}{s}}\geqslant\left(\mathbb{S}_{\mu}^{\otimes}\left[\frac{1}{(M+1)^s}(X+Y)^s\right]\right)^{\frac{1}{s}}\geqslant\frac{1}{M+1}\left(\mathbb{S}_{\mu}^{\otimes}\left[(X+Y)^s\right]\right)^{\frac{1}{s}}.\tag{4.9}$$

By (4.8) and (4.9), the following result holds

$$\left(\mathbb{S}_{\mu}^{\otimes}\left[X^{s}\right]\right)^{\frac{1}{s}}+\left(\mathbb{S}_{\mu}^{\otimes}\left[Y^{s}\right]\right)^{\frac{1}{s}} \geqslant K\left(\mathbb{S}_{\mu}^{\otimes}\left[(X+Y)^{s}\right]\right)^{\frac{1}{s}}.$$

This completes the Proof. □

Notice that if $\otimes$ is minimum (i.e., for Sugeno integral) in Theorem 4.3, then the following result holds.

Corollary 4.4. *Let X, Y be two non-negative measurable functions such that*

$$0<m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M \qquad \text{for all} \quad \omega \in \Omega.$$

Then for the Sugeno integral, the inequality

$$\left(\mathrm{Su}_{\mu}\left[X^{s}\right]\right)^{\frac{1}{s}}+\left(\mathrm{Su}_{\mu}\left[Y^{s}\right]\right)^{\frac{1}{s}} \geqslant K\left(\mathrm{Su}_{\mu}\left[(X+Y)^{s}\right]\right)^{\frac{1}{s}} \tag{4.10}$$

holds where $K=\frac{m(M+1)+(m+1)}{(m+1)(M+1)}$, $s>0$.

Remark 4.5. If $X(\omega)=\alpha Y(\omega)$, $\alpha>0$ in Corollary 4.4, then $m=M=\alpha$, $K=1$ and (4.10) reduces on the following inequality

$$\left(\mathrm{Su}_{\mu}\left[X^{s}\right]\right)^{\frac{1}{s}}+\left(\mathrm{Su}_{\mu}\left[Y^{s}\right]\right)^{\frac{1}{s}} \geqslant\left(\mathrm{Su}_{\mu}\left[(X+Y)^{s}\right]\right)^{\frac{1}{s}}.$$

The following example proves that the inequality of Corollary 4.4 is sharp.

***Example* 4.6.** Let $\Omega=[0,2]$, $X(\omega)=Y(\omega) \equiv 1$. Let $\mu(A)=\lambda(A)$ where λ is the Lebesgue measure on $\mathbb{R}$. Clearly $\frac{X(\omega)}{Y(\omega)}=1$. Then $m=M=1$. So, by Corollary 4.4, $K=1$ and

$$\mathrm{Su}_{\mu}[X+Y]=2, \qquad \mathrm{Su}_{\mu}[X]=\mathrm{Su}_{\mu}[Y]=1.$$

Therefore,

$$K \,\mathrm{Su}_{\mu}[X+Y]=2=\mathrm{Su}_{\mu}[X]+\mathrm{Su}_{\mu}[Y].$$

Notice that when working on $[0,1]$ in Theorem 4.3, then $\otimes=\circledast$ is semicopula (t-seminorm) and the following result holds.

Corollary 4.7. *Let $X, Y: \Omega \rightarrow[0,1]$ be two non-negative measurable functions such that*

$$0<m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M \qquad \text{for all} \quad \omega \in \Omega.$$

Let $s>0$. If semicopula $\circledast$ satisfying

$$(a \circledast b) \leqslant \min \left\{\frac{(m+1)^{s}}{m^{s}}\Big(\frac{m^{s} a}{(m+1)^{s}} \circledast b\Big),(M+1)^{s}\Big(\frac{a}{(M+1)^{s}} \circledast b\Big)\right\} \tag{4.11}$$

then for the $\mathbb{S}_{\mu}^{\circledast}$ integral (2.3), the inequality

$$\left(\mathbb{S}_{\mu}^{\circledast}\left[X^{s}\right]\right)^{\frac{1}{s}}+\left(\mathbb{S}_{\mu}^{\circledast}\left[Y^{s}\right]\right)^{\frac{1}{s}} \geqslant K\left(\mathbb{S}_{\mu}^{\circledast}\left[(X+Y)^{s}\right]\right)^{\frac{1}{s}}$$

holds where $K=\frac{m(M+1)+(m+1)}{(m+1)(M+1)}$.

The following example shows that the condition (4.11) in Corollary 4.7 cannot be omitted.

***Example* 4.8.** Let $\Omega = [0,1]$, $X(\omega) = Y(\omega) = \frac{\omega+1}{2}$. Let semicopula $\circledast$ be the Łukasiewicz t-norm T_L (i.e., $T_L(x,y) = \max\{(x+y-1), 0\}$) and $\mu(A) = \frac{\lambda(A)}{1+\lambda(A)}$ where λ is the Lebesgue measure on $\mathbb{R}$. Clearly $\frac{X(\omega)}{Y(\omega)} = 1$. Then $m = M = 1$. So, by Corollary 4.7, $K = 1$ and

$$\mathbb{S}_\mu^{T_L}[X+Y] = \bigvee_{\alpha\in[0,1]} T_L(\alpha, \mu(\{X+Y \geq \alpha\})) = \frac{1}{2},$$

$$\mathbb{S}_\mu^{T_L}[X] = \mathbb{S}_\mu^{T_L}[Y] = \bigvee_{\alpha\in[0,1]} T_L(\alpha, \mu(\{X \geq \alpha\}))$$

$$= \bigvee_{\alpha\in\left[\frac{1}{2},1\right]} T_L\left(\alpha, \frac{2\alpha-2}{2\alpha-3}\right) = \frac{3}{2} - \sqrt{2}.$$

Therefore,

$$\mathbb{S}_\mu^{\circledast}[X] + \mathbb{S}_\mu^{\circledast}[Y] = 3 - 2\sqrt{2} \ngeqslant \frac{1}{2} = K\left(\mathbb{S}_\mu^{\circledast}[X+Y]\right).$$

Note that if $\circledast$ is the usual product (i.e., for Shilkret integral) in Corollary 4.7, then we have the following result.

Corollary 4.9. *Let $X, Y\colon \Omega \to [0,1]$ be two non-negative measurable functions such that*

$$0 < m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M \qquad \text{for all} \quad \omega \in \Omega.$$

Then the inequality

$$(\mathrm{Sh}_\mu[X^s])^{\frac{1}{s}} + (\mathrm{Sh}_\mu[Y^s])^{\frac{1}{s}} \geqslant K(\mathrm{Sh}_\mu[(X+Y)^s])^{\frac{1}{s}}$$

holds where $K = \frac{m(M+1)+(m+1)}{(m+1)(M+1)}$, $s > 0$.

Finally, if $\circledast$ is minimum (i.e., for original Sugeno integral) in Corollary 4.7, then the following result holds.

Corollary 4.10. *Let $X, Y\colon \Omega \to [0,1]$ be two non-negative measurable functions such that*

$$0 < m \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant M \qquad \text{for all} \quad \omega \in \Omega.$$

Then the inequality

$$(\mathrm{Su}_\mu[X^s])^{\frac{1}{s}} + (\mathrm{Su}_\mu[Y^s])^{\frac{1}{s}} \geqslant K(\mathrm{Su}_\mu[(X+Y)^s])^{\frac{1}{s}}$$

holds where $K = \frac{m(M+1)+(m+1)}{(m+1)(M+1)}$, $s > 0$.

***Example* 4.11.** Let $\Omega = [0,1]$, $X(\omega) = \omega + \frac{1}{100}$ and $Y(\omega) = 1 - \frac{\omega}{2}$. Let $\mu(A) = \lambda^2(A)$ where λ is the Lebesgue measure on $\mathbb{R}$. Clearly $\frac{1}{100} \leqslant \frac{X(\omega)}{Y(\omega)} \leqslant \frac{101}{50}$. Then $m = \frac{1}{100}$ and $M = \frac{101}{50}$. So, by Corollary 4.10, $K = 0.341$,

$$\mathrm{Su}_\mu[X+Y] = 1, \quad \mathrm{Su}_\mu[X] = 0.3875 \quad \text{and} \quad \mathrm{Su}_\mu[Y] = 0.6096.$$

Therefore,

$$K\,\mathrm{Su}_\mu[X+Y] = 0.341 \leqslant 0.9971 = \mathrm{Su}_\mu[X] + \mathrm{Su}_\mu[Y].$$

5. Conclusions

This paper has successfully proved the Stolarsky inequality for Choquet-like expectation, which generalizes the pervious result of this inequality on Choquet expectation. Moreover, a new Minkowski's inequality without the comonotonicity condition for Choquet-like integrals is introduced. At first, we thoroughly described two classes of Choquet-like integrals. Then, we prepared extensions of these inequalities from the Choquet expectation and the Sugeno integral to the two classes of Choquet-like integrals. Observe that similarly some Chebyshev-type inequalities shown for the Choquet and Sugeno integrals in [3, 14] were generalized for the Choquet-like integrals in [5, 19].

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