

Stress-driven solution to rate-independent elasto-plasticity with damage at small strains and its computer implementation

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Received 03 June 2015; accepted 16 December 2015

Abstract

Quasistatic rate-independent damage combined with linearized plasticity with hardening at small strains is investigated. Fractional-step time discretization is devised with the purpose of obtaining a numerically efficient scheme, possibly converging to a physically relevant stress-driven solution, which however is to be verified a posteriori using a suitable integrated variant of the maximum-dissipation principle. Gradient theories both for damage and for plasticity are considered to make the scheme numerically stable with guaranteed convergence within the class of weak solutions. After finite-element approximation, this scheme is computationally implemented and illustrative 2-dimensional simulations are performed.

Keywords

Rate-independent systems, nonsmooth continuum mechanics, incomplete ductile damage, linearized plasticity with hardening, quasistatic rate-independent evolution, local solutions, maximum dissipation principle, fractional-step time discretization, quadratic programming

I. Introduction

A combination of plasticity and damage, also called *ductile damage*, opens colorful scenarios with important applications in civil or mechanical engineering. These have interesting mathematical problems, in particular when compared with plasticity or damage alone. Often, both plastification and damage processes are much faster than the rate of applied load and, in a basic scenario, any internal time scale is neglected and the mentioned inelastic processes are considered to be *rate independent*. The goal of this article is to devise a model together with its efficient computational approximation that would lead to a numerical stable and convergent scheme

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and, at least in particular situations, calculate physically relevant solutions of a stress-driven type, verifiable a posteriori by checking a suitable version of the maximum-dissipation principle.

We use the very standard *linearized, associative, plasticity* at small strain as presented in [1]. Simultaneously, we use also a rather standard *scalar (i.e. isotropic) damage* as presented in [2]. We have primarily in mind a conventional engineering model with *unidirectional* evolution of damage; in fact, healing will be allowed for analytical reasons, although is expected to be ineffective in the usual applications, see Remark 2.1 below. All rate-dependent phenomena (as inertia or heat conduction and thermo-coupling) are neglected; this means the problem is considered to be *quasistatic* and fully rate-independent. To avoid serious mathematical and computational difficulties, we have in mind an incomplete damage.

The aforementioned modeling simplification leading to a quasistatic rate-independent system, which reflects certain well-motivated asymptotics, brings serious questions and difficulties. This is because the class of reasonably general solutions is very wide if the governing energy is not convex (as necessary here) and involves solutions of a very different nature, some of which are physically not relevant [3,4]. In particular, to avoid the unwanted effects of unrealistically easy damage under subcritical stress, one cannot require energy conservation and thus cannot consider so-called energetic solutions [5]. This concept is, however, occasionally used for damage with plasticity in purely mathematically-focused literature [6–8]. This is related to the discussion on whether energy or stress is responsible for governing the evolution of rate-independent systems [9].

In contrast to the aforementioned energetic solutions (which allow for simpler analysis without considering gradient plasticity, but lead to recursive global-minimization problems which are difficult to realize and may slide to scenarios of unrealistic early damage), we will focus here on solutions that are stress-driven and that can be efficiently obtained numerically. We will rely on careful usage of a suitable integral-version of the maximum-dissipation principle, as devised in [10] and used, rather heuristically, in engineering models of damage with plasticity and hardening, see [11]. This brings specific difficulties with convergence (which requires the use of gradient plasticity) and specific a posteriori verification of a suitable approximate version of the aforementioned maximum-dissipation principle. This was suggested in [10, Remark 4.6] for damage itself, and is modified here for the combination of damage and plasticity (analogous to [12]) for a surface variant of the elasto-plasto-damage model. If the maximum-dissipation principle holds (at least with a good accuracy) we can claim that the numerically obtained solution is physically relevant as stress-driven (with a good accuracy).

A more physically justified and better motivated approach would be to involve a small viscosity to the damage variable or to the elastic and plastic strains, and then to pass these viscosities to zero. The limits obtained in this way are called vanishing-viscosity solutions to the original rate-independent system, and both their analysis and computer implementation are very difficult: for models without plasticity [13], for viscosity in damage or for viscosity in elastic strain [14].

In principle, there are two basic scenarios for how the material might respond to an increasing loading: either it will first plasticize and then go into damage due to hardening effects, or it will first go into damage and then plasticize; of course, various intermediate scenarios are possible too. The latter scenario needs a damage-influenced yield stress and must allow for no hardening (and in particular perfect plasticity), see [15]. Let us only remark that a damage-dependence of the yield stress in the fully rate-independent setting would make the dissipation state-dependent, which brings serious difficulties as seen in [4, Section 3.2] and, for the particular elasto-plasto-damage model, in [6–8]. In this paper, we will concern ourselves exclusively with the former scenario, in particular that damage does not influence the yield stress. Moreover, we will consider only kinematic hardening, although all the considerations could easily be augmented by isotropic hardening, too. Another essential difference from reference [15] is that, as already explained, the energy is intentionally not conserved here.

The plan of the paper is as follows: in Section 2 we devise the model in its classical formulation and then, in Section 3, its suitable weak formulation by discussing stress-driven solutions and the role of the maximum-dissipation principle. In Section 4, we propose a constructive time discretization method and prove its numerical stability (i.e. a priori estimates) and convergence towards weak solutions. After a further finite-element discretization outlined in Section 5, this allows for efficient computer implementation of the model, which is demonstrated on illustrative 2-dimensional examples in Section 6.

2. The model and its weak formulation

Hereafter, we suppose that the damageable elasto-plastic body occupies a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3. We denote \vec{n} the outward unit normal to $\partial\Omega$. We further suppose that the boundary of Ω splits as

$$\partial\Omega := \Gamma = \Gamma_D \cup \Gamma_N,$$

Table I. Summary of the basic notation used through the paper.

$d = 2, 3$ dimension of the problem,	$e_{\text{el}} : Q \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$ elastic strain,
$\mathbb{R}_{\text{sym}}^{d \times d} := \{A \in \mathbb{R}^{d \times d}; A = A^{\top}\},$	$e = e(u) = e_{\text{el}} + \pi = \frac{1}{2} \nabla u^{\top} + \frac{1}{2} \nabla u$ total
$\mathbb{R}_{\text{dev}}^{d \times d} := \{A \in \mathbb{R}_{\text{sym}}^{d \times d}; \text{tr } A = 0\},$	small-strain tensor,
$u : Q \rightarrow \mathbb{R}^d$ displacement,	$\mathbb{C} : [0, 1] \rightarrow \mathbb{R}^{3^4}$ elasticity tensor (dependent on ζ),
$\pi : Q \rightarrow \mathbb{R}_{\text{dev}}^{d \times d}$ plastic strain,	$\mathbb{H} \in \mathbb{R}^{3^4}$ hardening tensor (independent of ζ),
$\zeta : Q \rightarrow [0, 1]$ damage variable,	$S \subset \mathbb{R}_{\text{dev}}^{d \times d}$ the elastic domain (convex, $\text{int } S \ni 0$),
$a > 0$ activation energy for damage,	$w_D : \Sigma_D \rightarrow \mathbb{R}^d$ prescribed boundary displacement,
$f : \Sigma_N \rightarrow \mathbb{R}^d$ applied traction force,	$\kappa_1 > 0$ scale coefficient of the gradient of plasticity,
$g : Q \rightarrow \mathbb{R}^d$ applied bulk force (as gravity),	$\kappa_2 > 0$ scale coefficient of the gradient of damage,
$\sigma_{\text{el}} : Q \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$ elastic stress,	$b > 0$ activation energy for possible healing.

with Γ_D and Γ_N open subsets in the relative topology of $\partial\Omega$, disjoint one from each other and, up to $(d-1)$ -dimensional zero measure, covering $\partial\Omega$. Later, the Dirichlet or the Neumann boundary conditions will be prescribed on Γ_D and Γ_N , respectively. Considering $T > 0$ a fixed time horizon, we set

$$I := [0, T], \quad Q := (0, T) \times \Omega, \quad \Sigma := I \times \Gamma, \quad \Sigma_D := I \times \Gamma_D, \quad \Sigma_N := I \times \Gamma_N.$$

Further, $\mathbb{R}_{\text{sym}}^{d \times d}$ and $\mathbb{R}_{\text{dev}}^{d \times d}$ will denote the set of symmetric or symmetric trace-free (= deviatoric) $(d \times d)$ -matrices, respectively. For readers' convenience, let us summarize the basic notation used in Table 1.

The *state* is formed by the triple $q := (u, \pi, \zeta)$. The governing equation/inclusions read as

$$\text{div } \sigma_{\text{el}} + g = 0 \quad \text{with } \sigma_{\text{el}} = \mathbb{C}(\zeta)e_{\text{el}} \text{ and } e_{\text{el}} = e(u) - \pi, \quad (\text{momentum equilibrium}) \quad (2.1a)$$

$$\partial\delta_S^*(\dot{\pi}) \ni \text{dev } \sigma_{\text{el}} - \mathbb{H}\pi + \kappa_1\Delta\pi, \quad (\text{plastic flow rule}) \quad (2.1b)$$

$$\partial\delta_{[-a,b]}^*(\dot{\zeta}) \ni -\frac{1}{2}\mathbb{C}'(\zeta)e_{\text{el}} : e_{\text{el}} + \kappa_2 \text{div}(|\nabla\zeta|^{r-2}\nabla\zeta) - N_{[0,1]}(\zeta), \quad (\text{damage flow rule}) \quad (2.1c)$$

with δ_S the indicator function to S and δ_S^* its convex conjugate and with “dev” denoting the deviatoric part of a tensor, i.e. $\text{dev } A := A - \text{tr } A/d$. Here, $[\mathbb{C}(\zeta)e]_{ij}$ means $\sum_{k,l=1}^d \mathbb{C}_{ijkl}(\zeta)e_{kl}$.

Of course, (2.1) is to be completed by appropriate boundary conditions, e.g.

$$u = w_D \quad \text{on } \Gamma_D, \quad (2.2a)$$

$$\sigma_{\text{el}} \cdot \vec{n} = f \quad \text{on } \Gamma_N, \quad (2.2b)$$

$$\nabla\pi \cdot \vec{n} = 0 \quad \text{and} \quad \nabla\zeta \cdot \vec{n} = 0 \quad \text{on } \Gamma \quad (2.2c)$$

with \vec{n} denoting the unit outward normal to Ω . We will consider an initial-value problem for equations (2.1)–(2.2) by asking for

$$u(0) = u_0, \quad \pi(0) = \pi_0, \quad \text{and} \quad \zeta(0) = \zeta_0. \quad (2.3)$$

In fact, as \dot{u} does not occur in equation (2.1), u_0 is rather formal and its only qualification is to make $\mathcal{E}(0, u_0, \pi_0, \zeta_0)$ finite not to degrade the energy balance in equation (3.1d) on $[0, t]$.

Let us note that the homogeneous Neumann condition (2.2a) for both π and ζ is a standard first choice, reflecting the usual situation that the coefficients $\kappa_1 > 0$ and $\kappa_2 > 0$ are small and not reliably known, and should primarily determine the length scale inside the specimen (while even less is known about the surface). As for damage, prescribing a flux $j_{\text{dam}} = \kappa_2 |\nabla\zeta|^{r-2} \nabla\zeta \cdot \vec{n}$ on Γ is sometimes considered as a model for some erosion due to the influence of the outer severe environment. Alternatively, a Dirichlet condition $\zeta = 1$ on Γ would reflect that the damage may occur only inside while the surface is damage-free by some special technological treatment. Both modifications do not make particular difficulties for analysis or numerical implementation, the latter one making some aspects even simpler.

After considering an extension $u_{\text{D}} = u_{\text{D}}(t)$ of $w_{\text{D}}(t)$ from equation (2.2a) on the whole domain Ω , it is convenient to make a substitution of $u + u_{\text{D}}$ instead of u into equations (2.1)–(2.2), and we arrive at the problem with time-constant (even homogeneous) Dirichlet boundary conditions. More specifically,

$$e_{\text{el}} \text{ in equation (2.1b) replaced by } e_{\text{el}} = e(u+u_{\text{D}})-\pi, \text{ and} \quad (2.4a)$$

$$w_{\text{D}} \text{ in equation (2.2a) replaced by } 0. \quad (2.4b)$$

Assuming $(\mathbb{C}(\zeta)e(u_{\text{D}})) \cdot \vec{n} = 0$ on Γ_{N} for any admissible ζ , this transformation will keep f in equation (2.2b) unchanged.

Actually, equation (2.1b) represents the thermodynamical force-balance governing damage evolution, while the corresponding flow rule is written in the (equivalent) form

$$\dot{\pi} \in N_S(\text{dev } \sigma_{\text{el}} - \mathbb{H}\pi - \kappa_1 \Delta\pi) \quad \text{with } \sigma_{\text{el}} = \mathbb{C}(\zeta)e_{\text{el}} \quad (2.5)$$

with N_S denoting the set-valued normal-cone mapping to the convex set S . An analogous remark applies to equation (2.1c). The system (2.1) with the boundary conditions (2.2) has, in its weak formulation, the structure of an abstract Biot-type equation (or rather, here, inclusion, e.g. [4, 16, 17]):

$$\partial_q \mathcal{R}(\dot{q}) + \partial_q \mathcal{E}(t, q) \ni 0 \quad (2.6)$$

with suitable time-dependent stored-energy functional \mathcal{E} and the state-dependent potential of dissipative forces \mathcal{R} . Equally, as already used in equation (2.5), one can write equation (2.6) as a generalized gradient flow

$$\dot{q} \in \partial_{\xi} \mathcal{R}^*(-\partial \mathcal{E}(t, q)) \quad (2.7)$$

where \mathcal{R}^* denotes the conjugate functional. The governing functionals corresponding to equations (2.1)–(2.2) after the transformation (2.4) are

$$\begin{aligned} \mathcal{E}(t, u, \pi, \zeta) := & \int_{\Omega} \frac{1}{2} \mathbb{C}(\zeta)(e(u+u_{\text{D}}(t))-\pi) : (e(u+u_{\text{D}}(t))-\pi) + \frac{1}{2} \mathbb{H}\pi : \pi \\ & + \frac{\kappa_1}{2} |\nabla \pi|^2 + \frac{\kappa_2}{r} |\nabla \zeta|^r + \delta_{[0,1]}(\zeta) - g(t) \cdot u \, dx - \int_{\Gamma_{\text{N}}} f(t) \cdot u \, dS, \end{aligned} \quad (2.8a)$$

$$\mathcal{R}(\dot{\pi}, \dot{\zeta}) \equiv \mathcal{R}_1(\dot{\pi}) + \mathcal{R}_2(\dot{\zeta}) := \int_{\Omega} \delta_S^*(\dot{\pi}) + a\dot{\zeta}^- + b\dot{\zeta}^+ \, dx \quad (2.8b)$$

where $z^+ := \max(z, 0)$ and $z^- := \max(-z, 0) \geq 0$. Note that the damage does not affect the hardening, which reflects the idea that, on the microscopic level, damage in the material that underwent hardening develops by evolving microcracks and even a completely damaged material consists of micro-pieces that bear the hardening energy $\frac{1}{2} \mathbb{H}\pi : \pi$ stored before. This model preserves coercivity of hardening even under complete damage, however the analysis below admits only incomplete damage. If $\zeta \mapsto \mathbb{C}(\zeta)e:e$ is strictly convex for any $e \neq 0$, we speak about a cohesive damage which exhibits a certain hardening effect so that the required driving force increases when damage is to be accomplished. We can thus model quite a realistic response to various loading experiments, as shown schematically in Figure 1 for the case of a possible complete damage (whose analysis remains open, however). Note that, due to the “incompressibility” constraint $\text{tr } \pi = 0$, no plastification is triggered under a pure tension or compression loading.

Let us further note that $(u, \pi) \mapsto \mathcal{E}(t, u, \pi, \zeta)$ is smooth so that $\partial_q \mathcal{E} = \{\mathcal{E}'_u\} \times \{\mathcal{E}'_{\pi}\} \times \partial_{\zeta} \mathcal{E}$ with \mathcal{E}'_u and \mathcal{E}'_{π} denoting the respective partial Gâteaux derivatives. Equation (2.6) can thus be written more specifically as the system

$$\mathcal{E}'_u(t, u, \pi, \zeta) = 0, \quad (2.9a)$$

$$\partial \mathcal{R}_1(\dot{\pi}) + \mathcal{E}'_{\pi}(t, u, \pi, \zeta) \ni 0, \quad (2.9b)$$

$$\partial \mathcal{R}_2(\dot{\zeta}) + \partial_{\zeta} \mathcal{E}(t, u, \pi, \zeta) \ni 0. \quad (2.9c)$$

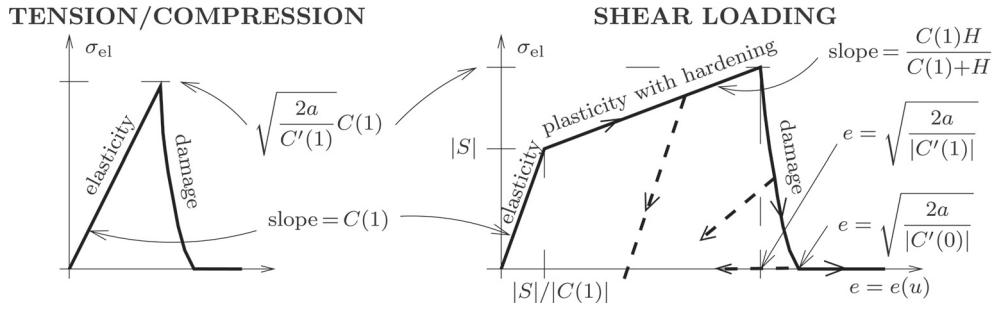


Figure 1. Schematic response of the stress σ_{el} to the total strain e during a ‘one-dimensional’ tension (left) or shear (right) loading experiment under a stress-driven scenario. The former option does not exhibit any plastification because the trace-free plastic strain (valued in $\mathbb{R}_{\text{dev}}^{d \times d}$) ignores tension/compression. The latter option combines plasticity with eventual complete damage. Dashed lines outline a response to unloading, $C = C(\zeta)$ refers to the Young’s modulus (left) or the shear modulus (right).

Remark 2.1 (Irreversible damage in engineering models). Usual engineering models consider $b = \infty$, i.e. no healing is allowed. In fact, due to an essentially missing driving force for healing, our modification $b < \infty$ would not have any influence on the evolution if it were not any $\nabla \zeta$ -term in the stored energy. Thus, if the healing threshold b is big and the gradient-term coefficient $\kappa_2 > 0$ is small, we expect to have essentially (usually desired) unidirectional evolution as far as the damage concerns.

Remark 2.2 (Surface variant of the damage/plasticity). A similar scenario distinguishing tension (which leads to damage without plastification) and shear (with plastifying the material before damage) as in Figure 1 was used in a surface variant to model an adhesive contact distinguishing delamination in the opening and in the shearing modes, devised in [18, 19] and later implemented by the fractional-step discretization with checking the approximate maximum-dissipation principle in [12, 20, 21]. An additional analytical difference is that, in contrast to our bulk model here, the surface variant allows for irreversible damage that does not need any gradient.

Remark 2.3 (Other material models). A separately convex stored energy $\mathcal{E}(t, \cdot)$ occurs also in other models. For example, some phenomenological models for *phase transformations* in (polycrystalline) *shape-memory materials* [22] gives ζ the meaning of a volume fraction (instead of damage) and π a transformation strain (or a combination of the plastic and the transformation strains). The total strain decomposes as $e(u) = e_{\text{el}} + \zeta \pi$ rather than that seen in equation (2.1a), or makes π dependent on ζ (which is then vector-valued). Considering the degree-1 homogeneous dissipation potential, most of the considerations in this paper can be applied to such a model, too; in fact, the only difference would be the nonsmoothness of \mathcal{E} with respect to the π variable. A similar (in general non-convex) model has been considered in [5, 23–26], although sometimes special choices of elastic moduli (leading to convex) were particularly under focus while the dissipation was made state-dependent.

3. Local solutions

We will use the standard notation $W^{1,p}(\Omega)$ for the Sobolev space of functions having the gradient in the Lebesgue space $L^p(\Omega; \mathbb{R}^d)$. If valued in \mathbb{R}^n with $n \geq 2$, we will write $W^{1,p}(\Omega; \mathbb{R}^n)$, and furthermore we use the shorthand notation $H^1(\Omega; \mathbb{R}^n) = W^{1,2}(\Omega; \mathbb{R}^n)$. We also use the notation of ‘‘ \cdot ’’ and ‘‘ $\cdot :$ ’’ for a scalar product of vectors and 2nd-order tensors, respectively, and later also ‘‘ $\cdot :$ ’’ for 3rd-order tensors. For a Banach space X , $L^p(I; X)$ will denote the Bochner space of X -valued Bochner measurable functions $u : I \rightarrow X$ with its norm $\|u(\cdot)\|$ in $L^p(I)$, here $\|\cdot\|$ stands for the norm in X . Further, $W^{1,p}(I; X)$ denotes the Banach space of mappings $u : I \rightarrow X$ whose distributional time derivative is in $L^p(I; X)$, while $BV(I; X)$ will denote the space of mappings $u : I \rightarrow X$ with a bounded variations, i.e. $\sup_{0 \leq t_0 < t_1 < \dots < t_{n-1} < t_n \leq T} \sum_{i=1}^n \|u(t_i) - u(t_{i-1})\| < \infty$ where the supremum is taken over all finite partitions of the interval $I = [0, T]$. By $B(I; X)$ we denote the space of bounded measurable (everywhere defined) mappings $I \rightarrow X$.

The concept of local solutions has been introduced for a special crack problem in [27] and independently also in [28], and further generally investigated in [3]. Here, we additionally combine it with the concept of semi-stability as invented in [29]. We adapt the general definition directly to our specific problem, which will lead to two semi-stability conditions for ζ and π , respectively:

Definition 3.1 (Local solutions). *We call a measurable mapping $(u, \pi, \zeta) : I \rightarrow H^1(\Omega; \mathbb{R}^d) \times H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}) \times W^{1,r}(\Omega)$ a local solution to the elasto-plasto-damage problem (2.1)–(2.3) if the initial conditions (2.3) are satisfied and, for some $J \subset I$ at most countable (containing time instances where the solution may possibly jump), it holds that:*

$$\forall t \in I \setminus J : \quad \mathcal{E}'_u(t, u(t), \pi(t), \zeta(t)) = 0, \quad (3.1a)$$

$$\forall t \in I \setminus J \quad \forall \tilde{\pi} \in H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}) : \quad \mathcal{E}(t, u(t), \pi(t), \zeta(t)) \leq \mathcal{E}(t, u(t), \tilde{\pi}, \zeta(t)) + \mathcal{R}_1(\tilde{\pi} - \pi(t)), \quad (3.1b)$$

$$\forall t \in I \setminus \{0\} \quad \forall \tilde{\zeta} \in W^{1,r}(\Omega), \quad 0 \leq \tilde{\zeta} \leq 1 :$$

$$\mathcal{E}(t, u(t), \pi(t), \zeta(t)) \leq \mathcal{E}(t, u(t), \pi(t), \tilde{\zeta}) + \mathcal{R}_2(\tilde{\zeta} - \zeta(t)), \quad (3.1c)$$

$$\begin{aligned} \forall 0 \leq t_1 \leq t_2 \leq T : \quad & \mathcal{E}(t_2, u(t_2), \pi(t_2), \zeta(t_2)) + \text{Diss}_{\mathcal{R}_1}(\pi; [t_1, t_2]) + \text{Diss}_{\mathcal{R}_2}(\zeta; [t_1, t_2]) \\ & \leq \mathcal{E}(t_1, u(t_1), \pi(t_1), \zeta(t_1)) + \int_{t_1}^{t_2} \mathcal{E}'_t(t, u(t), \pi(t), \zeta(t)) dt, \end{aligned} \quad (3.1d)$$

where

$$\text{Diss}_{\mathcal{R}_1}(\pi; [r, s]) := \sup_{\substack{N \in \mathbb{N} \\ r \leq t_0 < t_1 < \dots < t_{N-1} < t_N \leq s}} \sum_{j=1}^N \int_{\Omega} \delta_S^*(\pi(t_{j-1}) - \pi(t_j)) dx, \quad \text{and similarly} \quad (3.2a)$$

$$\text{Diss}_{\mathcal{R}_2}(\zeta; [r, s]) := \sup_{\substack{N \in \mathbb{N} \\ r \leq t_0 < t_1 < \dots < t_{N-1} < t_N \leq s}} \sum_{j=1}^N \int_{\Omega} a(\pi(t_{j-1}) - \pi(t_j))^+ + b(\pi(t_{j-1}) - \pi(t_j))^+ dx. \quad (3.2b)$$

Let us comment on the above definition briefly. Obviously, the momentum equilibrium (2.1a) after transforming the boundary condition (2.4) means precisely (3.1a), which in more detail means that $\int_{\Omega} \mathbb{C}(\zeta(t))(e(u(t) - w_D(t)) - \pi) : e(v) dx = \int_{\Omega} g \cdot v dx + \int_{\Gamma_D} f \cdot v dS$ for all $v \in H^1(\Omega; \mathbb{R}^d)$ with $v|_{\Gamma_D} = 0$, i.e. the weakly formulated Euler-Lagrange equation for displacement. Note that equation (3.1a) specifies also the boundary conditions for u , namely $u = 0$ on Γ_D because otherwise $\mathcal{E}(t, u, \pi, \zeta) = \infty$ would violate equation (3.1a) for v , which satisfies $v = 0$ on Γ_D , and also $\sigma_{\text{el}} \cdot \vec{n} = f$ on Γ_N can be proved by standard arguments based on Green's theorem. Equivalently, one can merge equation (3.1a) with (3.1b) by a single condition

$$\begin{aligned} \forall t \in I \setminus J \quad & \forall (\tilde{u}, \tilde{\pi}) \in H^1(\Omega; \mathbb{R}^d) \times H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}), \quad \tilde{u}|_{\Gamma_D} = 0 : \\ & \mathcal{E}(t, u(t), \pi(t), \zeta(t)) \leq \mathcal{E}(t, \tilde{u}, \tilde{\pi}, \zeta(t)) + \mathcal{R}_1(\tilde{\pi} - \pi(t)); \end{aligned} \quad (3.3)$$

which reveals that Definition 3.1 just copies the concept of local solutions from [27, 28], and is here generalized for the case of non-vanishing dissipation $\mathcal{R}_1 \neq 0$. As \mathcal{R}_1 is homogeneous degree-1, always $\partial \mathcal{R}_1(\dot{\pi}) \subset \partial \mathcal{R}_1(0)$ and thus equation (2.9b) implies $\partial \mathcal{R}_1(0) + \partial_{\pi} \mathcal{E}(u, \pi, \zeta) \ni 0$. From the convexity of \mathcal{R}_1 when taking into account that $\mathcal{R}_1(0) = 0$, the latter inclusion is equivalent to $\mathcal{R}_1(v) + \langle \partial_{\pi} \mathcal{E}(u(t), \pi(t), \zeta(t)), v \rangle \geq 0$ for any $v \in H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})$. Substituting $v = \tilde{z} - z(t)$ and using the convexity of $\mathcal{E}(t, u, \zeta, \cdot)$, we obtain the *semi-stability* (3.1b) of π at time t . Analogously, we obtain also equation (3.1c) from (2.9c); note that we do not require its validity at $t = 0$ so that we do not need to qualify the initial conditions as far as any (semi)stability concerns. Eventually, equation (3.1d) is the (im)balance of the mechanical energy. Note that, in view of equation (2.8a), the last term in equation (4.6d) involves

$$\mathcal{E}'_t(t, u, \pi, \zeta) = \int_{\Omega} \mathbb{C}(\zeta)(e(\dot{u}_D)) : (e(u + u_D) - \pi) - \dot{g}(t) \cdot u dx - \int_{\Gamma_N} \dot{f}(t) \cdot u dS.$$

This is equivalent (or, if $\mathcal{E}(t, \cdot, \cdot, \cdot)$ is not smooth, slightly generalizes) the standard definition of the *weak solution* to the initial-boundary-value problem (2.1)–(2.3), see [10, Proposition 2.3] for details.

To be more precise, the concept of local solutions as used in [3, 27] requires J only to have a zero Lebesgue measure. In addition, equation (3.1c) is valid only for a.a. t . On the other hand, conventional weak solutions allow even equation (3.1d) to hold only for a.a. t_1 and t_2 . Later, our approximation method will provide convergence to this stronger local solution, which motivates us to tailor Definition 3.1 to our results.

Actually, local solutions form essentially the largest reasonable class of solutions for rate-independent systems as (2.1)–(2.3) considered here. It includes the mentioned energetic solutions [3, 5], the vanishing-viscosity solutions, the balanced-viscosity (so-called BV) solutions, parametrized solutions (see [3, 4] for a survey), also stress-driven-like solutions obeying the maximum-dissipation principle in some sense, see Remark 3.2. The energetic solutions often have a tendency to undergo damage unphysically early, see [21] for a comparison on several computational experiments on a similar type of problem. The approximation method we will use in this article leads rather to the stress-driven option, see Remarks 3.2 and 4.3 below.

Remark 3.2 (Maximum-dissipation principle). The degree-1 homogeneity of \mathcal{R}_1 and \mathcal{R}_2 defined in equation (2.8b) allows for further interpretation of the flow rules (2.9b) and (2.9c). Using maximal-monotonicity of the subdifferential, equation (2.9b) means that $\langle \tilde{\xi} - \xi_{\text{plast}}, v - \dot{\pi} \rangle \geq 0$ for any v and any $\tilde{\xi} \in \partial\mathcal{R}_1(v)$ with the driving force $\xi_{\text{plast}} = -\mathcal{E}'_\pi(t, u, \pi, \zeta)$. In particular, for $v = 0$, defining the convex “elastic domain” $K_1 = \partial\mathcal{R}_1(0)$, one obtains

$$\langle \xi_{\text{plast}}(t), \dot{\pi}(t) \rangle = \max_{\tilde{\xi} \in K_1} \langle \tilde{\xi}, \dot{\pi}(t) \rangle \quad \text{with some } \xi_{\text{plast}}(t) = -\mathcal{E}'_\pi(t, u(t), \pi(t), \zeta(t)). \quad (3.4a)$$

To derive it, we have used that $\xi_{\text{plast}} \in \partial\mathcal{R}_1(\dot{\pi}) \subset \partial\mathcal{R}_1(0) = K_1$ thanks to the degree-0 homogeneity of $\partial\mathcal{R}_1$, so that $\langle \xi_{\text{plast}}, \dot{\pi} \rangle \leq \max_{\tilde{\xi} \in K_1} \langle \tilde{\xi}, \dot{\pi} \rangle$. The identity (3.4a) says that the dissipation due to the driving force ξ_{plast} is maximal provided that the order-parameter rate $\dot{\pi}$ is kept fixed, while the vector of possible driving forces $\tilde{\xi}$ varies freely over all admissible driving force from K_1 . This just resembles the so-called Hill’s *maximum-dissipation principle* articulated just for plasticity in [30]. Also it says that the rates are orthogonal to the elastic domain K_1 , known as an orthogonality principle [31]. Hill et al. [30] used it for a situation where $\mathcal{E}(t, \cdot)$ is convex while, in a general nonconvex case (as also here when damage is considered), it holds only along absolutely continuous paths (i.e. in stick or slip regimes) that are sufficiently regular (in the sense that $\dot{\pi}$ is valued not only in $L^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})$ but also in $H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})^*$, while it does not need to hold during jumps). Analogously it holds also for ζ , defining $K_2 := \partial\mathcal{R}_2(0)$, that

$$\langle \xi_{\text{dam}}(t), \dot{\zeta}(t) \rangle = \max_{\tilde{\xi} \in K_2} \langle \tilde{\xi}, \dot{\zeta}(t) \rangle \quad \text{with some } \xi_{\text{dam}}(t) \in -\partial_\zeta \mathcal{E}(t, u(t), \pi(t), \zeta(t)). \quad (3.4b)$$

Here, $\partial_\zeta \mathcal{E}(t, u, \pi, \zeta)$ is set-valued and its elements should be understood as “available” driving forces not necessarily falling into K_2 , while $\xi_{\text{dam}} \in K_2$ is in a position of an “actual” driving force realized during the actual evolution. As $\mathcal{E}(t, u, \cdot, \zeta)$ is smooth, the maximum-dissipation relation (3.4a) written in the form $\langle -\mathcal{E}'_\pi(t, u(t), \pi(t), \zeta(t)), \dot{\pi}(t) \rangle = \max \langle K_1, \dot{\pi}(t) \rangle = \mathcal{R}_1(\dot{\pi}(t))$ summed with the semistability (3.1b) which can be written in the form $\mathcal{R}_1(\tilde{\pi}) + \langle \mathcal{E}'_\pi(t, u(t), \pi(t), \zeta(t)), \tilde{\pi} \rangle \geq 0$ thanks to the convexity of $\mathcal{E}(t, u, \cdot, \zeta)$ yields

$$\mathcal{R}_1(\tilde{\pi}) + \langle \mathcal{E}'_\pi(t, u(t), \pi(t), \zeta(t)), \tilde{\pi} - \dot{\pi}(t) \rangle \geq \mathcal{R}_1(\dot{\pi}(t)) \quad (3.5)$$

for any $\tilde{\pi}$, which just means that $\xi_{\text{plast}}(t) = -\mathcal{E}'_\pi(t, u(t), \pi(t), \zeta(t)) \in \partial\mathcal{R}_1(\dot{\pi}(t))$, see equation (2.9b). This means that the evolution of π is governed by a thermodynamical driving force ξ_{plast} (we say that it is “stress-driven”) and it reveals the role of the maximum-dissipation principle in combination with semistability. Using the convexity of $\mathcal{E}(t, u, \pi, \cdot)$, a similar argument can be applied for equation (3.4b) in combination with semistability (3.1c) even if $\mathcal{E}(t, u, \pi, \cdot)$ is not smooth.

Remark 3.3 (Integrated maximum-dissipation principle). Let us emphasize that, in general, $\dot{\pi}$ and $\dot{\zeta}$ are measures possibly having singular parts concentrated at times when rupture occurs and the solution and also the driving forces need not be continuous. Even if $\dot{\pi}$ and $\dot{\zeta}$ are absolutely continuous, in our infinite-dimensional case the driving forces need not be in duality with them, as already mentioned in Remark 3.2. So, equation (3.4) is analytically not justified in any sense. For this reason, an *integrated* version of the *maximum-dissipation principle* (IMDP) was devised in [10] for a simpler case involving only one maximum-dissipation relation. Realizing that $\max_{\tilde{\xi} \in K_1} \langle \tilde{\xi}, \dot{\pi} \rangle = \mathcal{R}_1(\dot{\pi})$ and similarly $\max_{\tilde{\xi} \in K_2} \langle \tilde{\xi}, \dot{\zeta} \rangle = \mathcal{R}_2(\dot{\zeta})$, the integrated version of equation

(3.4) reads here as

$$\int_{t_1}^{t_2} \xi_{\text{plast}}(t) d\pi(t) = \int_{t_1}^{t_2} \mathcal{R}_1(\dot{\pi}) dt \quad \text{with } \xi_{\text{plast}}(t) = -\mathcal{E}'_\pi(t, u(t), \pi(t), \zeta(t)), \quad (3.6a)$$

$$\int_{t_1}^{t_2} \xi_{\text{dam}}(t) d\zeta(t) = \int_{t_1}^{t_2} \mathcal{R}_2(\dot{\zeta}) dt \quad \text{with some } \xi_{\text{dam}}(t) \in -\partial_\zeta \mathcal{E}(t, u(t), \pi(t), \zeta(t)) \quad (3.6b)$$

to be valid for any $0 \leq t_1 < t_2 \leq T$. This definition is inevitably a bit technical and, without sliding into too much detail, let us only mention that the left-hand-side integrals in equation (3.6) are the lower Riemann–Stieltjes integrals which are suitably generalized, and defined by limit superior of lower Darboux sums, i.e.

$$\int_r^s \xi(t) dz(t) := \limsup_{\substack{N \in \mathbb{N} \\ r=t_0 < t_1 < \dots < t_{N-1} < t_N = s}} \sum_{j=1}^N \inf_{t \in [t_{j-1}, t_j]} \langle \xi(t), z(t_j) - z(t_{j-1}) \rangle, \quad (3.7)$$

relying on the values of ξ being in duality with the values of z (but not necessarily of \dot{z}) and on that the collection of finite partitions of the interval $[r, s]$ forms a directed set when ordered by inclusion so that “limsup” in equation (3.7) is well defined. Let us mention that the conventional definition uses “sup” instead of “limsup” but restricts only to scalar-valued ζ and z with z non-decreasing. The limit-construction (3.7) is called a (here *lower*) *Moore–Pollard–Stieltjes* integral [32,33] used here for vector-valued functions in duality, which is a very special case of a so-called multilinear Stieltjes integral. As in the aforementioned classical scalar situation of the lower Riemann–Stieltjes integral using “sup” instead of “limsup”, the sub-additivity of the integral with respect to u and to v holds, in addition to additivity, which holds with respect to the domain.

The right-hand-side integrals in equation (3.6) are just the integrals of measures and equal to $\text{Diss}_{\mathcal{R}_1}(\pi; [t_1, t_2])$ and $\text{Diss}_{\mathcal{R}_2}(\zeta; [t_1, t_2])$, respectively. Equivalently, in view of the definition in equation (3.2), they can be also written as $\int_{t_1}^{t_2} \mathcal{R}_1 d\pi(t)$ and $\int_{t_1}^{t_2} \mathcal{R}_2 d\zeta(t)$, where the integrals can again be understood as the lower Moore–Pollard–Stieltjes integrals (or here as the aforementioned lower Riemann–Stieltjes integrals) modified for the case that the time-dependent linear functionals ξ are replaced by nonlinear but time-constant and 1-homogeneous convex functionals \mathcal{R} 's. Alternatively, though not equivalently, denoting the internal variables $z = (\pi, \zeta)$, the IMDP (3.6) can be written “more compactly” as

$$\int_{t_1}^{t_2} \xi(t) dz(t) = \int_{t_1}^{t_2} \mathcal{R} dz(t) \quad \text{with some } \xi(t) \in -\partial_z \mathcal{E}(t, u(t), z(t)). \quad (3.8)$$

Both IMDP (3.6) or (3.8) are satisfied on any interval $[t_1, t_2]$ where the solution to equation (2.9) is absolutely continuous with sufficiently regular time derivatives; then the integrals in equation (3.6) are the conventional Lebesgue integrals, in particular the left-hand sides in equation (3.6) are $\int_{t_1}^{t_2} \langle \xi_{\text{plast}}(t), \dot{\pi}(t) \rangle dt$ and $\int_{t_1}^{t_2} \langle \xi_{\text{dam}}(t), \dot{\zeta}(t) \rangle dt$, respectively. The particular importance of IMDP is especially seen at jumps, i.e. at times when abrupt damage can happen. It is shown in [4, 10] in various finite-dimensional examples of “damageable springs” that this IMDP can identify too early, rupturing local solutions when the driving force is obviously unphysically low (which occurs in particular within the energetic solutions of systems governed by nonconvex potentials, as here); its satisfaction for left-continuous local solutions indicates that the evolution is stress driven, as explained in Remark 3.2. On the other hand, it does not need to be satisfied even in physically well justified stress-driven local solutions. For example, this happens if two springs with different fracture toughness organized in parallel rupture at the same time, see [4, Example 4.3.40], although even in this situation our algorithm (4.2) below will give a correct approximate solution, see Figure 6. Therefore, even the IMDP (3.6) may serve only as a sufficient a posteriori condition whose satisfaction verifies the obtained local solution as physically relevant (in the sense that it is stress driven but its dissatisfaction does not mean anything). Eventually, let us realize that, as a consequence of the mentioned definitions, we have

$$\int_{t_1}^{t_2} \xi_{\text{plast}}(t) d\pi(t) + \int_{t_1}^{t_2} \xi_{\text{dam}}(t) d\zeta(t) \leq \int_{t_1}^{t_2} \xi(t) d(\pi, \zeta)(t) \quad \text{and} \quad (3.9a)$$

$$\int_{t_1}^{t_2} \mathcal{R}_1 d\pi(t) + \int_{t_1}^{t_2} \mathcal{R}_2 d\zeta(t) = \int_{t_1}^{t_2} (\mathcal{R}_1 + \mathcal{R}_2) d(\pi, \zeta)(t). \quad (3.9b)$$

As there is only inequality in equation (3.9a), the IMDP (3.8) is less selective than (3.6) in general. Moreover, we will rely rather on some approximation of IMDP, see Remarks 4.3 and 6.2.

4. Semi-implicit time discretization and its convergence

To prove the existence of local solutions, we use a constructive method relying on a suitable time discretization and the weak compactness of level sets of the minimization problems arising at each time level. When further discretized in space, it will later in Section 5 yield a computer-implementable efficient algorithm. Let us summarize the assumption on the data of the original continuous problem

$$\mathbb{C}(\cdot), \mathbb{H} \in \mathbb{R}^{d \times d \times d \times d} \text{ positive definite, symmetric, } \mathbb{C} : [0, 1] \rightarrow \mathbb{R}^{d \times d \times d \times d} \text{ continuous,} \quad (4.1a)$$

$$a, b, \kappa_1, \kappa_2 > 0, S \subset \mathbb{R}_{\text{dev}}^{d \times d} \text{ convex, bounded, closed, } \text{int } S \ni 0, \quad (4.1b)$$

$$w_{\text{D}} \in W^{1,1}(I; W^{1/2,2}(\Gamma_{\text{D}}; \mathbb{R}^d)), \quad (4.1c)$$

$$g \in W^{1,1}(I; L^p(\Omega; \mathbb{R}^d)) \text{ with } p \begin{cases} > 1 & \text{for } d = 2, \\ = 2d/(d+2) & \text{for } d \geq 3 \end{cases} \quad (4.1d)$$

$$f \in W^{1,1}(I; L^p(\Gamma_{\text{N}}; \mathbb{R}^d)) \text{ with } p \begin{cases} > 1 & \text{for } d = 2, \\ = 2 - 2/d & \text{for } d \geq 3, \end{cases} \quad (4.1e)$$

$$(u_0, \pi_0, \zeta_0) \in H^1(\Omega; \mathbb{R}^d) \times H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}) \times W^{1,r}(\Omega). \quad (4.1f)$$

The qualification (4.1c) allows for an extension u_{D} of w_{D} which belongs to $W^{1,1}(I; H^1(\Omega; \mathbb{R}^d))$; in what follows, we will consider some extension to this property.

For the aforementioned time discretization, we use an equidistant partition of the time interval $I = [0, T]$ with a time step $\tau > 0$, assuming $T/\tau \in \mathbb{N}$, and denote $\{u_{\tau}^k\}_{k=0}^{T/\tau}$ an approximation of the desired values $u(k\tau)$, and similarly ζ_{τ}^k is to approximate $\zeta(k\tau)$, etc.

We use a decoupled semi-implicit time discretization with the *fractional steps* based on the splitting of the state variables governed by the separately-convex character of $\mathcal{E}(t, \cdot, \cdot, \cdot)$. This will make the numerics considerably easier than any other splitting and simultaneously may lead to a physically relevant solution governed rather by stresses (if the maximum-dissipation principle holds at least approximately in the sense of Remark 4.3 below) than by energies and will prevent too-early debonding (as explained in Section 3). More specifically, exploiting the convexity of both $\mathcal{E}(t, \cdot, \cdot, \zeta)$ and $\mathcal{E}(t, u, \pi, \cdot)$ and the additivity $\mathcal{R} = \mathcal{R}_1(\dot{\pi}) + \mathcal{R}_2(\dot{\zeta})$, this splitting will be considered as (u, π) and ζ . This yields alternating convex minimization. Thus, for $(\pi_{\tau}^{k-1}, \zeta_{\tau}^{k-1})$ given, we obtain two minimization problems

$$\begin{aligned} & \text{minimize} && \mathcal{E}_{\tau}^k(u, \pi, \zeta_{\tau}^{k-1}) + \mathcal{R}_1(\pi - \pi_{\tau}^{k-1}) \\ & \text{subject to} && (u, \pi) \in H^1(\Omega; \mathbb{R}^d) \times H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}), \quad u|_{\Gamma_{\text{D}}} = 0, \end{aligned} \quad \left. \right\} \quad (4.2a)$$

with $\mathcal{E}_{\tau}^k := \mathcal{E}(k\tau, \cdot, \cdot, \cdot)$ and, denoting the unique solution as $(u_{\tau}^k, \pi_{\tau}^k)$

$$\begin{aligned} & \text{minimize} && \mathcal{E}_{\tau}^k(u_{\tau}^k, \pi_{\tau}^k, \zeta) + \mathcal{R}_2(\zeta - \zeta_{\tau}^{k-1}) \\ & \text{subject to} && \zeta \in W^{1,r}(\Omega), \quad 0 \leq \zeta \leq 1, \end{aligned} \quad \left. \right\} \quad (4.2b)$$

and denote its (possibly not unique) solution by ζ_{τ}^k . Existence of the discrete solutions $(u_{\tau}^k, \pi_{\tau}^k, \zeta_{\tau}^k)$ is straightforward by the aforementioned compactness arguments.

We define the piecewise-constant interpolants

$$\left. \begin{aligned} \bar{u}_{\tau}(t) &= u_{\tau}^k & \& \underline{u}_{\tau}(t) &= u_{\tau}^{k-1}, \\ \bar{\pi}_{\tau}(t) &= \pi_{\tau}^k & \& \underline{\pi}_{\tau}(t) &= \pi_{\tau}^{k-1}, \\ \bar{\zeta}_{\tau}(t) &= \zeta_{\tau}^k & \& \underline{\zeta}_{\tau}(t) &= \zeta_{\tau}^{k-1}, \\ \bar{\mathcal{E}}_{\tau}(t, u, \pi, \zeta) &= \mathcal{E}_{\tau}^k(u, \pi, \zeta) \end{aligned} \right\} \text{ for } (k-1)\tau < t \leq k\tau. \quad (4.3)$$

Later in Remark 4.3, we will also use the piecewise affine interpolants

$$\left. \begin{aligned} \pi_\tau(t) &= \frac{t-(k-1)\tau}{\tau} \pi_\tau^k + \frac{k\tau-t}{\tau} \pi_\tau^{k-1}, \\ \zeta_\tau(t) &= \frac{t-(k-1)\tau}{\tau} \zeta_\tau^k + \frac{k\tau-t}{\tau} \zeta_\tau^{k-1} \end{aligned} \right\} \text{ for } (k-1)\tau < t \leq k\tau. \quad (4.4)$$

The important attribute of the discretization (4.2) is also its numerical stability and satisfaction of a suitable discrete analog of (3.1), namely:

Proposition 4.1 (Stability of the time discretization). *Let equation (4.1) hold and, in terms of the interpolants (4.3), $(\bar{u}_\tau, \bar{\pi}_\tau, \zeta_\tau)$ be an approximate solution obtained by equation (4.2). Then, the following a priori estimates hold*

$$\|\bar{u}_\tau\|_{L^\infty(I; H^1(\Omega; \mathbb{R}^d))} \leq C, \quad (4.5a)$$

$$\|\bar{\pi}_\tau\|_{L^\infty(I; H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})) \cap \text{BV}(I; L^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))} \leq C, \quad (4.5b)$$

$$\|\bar{\zeta}_\tau\|_{L^\infty(\Omega) \cap \text{BV}(I; L^1(\Omega))} \leq C. \quad (4.5c)$$

Moreover, the obtained approximate solution satisfies for any $t \in I \setminus \{0\}$ the (weakly formulated) Euler-Lagrange equation for the displacement

$$\mathcal{E}'_u(t_\tau, \bar{u}_\tau(t), \bar{\pi}_\tau(t), \underline{\zeta}_\tau(t)) = 0, \quad (4.6a)$$

with $t_\tau := \min\{k\tau \geq t; k \in \mathbb{N}\}$, two separate semi-stability conditions for $\bar{\zeta}_\tau$ and $\bar{\pi}_\tau$

$$\forall \widetilde{\pi} \in H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}) : \mathcal{E}(t_\tau, \bar{u}_\tau(t), \bar{\pi}_\tau(t), \underline{\zeta}_\tau(t)) \leq \mathcal{E}(t_\tau, \bar{u}_\tau(t), \widetilde{\pi}, \underline{\zeta}_\tau(t)) + \mathcal{R}_1(\widetilde{\pi} - \bar{\pi}_\tau(t)), \quad (4.6b)$$

$$\forall \widetilde{\zeta} \in W^{1,r}(\Omega), 0 \leq \widetilde{\zeta} \leq 1 :$$

$$\mathcal{E}(t_\tau, \bar{u}_\tau(t), \bar{\pi}_\tau(t), \bar{\zeta}_\tau(t)) \leq \mathcal{E}(t_\tau, \bar{u}_\tau(t), \bar{\pi}_\tau(t), \widetilde{\zeta}) + \mathcal{R}_2(\widetilde{\zeta} - \bar{\zeta}_\tau(t)), \quad (4.6c)$$

and, for all $0 \leq t_1 < t_2 \leq T$ of the form $t_i = k_i\tau$ for some $k_i \in \mathbb{N}$, the energy (im)balance

$$\begin{aligned} \mathcal{E}(t_2, \bar{u}_\tau(t_2), \bar{\pi}_\tau(t_2), \bar{\zeta}_\tau(t_2)) + \text{Diss}_{\mathcal{R}_1}(\bar{\pi}_\tau; [t_1, t_2]) + \text{Diss}_{\mathcal{R}_2}(\bar{\zeta}_\tau; [t_1, t_2]) \\ \leq \mathcal{E}(t_1, \bar{u}_\tau(t_1), \bar{\pi}_\tau(t_1), \bar{\zeta}_\tau(t_1)) + \int_{t_1}^{t_2} \mathcal{E}'(t, \bar{u}_\tau(t), \bar{\pi}_\tau(t), \bar{\zeta}_\tau(t)) dt. \end{aligned} \quad (4.6d)$$

Sketch of the proof. Writing the optimality condition for equation (4.2a) in terms of u , one arrives at equation (4.6a), and by comparing the value of equation (4.2a) at (u_τ^k, π_τ^k) with its value at $(u_\tau^k, \widetilde{\pi})$ and using the degree-1 homogeneity of \mathcal{R}_1 , one arrives at equation (4.6b).

By comparing the value of equation (4.2b) at ζ_τ^k with its value at $\widetilde{\zeta}$ and using the degree-1 homogeneity of \mathcal{R}_2 , one arrives at equation (4.6c).

In obtaining equation (4.6d), we compare the value of equation (4.2a) at the minimizer (u_τ^k, π_τ^k) with the value at $(u_\tau^{k-1}, \pi_\tau^{k-1})$, and also the value of equation (4.2b) at the minimizer ζ_τ^k with the value at ζ_τ^{k-1} . We benefit from the cancellation of the terms $\pm \mathcal{E}(k\tau, u_\tau^k, \pi_\tau^k, \zeta_\tau^{k-1})$. We also use the discrete by-part integration (= summation) for the \mathcal{E}'_t -term.

Then, using equation (4.6d) for $t_1 = 0$ and the coercivity of $\mathcal{E}(t, \cdot, \cdot, \cdot)$ due to the assumptions (4.1), we obtain also the a priori estimates (4.5). \square

The cancellation effect mentioned in the above proof is typical in fractional-step methods, see for example [34, Remark 8.25]. Further, note that equation (4.6) is of a similar form as equation (3.1) and is thus prepared to make a limit passage for $\tau \rightarrow 0$:

Proposition 4.2 (Convergence towards local solutions). *Let (4.1) hold and let $(\bar{u}_\tau, \bar{\pi}_\tau, \bar{\zeta}_\tau)$ be an approximate solution obtained by the semi-implicit formula (4.2). Then there exists a subsequence (indexed again by τ for*

notational simplicity) and $u \in \mathbf{B}([0, T]; H^1(\Omega; \mathbb{R}^d))$ and $\pi \in \mathbf{B}([0, T]; H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})) \cap \mathbf{BV}([0, T]; L^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))$ and $\zeta \in \mathbf{B}([0, T]; W^{1,r}(\Omega)) \cap \mathbf{BV}([0, T]; L^1(\Omega))$ such that

$$\bar{u}_\tau(t) \rightarrow u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^d) \quad \text{for all } t \in [0, T], \quad (4.7a)$$

$$\bar{\pi}_\tau(t) \rightarrow \pi(t) \quad \text{in } H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}) \quad \text{for all } t \in [0, T], \quad (4.7b)$$

$$\bar{\zeta}_\tau(t) \rightarrow \zeta(t) \quad \text{in } W^{1,r}(\Omega) \quad \text{for all } t \in [0, T]. \quad (4.7c)$$

Moreover, any (u, π, ζ) obtained by this way is a local solution to the damage/plasticity problem in the sense of Definition 3.1.

Proof. By a (generalized) Helly's selection principle, see [3, 4], we choose a subsequence and $\pi \in \mathbf{B}([0, T]; H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})) \cap \mathbf{BV}([0, T]; L^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}))$ and $\zeta, \zeta_* \in \mathbf{B}([0, T]; W^{1,r}(\Omega)) \cap \mathbf{BV}([0, T]; L^1(\Omega))$ so that

$$\bar{\pi}_\tau(t) \rightharpoonup \pi(t) \quad \text{in } H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d}) \quad \text{for all } t \in [0, T], \quad (4.8a)$$

$$\bar{\zeta}_\tau(t) \rightharpoonup \zeta(t) \quad \& \quad \underline{\zeta}_\tau(t) \rightharpoonup \zeta_*(t) \quad \text{in } W^{1,r}(\Omega) \quad \text{for all } t \in [0, T]. \quad (4.8b)$$

Now, for a fixed $t \in [0, T]$, by Banach's selection principle, we select (for a moment) a further subsequence so that

$$\bar{u}_\tau(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^d). \quad (4.9)$$

We further use that $\bar{u}_\tau(t)$ minimizes $\mathcal{E}(t_\tau, \cdot, \bar{\pi}_\tau(t), \underline{\zeta}_\tau(t))$ with $t_\tau := \min\{k\tau \geq t; k \in \mathbb{N}\}$. Obviously, $t_\tau \rightarrow t$ for $\tau \rightarrow 0$ and, by the weak-lower-semicontinuity argument, we can easily see that $u(t)$ minimizes the strictly convex functional $\mathcal{E}(t, \cdot, \zeta_*(t), \pi(t))$; this is indeed simple to prove due to the compactness in both π and ζ due to the gradient theories involved. Thus $u(t)$ is determined uniquely so that, in fact, we did not need to make further selection of a subsequence, and this procedure can be performed for any t by using the same subsequence already selected for equation (4.8). Also, $u : [0, T] \rightarrow H^1(\Omega; \mathbb{R}^d)$ is measurable because π and ζ_* are measurable, and $\mathcal{E}'_u(t, u(t), \pi(t), \zeta_*(t)) = 0$ for all t .

The key ingredient is improvement of the weak convergence (4.8) and (4.9) for the strong convergence. For the strong convergence in u and π , we use the uniform convexity of the quadratic form induced by $\mathbb{C}(\zeta)$, \mathbb{H} , and κ_1 with the information we have at disposal from (4.6b) leading, when using the abbreviation $e_{\text{el}} = e(u - u_{\text{D}}) - \pi$ and $\bar{e}_{\text{el},\tau} = e(\bar{u}_\tau - \bar{u}_{\text{D},\tau}) - \bar{\pi}_\tau$, to the estimate

$$\begin{aligned} & \int_{\Omega} \mathbb{C}(\underline{\zeta}_\tau(t)) (\bar{e}_{\text{el},\tau}(t) - e_{\text{el}}(t)) : (\bar{e}_{\text{el},\tau}(t) - e_{\text{el}}(t)) \\ & + \mathbb{H}(\bar{\pi}_\tau(t) - \pi(t)) : (\bar{\pi}_\tau(t) - \pi(t)) + \frac{\kappa_1}{2} |\nabla \bar{\pi}_\tau(t) - \nabla \pi(t)|^2 \, dx \\ & \leq \int_{\Omega} -\mathbb{C}(\underline{\zeta}_\tau(t)) e_{\text{el}}(t) : (\bar{e}_{\text{el},\tau}(t) - e_{\text{el}}(t)) - (\mathbb{H} \pi(t) - \bar{\xi}_\tau(t)) : (\bar{\pi}_\tau(t) - \pi(t)) \\ & + \frac{\kappa_1}{2} \nabla \pi(t) : \nabla (\bar{\pi}_\tau(t) - \pi(t)) - \bar{f}_\tau(t) \cdot (\bar{u}_\tau(t) - u(t)) \, dx - \int_{\Gamma_N} \bar{g}_\tau(t) \cdot (\bar{u}_\tau(t) - u(t)) \, dS \rightarrow 0 \end{aligned}$$

where we use some $\bar{\xi}_\tau(t) \in \partial \delta_S^*(\bar{\pi}_\tau(t))$ which solves at time t in the weak sense the discrete plastic flow-rule $\bar{\xi}_\tau + \mathbb{H} \bar{\pi}_\tau - \text{dev} \bar{\sigma}_\tau = \kappa_1 \Delta \bar{\pi}_\tau$ with $\bar{\sigma}_\tau = \mathbb{C}(\underline{\zeta}_\tau) \bar{e}_{\text{el},\tau}$. Thus we proved

$$\bar{e}_{\text{el},\tau}(t) \rightarrow e_{\text{el}}(t) \quad \text{strongly in } L^2(\Omega; \mathbb{R}_{\text{sym}}^{d \times d})$$

together with equation (4.7b). Realizing that $e(\bar{u}_\tau(t)) = e(\bar{u}_{\text{D},\tau}(t)) + \bar{\pi}_\tau(t) + \bar{e}_{\text{el},\tau}(t)$, we obtain also $e(\bar{u}_\tau(t)) \rightarrow e(u(t))$ strongly in $L^2(\Omega; \mathbb{R}_{\text{sym}}^{d \times d})$, and thus also (4.7a). Note that we exploited the gradient theory for plasticity which ensures that the sequence $(\bar{\xi}_\tau)_{\tau>0}$, which is bounded in $L^\infty(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})$ because the plastic domain $S \subset \mathbb{R}_{\text{dev}}^{d \times d}$ is bounded, is relatively compact in $H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})^*$ so that the term $\int_{\Omega} \bar{\xi}_\tau(t) : (\bar{\pi}_\tau(t) - \pi(t)) \, dx$ indeed converges to zero because $\bar{\pi}_\tau(t) \rightharpoonup \pi(t)$ in $H^1(\Omega; \mathbb{R}_{\text{dev}}^{d \times d})$.

The convergence (4.7c) can be proved by using the uniform-like monotonicity of the set-valued mapping $\zeta \mapsto \partial\delta_{[0,1]}(\zeta) - \kappa_2 \operatorname{div}(|\nabla\zeta|^{r-2}\nabla\zeta) : W^{1,r}(\Omega) \rightrightarrows W^{1,r}(\Omega)^*$. Analogously to equation (2.1c), we can write the discrete damage flow rule after the shift (2.4) as

$$\bar{\xi}_{\text{dam},\tau} + \mathbb{C}'(\underline{\zeta}_\tau)\bar{e}_{\text{el},\tau} : \bar{e}_{\text{el},\tau} = \kappa_2 \operatorname{div}(|\nabla\underline{\zeta}_\tau|^{r-2}\nabla\underline{\zeta}_\tau) - \bar{\eta}_\tau \quad (4.10a)$$

$$\text{with some } \bar{\xi}_{\text{dam},\tau} \in \partial\delta_{[-a,b]}^*(\dot{\underline{\zeta}}_\tau) \text{ and } \bar{\eta}_\tau \in \partial\delta_{[0,1]}(\bar{\zeta}_\tau) \quad (4.10b)$$

with the boundary condition $\nabla\bar{\zeta}_\tau \cdot \vec{n} = 0$ on Σ ; in equation (4.10), $\bar{\xi}_{\text{dam},\tau}$ and $\bar{\eta}_\tau$ are considered piece-wise constants in time, consistently with our bar-notation. An important fact is that $\bar{\xi}_{\text{dam},\tau}(t)$ is valued in $[-b, a]$ and hence a priori bounded in $L^\infty(\Omega)$; here we vitally exploited the concept of the possible (small) healing allowed. We can rely on $\bar{\xi}_{\text{dam},\tau}(t) \xrightarrow{*} \xi_{\text{dam}}(t)$ in $L^\infty(\Omega)$ for some t -dependent subsequence and some $\xi_{\text{dam}}(t)$. Using that $\mathbb{C}'(\underline{\zeta}_\tau(t))\bar{e}_{\text{el},\tau}(t) : \bar{e}_{\text{el},\tau}(t)$ is bounded and, due to equations (4.7a,b), it has even been proved converging in $L^1(\Omega)$ which is a subspace of $W^{1,r}(\Omega)^*$ because $r > d$ is considered. By the standard theory for monotone variational inequalities, we can pass to the limit in equation (4.10) at time t to obtain, in the weak formulation

$$\xi_{\text{dam}}(t) + \mathbb{C}'(\zeta_*(t))e_{\text{el}}(t) : e_{\text{el}}(t) = \kappa_2 \operatorname{div}(|\nabla\zeta(t)|^{r-2}\nabla\zeta(t)) - \eta(t) \quad \text{with } \eta(t) \in \partial\delta_{[0,1]}(\zeta(t)). \quad (4.11)$$

Then, at any t , we can estimate

$$\begin{aligned} & \kappa_2 \limsup_{k \rightarrow \infty} (\|\nabla\underline{\zeta}_\tau(t)\|_{L'(\Omega;\mathbb{R}^d)}^{r-1} - \|\nabla\zeta(t)\|_{L'(\Omega;\mathbb{R}^d)}^{r-1}) (\|\nabla\bar{\zeta}_\tau(t)\|_{L'(\Omega;\mathbb{R}^d)} - \|\nabla\zeta(t)\|_{L'(\Omega;\mathbb{R}^d)}) \\ & \leq \limsup_{k \rightarrow \infty} \int_{\Omega} \kappa_2 (|\nabla\underline{\zeta}_\tau(t)|^{r-2}\nabla\bar{\zeta}_\tau(t) - |\nabla\zeta(t)|^{r-2}\nabla\zeta(t)) \cdot \nabla(\bar{\zeta}_\tau(t) - \zeta(t)) \\ & \quad + (\bar{\eta}_\tau(t) - \eta(t))(\bar{\zeta}_\tau(t) - \zeta(t)) \, dx \\ & = \lim_{k \rightarrow \infty} \int_{\Omega} \mathbb{C}'(\underline{\zeta}_\tau(t))\bar{e}_{\text{el},\tau}(t) : \bar{e}_{\text{el},\tau}(t)(\bar{\zeta}_\tau(t) - \zeta(t)) - \kappa_2 |\nabla\zeta(t)|^{r-2}\nabla\zeta(t) \cdot \nabla(\bar{\zeta}_\tau(t) - \zeta(t)) \\ & \quad - (\xi_{\text{dam}}(t) + \eta(t))(\bar{\zeta}_\tau(t) - \zeta(t)) \, dx = 0 \end{aligned} \quad (4.12)$$

where the last equality has exploited equation (4.11). The important fact used for equation (4.12) is that

$$\mathbb{C}'(\underline{\zeta}_\tau(t))\bar{e}_{\text{el},\tau}(t) : \bar{e}_{\text{el},\tau}(t)(\bar{\zeta}_\tau(t) - \zeta(t)) \rightarrow 0 \quad \text{weakly in } L^1(\Omega); \quad (4.13)$$

in fact, this convergence is even stronger when realizing that $\bar{\zeta}_\tau(t) \rightarrow \zeta(t)$ in $L^\infty(\Omega)$, for which $r > d$ is again exploited. From this, equation (4.7c) follows. Thus, from equation (4.12) we can see that $\|\nabla\bar{\zeta}_\tau(t)\|_{L'(\Omega;\mathbb{R}^d)} \rightarrow \|\nabla\zeta(t)\|_{L'(\Omega;\mathbb{R}^d)}$ and, from the uniform convexity of the Lebesgue space $L'(\Omega;\mathbb{R}^d)$, we eventually obtain equation (4.7c). Actually, the specific value $\xi_{\text{dam}}(t)$ of the limit of (a t -dependent subsequence of) $\{\xi_{\text{dam},\tau}(t)\}_{\tau>0}$ which is surely precompact in $W^{1,r}(\Omega)^*$ is not important and thus equation (4.7c) holds for the originally selected subsequence, too.

Having the strong convergences (4.7) proved, the limit passage from equation (4.6) towards equation (3.1) is simple. In particular, by continuity of both BV-functions $\zeta(\cdot)$ and $\zeta_*(\cdot)$ on $[0, T] \setminus J$ for some at most countable set J , we have also $\zeta_*(t) = \zeta(t)$ at any t except at most countable the set J . \square

Remark 4.3 (Approximate maximum-dissipation principle). One can devise the discrete analog of the integrated maximum-dissipation principle (3.6) straightforwardly for the left-continuous interpolants (4.3), which are required however to hold only asymptotically. More specifically, in an analog to equation (3.6) formulated equivalently for all $[0, t]$ instead of $[t_1, t_2]$, one can expect an *approximate maximum-dissipation principle* (AMDP) in the form

$$\int_0^t \bar{\xi}_{\text{plast},\tau} d\bar{\pi}_\tau \stackrel{?}{\sim} \text{Diss}_{\mathcal{R}_1}(\bar{\pi}_\tau; [0, t]) \quad \text{with } \bar{\xi}_{\text{plast},\tau} = -[\bar{\mathcal{E}}_\tau]'(\cdot, \bar{u}_\tau, \bar{\pi}_\tau, \underline{\zeta}_\tau), \quad (4.14a)$$

$$\int_0^t \bar{\xi}_{\text{dam},\tau} d\bar{\zeta}_\tau \stackrel{?}{\sim} \text{Diss}_{\mathcal{R}_2}(\bar{\zeta}_\tau; [0, t]) \quad \text{for some } \bar{\xi}_{\text{dam},\tau} \in -\partial_{\zeta} \bar{\mathcal{E}}_\tau(\cdot, \bar{u}_\tau, \bar{\pi}_\tau, \bar{\zeta}_\tau), \quad (4.14b)$$

or, analogously to equation (3.8)

$$\int_0^t \bar{\xi}_\tau \, d\bar{z}_\tau \stackrel{?}{\sim} \text{Diss}_{\mathcal{R}}(\bar{z}_\tau; [0, t]) \text{ with some } \bar{\xi}_\tau \in \{-[\bar{\mathcal{E}}_\tau]_\pi'(\cdot, \bar{u}_\tau, \bar{\pi}_\tau, \underline{\zeta}_\tau)\} \times -\partial_\zeta \bar{\mathcal{E}}_\tau(\cdot, \bar{u}_\tau, \bar{\pi}_\tau, \bar{\zeta}_\tau), \quad (4.15)$$

where the integrals are again the lower Moore–Pollard–Stieltjes integrals as in equation (3.6) and where $\bar{\mathcal{E}}_\tau(\cdot, u, \pi, \zeta)$ is the left-continuous piecewise-constant interpolant of the values $\mathcal{E}(k\tau, u, \pi, \zeta)$, $k = 0, 1, \dots, T/\tau$. Moreover, “ $\stackrel{?}{\sim}$ ” in equation (4.14) means that the equality holds possibly only asymptotically for $\tau \rightarrow 0$, but even this is only desirable and not always valid. Loadings which, under the given geometry of the specimen, lead to rate-independent slides where the solution is absolutely continuous will always comply with AMDP (4.14). Also, some finite-dimensional examples of “damageable springs” in [4, 10] show that this AMDP can detect rupturing local solutions too early (in particular the energetic ones), while it generically holds for solutions obtained by the algorithm (4.2). Generally speaking, equation (4.14) should be checked a posteriori to justify the (otherwise not physically based) simple and numerically efficient fractional-step-type semi-implicit algorithm (4.2) from the perspective of the stress-driven solutions in particular situations, and possibly to provide valuable information that can be exploited to adapt time or space discretization towards better accuracy in equation (4.14) (and thus close towards the stress-driven scenario). Actually, for the piecewise-constant interpolants, we can simply evaluate the integrals explicitly, so that AMDP (4.15) reads

$$\sum_{k=1}^K \int_{\Omega} \delta_S^*(\pi(t_{j-1}) - \pi(t_j)) + a(\zeta_\tau^k - \zeta_\tau^{k-1})^- + b(\zeta_\tau^k - \zeta_\tau^{k-1})^+ \, dx \stackrel{?}{=} \varepsilon_\tau \stackrel{?}{\searrow} 0 \quad (4.16)$$

where $\xi_{\text{plast},\tau}^k = -[\mathcal{E}_\tau^k]_\pi'(u_\tau^k, \pi_\tau^k, \zeta_\tau^k)$ and $\xi_{\text{dam},\tau}^k \in -\partial_\zeta \mathcal{E}_\tau^k(u_\tau^k, \pi_\tau^k, \zeta_\tau^k)$

where $K = \max\{k \in \mathbb{N}; k\tau \leq t\}$ and the notation “ $\stackrel{?}{\searrow}$ ” means that it is only a desired but not granted convergence. Notably, in contrast to equations (3.6) and (3.8), the AMDP (4.14) and (4.15) are equivalent to each other as the limsup’s (see definition (3.7)) in all involved integrals is attained on the equidistant partitions with the time step τ , and the “inf” in the Darboux sums is redundant. Evaluating the dualities, the residuum ε_τ in equation (4.16) can be written more explicitly as $\varepsilon_\tau = \int_{\Omega} R_\tau^K \, dx \geq 0$ with the local residuum

$$\begin{aligned} R_\tau^K := & \sum_{k=1}^K \left(\delta_S^*(\pi(t_{j-1}) - \pi(t_j)) + a(\zeta_\tau^k - \zeta_\tau^{k-1})^- + b(\zeta_\tau^k - \zeta_\tau^{k-1})^+ \right. \\ & - \left(\mathbb{C}(\zeta_\tau^{k-2})(\pi_\tau^{k-1} - e(u_\tau^{k-1} + u_{\text{d},\tau}^{k-1})) + \mathbb{H}\pi_\tau^{k-1} \right) : (\pi_\tau^k - \pi_\tau^{k-1}) \\ & - \left(\frac{1}{2} \mathbb{C}'(\zeta_\tau^{k-1})(e(u_\tau^{k-1} + u_{\text{d},\tau}^{k-1}) - \pi_\tau^{k-1}) : (e(u_\tau^{k-1} + u_{\text{d},\tau}^{k-1}) - \pi_\tau^{k-1}) + \xi_{\text{const},\tau}^{k-1} \right) (\zeta_\tau^k - \zeta_\tau^{k-1}) \\ & \left. - \kappa_1 \nabla \pi_\tau^{k-1} : \nabla (\pi_\tau^k - \pi_\tau^{k-1}) - \kappa_2 |\nabla \zeta_\tau^{k-1}|^{r-2} \nabla \zeta_\tau^{k-1} \cdot \nabla (\zeta_\tau^k - \zeta_\tau^{k-1}) \right) \end{aligned} \quad (4.17)$$

with some multiplier $\xi_{\text{const},\tau}^k \in N_{[0,1]}(\zeta_\tau^k)$ and with ζ_τ^{k-2} for $k = 1$ equal to ζ_0 . Note that R_τ^K cannot be guaranteed to be non-negative pointwise on Ω , only their integrals over Ω are non-negative. One can check the residua a posteriori depending on t or possibly also on space, see also [12, 21] for a surface variant of such a model or Figures 4–7 below.

5. Implementation of the discrete model

To implement the model computationally, we need to make a *spatial discretization* of the variables from the semi-implicit time discretization of Section 4. Essentially, we apply conformal Galerkin (also called Ritz) method to the minimization problems (4.2a) and (4.2b) which are then restricted to the corresponding finite-dimensional subspaces. These subspaces are constructed by the *finite-element method* (FEM), and the solution thus obtained is denoted by

$$q_{\tau h}^k := (u_{\tau h}^k, \pi_{\tau h}^k, \zeta_{\tau h}^k),$$

with $h > 0$ denoting the mesh size of the triangulation, let us denote it by \mathcal{T}_h , of the domain Ω considered polyhedral here; later in Section 6 we consider $d = 2$ and henceforth a polyhedral domain. In this way, we obtain also the piecewise constant and the piecewise affine interpolants in time, denoted respectively by \bar{u}_{th} and u_{th} , $\bar{\pi}_{th}$ and π_{th} , and eventually $\bar{\zeta}_{th}$ and ζ_{th} . The simplest option is to consider the lowest-order conformal FEM, i.e. P1-elements for u , ζ , and π . In Section 6, only the case $d = 2$ will be treated, so the previous analytical parts have required $r > 2$ and we make an (indeed small) shortcut by considering $r = 2$. Moreover, we will not consider the loading on Γ_n , so $f = 0$.

The material is assumed to be isotropic with properties linearly dependent on damage. The isotropic elasticity tensor is assumed to be

$$\mathbb{C}_{ijkl}(\zeta) := [(\lambda_1 - \lambda_0)\zeta + \lambda_0]\delta_{ij}\delta_{kl} + [(\mu_1 - \mu_0)\zeta + \mu_0](\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (5.1)$$

where λ_1, μ_1 and λ_0, μ_0 are two sets of Lamé parameters satisfying $\lambda_1 \geq \lambda_0 \geq 0$ and $\mu_1 \geq \mu_0 > 0$. Here, δ denotes the Kronecker symbol. This choice implies that the elastic-moduli tensor is positively-definite-valued (and therefore is invertible). The elastic domain S is assumed to satisfy

$$S = \{\sigma \in \mathbb{R}_{dev}^{d \times d}; |\sigma| \leq \sigma_y\}, \quad (5.2)$$

where $\sigma_y > 0$ is a given plastic yield stress. More specifically, the minimization problems (4.2) after spatial discretization can be rewritten as

$$(u_{th}^k, \pi_{th}^k) = \operatorname{argmin}_{\substack{u \in W^{1,\infty}(\Omega; \mathbb{R}^d) \\ u, \pi \text{ elementwise affine on } \mathcal{T}_h}} \int_{\Omega} \left(\frac{1}{2} \mathbb{C}(\zeta_{th}^{k-1})(e(u + u_{th}^k) - \pi) : (e(u + u_{th}^k) - \pi) \right. \\ \left. + \frac{1}{2} \mathbb{H}\pi : \pi + \frac{\kappa_1}{2} |\nabla \pi|^2 - g_{th}^k \cdot u + \sigma_y |\pi - \pi_{th}^{k-1}| \right) dx, \quad (5.3a)$$

$$\zeta_{th}^k = \operatorname{argmin}_{\substack{\zeta \in W^{1,\infty}(\Omega) \\ 0 \leq \zeta \leq 1 \text{ on } \Omega \\ \zeta \text{ elementwise affine on } \mathcal{T}_h}} \int_{\Omega} \left(\frac{1}{2} \mathbb{C}(\zeta)(e(u_{th}^k + u_{th}^k) - \pi_{th}^k) : (e(u_{th}^k + u_{th}^k) - \pi_{th}^k) \right. \\ \left. + \frac{\kappa_2}{2} |\nabla \zeta|^2 + a(\zeta - \zeta_{th}^{k-1})^- + b(\zeta - \zeta_{th}^{k-1})^+ \right) dx. \quad (5.3b)$$

The damage problem (5.3b) represents a minimization of a nonsmooth but strictly convex functional. To facilitate its numerical solution, we still modify it a bit, namely

$$\operatorname{argmin}_{\substack{\zeta, \zeta^\Delta, \zeta^\nabla \in W^{1,\infty}(\Omega) \\ 0 \leq \zeta \leq 1 \text{ on } \Omega \\ \zeta, \zeta^\Delta, \zeta^\nabla \text{ elementwise affine on } \mathcal{T}_h}} \int_{\Omega} \left(\frac{1}{2} \mathbb{C}(\zeta)(e(u_{th}^k + u_{th}^k) - \pi_{th}^k) : (e(u_{th}^k + u_{th}^k) - \pi_{th}^k) \right. \\ \left. + \frac{\kappa_2}{2} |\nabla \zeta|^2 + a\zeta^\Delta + b\zeta^\nabla \right) dx, \quad (5.4a)$$

$$\text{where } \zeta^\Delta = (\zeta - \zeta_{th}^{k-1})^+ \text{ and } \zeta^\nabla = (\zeta - \zeta_{th}^{k-1})^- \text{ at all nodal points.} \quad (5.4b)$$

We used additional auxiliary ‘update’ variables ζ^Δ and ζ^∇ which are also considered as P1-functions. This modification can also be understood as a certain specific numerical integration applied to the original minimization problem (5.3b). It should be noted that ζ and ζ_{th}^{k-1} are P1-functions and, if we would require equation (5.4b) to be valid everywhere on Ω , ζ^Δ and ζ^∇ could not be P1-functions in general on elements where nodal values of $\zeta - \zeta_{th}^{k-1}$ alternate signs. The important advantage of equation (5.4b) required only at nodal points (while at remaining points it is fulfilled only approximately (depending on h)) is that equation (5.4a) actually represents a conventional *quadratic-programming* problem (QP) involving the linear and the box constraints

$$\zeta = \zeta_{th}^{k-1} + \zeta^\Delta - \zeta^\nabla, \quad 0 \leq \zeta^\Delta \leq 1 - \zeta_{th}^{k-1}, \quad 0 \leq \zeta^\nabla \leq \zeta_{th}^{k-1}. \quad (5.5)$$

A convex quadratic cost functional of this QP problem has only a positive-semidefinite Jacobian, since there are no Dirichlet boundary conditions on the damage variable ζ . Note that the optimal pair $(\zeta^\Delta, \zeta^\nabla)$ must satisfy $\zeta^\Delta \zeta^\nabla = 0$ in all nodes, i.e. both variables cannot be positive. This can be easily seen by contradiction: If $\zeta^\Delta \zeta^\nabla > 0$ in some node, then a different pair $(\zeta^\Delta - \min\{\zeta^\Delta, \zeta^\nabla\}, \zeta^\nabla - \min\{\zeta^\Delta, \zeta^\nabla\})$ would again satisfy the constraints (5.5) but would provide a smaller energy value in (5.4a).

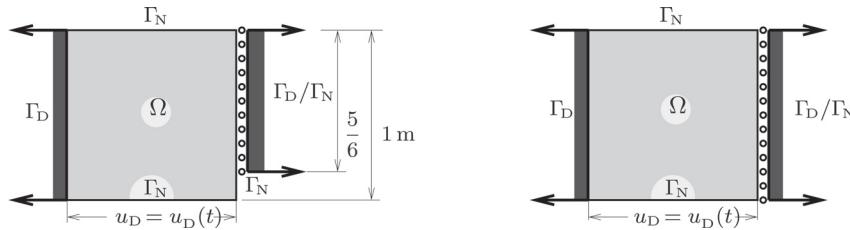


Figure 2. Geometry of a 2-dimensional square-shaped specimen subjected to two tension-loading experiments; the right-hand side of the rectangle Ω combines Dirichlet condition in the horizontal direction and homogeneous Neumann condition in the vertical direction.

As we have a priori bounds of ζ in $W^{1,r}(\Omega)$ uniformly in t , τ , and h also if the modified problem (5.4b) is considered (disregarding that we used $r = 2$ above), we have estimates also in Hölder spaces for ζ^Δ and ζ^∇ and can show that the constraints (5.4b) are valid everywhere on Ω in the limit for $h \rightarrow 0$. Thus, an analogy of Proposition 4.2 for a successive limit passage $h \rightarrow 0$ and then $\tau \rightarrow 0$ might be obtained, although it does not have much practical importance for situations when $(h, \tau) \rightarrow (0, 0)$ simultaneously.

A similar modification can be used also for equation (5.3a). In addition, one can then exploit the structure of the cost functional being the sum of a quadratic functional and a nonsmooth convex functional with the epi-graph having a “ice-cream-cone” shape. After introduction of auxiliary variables at each element, it can be transformed to a so-called second-order cone programming problem (SOCP), see [4, Section 3.6.3], for which efficient codes exist.

Other way is to use simply the quasi-Newton iterative method. This option was used also here.

6. Illustrative 2-dimensional examples

Finally, we demonstrate both the relevance of the model together with the solution concept from Sections 2 and 3 and the efficiency and convergence of the discretization scheme from Section 4 together with the implementation from Section 5 on a two-dimensional example.

The material: We consider an isotropic homogeneous material with the elastic properties given by Young’s modulus $E_{\text{Young}} = 27 \text{ GPa}$ and Poisson’s ratio $\nu = 0.2$ in the non-damaged state, which means that the elastic-moduli tensor in the form (5.1) takes $\lambda_1 = 7.5 \text{ GPa}$ and $\mu_1 = 11.25 \text{ GPa}$, while the damaged material uses 10^7 -times smaller moduli, i.e. $\lambda_0 = 750 \text{ Pa}$ and $\mu_0 = 112.5 \text{ Pa}$ in (5.1). The yield stress from equation (5.2) and the kinematic hardening parameter are chosen as $\sigma_y = 2 \text{ MPa}$ and $\mathbb{H} = (E_{\text{Young}}/20)\mathbb{I}$. The activation energy for damage is $a = 1.2 \text{ kPa}$ and the damage length-scale coefficient is $\kappa_2 = 0.001 \text{ J/m}$; the healing (used before for analytical reasons) was effectively not considered, see Remark 2.1.

The specimen and its loadings: We consider a 2-dimensional square-shaped specimen subjected to two slightly different loading regimes. Both of them consist of a pure “hard-devise” horizontal load by Dirichlet boundary conditions with the left-hand side Γ_D fully fixed while the right-hand side Γ_D/Γ_N combines a time-varying Dirichlet condition in the horizontal direction with the Neumann condition in the vertical direction. The only (intentionally small) difference is in keeping a small bottom part of this vertical side free (see Figure 2 (left)) or not (see Figure 2 (right)). As our model is fully rate-independent, the time scale is irrelevant and we thus consider a dimensionless process time $t \in [0, 80]$ controlling the linearly growing hard-devise (=Dirichlet) load until the maximal horizontal shift 80 mm of the right-hand side Γ_D/Γ_N .

The discretization: In comparison with Section 5, we dare make a shortcut by neglecting the gradient term $\nabla\pi$ in the stored energy (2.8a) by putting $\kappa_1 = 0$, which allows for using only P0-elements for π . It also allows for transformation of the cost functional of equation (5.3a) to a functional of the variable u only by substituting the elementwise dependency of π on u , see [35, 36] for more details. Then, the quasi-Newton iterative method mentioned in Section 5 is applied to solve $u_{\tau h}^k$ while $\pi_{\tau h}^k$ is reconstructed from it. More details on this specific elasto-plasticity solver can be found in [35, 36, 38]. Here, the spatial P1/P0 FEM discretization of the rectangular domain Ω uses a uniform triangular mesh with 2304 elements and 1201 nodes. The code was implemented in Matlab, being available for download and testing at Matlab Central as a package *Continuum undergoing combined elasto-plasto-damage transformation*, [39]. It is based on an original elastoplasticity code related to multi-threshold models [40], here simplified for a single-threshold case. It partially utilizes the vectorization

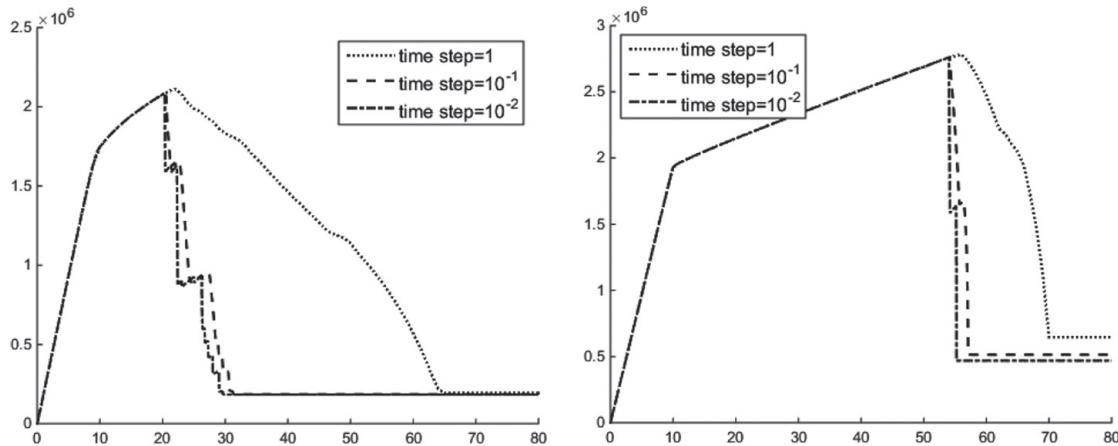


Figure 3. Evolution of averaged elastic von Mises stresses $\int_{\Omega} |\operatorname{dev} \sigma_{\text{el}}| dx$ over time for the two experiments from Figure 2 for three different time discretizations. The left experiment ruptures earlier under less stretch and leaving less plastification and remaining stress comparing to the right experiment. In both experiments, the convergence theoretically supported by Proposition 4.2 is well documented.

techniques of [41] and works reasonably fast also for finer triangular meshes. In contrast to the fixed spatial discretization, we consider three time discretization to document the convergence (theoretically stated only for unspecified subsequences in Proposition 4.2) on particular computational experiments. More specifically, we used three time steps $\tau = 1, 0.1$, or 0.01 , i.e. the equidistant partition of the time interval $[0, 80]$ to 80, 800, or 8000 time steps, respectively.

Simulation results: The averaged stress/stain (or rather force/stretch) response is depicted in Figure 3. Notably, after damage is completed, some stress still remains (as is nearly independent on further stretch because the elastic moduli λ_0 and μ_0 are considered very small). These remaining stresses are caused by non-uniform plastification of the specimen during the previous phases of the loading. One can also note that Figure 3 (right) imitates quite well the scenario from Figure 1 (right) while Figure 3 (left) is rather a mixture of both regimes from Figure 1 and, interestingly, the rupture proceeds in three stages. The respective spacial distribution of the evolving state variable is depicted at few selected instants on Figures 4 and 5. It is seen how a relatively small variation of geometry in Figure 2 dramatically changes the spatial scenario and triggers damage in very different spots of the specimen. This is an expected notch-effect causing stress concentration and relatively early initiation of cracks at such spots, i.e. here such a notch is the point of the transition Γ_N to Γ_D/Γ_N in Figure 2 (left). The AMDP suggested in Remark 4.3 is depicted in Figures 6 and 7. It should be emphasized that the maximum-dissipation principle (as devised originally by Hill [30]) is reliably satisfied only for convex stored energies as occurs during a mere plastification phase, as also seen in Figure 7. In general it does not need to be satisfied even in obviously physically relevant stress-driven evolutions, as already mentioned in Remark 3.3, and which can be expected even here during massive fast rupture of a wider region (in spite of this, Figure 6 shows a good satisfaction of AMDP even during such rupture phases and in some sense demonstrates a good applicability of the model and solution concept and its algorithmic realization).

Remark 6.1 (Symmetry issue). Actually, one could understand the square 1×1 in Figure 2 as one half of a rectangle with sides 2×1 , with the right-hand side of the 1×1 square being the symmetry axis of the 2×1 rectangle, which is then loaded from the vertical sides fully symmetrically. Engineers actually routinely assume that a symmetry of this geometry would be inherited by all (or at least by one) solution(s), and therefore would use the reduced geometries on Figure 2 for calculations of the full 2×1 -rectangle. We intentionally did not use this interpretation because, in fact, one can only say that the set of all solutions inherits the (possible) symmetry of the specimen and its loading but not particular solutions. It may be that there is no solution inheriting this symmetry or that experimental evidence shows preferences for nonsymmetric solutions, see the discussion in [42, 43]. In addition, the geometry in Figure 2 (left) would lead to a 2×1 rectangle with a partial ‘cut’ in the mid-bottom side, which is not a Lipschitz domain.

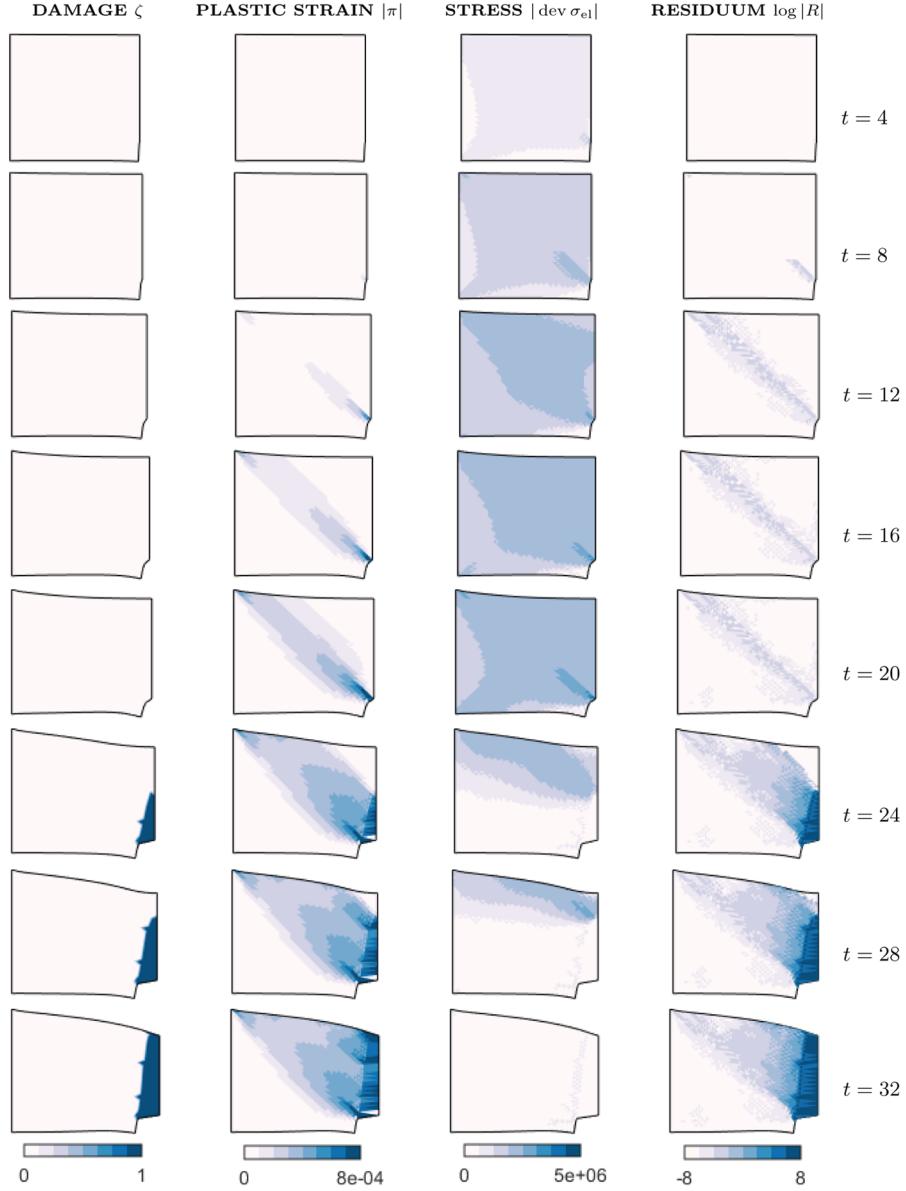


Figure 4. Evolution of the spatial distribution of the state (u, π, ζ) with the von Mises stress and the residuum R from equation (4.17) at (equidistantly) selected instants for the asymmetric geometry from Figure 2 (left). The deformation is visualized by a displacement u magnified by $250 \times$, with $\tau = 0.1$ used. Damage occurs relatively early on in the right-hand side due to the stress concentration, and propagates in several partial steps, see Figure 3 (left).

Remark 6.2 (Recovery of the integrated maximum-dissipation principle IMDP). It should be emphasized that, even if the intuitively straightforward AMDP is asymptotically satisfied, the recovery of even the less-selective IMDP (3.8) for $\tau \rightarrow 0$ is not clear. This is obviously related to the instability of IMDP under data perturbation if $\mathcal{E}(t, \cdot)$ is not convex. Here, to recover the IMDP on I , it would suffice to show that for all $\varepsilon > 0$ there is $\tau_\varepsilon > 0$ such that for any $0 < \tau \leq \tau_\varepsilon$ it holds

$$\sum_{k=1}^{T/\tau} \inf_{t \in [k\tau - \tau, k\tau]} \langle \xi(t), z(k\tau) - z(k\tau - \tau) \rangle - \mathcal{R}(z(k\tau) - z(k\tau - \tau)) \geq -\varepsilon \quad (6.1)$$

for some selection $\xi(t) \in -\partial_z \mathcal{E}(t, u(t), z(t))$, see definition (3.7). The equi-distant partitions are cofinal in all partitions of I . This can be guaranteed only under rather strong conditions, namely if, for all $\varepsilon > 0$, there is

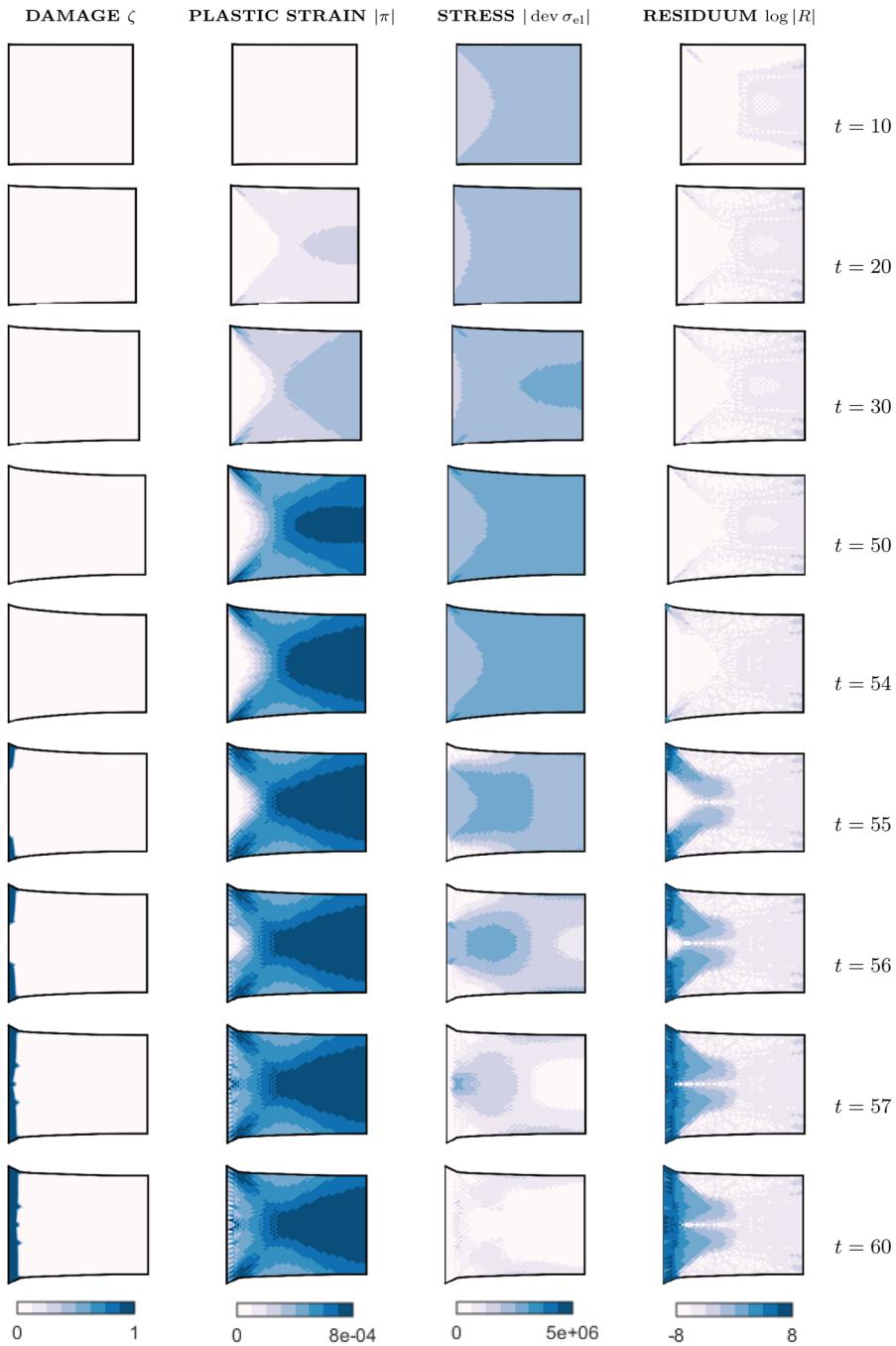


Figure 5. Evolution of spatial distribution of the state (u, π, ξ) with the von Mises stress and the residuum R from equation (4.17) at selected instants for the symmetric geometry from Figure 2 (right). The deformation is visualized by a displacement u magnified by $250 \times$, where $\tau = 0.1$ was used. The process inherits the symmetry of the specimen and loading. In contrast to Figure 4, damage occurs rather later on in the left-bottom corner and propagates quickly, hence the snapshots are not selected in an equidistant way here.

$\tau_\varepsilon > 0$ such that for any $0 < \tau \leq \tau_\varepsilon$, the following strengthened version of the AMDP

$$\sum_{k=1}^{T/\tau_\varepsilon} \mathcal{R}(\bar{z}_\tau(t_k) - \bar{z}_\tau(t_{k-1})) - \langle \bar{\xi}_\tau(t), \bar{z}_\tau(t_k) - \bar{z}_\tau(t_{k-1}) \rangle \leq \varepsilon \quad (6.2)$$

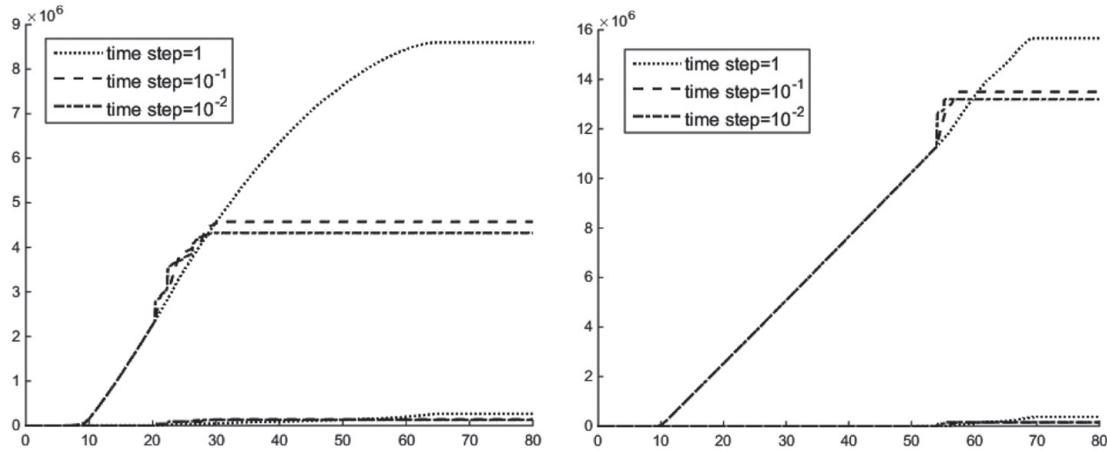


Figure 6. Evolution of the dissipated energy $\text{Diss}_{\mathcal{R}}(z; [0, t])$ (top 3 curves) and integrated residua $\int_0^t \int_{\Omega} R \, dx \, dt$ (bottom 3 curves) in two experiments from Figure 2. In particular, a good tendency of convergence is again seen and, moreover, the violation of the approximate maximum-dissipation principle is small with respect to the overall dissipated energy. The evolution was stress driven, with a good accuracy of about 1-2%.

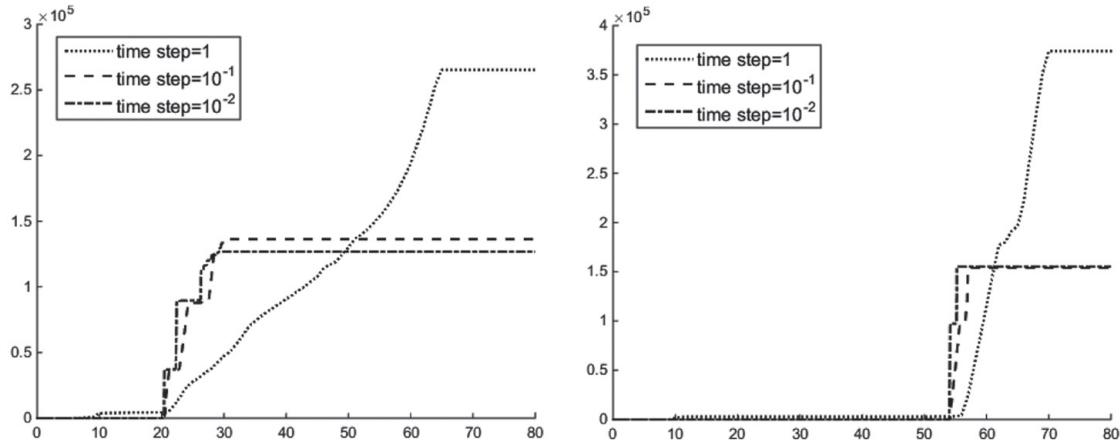


Figure 7. A detailed scaling of the bottom 3 curves (=the residua in AMDP) from Figure 6. A good convergence to zero is seen in the plastification period while some residuum is generated during the damage period, where the nonconvexity of the stored energy truly comes into effect.

holds for $t_k = k\tau_\varepsilon$, any $t_{k-1} \leq t \leq t_k$, and some $\bar{\xi}_\tau(t) \in -\partial_z \mathcal{E}(t, \bar{u}_\tau(t), \bar{z}_\tau(t))$, and if $\bar{\xi}_\tau(t) \rightharpoonup \xi(t)$, which can be assumed due to the available a priori estimates used in the proof of Proposition 4.2. Using equation (4.7) and the (norm,weak)-upper semicontinuity of $\partial_z \mathcal{E}(t, \cdot, \cdot)$, in the limit for $\tau \rightarrow 0$, from such a strengthened AMDP, one can read $\sum_{k=1}^{T/\tau_\varepsilon} \mathcal{R}(z(t_k) - z(t_{k-1})) - \langle \xi(t), z(t_k) - z(t_{k-1}) \rangle \leq \varepsilon$ for any $t_{k-1} \leq t \leq t_k$ and for some $\xi(t) \in -\partial_z \mathcal{E}(t, u(t), z(t))$, from which equation (6.1) indeed follows. In fact, our intuitive version of AMDP from Remark 4.3 computationally verified equation (6.2) in Figure 7 in particular examples for $\tau = \tau_\varepsilon$ only. In addition, we would need equation (6.2) to be shown rather for $\xi_\tau \in (-\partial_\pi \mathcal{E}(t, \bar{u}_\tau(t), \bar{z}_\tau(t-\tau)), -\partial_\zeta \mathcal{E}(t, \bar{u}_\tau(t), \bar{z}_\tau(t)))$.

Conflict of interest

None declared.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This research has been supported by GA ČR through the projects 13-18652S “Computational modeling of damage and transport processes in quasi-brittle materials” and 14-15264S “Experimentally justified multiscale modelling of shape memory alloys” with also the also institutional support RVO:61388998 (ČR).

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