Computing Superdifferentials of Lovász Extension with Application to Coalitional Games

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Abstract. Every coalitional game can be extended from the powerset onto the real unit cube. One of possible approaches is the Lovász extension, which is the same as the discrete Choquet integral with respect to the coalitional game. We will study some solution concepts for coalitional games (core, Weber set) using superdifferentials developed in non-smooth analysis. It has been shown that the core coincides with Fréchet superdifferential and the Weber set with Clarke superdifferential for the Lovász extension, respectively. We introduce the intermediate set as the limiting superdifferential and show that it always lies between the core and the Weber set. From the game-theoretic point of view, the intermediate set is a non-convex solution containing the Pareto optimal payoff vectors, which depend on some ordered partition of the players and the marginal coalitional contributions with respect to the order.

Keywords: Coalitional game · Lovász extension · Choquet integral · Core · Weber set · Superdifferential

1 Introduction

Many important solution concepts for transferable-utility n-person coalitional games can be expressed in terms of formulas involving gradients or generalized gradients of a suitable extension of the game. The purpose of such a "differential representation" is not only computational, but it is also to provide a new interpretation of the corresponding payoff vectors, which usually revolves around the idea of marginal contributions to a given coalition.

In this contribution we will build a bridge between the class of solution concepts involving the core and the Weber set by applying certain generalized derivatives, namely the supergradients, which are studied in variational analysis [8,12]. Among the main superdifferentials count the Fréchet, the limiting and the Clarke superdifferential, respectively. By adopting the idea proposed in [13] we employ the limiting superdifferential to define directly a new solution concept for coalitional games, the so-called intermediate set. Specifically, the

J.P. Carvalho et al. (Eds.): IPMU 2016, Part I, CCIS 610, pp. 35–45, 2016.
DOI: 10.1007/978-3-319-40596-4_4

intermediate set is the limiting superdifferential of the Lovász extension [6] of the game v (or, equivalently, the discrete Choquet integral with respect to v [4]) calculated at the grand coalition. The associated payoff vectors are thus marginal contributions to the grand coalition in the sense conveyed by the limiting superdifferential.

It turns out that the newly constructed solution is meaningful and interesting from many viewpoints. The intermediate set can be seen as a nonempty interpolant between the core and the Weber set, which makes it applicable especially when the former is empty or small and the latter is huge. Theorem 2 provides a combinatorial description of the payoff vectors from the intermediate set in the following sense. For some ordered partition of the player set, each such vector is a Weber-style marginal vector on the level of blocks of coalitions and, at the same time, no coalition inside each block can improve upon this payoff vector in the sense of marginal coalitional contributions. The intermediate set is thus a solution concept that looks globally like the Weber set, but behaves locally like the core concept.

The paper is structured as follows. Section 2 introduces the basic notions and results from cooperative game theory and non-smooth analysis needed throughout the paper. The intermediate set is introduced in Sect. 3, where we formulate its equivalent characterization using ordered partitions of the player set and discuss its properties together with some examples.

The proofs are omitted for the space restrictions in this paper. The interested reader is invited to consult the authors' paper [1], which provides full details and further arguments in favor of the solution concept presented in this proceedings paper.

2 Basic Notions

We recall basic notions and results from cooperative game theory [10] and non-smooth variational analysis [8,12].

2.1 Coalitional Games

Let $N = \{1, \dots, n\}$ be a finite set of *players*, where n is a positive integer. By 2^N we denote the powerset of N whose elements $A \subseteq N$ are called *coalitions*. A *(transferable utility coalitional) game* is a function $v \colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. Any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is called a *payoff vector*. We introduce the following notation:

$$\mathbf{x}(A) = \sum_{i \in A} x_i, \quad \text{for every } A \subseteq N.$$

We say that a payoff vector $\mathbf{x}$ is *feasible* in a game v whenever $\mathbf{x}(N) \le v(N)$. The set of all feasible payoff vectors in v is denoted by $\mathcal{F}(v)$.

Let $\Gamma(N)$ be the set of all games and $\Omega \subseteq \Gamma(N)$. A *solution* on Ω is a set-valued mapping $\sigma \colon \Omega \to 2^{\mathbb{R}^n}$ that maps every game $v \in \Omega$ to a set $\sigma(v) \subseteq \mathcal{F}(v)$.

We recall the core solution and the Weber set. The *core* of a game v is the convex polytope $\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N),\ \mathbf{x}(A) \geq v(A) \text{ for every } A \subseteq N\}$.

Let Π_n be the set of all the permutations π of the player set N. Let $v \in \Gamma(N)$ and $\pi \in \Pi_n$. A *marginal vector* of a game v with respect to π is the payoff vector $\mathbf{x}^v(\pi) \in \mathbb{R}^n$ with coordinates

$$x_i^v(\pi) = v\left(\bigcup_{j \leq \pi^{-1}(i)} \{\pi(j)\}\right) - v\left(\bigcup_{j < \pi^{-1}(i)} \{\pi(j)\}\right), \quad i \in N. \tag{1}$$

The *Weber set* of v is defined as

$$\mathcal{W}(v) = \operatorname{conv}\{\mathbf{x}^v(\pi) \mid \pi \in \Pi_n\}.$$

Since $\mathbf{x}^v(\pi)(N) = v(N)$, the Weber set is a solution on $\Gamma(N)$ in the sense defined above. Moreover, it always contains the core solution; see [15, Theorem 14].

Proposition 1. *$\mathcal{C}(v) \subseteq \mathcal{W}(v)$ for every $v \in \Gamma(N)$.*

The fundamental tool in this paper is the concept of Lovász extension [6]. For every set $A \subseteq N$ let χ_A denote the incidence vector in $\mathbb{R}^n$ whose coordinates are given by

$$(\chi_A)_i = \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{otherwise.} \end{cases}$$

We write 0 in place of $\chi_\emptyset$. The embedding of 2^N into $\mathbb{R}^n$ by means of the mapping $A \mapsto \chi_A$ makes it possible to interpret a game on 2^N as a real function on $\{0,1\}^n$. Indeed, it suffices to define $\hat{v}(\chi_A) = v(A)$, for every $A \subseteq N$. We will extend the function $\hat{v}$ onto $\mathbb{R}^n$. For every $\mathbf{x} \in \mathbb{R}^n$, put

$$\Pi(\mathbf{x}) = \{\pi \in \Pi_n \mid x_{\pi(1)} \geq \cdots \geq x_{\pi(n)}\}.$$

Given $i \in N$ and $\pi \in \Pi(\mathbf{x})$, define $V_i^\pi(\mathbf{x}) = \{j \in N \mid x_j \geq x_{\pi(i)}\}$. Note that $V_i^\pi(\mathbf{x}) = V_i^\rho(\mathbf{x})$ for every $\pi, \rho \in \Pi(\mathbf{x})$. This implies that any vector $\mathbf{x} \in \mathbb{R}^n$ can be unambiguously written as a linear combination

$$\mathbf{x} = \sum_{i=1}^{n-1} (x_{\pi(i)} - x_{\pi(i+1)}) \cdot \chi_{V_i^\pi(\mathbf{x})} + x_{\pi(n)} \cdot \chi_N. \tag{2}$$

Using the convention $V_0^\pi(\mathbf{x}) = \emptyset$, we can rewrite (2) as

$$\mathbf{x} = \sum_{i=1}^{n} x_{\pi(i)} \cdot \left(\chi_{V_i^\pi(\mathbf{x})} - \chi_{V_{i-1}^\pi(\mathbf{x})}\right). \tag{3}$$

The *Lovász extension* $\hat{v}$ of $v \in \Gamma(N)$ is the function $\mathbb{R}^n \to \mathbb{R}$ defined linearly with respect to the decomposition (3):

$$\hat{v}(\mathbf{x}) = \sum_{i=1}^{n} x_{\pi(i)} \cdot \left(v(V_i^\pi(\mathbf{x})) - v(V_{i-1}^\pi(\mathbf{x}))\right), \quad \text{for any } \mathbf{x} \in \mathbb{R}^n.$$

Observe that the definition of $\hat{v}(\mathbf{x})$ is independent on the choice of $\pi \in \Pi(\mathbf{x})$. Clearly $\hat{v}(\chi_A) = v(A)$ for every coalition $A \subseteq N$. It is easy to see that the Lovász extension $\hat{v}$ of any game v fulfills these properties:

- $\hat{v}$ is continuous and piecewise affine on $\mathbb{R}^n$;
- $\hat{v}$ is positively homogeneous: $\hat{v}(\lambda \cdot \mathbf{x}) = \lambda \cdot \hat{v}(\mathbf{x})$ for every $\lambda \geq 0$ and $\mathbf{x} \in \mathbb{R}^n$;
- the mapping $v \in \Gamma(N) \mapsto \hat{v}$ is linear.

The following lemma says that the local behavior of $\hat{v}$ is the same around χ_N as in the neighborhood of 0.

Lemma 1. *For any $\mathbf{x} \in \mathbb{R}^n$ it holds true that $\hat{v}(\mathbf{x} + \chi_N) = \hat{v}(\mathbf{x}) + \hat{v}(\chi_N)$.*

A game $v \in \Gamma(N)$ is called *supermodular* (or *convex*) if the following inequality is satisfied: $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$, for every $A, B \subseteq N$. A *submodular* game v is such that $-v$ is supermodular. A game v is called *additive* when $v(A \cup B) = v(A) + v(B)$ for every $A, B \subseteq N$ with $A \cap B = \emptyset$. We will make an ample use of several characterizations of supermodular games appearing in the literature; see [5,6,14,15].

Theorem 1. *Let $v \in \Gamma(N)$. Then the following assertions are equivalent:*

1. *v is supermodular;*
2. *$\{\mathbf{x}^v(\pi) \mid \pi \in \Pi_n\} \subseteq \mathcal{C}(v)$;*
3. *$\mathcal{C}(v) = \mathcal{W}(v)$;*
4. *The Lovász extension $\hat{v}$ of v is a concave function.*

2.2 Superdifferentials

In this section we will define the selected concepts of variational (nonsmooth) analysis, namely various superdifferentials which generalize the superdifferential of convex functions. Since the superdifferentials will be computed only for the Lovász extension, we will confine to defining superdifferentials only for piecewise affine functions at a point $\bar{\mathbf{x}} \in \mathbb{R}^n$. This assumption enables us to neglect the term $o(\|\mathbf{x} - \bar{\mathbf{x}}\|)$ present in the more general definitions; see [12, Definition 8.3], for example. We refer the reader to [12] for the general framework involving upper semicontinuous functions.

While the standard monographs on variational analysis [8,11,12] deal with subdifferentials instead of superdifferentials, most of the results can be readily transformed to the setting of superdifferentials, usually by reversing inequalities only.

Definition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a piecewise affine function and $\bar{\mathbf{x}} \in \mathbb{R}^n$. We say that $\mathbf{x}^* \in \mathbb{R}^n$ is a*

1. Fréchet supergradient *of f at $\bar{\mathbf{x}}$ if there exists neighborhood $\mathcal{X}$ of $\bar{\mathbf{x}}$ such that for all $\mathbf{x} \in \mathcal{X}$ we have*

$$f(\mathbf{x}) - f(\bar{\mathbf{x}}) \leq \langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle;$$

2. limiting supergradient *of* f *at* $\bar{\mathbf{x}}$ *if for every neighborhood* $\mathcal{X}$ *of* $\bar{\mathbf{x}}$ *there exists* $\mathbf{x} \in \mathcal{X}$ *such that* $\mathbf{x}^*$ *is a Fréchet supergradient of* f *at* $\mathbf{x}$*;*
3. Clarke supergradient *of* f *at* $\bar{\mathbf{x}}$ *if*

$$\mathbf{x}^* \in \operatorname{conv}\{\mathbf{y} |\ \forall \textit{ neighborhood } \mathcal{X} \textit{ of } \bar{\mathbf{x}}\ \exists \mathbf{x} \in \mathcal{X} \cap D \textit{ with } \mathbf{y} = \nabla f(\mathbf{x})\},$$

where $D := \{\mathbf{x} \in \mathbb{R}^n |\ f$ *is differentiable at* $\mathbf{x}\}$*.*

The collection of all (Fréchet, limiting, Clarke) supergradients of f *at* $\bar{\mathbf{x}}$ *is called* (Fréchet, limiting, Clarke) superdifferential *and it is denoted by* $\hat{\partial} f(\bar{\mathbf{x}})$*,* $\partial f(\bar{\mathbf{x}})$ *and* $\overline{\partial} f(\bar{\mathbf{x}})$*, respectively.*

It is easy to see that

$$\hat{\partial} f(\bar{\mathbf{x}}) \subseteq \partial f(\bar{\mathbf{x}}) \subseteq \overline{\partial} f(\bar{\mathbf{x}}), \quad \bar{\mathbf{x}} \in \mathbb{R}^n,$$

where all the inequalities may be strict. Moreover, [12, Theorem 8.49] yields that the limiting and the Clarke superdifferential of a piecewise affine function f are related as follows: $\overline{\partial} f(\bar{\mathbf{x}}) = \operatorname{conv} \partial f(\bar{\mathbf{x}})$. The following two examples show that the three superdifferentials can differ significantly.

Example 1 ([2, Example 10.28]). Consider the function $\mathbb{R}^2 \to \mathbb{R}$ defined as $f(x_1, x_2) = \max(\min(2x_1 + x_2, x_1), 2x_2)$. This function is piecewise affine and it can be expressed as follows:

$$f(x_1, x_2) = \begin{cases} 2x_1 + x_2 & \text{if } x_2 \leq 2x_1 \text{ and } x_2 \leq -x_1, \\ x_1 & \text{if } x_2 \leq \frac{x_1}{2} \text{ and } x_2 \geq -x_1, \\ 2x_2 & \text{if } x_2 \geq 2x_1 \text{ or } x_2 \geq \frac{x_1}{2}. \end{cases}$$

Let us compute all the three superdifferentials of f at $\bar{\mathbf{x}} = 0$:

$$\begin{aligned} &\hat{\partial} f(\bar{x}) = \emptyset, \\ &\partial f(\bar{x}) = \operatorname{conv}\{(2,1), (1,0)\} \cup \{(0,2)\}, \\ &\overline{\partial} f(\bar{x}) = \operatorname{conv}\{(2,1), (1,0), (0,2)\}. \end{aligned}$$

Example 2. Let

$$g(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 \leq 0 \text{ or } x_2 \leq 0, \\ -x_1 & \text{if } x_2 \geq x_1 \geq 0, \\ -x_2 & \text{if } x_1 \geq x_2 \geq 0, \end{cases} \qquad \text{for every } (x_1, x_2) \in \mathbb{R}^2.$$

Function g is piecewise affine and the three superdifferentials of g at $\bar{\mathbf{x}} = 0$ are, respectively,

$$\begin{aligned} &\hat{\partial} g(\bar{x}) = \{(0,0)\}, \\ &\partial g(\bar{x}) = \operatorname{conv}\{(0,0), (-1,0)\} \cup \operatorname{conv}\{(0,0), (0,-1)\}, \\ &\overline{\partial} g(\bar{x}) = \operatorname{conv}\{(0,0), (-1,0), (0,-1)\}. \end{aligned}$$

3 Intermediate Set

The Lovász extension $\hat{v}$ of a coalitional game v is instrumental in characterizing the core solution and the Weber set by the tools of nonsmooth calculus. Specifically, it was shown that the core coincides with the Fréchet superdifferential of $\hat{v}$ at 0 [3, Proposition 3] and that the Weber set is the Clarke superdifferential of $\hat{v}$ at 0 [13, Proposition 4.1]. It may be more natural to use the grand coalition N in place of the empty coalition $\emptyset$ in those formulas. Lemma 1 says that this is always possible.

Proposition 2. *For every game* $v \in \Gamma(N)$, $\mathcal{C}(v) = \hat{\partial}\hat{v}(\chi_N) = \hat{\partial}\hat{v}(0)$ *and* $\mathcal{W}(v) = \overline{\partial}\hat{v}(\chi_N) = \overline{\partial}\hat{v}(0)$.

It can easily be shown that the gap between the core and the Weber set can be too large. Indeed, the core can be empty, while the Weber set can be a large convex polytope. Taking into account the hierarchy of superdifferentials introduced in the previous section, we will pursue an idea mentioned in [13] and by analogy with Proposition 2 we define a new solution concept as $\partial\hat{v}(\chi_N)$, where ∂ is the limiting superdifferential. This leads to the following notion.

Definition 2. *The* intermediate set $\mathcal{M}(v)$ *of* $v \in \Gamma(N)$ *is the set*

$$\mathcal{M}(v) := \partial\hat{v}(\chi_N).$$

Similarly as in Proposition 2, we can show that for every game $v \in \Gamma(N)$, $\mathcal{M}(v) = \partial\hat{v}(0)$. Lemma 2 explains why the solution concept $\mathcal{M}(v)$ was termed the "intermediate set".

Lemma 2. *Let* $v \in \Gamma(N)$. *Then:*

1. $\mathcal{M}(v) \neq \emptyset$.
2. *We have*
$$\mathcal{C}(v) \subseteq \mathcal{M}(v) \subseteq \mathcal{W}(v),$$
where both inclusions may be strict.
3. $\mathcal{W}(v) = \operatorname{conv} \mathcal{M}(v)$.
4. v *is supermodular if and only if* $\mathcal{C}(v) = \mathcal{M}(v) = \mathcal{W}(v)$.

Example 3. [3-player glove game] Let $N = \{1, 2, 3\}$. The first player owns a single left glove and the remaining two players possess one right glove each. The profit of a coalition is a total of glove pairs the coalition owns:

$$v(A) = \begin{cases} 1 & \text{if } A \in \{\{1,2\},\{1,3\},N\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to compute $\mathcal{C}(v), \mathcal{M}(v)$ and $\mathcal{W}(v)$ directly:

$$\begin{aligned} \mathcal{C}(v) &= \{(1,0,0)\}, \\ \mathcal{M}(v) &= \operatorname{conv}\{(1,0,0),(0,1,0)\} \cup \operatorname{conv}\{(1,0,0),(0,0,1)\}, \\ \mathcal{W}(v) &= \operatorname{conv}\{(1,0,0),(0,1,0),(0,0,1)\}. \end{aligned}$$

3.1 Characterization by Ordered Partitions

In this section we are going to show an alternative expression for the intermediate set using the concept of an ordered partition. Thus the purely analytic definition of intermediate set can be equivalently stated in terms of the combinatorial and order-theoretic properties of a coalitional game.

Let $K \geq 1$. An *ordered partition* of the player set N is a K-tuple

$$P := (B_1, \ldots, B_K)$$

of coalitions $\emptyset \neq B_i \subseteq N$ such that $B_i \cap B_j = \emptyset$ $(i \neq j)$ and $B_1 \cup \cdots \cup B_K = N$. Let

$$\mathcal{P} = \{P \mid P \text{ is an ordered partition of } N\}.$$

The family $\mathcal{P}$ is associated with the following scheme of allocating profits $\mathbf{x}$ among the players in a game v:

1. The players may be split into any ordered partition $P = (B_1, \ldots, B_K) \in \mathcal{P}$.
2. Each block of players B_k can distribute the total amount

$$\mathbf{x}(B_k) = v(B_1 \cup \cdots \cup B_{k-1} \cup B_k) - v(B_1 \cup \cdots \cup B_{k-1})$$

   to its members, which can be interpreted as the marginal contribution of coalition B_k to the coalition $B_1 \cup \cdots \cup B_{k-1}$ with respect to P.
3. No coalition B in a block B_k may improve upon $\mathbf{x}$, while respecting the given order of coalition blocks, that is,

$$\mathbf{x}(B) \geq v(B_1 \cup \cdots \cup B_{k-1} \cup B) - v(B_1 \cup \cdots \cup B_{k-1}).$$

Note that the players share total of $v(N)$ among them as a consequence of the second principle. The distribution procedure explained above has two extreme cases. Assume that the ordered partition P is the finest possible, that is, $P = (\{\pi(1)\}, \ldots, \{\pi(n)\})$ for some permutation $\pi \in \Pi_n$. In this case the allocation scheme in a game v leads to the marginal vectors $\mathbf{x}^v(\pi)$ defined by (1). On the contrary, if the partition contains one block only, $P = (N)$, then all the players (and coalitions) are treated equally, which results in distributing payoffs according to the definition of core. Any ordered partition $P = (B_1, \ldots, B_K)$ different from the two extreme cases generates a combination of the principle of marginal distribution on the level of blocks with the core-like stability inside each block of the partition, while respecting the given order of coalitions. Such a distribution process is thus always a mixture of the considerations endogenous to B_i and those which are exogenous to B_i.

Our characterization says that $\mathbf{x} \in \mathcal{M}(v)$ if and only if there is an ordered partition P such that $\mathbf{x}$ is allocated to the players according to the above distribution principles.

Theorem 2. *For every game $v \in \Gamma(N)$,*

$$\mathcal{M}(v) = \bigcup_{P \in \mathcal{P}} \mathcal{M}_P(v),$$

where $\mathcal{M}_P(v)$ *with* $P = (B_1, \ldots, B_K)$ *is the set of all* $\mathbf{x} \in \mathbb{R}^n$ *such that the following two conditions hold for every* $k = 1, \ldots, K$ *and for each* $B \subseteq B_k$*:*

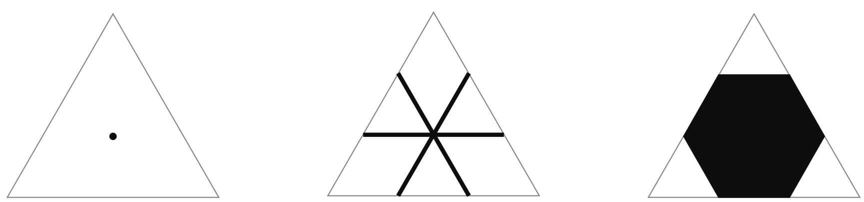
$$\mathbf{x}(B_k) = v(B_1 \cup \cdots \cup B_{k-1} \cup B_k) - v(B_1 \cup \cdots \cup B_{k-1}),$$
$$\mathbf{x}(B) \geq v(B_1 \cup \cdots \cup B_{k-1} \cup B) - v(B_1 \cup \cdots \cup B_{k-1}).$$

Example 4. Let $N = \{1, 2, 3\}$ and

$$v(A) = \begin{cases} 0 & \text{if } |A| = 1, \\ 2 & \text{if } |A| = 2, \\ 3 & \text{if } A = N. \end{cases}$$

It is easy to see that v is not supermodular but only superadditive, that is, $v(A \cup B) \geq v(A) + v(B)$ for every $A, B \subseteq N$ with $A \cap B = \emptyset$.

The core of this game is $\mathcal{C}(v) = \{(1, 1, 1)\}$, while the Weber set $\mathcal{W}(v)$ coincides with the hexagon whose 6 vertices are all the permutations of the payoff vector $(0, 1, 2)$. The intermediate set is the union of three line segments; see Fig. 1. We obtain that $\mathcal{M}_{(i,jk)}(v) = \emptyset$ for every ordered partition (i, jk) of N.[1] On the other hand, $\mathcal{M}_{(ij,k)}(v)$ is the line segment whose endpoints are the two marginal vectors $\mathbf{x}$ with $x_k = 1$. Thus $\mathbf{x} \in \mathcal{M}(v)$ iff it belongs to $\mathcal{M}_{(ij,k)}(v)$ for some ordered partition (ij, k) of N. The example shows that, in general, the intermediate set is not a union of selected faces of the Weber set.



(a) Core (b) Intermediate set (c) Weber set

Fig. 1. The solutions from Example 4 in the barycentric coordinates

3.2 Properties

It was proved in [1] that the core, the intermediate set, and the Weber set share many properties of solution concepts for coalitional games. Namely each of the three solutions is Pareto optimal, anonymous, covariant, and has both the null player property and the dummy player property.

In sharp contrast to the core and the Weber set, the previous examples showed that the intermediate set is typically non-convex. Indeed, the Weber set

[1] We may occasionally switch to a simplified notation for coalitions, writing ij in place of $\{i, j\}$ and similarly.

is always the convex hull of the intermediate set. Moreover, the core can be void, while the intermediate set is always non-empty. Individual rationality is not fulfilled by the intermediate set, in general. However, the intermediate set satisfies this property on the class of all weakly superadditive games, that is, the coalitional games v for which the following property holds true:

$$v(A \cup \{i\}) \geq v(A) + v(\{i\}), \quad \text{for every } A \subseteq N \text{ and } i \in N \setminus A.$$

3.3 Example: Simple Games

We will compute the intermediate set for the class of all simple games and compare the achieved results to the shape of the core. A game $v \in \Gamma(N)$ is *monotone* if $v(A) \leq v(B)$ whenever $A \subseteq B \subseteq N$. A *simple game* is a monotone game v with $v(A) \in \{0, 1\}$ and $v(N) = 1$. Every simple game v over the player set N can be identified with the family of *winning coalitions*

$$\mathcal{V} = \{A \subseteq N \mid v(A) = 1\}.$$

Conversely, any system of coalitions $\mathcal{V}$ such that

1. $N \in \mathcal{V}$, $\emptyset \notin \mathcal{V}$ and
2. $A \subseteq B \subseteq N,\ A \in \mathcal{V} \Rightarrow B \in \mathcal{V}$,

gives rise to a simple game v by putting $v(A) = 1$ if $A \in \mathcal{V}$ and $v(A) = 0$, otherwise. The family of *minimal winning coalitions* in v is given by

$$\mathcal{V}^m = \{A \in \mathcal{V} \mid B \subsetneq A \Rightarrow B \notin \mathcal{V},\ \text{for every } B \subseteq N\}.$$

The core of a simple game v is fully determined by the minimal winning coalitions in v. Indeed, it is well-known that

$$\mathcal{C}(v) = \bigcap_{E \in \mathcal{V}^m} \{\mathbf{x} \in \mathcal{I}(v) \mid x_i = 0 \text{ for every } i \in N \setminus E\},$$

where $\mathcal{I}(v) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(N) = v(N),\, x_i \geq v(i),\, i \in N\}$ is the set of imputations in v. Using our Theorem 2 we can show that an analogous formula exists for the intermediate set. It states that $\mathcal{M}(v)$ arises as a union of faces of the standard simplex, where each face corresponds to one minimal winning coalition.

Theorem 3. *Let $v \in \Gamma(N)$ be a simple game. Then*

$$\mathcal{M}(v) = \bigcup_{E \in \mathcal{V}^m} \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_i = 0 & \text{if } i \in N \setminus E \\ x_i \geq 0 & \text{if } i \in E \\ \sum_{i \in E} x_i = 1 & \end{array} \right\}.$$

As an example we will compute the intermediate set of the UN Security Council voting scheme.

Example 5. The UN Security Council contains 5 permanent members with veto power and 10 non–permanent members. To pass a resolution, all the permanent members and at least 4 non–permanent members have to vote for the proposal. This is a mildly simplified version of the real voting process, in which abstention of a permanent member is not usually regarded as a veto. However, this assumption is usually accepted in game-theoretic literature; see e.g. [9, Example XI.2.9] or [7, Example 16.1.3].

We assume that the players $N = \{1, \ldots, 15\}$ are ordered in such a way that the first five are the permanent members and the last ten are the non–permanent members. Then it is easy to show that the core and and the Weber set of the corresponding simple game v are, respectively,

$$\mathcal{C}(v) = \left\{ \mathbf{x} \in \mathbb{R}^{15} \;\middle|\; \mathbf{x} \geq 0,\ \sum_{i=1}^{5} x_i = 1,\ x_i = 0 \text{ for } i = 6, \ldots, 15 \right\} \text{ and}$$

$$\mathcal{W}(v) = \left\{ \mathbf{x} \in \mathbb{R}^{15} \;\middle|\; \mathbf{x} \geq 0,\ \sum_{i=1}^{15} x_i = 1 \right\}.$$

Since core allocations are stable, any payoff $\mathbf{x} \in \mathcal{C}(v)$ is distributed only among the permanent members (the vetoers). By contrast, the Weber set is the whole 14-dimensional standard simplex in $\mathbb{R}^{15}$, which contains some payoff vectors whose meaning is problematic. For instance, it is not entirely clear how to interpret a vector $\left(0, \ldots, 0, \frac{1}{10}, \ldots, \frac{1}{10}\right) \in \mathcal{W}(v)$. As we will see, this vector is not contained in $\mathcal{M}(v)$.

Given $i \in N$, denote by $\mathbf{e}_i \in \mathbb{R}^{15}$ the vector whose coordinates are $e_j = 1$ if $j = i$ and $e_j = 0$ otherwise. Put $\mathcal{D} = \{D \subseteq \{6, \ldots, 15\} \mid |D| = 4\}$. Theorem 3 yields

$$\mathcal{M}(v) = \bigcup_{D \in \mathcal{D}} \operatorname{conv}\left(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} \cup \{\mathbf{e}_i \mid i \in D\}\right).$$

In other words, $\mathcal{M}(v)$ is a union of $\binom{10}{4}$ 8-dimensional standard simplices, each of which is a convex hull of $\mathbf{e}_i$s corresponding to the five permanent members and four other non–permanent members. Each such simplex is associated with the ordered partition having two blocks, $(\{1, \ldots, 5\} \cup D, N \setminus (\{1, \ldots, 5\} \cup D))$ where $D \in \mathcal{D}$.

4 Conclusions

Not every solution concept is usually suitable for the entire class of coalitional games. In our future research we plan to study if the intermediate set is well-tailored for some subclass of games. The intuition says that such a class of games has the small core and the large Weber set since this makes the interpolation by the intermediate set between the two solutions especially appealing.

An interesting open question is based on the behavior of the core and the components of the intermediate set $\mathcal{M}_P(v)$ observed in Example 4 and Theorem 3: Can we recover the core of a coalitional game v as an intersection of selected components $\mathcal{M}_P(v)$?

Acknowledgments. L. Adam gratefully acknowledges the support from the Grant Agency of the Czech Republic (15-00735S). The work of T. Kroupa was supported by Marie Curie Intra-European Fellowship OASIG (PIEF-GA-2013-622645).

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