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Structured Lyapunov functions for synchronization of identical affine-in-control agents—Unified approach ☆

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Abstract

This paper brings structured Lyapunov functions guaranteeing cooperative state synchronization of identical agents. Versatile synchronizing region methods for identical linear systems motivate the structure of proposed Lyapunov functions. The obtained structured functions are applied to cooperative synchronization problems for affine-in-control nonlinear agents. For irreducible graphs a virtual leader is used to analyze synchronization. For reducible graphs a combination of cooperative tracking and irreducible graph cooperative synchronization is used to address cooperative dynamics by Lyapunov methods. This provides a connection between the synchronizing region analysis, incremental stability and Lyapunov cooperative stability conditions. A class of affine-in-control systems is singled out based on their contraction properties that allow for cooperative stability *via* the presented Lyapunov designs.
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1. Introduction

The last two decades have witnessed an increasing interest in multi-agent networked cooperative systems [1,5,10,11,15–17,21,26]. Early work [5,15–17,21] refers to *consensus* without a leader. We term this the *cooperative regulator* problem. There the asymptotic

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consensus state depends on precise initial conditions of an entire system. By adding a command generator leader that pins to a group of agents one can have *synchronization* to the leader's reference trajectory for all initial conditions; this is termed *pinning control* [2,8,24,26,27,31]. There, pinning to all root nodes of a spanning forest is necessary for synchronization [31]. We call this the *cooperative tracker problem*.

Necessary and sufficient conditions for synchronization are given by master stability functions [20,24,33] and the related concept of synchronizing regions [4,24–26]. This guarantees local stability. For linear systems, however, local and global stability coincide; hence the synchronizing region approach yields global results. Synchronizing region and pinning control papers often *a priori* assume inner coupling functions having special properties [2,4,12,24,26], thereby disregarding the controllability properties inherent to single-agents.

Global results for nonlinear systems are generally obtained by Lyapunov methods [27,30,32] or contraction analysis, *i.e.* incremental stability [13,18,22]. Especially interesting for interconnected systems are the results involving incremental stability [13,22,33,34] and incremental passivity [19]. Often Lyapunov methods in the literature either assume certain *a priori* forms of the drift dynamics and inner coupling functions or restrict their considerations to undirected or balanced graphs. For example, special drift dynamics (QUAD) is assumed in [2,12] to guarantee a quadratic bound on the pertaining contribution to dissipation, and the distributed control is taken as all-state direct feedback in [2,23,31,32] to completely dominate the bounded effect of the drift dynamics. Other special properties of the inner coupling function are assumed in [12,26,31,32,43]; *e.g.* diagonal form [12,26,43], positive definiteness [32], or positive definite contribution to dissipation [31]. In [40] vector double integrator agents are considered and the underlying graph topologies, although allowed to be switching, are assumed undirected. The approach in [40] relies on joint Lyapunov functions. Similarly [41], although considering a different notion of consensus, also assumes undirected graphs and double integrator agents. Consistent with restricting attention to double integrators, the leader's reference signal in both [41,42] is constant. Developments of [41] use Laplacian potentials for undirected graphs [27]. More general Laplacian potentials, in part, motivate the approach of this paper as well. The contraction approach [13,18,32,33,37], in contrast, occupies middle ground between the local linearization results of synchronizing regions and global Lyapunov conclusions, in the sense that linearized dynamics is used but stability requirements hold uniformly, implying global results [13,36]. However, in [32,33] also *a priori* assumptions on inner coupling functions are made without considering how to guarantee them for a given system.

Any *a priori* conditions on inner coupling functions disregard the given controllability properties of single-agents. Realistic systems are characterized by their controllability structure and possibly the constraints of output-feedback. This restricts the feasible distributed controls, and must be accounted for in the control design. Furthermore, *a priori* choices of inner coupling matrix and drift dynamics appear somewhat artificial in light of the fact that it is possible, within the synchronizing region approach, to obtain the required properties by design. For example, in [8,11,25,27] the inner coupling matrix is designed by considering the given single-agent systems. The resulting feedback interplays with the drift dynamics to guarantee cooperative stability. Namely, for linear time-invariant (LTI) agent synchronization [11,25,27] use single-agent optimal feedback derived from algebraic Riccati equations (ARE). Such control guarantees an unbounded right-half plane synchronizing region. Hence, state synchronization is achieved under mild requirements on directed communication topology, utilizing given stabilizability properties of individual agents rather than imposing them *a priori* by assumptions. Apart from accounting for the controllability relations, synchronizing region approach [20,25,26,28] also treats cases of output-feedback on similar grounds [9]. Algebraic matrix equations give single-agent Lyapunov functions [8,9,25,27] that provide a guaranteed synchronizing region. Control design is thus based on single-agent systems and graph effects are dealt with through robustness of the stabilizing feedback gains. Therefore, *a priori* assumptions on single-agents, often found in Lyapunov and contraction approaches, are certainly at odds with the versatility of synchronizing region results. A similar remark holds for restricting Lyapunov analysis to undirected graphs [40–42] while the synchronizing region approach requires only a spanning tree.

This paper aims to provide a generalization of the synchronizing region approach which would apply to affine-in-control systems globally and be as versatile as the synchronizing region methods in the sense that it would treat both state and output-feedback similarly, would not require *a priori* assumptions on the inner coupling functions and would permit general directed graphs with a spanning tree. For this purpose we undertake a systematic construction of structured cooperative Lyapunov functions and design controls in relation to these functions. Single-agent Lyapunov functions, familiar from the synchronizing region approach, hint at the appropriate structure of Lyapunov functions for cooperative stability. We are motivated to remove *a priori* assumptions not appearing in the synchronizing region approach, or rather to guarantee the required properties by design. That is, we are emulating the synchronizing region approach in our Lyapunov analysis to account for the given single-agent structure. Moreover, while Lyapunov results in the literature consider mainly pinning control or irreducible graphs [3,27,29], the synchronizing regions treat in a unified way all graphs containing a spanning tree [11,28], which is indeed a necessary prerequisite for synchronization. Our goal here is to achieve the same level of generality by Lyapunov methods. This in turn opens new ways to address robust and adaptive cooperative control problems. Lyapunov functions presented here account systematically for single-agent controllability and output-feedback restrictions. These structured functions are applicable both to linear and nonlinear systems, yielding global stability results. Understood as sufficient cooperative stability conditions, it is shown that the synchronizing region and the proposed Lyapunov approaches can be used for linear systems equivalently, while for nonlinear systems Lyapunov functions offer global conclusions. Stronger assumptions on drift dynamics allow for weaker assumptions on distributed control. In contrast to [40–42] this paper considers affine-in-control systems and requires constant, but otherwise general, directed graphs with a spanning tree. Our structured Lyapunov functions can be seen as a systematic generalization of those appearing in [40,42] to directed graphs with a spanning tree.

First contribution of this paper is in revealing the appropriate structure of Lyapunov functions for cooperative stability. The proposed functions are found to depend separately on network topology and single-agents. These Lyapunov functions are subsequently applied to cooperative regulator and tracker problems for affine-in-control systems yielding cooperative control design prescriptions which generalize the algebraic Riccati equations of the synchronizing region approach [9,25,27]. The resulting distributed controls account for controllability properties of single-agents and interplay with the drift dynamics to achieve synchronization. The second contribution is in bringing a Lyapunov treatment of reducible graphs containing a spanning tree without an isolated leader. To the best knowledge of the authors the literature mainly brings Lyapunov analysis of either irreducible graphs [3,27,43] or pinning control with an isolated leader [24,26,32]. Based on [31] reducible graphs with a spanning tree, but without an isolated leader, are treated here as hybrids between irreducible graph cooperative regulator and single leader cooperative tracker. This should be contrasted with the treatment of reducible graphs in [43] which is based solely on analysis of the graph's irreducible components. Thus all cases naturally covered by the synchronizing regions are addressed here by Lyapunov analysis. The third contribution is an introduction of a novel special class of affine-in-control systems characterized by their contraction properties. The defining contraction properties are required for the presented cooperative stability Lyapunov theorems. These properties are stronger than the conventional QUAD [12,32,43], and allow for an interplay of single-agent drift dynamics with distributed linear feedbacks, similarly as in synchronizing region approach for LTI systems [9,27]. A sufficient condition is given for the required contraction properties, avoiding the need for *a priori* assumptions on distributed feedback, as *e.g*. those in [2,4,12,24,26,31–33], and generalizing algebraic matrix equations from the synchronizing region approach to affine-in-control systems.

The layout of the paper is as follows. Section 2 gives graph preliminaries and notational conventions. Section 3 introduces the considered systems and the control goals. A proposition is given bringing differential geometric considerations highlighting the properties of considered systems. Section 4 outlines the synchronizing region approach and develops the structured Lyapunov functions for linear systems. Separate Lyapunov constructions are needed for different types of interconnection graph topologies. First is considered the cooperative tracker with an isolated leader, then a cooperative regulator on irreducible graphs, followed by their combination appropriate for general reducible graphs containing a spanning tree. Section 5 applies similar Lyapunov functions to a special class of affine-in-control systems along the lines of Section 4. In summary, Theorems 2 and 5 require isolated leaders, Theorems 3 and 6 require strongly connected graphs and Theorems 4 and 7 require only spanning trees. Section 6 presents a numerical example validating the proposed designs, cooperative stability conditions and Lyapunov functions. Section 7 concludes the paper.

2. Graph theory preliminaries and notational conventions

Consider a graph $\mathcal{G}=(\mathcal{V},\mathscr{E})$ with a nonempty finite set of N nodes $\mathcal{V}=\{v_1,\cdots,v_N\}$ and a set of edges $\mathscr{E}\subseteq\mathcal{V}\times\mathcal{V}$. It is assumed that the graph is simple, *i.e.* there are no repeated edges or self-loops. Directed graphs are considered, and information propagates through the graph along the edges. Two nodes v_j, v_k connected by an edge $(v_j,v_k)\in\mathscr{E}$ are termed *parent* node v_k and *child* node v_j, *i.e.* the edge leaves the parent node and connects into the child node. Denote the adjacency matrix as $E=\left[e_{ij}\right]$ with $e_{ij}>0$ if $(v_i,v_j)\in\mathscr{E}$ and $e_{ij}=0$ otherwise. Note that diagonal elements satisfy $e_{ii}=0$. The set of neighbors of a node v_i is $\mathcal{N}_i=\{v_j:(v_i,v_j)\in\mathscr{E}\}$, *i.e.* the set of nodes with edges connecting into v_i. Define the (weighted) *in-degree* matrix as a diagonal matrix $D=diag(d_1\ldots d_N)$, where $d_i=\sum_j e_{ij}$ is the (weighted) in-degree of a node i, *i.e.* a row sum od E. The weighted *out-degree* is defined as $d_i^o=\sum_j e_{ji}$, *i.e.* a column sum of E. Define the graph Laplacian matrix as $L=D-E$, which has all row sums equal to zero. A graph is *balanced* if in-degrees of all the nodes equal their out-degrees. A graph is *detailed balanced* if there exists a positive diagonal matrix which premultiplying its Laplacian results in a symmetric matrix [7]. A *directed path* is a sequence of edges joining two nodes. A graph is said to be *strongly connected* if any two nodes can be joined by a directed path. A node is termed *isolated* if it has no incoming edges. Hence, in strongly connected graphs there are no isolated nodes. A *directed tree* is a subgraph having a single isolated node v_0, such that all other nodes except v_0 have only one parent and are joined to v_0 by a directed path. Node v_0 is called a *root node*. A graph is said to contain a directed *spanning tree* if there exists a directed tree containing all nodes in the graph. The Laplacian matrix L has a simple zero eigenvalue if and only if its directed graph contains a spanning tree. A graph is *quasi-strongly connected* if for every pair of nodes there exists a distinct node such that there is a directed path from it to both nodes of the original pair. If the graph has a root node then it is quasi-strongly connected. Therefore a spanning tree implies quasi-strong connectivity.

Spanning forest is a set of directed trees such that a set of all nodes of these trees equals $\mathcal{V}$. A graph is said to be *reducible* if its Laplacian matrix is cogredient, *i.e.* it can be transformed by permuting the nodes, to the block triangular form

$$T^T LT = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix}, \tag{1}$$

where T is a permutation matrix. If the graph is not reducible it is said to be *irreducible*. A directed graph is irreducible if and only if it is strongly connected. Let the directed graph be reducible and let it contain a spanning forest. Then the Laplacian of the graph can be reduced by node permutation to the Frobenius normal form [31]. If the graph contains a single spanning tree the Frobenius normal form equals

$$T^T LT = \begin{bmatrix} \overbrace{L_{11} \quad \cdots \quad L_{1m}}^{\tilde{L}+\tilde{G}} & L_{1m+1} \\ \quad\quad \ddots \quad\quad \vdots & \vdots \\ 0 \quad\quad\quad L_{mm} & L_{mm+1} \\ 0 & L_{m+1m+1} \end{bmatrix} \tag{2}$$

where all L_{ii} blocks are irreducible. This paper is concerned with graphs having a single spanning tree. Such graphs can either be irreducible or have a single leader group, L_{m+1m+1}. An important special case of a leader group is a single isolated leader. Note that due to existence of a spanning tree there can be at most one irreducible leader group, or in particular, at most one isolated leader.

Undirected graphs present a special simpler case of balanced graphs. Those are balanced by definition and irreducible if and only if connected. Undirected graph contains a spanning tree if and only if it is connected. As a special case they are not specifically addressed in this paper, rather the results presented for irreducible graphs naturally specialize to connected undirected graphs.

The following lemma on singular and nonsingular M-matrices [29], is useful in constructing Lyapunov functions for cooperative control [27]. This well-known result is given here, without proof, for the sake of completeness

Lemma 1. [29] *For strongly connected, irreducible, graphs there exists a positive diagonal matrix* $\Theta = diag(\vartheta_1 \ldots \vartheta_N) > 0$ *such that the graph Laplacian matrix L satisfies*

$$L^T \Theta + \Theta L \geq 0. \tag{3}$$

The diagonal elements ϑ_i are the components of the Laplacian's left eigenvector for the eigenvalue 0, $\vartheta^T = [\vartheta_1 \ldots \vartheta_N]$, $\vartheta^T L = 0$. If the graph contains a spanning tree with a nonzero g_i for a root node v_i, then there exist a positive diagonal matrix Ξ such that

$$(L+G)^T \Xi + \Xi(L+G) > 0, \tag{4}$$

where $G = diag(g_1 \ldots g_N) \geq 0$ is a diagonal matrix.

Proof. The first part of the proof is found in [29], the second part follows straightforwardly from [29] since $L+G$ is a nonsingular M-matrix [27]. □

Lemma 2. [7] *For a positive semidefinite symmetric matrix, $M = M^T \geq 0$, given a vector of appropriate dimension v, $v^T M v = 0 \Leftrightarrow Mv = 0$.*

Although Lemma 2 seems trivial it does not hold in general for asymmetric or indefinite matrices. Complex conjugation of a scalar $\sigma \in \mathbb{C}$ is denoted by $\overline{\sigma} \in \mathbb{C}$, and spectrum of a matrix A by $spec(A)$. The symbol $\underline{1}_N$ stands for a column vector of ones $[1...1]^T$, the *consensus vector*, of dimension N.

3. System description

3.1. System dynamics

Let the multi-agent system comprise N identical agents whose dynamics is given as

$$\begin{aligned} \dot{x}_i &= f(t, x_i) + Bu_i, \\ y_i &= Cx_i, \end{aligned} \tag{5}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$. The term $f(t, x_i)$ is referred to as the *drift term*. Systems (5) are a special case of affine-in-control systems, with linear input B and output C relations. It is assumed that B and C have full column and row rank respectively, implying no redundancy in controls or measurements. The form of single-agents (5) is not overly restrictive, as detailed in the following proposition.

Proposition 1. *An affine-in-control dynamical system*

$$\dot{x} = f(t, x) + g(x)u \tag{6}$$

can be transformed locally to dynamics

$$\dot{z} = \tilde{f}(t, z) + Bu \tag{7}$$

if and only if the columns of $n \times m$ matrix $g(x)$, $g_k(x), k = 1 \ldots m$, construed as vector fields on $\mathbb{R}^n$, satisfy

i. $\{g_1(x), ..., g_m(x)\}$ are linearly independent vector fields
ii. $[g_j(x), g_k(x)] \equiv 0$, $j, k = 1...m$, where [,] denotes a Lie bracket of vector fields.

Furthermore, a transforming diffeomorphism can be constructed as

$$x = T(z) = \Phi_{z_1}^{g_1} \circ \Phi_{z_2}^{g_2} \circ \cdots \circ \Phi_{z_m}^{g_m} \circ \Phi_{z_{m+1}}^{h_{m+1}} \circ \cdots \Phi_{z_n}^{h_n}(x_0), \tag{8}$$

where $h_{m+1} \ldots h_n$ are any smooth vector fields that satisfy $rank\{g_1(x), \ldots, g_m(x), h_{m+1}(x), \ldots, h_n(x)\} = n$, and $\Phi_t^{g_i}(x_0)$ are flows of the vector fields $g_i(x)$, with $t = z_i$ respectively.

Proof. First note that Eq. (6) is transformed to Eq. (7) by a diffeomorphism $x = T(z)$ if and only if

$$\left[\frac{\partial T(z)}{\partial z}\right]^{-1} g(T(z)) \equiv B\,. \tag{9}$$

Therefore one has to prove that Eq. (9) holds if and only if assertions (i) and (ii) of the Proposition statement hold.

Sufficiency: It is well known [39], that

$$\Phi_{z_j}^{g_j} \circ \Phi_{z_k}^{g_k} = \Phi_{z_k}^{g_k} \circ \Phi_{z_j}^{g_j}, \forall z_j, z_k \Leftrightarrow \left[g_j(x), g_k(x)\right] \equiv 0.$$

Therefore by Eq. (8) and (*ii*) one has for $k = 1\ldots m$

$$x = T(z) = \Phi_{z_k}^{g_k} \circ \Phi_{z_1}^{g_1} \circ \cdots \circ \Phi_{z_{k-1}}^{g_{k-1}} \circ \Phi_{z_{k+1}}^{g_{k+1}} \circ \ldots \circ \Phi_{z_m}^{g_m} \circ \Phi_{z_{m+1}}^{h_{m+1}} \circ \cdots \Phi_{z_n}^{h_n}(x_0).$$

As a consequence, by definition of the flow,

$$\frac{\partial T(z)}{\partial z_k} = g_k(T(z)), \; k = 1\ldots m.$$

In other words, by matrix inverse one finds

$$\left[\frac{\partial T(z)}{\partial z}\right]^{-1} g_k(T(z)) = \left[\frac{\partial T(z)}{\partial z}\right]^{-1} \frac{\partial T(z)}{\partial z_k} = w_k, \quad k = 1\ldots m,$$

where w_k is a k-th canonical basis vector of $\mathbb{R}^n$. Whence it follows that $\left[\frac{\partial T(z)}{\partial z}\right]^{-1} g(T(z)) \equiv B$, with $B = [w_1 \ldots w_m]$. This proves sufficiency.

Necessity: Assume that Eq. (6) is transformed by $x = T(z)$ into the form (7). Then

$$w_j = \left[\frac{\partial T(z)}{\partial z}\right]^{-1} g_j(T(z)), \quad w_k = \left[\frac{\partial T(z)}{\partial z}\right]^{-1} g_k(T(z)),$$

where $B = [w_1 \ldots w_m]$. Constant column vectors in B, $w_1, \ldots, w_m$, are linearly independent by definition and their Lie brackets vanish identically, $\left[w_j, w_k\right] \equiv 0$. Hence by coordinate invariance the vector fields $g_1, \ldots, g_m$ are linearly independent, *i*), and their Lie brackets vanish identically $\left[g_j, g_k\right] \equiv 0$, *ii*). This proves necessity, thus completing the proof of the Proposition. □

Moreover, the results of Proposition 1 hold globally, in the sense of [38], if the vector fields $\left\{g_1(x), \ldots, g_m(x), h_{m+1}(x), \ldots, h_n(x)\right\}$ are complete and the image of $\Phi_{z_1}^{g_1} \circ \Phi_{z_2}^{g_2} \circ \cdots \circ \Phi_{z_m}^{g_m} \circ \Phi_{z_{m+1}}^{h_{m+1}} \circ \cdots \Phi_{z_n}^{h_n}(x_0)$ is simply connected. As for linear output relation in Eq. (5), it holds provided outputs y_i for Eq. (6) in terms of x_i are chosen as $y_i = CT^{-1}(x_i)$. Transformation (8) is not unique, in fact Eq. (8) composed with any linear transformation serves the same purpose.

Cooperative stability in transformed z coordinates, pertaining to Proposition 1, is equivalent to cooperative stability in the original x coordinates if $T(z)$ satisfies a uniform bound

$$\underline{\alpha}(\|z_1 - z_2\|) \leq \|T(z_1) - T(z_2)\| \leq \overline{\alpha}(\|z_1 - z_2\|). \tag{10}$$

for some $\mathcal{K}-$ class functions $\underline{\alpha}, \overline{\alpha}$. With the bound (10) a coordinate transform T preserves the sense of total state's distance from the consensus manifold, and asymptotic partial stability of a total multi-agent system [35].

3.2. Isolated and collective leaders

The *leader* is an agent at the isolated root of a spanning tree or an external reference generator pinning to root nodes of all trees in a spanning forest [31]. In the latter instance, the augmented graph containing the external leader as a node reduces to the former case. Leader's system is taken as

$$\begin{aligned} \dot{x}_0 &= f(t, x_0), \\ y_0 &= Cx_0. \end{aligned} \tag{11}$$

Irreducible graphs have no isolated nodes, hence no leaders in the sense of Eq. (11). The graph Laplacian L of an irreducible graph has the left eigenvector for eigenvalue 0, $p^T L = 0$, $p = [p_1 \cdots p_N]$, $p_i > 0,\ \forall i$ [27,29,43]. This positive vector p is used to define a *collective leader* state as a weighted average of all the agents' states [43],

$$x^* = \left(\sum_i p_i\right)^{-1} \sum_i p_i x_i. \tag{12}$$

At consensus one has $\forall (i,j)\ \ x_i = x_j = x^*$. The collective leader (12), as opposed to an isolated leader (11), depends on all agents and thereby presents centralized information. Its state, therefore, serves for analysis only and is not used directly as a reference signal for distributed control. Note that in balanced, and in particular undirected, graphs the collective leader's state (12) is an average of states of all agents. For reducible graphs the corresponding left eigenvector satisfies $p_i \geq 0,\ \forall i$ and there necessarily exists an i such that $p_i = 0$ [43], hence Eq. (12) cannot be used in the same way to describe the collective dynamics on such graphs.

3.3. Synchronization errors and control problems

For cooperative stability analysis one defines synchronization errors in terms of isolated and collective leaders

$$\delta_{1i} = x_i - x_0, \tag{13}$$

$$\delta_{2i} = x_i - x^*, \tag{14}$$

respectively. Consequently, one distinguishes between the *cooperative tracker*, *i.e. synchronization*, and the *cooperative regulator*, *i.e. consensus* problem. In the former the asymptotic final states of all agents equal the state of the reference generator $x_0(t)$, while in the latter their asymptotic final states $x^*(t)$ depend on precise initial conditions of an entire system. Hence, from dynamical systems' point of view the difference is that in cooperative tracker one has an isolated leader, either as an external reference generator or an isolated root node, which determines the asymptotic behavior of an entire system.

The control goal in cooperative tracker problem is to find distributed controls for all agents u_i, such that $\|x_i - x_0\| \to 0$ as $t \to \infty,\ \forall i$. The control goal in cooperative regulator problem is to find distributed controls for all agents u_i, such that $\|x_i - x_j\| \to 0$ as $t \to \infty,\ \ \forall i,j$, or equivalently $\|x_i - x^*\| \to 0,\ \ \forall i$. These two convergences are equivalent since $x_i - x^* = \ x_i - \left(\sum_j p_j\right)^{-1} \sum_j p_j x_j = \left(\sum_j p_j\right)^{-1} \sum_j p_j (x_i - x_j)$, hence $\|x_i - x_j\| \to 0\ \forall (i,j) \Rightarrow \|x_i - x^*\| \to 0,\ \ \forall i$. The reverse implication follows by Cauchy-Schwarz, $\|x_i - x_j\| \leq \|x_i - x^*\| + \|x_j - x^*\|$.

Synchronization errors δ_{2i} (14) have a crucial property with respect to Laplacian's left eigenvector p [43],

$$\sum_i p_i \delta_{2i} = \sum_i p_i x_i - \left(\sum_i p_i\right)\left(\sum_i p_i\right)^{-1} \sum_i p_i x_i = 0. \tag{15}$$

Constraint (15) implies that $\forall (i,j)\ \ \delta_{2i} = \delta_{2j} \Rightarrow \delta_{2i} = 0\ \ \forall i$, *i.e.* in $\delta_2 -$consensus all δ_{2i} s vanish identically.

3.4. Local neighborhood errors and the form of distributed controls

Define the local neighborhood state error in case of an isolated leader, *i.e.* cooperative tracker, as

$$\varepsilon_{1i} = \sum_j e_{ij}(x_j - x_i) + g_i(x_0 - x_i), \tag{16}$$

where $g_i \geq 0$ are the pinning gains, nonzero only for a small fraction of nodes having a direct connection to the leader. For cooperative regulator, without an isolated leader, define the local neighborhood error as

$$\varepsilon_{2i} = \sum_j e_{ij}(x_j - x_i). \tag{17}$$

Remark 1. If an external leader pins into all roots of a spanning forest the augmented graph containing such a leader as an isolated root node of a spanning tree is not strongly connected. This spanning tree in an augmented graph is necessary for synchronization [11,27,31]. Augmented graph Laplacian then equals

$$\begin{bmatrix} L+G & -\overline{g} \\ 0 & 0 \end{bmatrix}, \tag{18}$$

where $\overline{g} = vec(g_1 \ldots g_N)$ is a vector of pinning gains, and $G = diag(g_1, \ldots, g_N)$ is a matrix of pinning gains. Comparing Eq. (18) with Eq. (2), if an irreducible leader group is a singleton, then its associated Laplacian matrix block L_{m+1m+1} reduces to a 1×1 zero matrix as in Eq. (18).

If neighbors' full states are available both for cooperative tracker, labeled by subscript 1, and regulator, labeled by subscript 2, then the distributed feedback is chosen as a linear state-feedback with gain matrix K, which is designed later, as

$$u_i = cK\varepsilon_{1,2i}, \tag{19}$$

where $c > 0$ is a coupling gain detailed first in Section 4.1. In total state-space form, $x = \left[x_1^T \cdots x_N^T\right]^T \in \mathbb{R}^{Nn}$, $\overline{x}_0 = \left[x_0^T \cdots x_0^T\right]^T \in \mathbb{R}^{Nn}$. With this notational convention applied to all vectors, the local neighborhood state errors (16) and (17) in the total form are

$$\varepsilon_1 = -(L+G) \otimes I_n(x - \overline{x}_0) = -(L+G) \otimes I_n\delta_1, \tag{20}$$

$$\varepsilon_2 = -L \otimes I_n x = -L \otimes I_n \delta_2, \tag{21}$$

for cooperative tracker and regulator respectively. Note that the ε_2 vector (21) also satisfies the constraint (15) of δ_2, that $\sum_i p_i \varepsilon_{2i} = 0$, forcing $\varepsilon_{2i} = \varepsilon_{2j}\ \forall (i,j) \Rightarrow \varepsilon_{2i} = 0,\ \forall i$.

If only neighbors' outputs are available for distributed feedback control, the local neighborhood output errors are used, $\varepsilon_{1yi} = C\varepsilon_{1i}$, $\varepsilon_{2yi} = C\varepsilon_{2i}$. The control is then chosen as a distributed output-feedback, subscripts 1 and 2 denoting cooperative tracker and regulator respectively,

$$u_i = cKC\varepsilon_{1,2i}. \tag{22}$$

Total form of local neighborhood output errors $\varepsilon_{y1,2}$ differs from their full-state counterparts (20) and (21) by matrix C instead of I_n appearing in the Kronecker product with the graph matrix [9].

With local neighborhood output errors in distributed feedback the single-agent closed-loop systems equal

$$\dot{x}_i = f(t,x_i) + cBKC\varepsilon_{1,2i}, \tag{23}$$

for cooperative tracker and regulator respectively. Linear distributed feedback (22) in cooperative regulators on strongly connected graphs leads to the collective leader dynamics

$$\dot{x}^* = \left(\sum_i p_i\right)^{-1} \sum_i p_i \dot{x}_i. \tag{24}$$

The sum $\sum_i p_i \dot{x}_i$ in Eq. (24) equals $\sum_i p_i \dot{x}_i = \sum_i p_i f(t,x_i) + cBKC \sum_i p_i \varepsilon_{2i} = \sum_i p_i f(t,x_i)$, from where one has

$$\dot{x}^* = \left(\sum_i p_i\right)^{-1} \sum_i p_i f(t,x_i). \tag{25}$$

Since generally $\left(\sum_i p_i\right)^{-1} \sum_i p_i f(t,x_i) \neq f(t,x^*)$, the collective leader, contrary to the isolated leader (11), does not follow the single-agent drift dynamics f. However, on synchronization manifold $\forall (i,j)\ \ x_i = x_j = x^*$ and one still has $\dot{x}^* = f(t,x^*)$.

4. Linear systems-synchronizing region and Lyapunov approach

4.1. LTI systems dynamics

Assume that the single-agent drift dynamics is a constant linear function of agent's states, $f(t,x_i) = Ax_i$. The system dynamics in total form is then given as

$$\dot{x} = (I_N \otimes A)x - c(L+G) \otimes BKC\delta_1, \tag{26}$$

$$\dot{x} = (I_N \otimes A - cL \otimes BKC)x, \tag{27}$$

for cooperative tracker and regulator respectively, revealing the Kronecker product structure. As a consequence of the linear drift dynamics, the collective leader (12) in this special case follows the single-agent drift dynamics,

$$\dot{x}^* = \left(\sum_j p_j\right)^{-1} \sum_j p_j A x_j = Ax^*. \tag{28}$$

Hence for LTI agents the actual and the collective leader x_0, x^* behave in the same way. This yields similar total state-space forms for respective closed-loop synchronization error dynamics,

$$\dot{\delta}_1 = (I_N \otimes A - c(L+G) \otimes BKC)\delta_1, \tag{29}$$

$$\dot{\delta}_2 = (I_N \otimes A - cL \otimes BKC)\delta_2. \tag{30}$$

Cooperative tracker and regulator are equivalent to asymptotic stability of synchronization errors, $\delta_{1,2} \to 0$. Dynamics of the collective leader (28) provides a straightforward dependence of asymptotically synchronized state on the initial conditions of the cooperative regulator problem [23].

Autonomous systems (29) and (30) are analyzed both by synchronizing regions and Lyapunov methods. The following sections present an outline of the synchronizing region approach, for general distributed output-feedback [9], followed by the construction of structured Lyapunov functions guaranteeing cooperative stability for LTI cooperative tracker and regulator problems under the same set of conditions.

4.2. Synchronizing region approach

This section outlines the versatile synchronizing region approach as it applies to linear systems. The Lyapunov results introduced subsequently are compared against its desirable features. Although this approach is certainly applicable locally to nonlinear systems as well, this is not the point we wish to make here. Rather, the main point we emphasize is the control design based on single-agent algebraic Riccati equations, and the associated single-agent Lyapunov functions.

For autonomous error systems (29) and (30) a synchronizing region approach relies on robust stabilizability of single-agent systems to account for the effects of distributed communication. The synchronizing region approach uses a structured total state transformation given by matrix $T \otimes I_n$, such that $T^{-1}(L+G)T = \Lambda$ for cooperative tracker, or $T^{-1}LT = \Lambda$ for cooperative regulator, where Λ is a triangular matrix. The transformed system matrix in Eqs. (29) and (30) then equals

$$I_N \otimes A - c\Lambda \otimes BKC, \tag{31}$$

and the overall system's stability is determined by the stability of diagonal blocks, $\lambda_j = \Lambda_{jj}$,

$$A - c\lambda_j BKC, \tag{32}$$

$\forall \lambda_j$ for the cooperative tracker, and $\forall \lambda_j \neq 0$ for the cooperative regulator. For cooperative regulator, stability of diagonal blocks (32) $\forall \lambda_j \neq 0$ describes the stability with respect to $\lambda = 0$ invariant subspace of Eq. (27), which, under the existence of a spanning tree, is precisely the synchronization manifold. For cooperative tracker, stability of diagonal blocks $\forall \lambda_j$ describes the stability of origin in Eq. (29). This robust stability for all λ_j in $spec(L)/\{0\}$ or $spec(L+G)$ is addressed through the synchronizing region in the complex plane [7,20].

Definition 1. A synchronizing region of the matrix pencil

$$A - \sigma BKC \tag{33}$$

is a subset of the complex plane $S = \{\sigma \in \mathbb{C} : A - \sigma BKC \ \ is \ \ Hurwitz\}$.

Hence the total system (29) or (30) synchronizes if and only if there exists a coupling gain $c > 0$ such that the relevant eigenvalues satisfy $\forall i : c\lambda_i \in S$. One way of assessing the synchronizing region is provided by quadratic Lyapunov functions determined by real matrices, $P = P^T > 0$ [9,27]. The Lyapunov stability condition,

$$(A - \sigma BKC)^{\dagger} P + P(A - \sigma BKC) < 0, \tag{34}$$

guarantees a synchronizing region [9]. The following theorem brings the pertaining design of feedback gain K.

Theorem 1. [9] *Let the graph contain a spanning tree, for the cooperative regulator, and let the graph contain a spanning tree with pinning into a root node, for the cooperative tracker. Let the system be given as Eq.* (29) *or Eq.* (30). *Let the single-agent feedback gain K satisfy*

$$KC = R^{-1}(B^T P + M), \tag{35}$$

for some M, where $R = R^T > 0$ and matrix P is a solution of the algebraic matrix equation

$$A^T P + PA + Q - PBR^{-1}B^T P + M^T R^{-1} M = 0. \tag{36}$$

with design matrices $Q = Q^T > 0$ and M. Let M further satisfy

$$PBR^{-1}B^TP - M^TR^{-1}M \geq 0. \tag{37}$$

Then the single-agent feedback gain (35) guarantees an unbounded conical sector synchronizing region. This ensures synchronization for a coupling gain $c > 0$ sufficiently large if arguments of the relevant graph matrix eigenvalues, $\arg\lambda_j$, are bounded by the angle of the cone.

Remark 2. The simple zero eigenvalue of L in the cooperative regulator and the nonsingular $L + G$ in the cooperative tracker are necessary for synchronization. These are guaranteed by a spanning tree in the former and the existence of a spanning tree with pinning to a root node in the latter. Also note that BKC, which amounts to the inner coupling matrix, is here designed with respect to (A, B, C) instead of being *a priori* assumed.

In the case of full-state feedback $C = I_n$ and $M = 0$, Eq. (36) reduces to the conventional ARE [25,27].

Corollary 1. [9] *Let the conditions of* Theorem 1 *be satisfied. If* $C = I_n$ *the choice* $M = 0$ *gives the local feedback gain*

$$K = R^{-1}B^TP, \tag{38}$$

where $P > 0$ is a solution of the algebraic Riccati equation

$$A^TP + PA + Q - PBR^{-1}B^TP = 0. \tag{39}$$

This guarantees an unbounded half-plane synchronizing region, $\mathrm{Re}\sigma > 1/2$, and allows for synchronization under sufficiently large coupling gains $c > 0$ for all graphs containing a spanning tree.

Remark 3. The byproduct of Theorem 1 and Corollary 1 is a single-agent Lyapunov function kernel P guaranteeing robust stability of the closed-loop single-agent system with the optimal feedback gain (35) or (38), providing in turn a guaranteed synchronizing region in the complex plane. This single-agent Lyapunov approach is not appropriate for nonlinear systems globally, though it remains applicable locally [20]. Note the judicious choice of P, and consequently K, that makes the drift dynamics stabilize the error system (29) or (30) whenever the control vanishes outside the synchronization manifold. Furthermore, in Eq. (35) the matrix M provides an additional design freedom needed for output-feedback stabilization [6,9]. This circumvents the restrictive conditions on output-feedback in [14] and the requirement of G-passivity in [30].

4.3. Lyapunov approach

To extend favorable aspects of the synchronizing region approach to nonlinear systems globally the control problems of Section 3 are revisited here *via* Lyapunov analysis with structured Lyapunov functions,

$$V = \delta_1^T(P_1 \otimes P_2)\delta_1, \quad V = \delta_2^T(P_1 \otimes P_2)\delta_2,$$

for cooperative tracker 1 and regulator 2, respectively. Matrices P_1 and P_2 are symmetric positive definite matrices, P_1 depending on the graph topology and P_2 depending on the single-agent systems.

The following sections bring Lyapunov cooperative stability conditions. Presented Lyapunov constructions particularly depend on graph topology types. This motivates separate elaboration of Lyapunov results as they apply to distinct types of graphs, which are all naturally covered by the synchronizing region approach in a unified way [11,28]. One single Lyapunov construction here, with all its peculiarities, does not apply to the most general graph topology. Hence, first is considered a graph containing a spanning tree with an isolated leader at the root node. This is equivalent to pinning with an external leader to all roots of a spanning forest and this graph is not strongly connected, Remark 1. Secondly a strongly connected, *i.e.* irreducible, graph is considered using Lemma 1, and finally a non-strongly connected, *i.e.* reducible, graph containing a spanning tree, but without an isolated leader, is presented as a hybrid of the preceding two simpler topologies.

4.3.1. Cooperative tracker

This section considers an external leader pinning to root nodes of all trees in a spanning forest. An isolated node at the root of a spanning tree reduces to this form if it is considered excluded from the graph, Remark 1. Due to the spanning tree there can be at most one such isolated leader.

Theorem 2. *Let the graph contain a spanning forest with a leader pinning into root nodes of all trees. Let the total error dynamics be given as*

$$\dot{\delta}_1 = (I_N \otimes A - c(L+G) \otimes BKC)\delta_1. \tag{40}$$

Let K be chosen to satisfy

$$KC = R^{-1}(B^T P_2 + M), \tag{41}$$

where $R = R^T > 0$ and P_2 solves the algebraic Riccati equation

$$A^T P_2 + P_2 A + Q - P_2 B R^{-1} B^T P_2 + M^T R^{-1} M = 0, \tag{42}$$

with design matrices $Q = Q^T > 0$ and M, where M further satisfies

$$P_2 B R^{-1} B^T P_2 - M^T R^{-1} M \geq 0. \tag{43}$$

Let P_1 be chosen in dependence of M as

(1) $M = 0$: P_1 is chosen such that $P_1(L+G) + (L+G)^T P_1 > 0$
(2) $M \neq 0$: P_1 is chosen such that $P_1(L+G) = (L+G)^T P_1 > 0$

Take the Lyapunov function as

$$V = \delta_1^T (P_1 \otimes P_2)\delta_1. \tag{44}$$

Then for the coupling gain $c > 0$ sufficiently large the structured Lyapunov function (44) guarantees asymptotic cooperative tracking, $\delta_1 \to 0$.

Proof: Given total error dynamics (40), the time-derivative of the Lyapunov function (44) is given by the quadratic form of a matrix

$$P_1 \otimes (P_2 A + A^T P_2) - cP_1(L+G) \otimes P_2 BKC - c(L+G)^T P_1 \otimes C^T K^T B^T P_2. \tag{45}$$

If Eq. (41) holds for P_2 satisfying the algebraic matrix Eq. (42), then Eq. (45) equals

$$P_1 \otimes (P_2A + A^TP_2) - c(P_1(L+G) + (L+G)^TP_1) \otimes P_2BR^{-1}B^TP_2 - cP_1(L+G) \otimes P_2BR^{-1}M - c(L+G)^TP_1 \otimes M^TR^{-1}B^TP_2 \tag{46}$$

(1) For $M=0$ the choice 1 applies, yielding

$$P_1 \otimes (P_2A + A^TP_2) - c(P_1(L+G) + (L+G)^TP_1) \otimes P_2BR^{-1}B^TP_2. \tag{47}$$

Outside the origin of the δ_1 − space the contribution of the second term in Eq. (47),

$$\delta_1^T c(P_1(L+G) + (L+G)^TP_1) \otimes P_2BR^{-1}B^TP_2\delta_1,$$

it being a positive semi-definite symmetric matrix, *c.f.* Lemma 2, is zero if and only if $\forall i$ each δ_{1i} is in the kernel of $P_2BR^{-1}B^TP_2$. Then one has that the contribution of drift dynamics

$$\delta_1^TP_1 \otimes (P_2A + A^TP_2)\delta_1$$

equals, by algebraic matrix Eq. (42),

$$\delta_1^TP_1 \otimes (-Q + P_2BR^{-1}B^TP_2)\delta_1 = -\delta_1^TP_1 \otimes Q\delta_1 < 0.$$

Hence on the whole δ_1-space the time-derivative of the Lyapunov function (44) is negative definite for $c>0$ so large that the negative semidefinite term in Eq. (47) dominates the indefinite first term.

(2) For $M \neq 0$ the last two terms in Eq. (46) are indefinite, and the choice 2 for P_1 applies, leading to $P_1(L+G) = R_1 = R_1^T > 0$ rendering the time-derivative of Lyapunov function (44), given by Eq. (45), equal to

$$\begin{aligned} &P_1 \otimes (P_2A + A^TP_2) - 2cR_1 \otimes P_2BR^{-1}B^TP_2 - cR_1 \otimes (P_2BR^{-1}M + M^TR^{-1}B^TP_2) \\ &\quad = P_1 \otimes (P_2A + A^TP_2) - 2cR_1 \otimes P_2BR^{-1}B^TP_2 - cR_1 \\ &\quad \otimes (C^TK^TRKC - P_2BR^{-1}B^TP_2 - M^TR^{-1}M) \\ &\quad = P_1 \otimes (P_2A + A^TP_2) - cR_1 \otimes (P_2BR^{-1}B^TP_2 - M^TR^{-1}M) - cR_1 \otimes C^TK^TRKC \end{aligned} \tag{48}$$

Since $P_2BR^{-1}B^TP_2 - M^TR^{-1}M \geq 0$, outside the origin of δ_1 − space the time-derivative of the Lyapunov function (44) is negative definite, for $c>0$ so large that the negative semidefinite terms in Eq.(48) dominate the indefinite first term. This guarantees synchronization since the contribution of negative semi-definite terms in Eq. (48),

$$\delta_1^TcR_1 \otimes (P_2BR^{-1}B^TP_2 - M^TR^{-1}M)\delta_1, \;\; \delta_1^TcR_1 \otimes C^TK^TRKC\delta_1,$$

vanishes if and only if $\forall i$ δ_{1i} is in $\ker(P_2BR^{-1}B^TP_2 - M^TR^{-1}M)$ and in $\ker(C^TK^TRKC)$, but then, by algebraic matrix Eq. (42), the contribution of drift dynamics $\delta_1^TP_1 \otimes (P_2A + A^TP_2)\delta_1 = -\delta_1^TP_1 \otimes (Q - P_2BR^{-1}B^TP_2 + M^TR^{-1}M)\delta_1 = -\delta_1^TP_1 \otimes Q\delta_1 < 0$. This concludes the proof. □

4.3.2. Cooperative regulator for irreducible graphs

This section is concerned with the cooperative regulator problem. Graph is assumed irreducible, which facilitates construction of structured Lyapunov functions, using Lemma 1. An auxiliary lemma is required for the subsequent development.

Lemma 3. *If* $\sum_i p_i \delta_{2i} = 0$ *with* $p_i > 0$, $\forall i$ *then for a matrix* L *having a simple zero eigenvalue with the eigenvector* $\underline{1}$, $L \otimes M\delta_2 = 0 \Leftrightarrow \delta_{2i} \in \ker M\ \forall i$.

Proof. The reverse implication $\Leftarrow$ is obvious. To prove the $\Rightarrow$ implication assume $L \otimes M\delta_2 = 0$ but $\delta_{2i} \notin \ker M$ for at least one i. With a one dimensional kernel of L, $span(\underline{1})$, this generally means $\delta_{2i} = \alpha + v_i$ where $v_i \in \ker M, \forall i$, but $\alpha \notin \ker M$. Hence $\delta_{2i} \notin \ker M, \forall i$. Then, since $\sum_i p_i \delta_{2i} = 0$, one has $\alpha \sum_i p_i + \sum_i p_i v_i = 0$ whence, with $\sum_i p_i > 0$, it follows that $\alpha \in span(v_1 \ldots v_N)$, so $\alpha \in \ker M$, implying $\delta_{2i} \in \ker M, \forall i$; a contradiction to the original assumption. Hence δ_{2i} is in $\ker M, \forall i$, concluding the proof. □

Theorem 3. *Let the graph be strongly connected. Let the total error dynamics be given as*

$$\dot{\delta}_2 = (I_N \otimes A - cL \otimes BKC)\delta_2. \tag{49}$$

Let K be chosen to satisfy

$$KC = R^{-1}(B^T P_2 + M), \tag{50}$$

where $R = R^T > 0$ and P_2 solves

$$A^T P_2 + P_2 A + Q - P_2 B R^{-1} B^T P_2 + M^T R^{-1} M = 0, \tag{51}$$

with design matrices $Q = Q^T > 0$ and M, where M further satisfies

$$P_2 B R^{-1} B^T P_2 - M^T R^{-1} M \geq 0, \tag{52}$$

Let P_1 is chosen in dependence on M as

(1) $M = 0$: P_1 is chosen such that $P_1 L + L^T P_1 \geq 0$,
(2) $M \neq 0$: P_1 is chosen such that $P_1 L = L^T P_1 \geq 0$.

Take the Lyapunov function as

$$V = \delta_2^T (P_1 \otimes P_2)\delta_2. \tag{53}$$

Then for the coupling gain $c > 0$ sufficiently large the structured Lyapunov function (53) guarantees asymptotic cooperative stability, $\delta_2 \to 0$.

Proof. The proof follows similarly as that for Theorem 2, except with a singular graph matrix L instead of nonsingular $L + G$. Lemma 3 provides, however, a relationship similar to that for nonsingular $L + G$, forcing $\delta_{2i} \in \ker P_2 B R^{-1} B^T P_2, \forall i$ if the control contribution vanishes outside the target set. Given total error dynamics (49), the time-derivative of the Lyapunov function (53) is given by quadratic form of the matrix

$$P_1 \otimes (P_2 A + A^T P_2) - cP_1 L \otimes P_2 BKC - cL^T P_1 \otimes C^T K^T B^T P_2. \tag{54}$$

If Eq. (50) holds for P_2 satisfying Eq. (51), then Eq. (54) equals

$$P_1 \otimes (P_2A + A^T P_2) - c(P_1 L + L^T P_1) \otimes P_2 B R^{-1} B^T P_2 - cP_1 L \otimes P_2 B R^{-1} M - cL^T P_1 \otimes M^T R^{-1} B^T P_2. \tag{55}$$

(1) For $M=0$ the choice 1 applies, yielding

$$P_1 \otimes (P_2A + A^T P_2) - c(P_1 L + L^T P_1) \otimes P_2 B R^{-1} B^T P_2, \tag{56}$$

Outside the $\delta_2 -$ synchronization manifold the contribution of the second term in (56),

$$\delta_2^T c(P_1 L + L^T P_1) \otimes P_2 B R^{-1} B^T P_2 \delta_2,$$

it being a quadratic form of a positive semi-definite symmetric matrix, with $P_1 L + L^T P_1$ having one-dimensional kernel $span(\underline{1})$, (by Lemmas 1, 2), is zero by Lemma 3 if and only if $\forall i\ \delta_{2i}$ is in the kernel of $P_2 B R^{-1} B^T P_2$. Then one has that the contribution of drift dynamics

$$\delta_2^T P_1 \otimes (P_2 A + A^T P_2)\delta_2$$

equals, by algebraic matrix equation (51),

$$\delta_2^T P_1 \otimes (-Q + P_2 B R^{-1} B^T P_2)\delta_2 = -\delta_2^T P_1 \otimes Q\delta_2 < 0,$$

since each δ_{2i} is in the kernel of $P_2 B R^{-1} B^T P_2$. Hence outside the δ_2-synchronization manifold the time-derivative of the Lyapunov function is negative definite for $c>0$ so large that the negative semidefinite term in Eq. (56) dominates the indefinite first term.

(2) For $M \neq 0$ the last two terms in Eq. (55) are indefinite, and the choice 2 for P_1 applies, leading to $P_1 L = R_1 = R_1^T \geq 0$ rendering the time-derivative of Lyapunov function (53), given by Eq. (54), equal to

$$\begin{aligned} &P_1 \otimes (P_2A + A^T P_2) - 2cR_1 \otimes P_2 B R^{-1} B^T P_2 - cR_1 \otimes (P_2 B R^{-1} M + M^T R^{-1} B^T P_2) \\ &= P_1 \otimes (P_2A + A^T P_2) - 2cR_1 \otimes P_2 B R^{-1} B^T P_2 - cR_1 \\ &\quad \otimes (C^T K^T R K C - P_2 B R^{-1} B^T P_2 - M^T R^{-1} M) \\ &= P_1 \otimes (P_2A + A^T P_2) - cR_1 \otimes (P_2 B R^{-1} B^T P_2 - M^T R^{-1} M) - cR_1 \otimes C^T K^T R K C \end{aligned} \tag{57}$$

Since $P_2 B R^{-1} B^T P_2 - M^T R^{-1} M \geq 0$, outside the δ_2-synchronization manifold the time-derivative of the Lyapunov function is negative definite for $c>0$ so large that the negative semidefinite terms in Eq. (57) dominate the indefinite first term. This guarantees synchronization. Namely, the contribution of negative semidefinite terms in Eq. (57),

$$\delta_2^T cR_1 \otimes (P_2 B R^{-1} B^T P_2 - M^T R^{-1} M)\delta_2,\ \ \delta_2^T cR_1 \otimes C^T K^T R K C \delta_2,$$

with $\ker R_1 = span(\underline{1})$, vanishes outside the $\delta_2 -$ synchronization manifold if and only if $\forall i\ \delta_{2i}$ is in $\ker(P_2 B R^{-1} B^T P_2 - M^T R^{-1} M)$ and in $\ker(C^T K^T R K C)$, but then, by the algebraic matrix equation (51), the contribution of drift dynamics satisfies $\delta_2^T P_1 \otimes (P_2 A + A^T P_2)\delta_2 = -\delta_2^T P_1 \otimes (Q + P_2 B R^{-1} B^T P_2 - M^T R^{-1} M)\delta_2 = -\delta_2^T P_1 \otimes Q\delta_2 < 0$. On the $\delta_2 -$ synchronization manifold both the Lyapunov function (53) and its time-derivative (54) equal zero as δ_2 vanishes there identically by Eq. (15). This concludes the proof. □

The requirement of graph irreducibility is crucial for the above presented result as in reducible graphs there necessarily exists at least one $p_i = 0$ [43], which would invalidate the Lyapunov construction of Theorem 3.

Remark 4. By Lemma 1, a diagonal P_1 of the choice 1 in Theorems 2 and 3 can under conditions of Theorems 2 and 3 always be found. The choice 1, $M = 0$, is more appropriate, however, for the full-state rather than output distributed feedback. The choice 2 can be made only for simple graph matrices [7], resulting in a symmetric P_1. For detailed balanced graphs the P_1 of choice 2 may be a diagonal matrix [7]. In that case, the diagonal entries of P_1 symmetrizing L in Theorem 3 equal the components of its left zero-eigenvector. Note that if the choice 2 can be made it also satisfies the graph condition of the choice 1.

The crucial property of structured Lyapunov functions (44), (53) and the chosen feedback gains, as revealed in proofs of Theorems 2 and 3 is that the drift dynamics' contribution is stabilizing whenever the contribution of the control vanishes outside the target set. Note also that $\varepsilon_1, \varepsilon_2$ can be used in Lyapunov functions (44) and (53) instead of δ_1, δ_2, without any significant change. Both ε_1 and ε_2 for linear systems satisfy the same dynamics as the corresponding synchronization errors (29) and (30). Errors ε_2 also satisfy constraint (15) and Lemma 3 consequently applies. Under stipulations on the graph topology, the vanishing of local neighborhood errors $\varepsilon_{1,2} = 0$ is equivalent to synchronization $\delta_{1,2} = 0$. However, for nonlinear systems it is more convenient to use synchronization errors δ_1, δ_2, *c.f.* Section 5.

Remark 5. The condition $P_2 B R^{-1} B^T P_2 - M^T R^{-1} M \geq 0$ is to be compared with conditions in [9], guaranteeing the unbounded conical sector synchronizing region. Note that this is automatically satisfied for $M = 0$.

4.3.3. Cooperative regulator for reducible graphs with a leader group

Theorems 2 and 3 deal with cooperative trackers having isolated leaders and cooperative regulators on irreducible graphs. Note, that an isolated leader at the root node is equivalent to external pinning to roots of all trees in a remaining spanning forest, *c.f.* Remark 1. Contrasted with Theorem 1, which assumes only a spanning tree for cooperative regulator, Theorem 3, requires a stronger property of irreducibility. This ostensibly leaves out cooperative regulators on reducible graphs containing a spanning tree, but not having an isolated leader. This instance is naturally included in Theorem 1 under the existence of a spanning tree, while here it needs to be considered separately.

Consider a reducible graph containing a spanning tree, but not having an isolated leader. Then one can re-label the agents so that the Laplacian appears in normal Frobenius form (2). The L_{m+1m+1} block of size N' represents an *irreducible autonomous leader group*. Its cooperative regulator dynamics is separated from the rest of the system

$$\begin{aligned} \dot{x}_k &= Ax_k + cBKC\sum\nolimits_l e_{kl}(x_l - x_k), \\ \dot{x}_k &= Ax_k + cBKC\varepsilon_{2k}. \end{aligned} \tag{58}$$

With a slight abuse of notation indices *k,l* in Eq. (58) label the N' nodes in L_{m+1m+1}, while indices *i,j* are reserved for the remaining *N*-N' graph's nodes. The remainder of the system follows the cooperative regulator dynamics

$$\dot{x}_i = Ax_i + cBKC\sum_j e_{ij}(x_j - x_i) + cBKC\sum_k g_{ik}(x_k - x_i), \tag{59}$$

where e_{ij} are the adjacency matrix elements pertaining to blocks L_{ij}, with $i,j \le m$, while the g_{ik} describe connections between the leader group and the rest of the graph, as given by blocks L_{im+1}, $i \le m$.

With the collective leader for an autonomous irreducible leader group (12),

$$\dot{x}^* = Ax^*, \tag{60}$$

and pertaining errors $\delta_{2k} = x_k - x^*$, the total synchronization error dynamics of the autonomous leader group is

$$\dot{\delta}_2 = (I_{N'} \otimes A - cL_{m+1m+1} \otimes BKC)\delta_2. \tag{61}$$

The remainder of the dynamics (59) can then be written as

$$\dot{x}_i = Ax_i + cBKC\sum_j e_{ij}(x_j - x_i) + cBKC\sum_k g_{ik}(x^* - x_i) + cBKC\sum_k g_{ik}(x_k - x^*). \tag{62}$$

From Eq. (2), for a leader group in consensus, *e.g.* with $v_i \in \mathbb{R}, v_0 \in \mathbb{R}$, one has

$$\begin{bmatrix} L_{11} & \cdots & L_{1m} & L_{1m+1} \\ & \ddots & \vdots & \vdots \\ & & L_{mm} & L_{mm+1} \\ & & & L_{m+1m+1} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{m'} \\ \underline{1}_l v_0 \end{bmatrix} = \begin{bmatrix} L_{11} & \cdots & L_{1m} & L_{1m+1}\underline{1}_l \\ & \ddots & \vdots & \vdots \\ & & L_{mm} & L_{mm+1}\underline{1}_l \\ & & & 0_l \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{m'} \\ v_0 \end{bmatrix} \tag{63}$$

where $\underline{1}_l$ denotes a vector of ones of length l for the leader group of size l whose topology is given by L_{m+1m+1}, and 0_l is a zero column vector of length l. Therefore the leader group in consensus has the same effect on the remainder of the graph as a single isolated leader, with the pinning gains $\tilde{g}_i = L_{im}\underline{1}_l$.

Motivated by the above, the collective leader of the irreducible leader group x^* is construed as a single isolated leader x_0 pinning into the remainder of the graph. One then has the dynamics (62) as

$$\dot{x}_i = Ax_i + cBKC\left(\sum_j e_{ij}(x_j - x_i) + \left(\sum_k g_{ik}\right)(x^* - x_i)\right) + cBKC\sum_k g_{ik}\delta_{2k}. \tag{64}$$

With definition of pinning gains

$$\tilde{g}_i : = \sum_k g_{ik}, \tag{65}$$

one recognizes in Eq. (64) the cooperative tracker local neighborhood error

$$\varepsilon_{1i} = \sum_j e_{ij}(x_j - x_i) + \tilde{g}_i(x^* - x_i).$$

Hence, with synchronization errors $\delta_{1i} = x_i - x^*$, pinning gain matrix $\tilde{G} = diag(\tilde{g}_1 \ldots \tilde{g}_N)$ and $G' = [g_{ik}]$, the global δ_1 synchronization error dynamics for the remainder of the graph (60), (64) equals

$$\dot{\delta}_1 = (I_{N-N'} \otimes A - c(\tilde{L} + \tilde{G}) \otimes BKC)\delta_1 + cG' \otimes BKC\delta_2. \tag{66}$$

Eqs. (61) and (66) describe a hierarchically coupled cooperative regulator and tracker, motivating the following.

Theorem 4. *Let the graph contain a spanning tree. Let the conditions of* Theorems 3 and 2 *apply separately to the irreducible leader group* (58) *and the remainder of the graph* (66), *construed as pinned by a single isolated leader with pinning gains defined by Eq.* (65). *Then the synchronization of the hierarchical system* (61) and (66) *is guaranteed by the composite Lyapunov function*

$$V = \delta_1^T P_{t1} \otimes P_2 \delta_1 + \alpha \delta_2^T P_{r1} \otimes P_2 \delta_2, \tag{67}$$

where matrices P_{t1}, P_{r1} are chosen for $\tilde{L} + \tilde{G}$ and L_{m+1m+1} respectively according to Theorems 2 and 3, and $\alpha > 0$ is a sufficiently large scaling constant.

Proof: Construct a composite Lyapunov function (67) for the coupled system as *per* Theorems 2 and 3. Indices r, t in P_1 stand for cooperative regulator and tracker. Since the single-agent systems are identical the same P_2 is chosen for cooperative regulator and tracker subsystems. Based on Theorems 2 and 3 for coupling gain $c > 0$ sufficiently large one has that

$$\dot{V} \leq -\delta_1^T Q_t \delta_1 - \alpha \delta_2^T Q_r \delta_2 + c \delta_1^T P_{t1} G' \otimes P_2 BKC \delta_2, \tag{68}$$

where positive definite Q_t, Q_r give the time-derivatives of Lyapunov functions as in Theorems 2 and 3. By assumption of a spanning tree, zero is the simple eigenvalue of the entire graph. By irreducibility, zero is the simple eigenvalue of the irreducible Laplacian L_{m+1m+1}, hence matrix $\tilde{L} + \tilde{G}$ is nonsingular, and by diagonal dominance it is an M-matrix, hence Theorem 2 applies. The expression (68) is written concisely as

$$\dot{V} \leq -\begin{bmatrix} \delta_1^T & \delta_2^T \end{bmatrix} \begin{bmatrix} Q_t & -\frac{c}{2} P_{t1} G' \otimes P_2 BKC \\ -\frac{c}{2} c G'^T P_{t1} \otimes C^T K^T B^T P_2 & \alpha Q_r \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \tag{69}$$

which is negative definite by Schur complement for $\alpha > 0$ sufficiently large, thus proving asymptotic stability of δ_1, δ_2 and guaranteeing synchronization of the total system. □

Remark 6. The structured Lyapunov functions of Theorems 2, 3 and 4 guarantee cooperative stability in all cases covered by the synchronizing region approach of Theorem 1; cooperative regulator with a spanning tree and cooperative tracker with a spanning forest having all trees pinned into at their root nodes. Hence for linear systems both synchronizing region and Lyapunov methods of this section can be used equivalently as sufficient conditions for cooperative stability. Therefore one finds here an appropriate alternative to the synchronizing region approach. In Section 5 we apply this alternative to a class of nonlinear systems.

Analysis given here is somewhat similar to that in [43] where it concerns reducible graphs. However, here we use results on nonsingular M-matrices [29], Lemma 2, whereas [43] applies results for irreducible graphs to reducible graph's irreducible components. In comparison, our decomposition of the total system's dynamics considers only one irreducible component, the autonomous leader group and the remainder of the graph, rather than considering all irreducible components. This difference is reflected in a different choice of the matrix P_1.

Remark 7. Special cases of structured Lyapunov functions (44) and (53) given by kernels $P_1 \otimes I_n, I_N \otimes P_2$ are appropriate for special graphs or controllability properties [23,30,40,42]. If $B = C = I_n$, choose $K = I_n$. Then the Lyapunov functions (44) and (53) with kernel $P_1 \otimes I_n$

guarantee cooperative stability. Namely for $P_2 = I_n$ the time-derivatives are given by quadratic forms of

$$P_1 \otimes (A^T + A) - c(P_1(L+G) + (L+G)^T P_1) \otimes I_n \tag{70}$$

for cooperative tracker, or

$$P_1 \otimes (A^T + A) - c(P_1 L + L^T P_1) \otimes I_n \tag{71}$$

for cooperative regulator. In Eqs. (70) and (71) for $c>0$ sufficiently large the second term dominates the first just as in the proofs of Theorems 2 and 3. On the other hand, if the graph topology is balanced or undirected choose $K = R^{-1} B^T P_2$ where P_2 solves (39). Then the Lyapunov functions (44) and (53) with kernel $I_N \otimes P_2$ guarantee cooperative stability [30,42]. Namely for $P_1 = I_N$ the time-derivatives are given by quadratic forms of

$$I_N \otimes (A^T P_2 + P_2 A) - c(L + G + (L+G)^T) \otimes P_2 B R^{-1} B^T P_2, \tag{72}$$

for cooperative tracker, or

$$I_N \otimes (A^T P_2 + P_2 A) - c(L + L^T) \otimes P_2 B R^{-1} B^T P_2 \tag{73}$$

for cooperative regulator. For undirected graphs a state transformation further leads to block diagonal structure with blocks $A^T P_2 + P_2 A - c\mu_j P_2 B R^{-1} B^T P_2$, akin to expression (39), where μ_j are the eigenvalues of the original symmetric positive semi-definite graph matrices, providing a connection with synchronizing regions [30,42].

From this it is evident that the choice $P_2 = I_n$ disregards the controllability relations inherent to single-agent systems, or simplifies them to all-state direct full-state feedback, while $P_1 = I_N$ restricts the graph topology. These special cases clearly reveal the connection of graph topology with the choice of P_1 and that of single-agent properties with the choice of P_2. This paper bridges the gap between local synchronizing region approach and global Lyapunov approach by structured quadratic Lyapunov functions, $P_1 \otimes P_2$, thus supplementing relations between local synchronizing region and global stability approaches studied in [33].

5. Nonlinear systems

This section applies Lyapunov functions, structured as in Section 4, to original nonlinear systems (5). For nonlinear cooperative regulators on irreducible graphs the diagonal form of P_1 is crucial [32]. Since this restriction generally comes at odds with the choice 2 of $M \neq 0$ in Theorems 2 and 3 one takes here $C = I_n$, $M = 0$, appropriate for the full-state feedback. The choice of $M \neq 0$ can nevertheless be made in nonlinear case for detailed balanced graphs, (*c.f.* Remark 4). Two definitions detailing incremental properties of the drift dynamics are needed in the sequel.

Definition 2. The drift dynamics $f(t,x)$ is termed QUAD or *quadratically bounded* uniformly, if there exists a matrix $P_2 = P_2^T > 0$ and some matrix $Q_2 = Q_2^T$ such that $\forall x, y$, uniformly in t,

$$(x-y)^T P_2 (f(t,x) - f(t,y)) \leq (x-y)^T Q_2 (x-y). \tag{74}$$

The QUAD property, as usually defined in the literature [12,32,43], uses diagonal Q_2 matrices, but here QUAD is generalized to symmetric matrices, more in keeping with the contraction approach [13,18,34]. The QUAD property provides quadratic bounds on the effect of drift dynamics on the time-derivative of a quadratic incremental stability Lyapunov function [34].

Definition 3. The drift dynamics $f(t,x)$ is termed special QUAD (sQUAD) with respect to the given linear state feedback control, $u=Kx$, if there exists a matrix $P_2=P_2^T>0$ such that $\forall x,y$, uniformly in t,

$$(x-y)^T P_2(f(t,x)-f(t,y)) \le (x-y)^T Q_2(x-y), \tag{75}$$

with $Q_2=Q_2^T$ negative definite on the linear subspace where the feedback control vanishes; $\ker K$.

In addition to QUAD of Definition 2 the extra structure of Q_2, with respect to the a linear feedback control, makes this a sQUAD property. Note that sQUAD depends not only on the drift dynamics but also on the chosen linear feedback. It does not imply that the drift dynamics is contracting [32], but if the drift dynamics is contracting, in the sense of V-uniformly decreasing [31], then it is sQUAD.

For nonlinear system (5), the linear feedback gain $K=R^{-1}B^TP_2$, where $R=R^T>0$ and P_2 satisfies

$$(x-y)^T P_2(f(t,x)-f(t,y)) - \frac{1}{2}(x-y)^T P_2BR^{-1}B^TP_2(x-y) \le -\frac{1}{2}(x-y)^T Q_2(x-y) \le 0, \tag{76}$$

uniformly in t,x,y, guarantees sQUAD with respect to the linear state-feedback $u=Kx$. Clearly, global exponential incremental stabilizability of Eq. (5) by a linear feedback is a necessary condition for solution of Eq. (76) to exist. Note that Eq. (76) applied to LTI systems reduces to algebraic Riccati equation (39).

Exposition of the main results of this section proceeds along the lines of Section 4, due to a similar dependence of Lyapunov constructions on the underlying graph topology types.

5.1. Cooperative tracker

Similarly to Theorem 2 we have the following result for cooperative trackers with affine-in-control systems (5).

Theorem 5. *Let the graph contain a spanning forest with a leader pinning into root nodes of all trees. Let the single-agent closed-loop system be given as*

$$\dot{x}_i = f(t,x_i) + cBK\varepsilon_{1i}, \tag{77}$$

with the linear feedback gain chosen as

$$K = R^{-1}B^TP_2, \tag{78}$$

where $R=R^T>0$ and P_2 satisfies

$$\delta_{1i}^T P_2(f(t,x_i)-f(t,x_0)) \le \delta_{1i}^T Q_2 \delta_{1i} \tag{79}$$

uniformly in t,x_i,x_0, with $Q_2=Q_2^T<0$ on $\ker P_2BR^{-1}B^TP_2$. Let P_1 be chosen for $L+G$ according to Lemma 1. Then the structured Lyapunov function,

$$V = \delta_1^T P_1 \otimes P_2 \delta_1 = \sum_i p_i \delta_{1i}^T P_2 \delta_{1i}, \tag{80}$$

for the coupling gain $c>0$ sufficiently large guarantees asymptotic cooperative tracking, $\delta_1 \to 0$.

Proof. The time-derivative of the Lyapunov function (80) is

$$\dot{V} = \sum_i p_i \delta_{1i}^T P_2 (f(t,x_i) - f(t,x_0) + cBK \sum_j e_{ij}(x_j - x_i)) = \delta_1^T P_1 \otimes P_2 (F(t,x) - \bar{f}(t,x_0))$$
$$- c\delta_1^T P_1 (L+G) \otimes P_2 BK \delta_1, \tag{81}$$

where $F(t,x) = \left[f^T(t,x_1) \cdots f^T(t,x_N)\right]^T \in \mathbb{R}^{Nn}$, and $\bar{f}(t,x_0) = \left[f^T(t,x_0) \cdots f^T(t,x_0)\right]^T \in \mathbb{R}^{Nn}$. Under conditions of the Theorem, the contribution of the second term in Eq. (81),

$$\frac{1}{2}\delta_1^T c(P_1(L+G) + (L+G)^T P_1) \otimes P_2 B R^{-1} B^T P_2 \delta_1, \tag{82}$$

is negative semidefinite, vanishing only if $\forall i,\ \delta_{1i} \in \ker P_2 B R^{-1} B^T P_2 \equiv \ker B^T P_2$, by the full column rank of B, and regularity of R. In that case one is left with the contribution of the drift dynamics

$$\delta_1^T P_1 \otimes P_2 (F(t,x) - \bar{f}(t,x_0)) = \sum\nolimits_i p_i \delta_{1i}^T P_2 (f(t,x_i) - f(t,x_0)), \tag{83}$$

which is rendered negative by the positivity of p_i (Lemma 1) and the properties of drift dynamics assumed in the theorem (79);

$$\sum\nolimits_i p_i \delta_{1i}^T P_2 (f(t,x_i) - f(t,x_0)) \le \sum\nolimits_i p_i \delta_{1i}^T Q_2 \delta_{1i}^T = \delta_1^T P_1 \otimes Q_2 \delta_1^T < 0,$$

for $\delta_{1i} \in \ker P_2 B R^{-1} B^T P_2\ \forall i$. Whenever the term (82) does not vanish $c > 0$ needs to be so large that it dominates the contribution of the generally indefinite first term, quadratically bounded by $\delta_1^T P_1 \otimes Q_2 \delta_1$ □

The sQUAD condition (79) appearing in Theorem 5 serves the purpose of and generalizes the ARE of LTI systems, Theorem 2. Compared with [43], which considers only diagonal inner coupling matrices and diagonal P_2, Q_2 matrices characterizing the QUAD property, our approach accounts for the controllability properties of single-agents which result in more general inner coupling matrices, requiring more general symmetric P_2, Q_2 matrices. Also, cooperative stability criteria in [43] require checking numerous conditions that are here satisfied by design.

5.2. *Cooperative regulator for irreducible graphs*

Similarly to Theorem 3 we have the following result for cooperative regulators with affine-in-control system (5) on irreducible graphs.

Theorem 6. *Let the graph be strongly connected. Let the closed-loop systems be given as*

$$\dot{x}_i = f(t,x_i) + cBK\varepsilon_{2i}, \tag{84}$$

with the linear feedback gain chosen as $K = R^{-1} B^T P_2$ where $R = R^T > 0$ and P_2 satisfies

$$\delta_{2i}^T P_2 (f(t,x_i) - f(t,x^*) \le \delta_{2i}^T Q_2 \delta_{2i}, \tag{85}$$

uniformly in t, x_i, x^* with $Q_2 = Q_2^T < 0$ on $\ker P_2 B R^{-1} B^T P_2$. Let P_1 be chosen according to Lemma 1, as $diag(p_1 ... p_N)$. Then the structured Lyapunov function,

$$V = \delta_2^T P_1 \otimes P_2 \delta_2 = \sum_i p_i \delta_{2i}^T P_2 \delta_{2i} \tag{86}$$

for the coupling gain $c > 0$ sufficiently large guarantees asymptotic cooperative stability, $\delta_2 \to 0$.

Proof. The time-derivative of the Lyapunov function (86) is

$$\dot{V} = \sum_i p_i \delta_{2i}^T P_2 (f(t,x_i) - \dot{x}^*) + cBK \sum_j e_{ij}(x_j - x_i)) = \delta_2^T P_1 \otimes P_2 (F(t,x) - \overline{f}(t,x^*))$$
$$-c\delta_2^T P_1 L \otimes P_2 BK \delta_2, \tag{87}$$

where $F(t,x) = \left[f^T(t,x_1)\cdots f^T(t,x_N)\right]^T \in \mathbb{R}^{Nn}$, and $\overline{f}(t,x^*) = \left[f^T(t,x^*)\cdots f^T(t,x^*)\right]^T \in \mathbb{R}^{Nn}$. Note that in Eq. (87) $\dot{x}^* \neq f(t,x^*)$ but because of the crucial constraint (15) one has that $\sum_i p_i \delta_{2i}^T P_2 a = \left(\sum_i p_i \delta_{2i}^T\right) P_2 a = 0$ for any vector a, and in our case $\dot{x}^*$ appearing in each entry i can be replaced by any vector *e.g.* $f(t,x^*)$. The contribution of the second term in Eq. (87),

$$\frac{1}{2}\delta_2^T c(P_1 L + L^T P_1) \otimes P_2 BR^{-1} B^T P_2 \delta_2 \tag{88}$$

is negative semidefinite, vanishing outside the $\delta_2 -$ synchronization manifold if and only if $\forall i$, $\delta_{2i} \in \ker B^T P_2$, by the full column rank of B and regularity of R. In that case one is left with the contribution of drift dynamics

$$\delta_2^T P_1 \otimes P_2 (F(t,x) - \overline{f}(t,x^*)) = \sum_i p_i \delta_{2i}^T P_2 (f(t,x_i) - f(t,x^*)), \tag{89}$$

which is rendered negative by the positivity of p_i (Lemma 1) and the properties of drift dynamics assumed in the theorem (84),

$$\sum_i p_i \delta_{2i}^T P_2 (f(t,x_i) - f(t,x^*)) \le \sum_i p_i \delta_{2i}^T Q_2 \delta_{2i}^T = \delta_2^T P_1 \otimes Q_2 \delta_2 < 0,$$

for $\delta_{2i} \in \ker P_2 BR^{-1} B^T P_2 \ \forall i$. Where Eq. (88) does not vanish $c > 0$ needs to be sufficiently large so that it dominates the contribution of the generally indefinite first term, quadratically bounded by $\delta_2^T P_1 \otimes Q_2 \delta_2$. □

As in Theorem 3, graph irreducibility is crucial for the above result since in reducible graphs there necessarily exists at least one $p_i = 0$ [43], which would invalidate the Lyapunov construction of Theorem 6.

5.3. *Cooperative regulator for reducible graphs with a leader group*

Cooperative regulators on reducible graphs containing a spanning tree without an isolated leader are treated similarly as in Section 4. The proof however differs in a crucial requirement on single-agent drift dynamics. Similar to Section 4.3.3 one has cooperative regulator dynamics of the autonomous leader group

$$\dot{x}_k = f(t,x_k) + cBK \sum_l e_{kl}(x_l - x_k),$$
$$\dot{x}_k = f(t,x_k) + cBK \varepsilon_{2k}. \tag{90}$$

The remainder of the system follows cooperative regulator dynamics

$$\dot{x}_i = f(t,x_i) + cBK \sum_j e_{ij}(x_j - x_i) + cBK \sum_k g_{ik}(x_k - x_i), \tag{91}$$

where e_{ij} are adjacency matrix elements pertaining to the blocks L_{ij}, with $i,j \le m$, while the g_{ik} describe the connections between the leader group and the remainder of the graph, as given by blocks L_{im+1}, $i \le m$. With a collective leader for the irreducible autonomous leader group, as detailed in Theorem 4, one has

$$\dot{x}^* = \left(\sum_k p_k\right)^{-1} \sum_k p_k f(t,x_k). \tag{92}$$

With $\delta_{2k} = x_k - x^*$ the synchronization error dynamics for the autonomous leader group is

$$\dot{\delta}_{2k} = f(t, x_k) - \dot{x}^* + cBK\sum_l e_{kl}(\delta_{2l} - \delta_{2k}). \tag{93}$$

The remainder of the dynamics (91) can then be written as

$$\dot{x}_i = f(t, x_i) + cBK\sum_j e_{ij}(x_j - x_i) + cBK\sum_k g_{ik}(x^* - x_i) + cBK\sum_k g_{ik}(x_k - x^*) \tag{94}$$

With x^* construed as the single isolated leader x_0 pinning into the remainder of the graph one has

$$\dot{x}_i = f(t, x_i) + cBK\left(\sum_j e_{ij}(x_j - x_i) + \left(\sum_k g_{ik}\right)(x^* - x_i)\right) + cBK\sum_k g_{ik}\delta_{2k}. \tag{95}$$

For synchronization errors $\delta_{1i} = x_i - x^*$, similarly as in Theorem 4, this leads to

$$\dot{\delta}_1 = F(t, x) - \overline{\dot{x}}^* - c(\tilde{L} + \tilde{G}) \otimes BK\delta_1 + cG' \otimes BK\delta_2,$$
$$\dot{\delta}_1 = F(t, x) - \overline{f}(t, \dot{x}^*) - c(\tilde{L} + \tilde{G}) \otimes BK\delta_1 + cG' \otimes BK\delta_2 + \overline{f}(t, \dot{x}^*) - \overline{\dot{x}}^*. \tag{96}$$

Eqs. (93) and (96) describe a hierarchically coupled system, with the caveat that $\overline{f}(t, \dot{x}^*) - \overline{\dot{x}}^* \neq 0$ outside the synchronization manifold for the autonomous leader group. Still one has the following result.

Theorem 7. *Let the graph contain a spanning tree. Let the conditions of* Theorems 6 and 5 *apply separately to the autonomous irreducible leader group* (90), *and the remainder of the graph* (91), *construed as pinned by a single isolated leader. If the single-agent drift dynamics $f(t, x)$ is globally Lipschitz in x, uniformly in t, then the synchronization of hierarchical system* (93) and (96) *for a coupling gain $c > 0$ large enough is guaranteed by the composite Lyapunov function*

$$V = \delta_1^T P_{t1} \otimes P_2\delta_1 + \alpha\delta_2^T P_{r1} \otimes P_2\delta_2, \tag{97}$$

where matrices P_{t1}, P_{r1} are chosen for $\tilde{L} + \tilde{G}$ and L_{m+1m+1} according to Theorems 5 and 6, and $\alpha > 0$ is a sufficiently large scaling constant.

Proof. Construct a Lyapunov function (97) where indices r, t in P_1 stand for cooperative regulator and tracker respectively. Matrices P_{t1}, P_{r1} are chosen for $\tilde{L} + \tilde{G}$ and L_{m+1m+1} according to Theorems 5 and 6. Since the single-agent systems are identical the same P_2 is chosen for cooperative regulator and tracker subsystems. Based on the results of Theorems 5 and 6, for the coupling gain $c > 0$ sufficiently large one has that

$$\dot{V} \leq -\delta_1^T Q_t\delta_1 - \alpha\delta_2^T Q_r\delta_2 + \delta_1^T cP_{t1}G' \otimes P_2BK\delta_2 + \delta_1^T P_{t1} \otimes P_2(\overline{f}(t, \dot{x}^*) - \overline{\dot{x}}^*). \tag{98}$$

where positive definite Q_t, Q_r characterize the time-derivatives of Lyapunov functions as in Theorems 5 and 6. The last term in Eq. (98) requires special attention. It equals

$$\delta_1^T P_{t1} \otimes P_2(\overline{f}(t, \dot{x}^*) - \overline{\dot{x}}^*) = \sum_k p_k\delta_{1k}^T P_2(f(t, \dot{x}^*) - \dot{x}^*). \tag{99}$$

Furthermore, one has that

$$f(t, x^*) - \dot{x}^* = f(t, x^*) - \left(\sum_j p_j\right)^{-1}\sum_j p_j f(t, x_j) = \left(\sum_j p_j\right)^{-1}\sum_j p_j(f(t, x^*) - f(t, x_j)), \tag{100}$$

whence it follows by Cauchy–Schwartz that $\|f(t,x^*)-\dot{x}^*\| \le \left(\sum_j p_j\right)^{-1} \sum_j p_j \|f(t,x^*)-f(t,x_j)\|$. Global Lipschitz bound on the drift dynamics in x uniformly in t, $\|f(t,x^*)-f(t,x_j)\| \le C\|x_j - x^*\| \le C\|\delta_2\|$, implies $\|f(t,x^*)-\dot{x}^*\| \le C\|\delta_2\|$, therefore $\|\overline{f}(t,x^*)-\overline{\dot{x}}^*\| \le C\sqrt{N}\|\delta_2\|$. Hence inequality (98) implies the following:

$$\dot{V} \le -\begin{bmatrix} \|\delta_1\| & \|\delta_2\| \end{bmatrix} \begin{bmatrix} \underline{\sigma}(Q_t) & -\frac{1}{2}(c\overline{\sigma}(P_{t1}G' \otimes P_2BK) + \overline{\sigma}(P_{t1} \otimes P_2)\sqrt{N}C) \\ -\frac{1}{2}(c\overline{\sigma}(P_{t1}G' \otimes P_2BK) + \overline{\sigma}(P_{t1} \otimes P_2)\sqrt{N}C) & \alpha\underline{\sigma}(Q_r) \end{bmatrix} \times \begin{bmatrix} \|\delta_1\| \\ \|\delta_2\| \end{bmatrix}. \tag{101}$$

For $\alpha > 0$ sufficiently large (101) is negative definite, by Schur complement, thus proving asymptotic stability of δ_1, δ_2, guaranteeing synchronization of the total system. □

Remark 8. Notions related to sQUAD appearing in the literature [31–33,43] guarantee distributed synchronization. However it is often *a priori* assumed that the feedback has certain sufficient, but not necessary properties. In [43] the inner coupling matrix is assumed diagonal, for which the diagonal matrices characterizing the QUAD property can be used. This does not represent realistic controllability properties, whereas the inner coupling matrices designed here with respect to single-agents do. In [31] the inner coupling matrix is assumed such that its contribution in the dissipation relation is symmetric and positive definite. Here the corresponding contribution is given by $PBR^{-1}B^TP$ which is symmetric, but only positive semi-definite. Reference [32] at first assumes all-state direct feedback, as appearing also in [2], and subsequently weakens it by assuming the coupling function h satisfying $\partial h/\partial x > 0$. Here this corresponds to $BKC > 0$. Both assumptions are weakened by requiring sQUAD. In [33] synchronization is approached *via* incremental stability of $f(t,x) - \alpha h(x)$ system, which is stated as an assumption. Instead of assuming that $h(x)$ yields incremental stability, as in [33], the sQUAD requirement here, by virtue of (76), guarantees that the corresponding closed-loop system $f(t,x) - BR^{-1}B^TPx$ is indeed incrementally stable by design. General quadratic bounds on the contribution of drift dynamics, *e.g.* requiring contraction of $f(t,x) - \alpha x$ for some positive α in [31,32], or drift dynamics being V-uniformly decreasing in [31], obscure special structural properties relating the drift dynamics $f(t,x)$ and the input matrix B, explicitly emphasized in Definition 3.

6. Numerical example

This section brings a network of 11 nonlinear oscillators with dynamics given in input–output form as

$$\ddot{y}_i + 0.2\frac{y_i^2 - 1}{5(1 + y_i^2 + \dot{y}_i^2)}\dot{y}_i + (1 + 0.2 \sin 0.6t)y_i = u_i.$$

Their drift dynamics is a Lienard system possessing a limit cycle. The state-space model is chosen as

$$\frac{d}{dt}\begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix}\begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2\frac{4x_{i1}^2 + 5x_{i2}^2 + 6}{5(1 + x_{i1}^2 + x_{i2}^2)}x_{i2} - 0.2x_{i1} \sin 0.6t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_i.$$

The time-varying and nonlinear part is globally Lipschitz, uniformly in time, with globally bounded gradient, and the single-agent drift dynamics is sQUAD with

$$P_2 = \begin{bmatrix} 3.042 & 0.732 \\ 0.732 & 1.672 \end{bmatrix}, \quad Q_2 = 0.13 I_2.$$

The control gain is designed *via* Eq. (76), $K = B^T P_2 = \begin{bmatrix} 0.732 & 1.672 \end{bmatrix}$. The interconnecting communication graph topology is given by its Laplacian matrix and is depicted on Fig. 5.

$$L = \left[\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & & & 1 & -1 & 0 \\ & & & & 0 & & & & 0 & 2 & -2 \\ & & & & & & & & -1 & 0 & 1 \end{array}\right].$$

This graph is reducible, contains a spanning tree, but it does not have an isolated leader; rather it has an autonomous leader group. It is found that $P_{t1} = diag(1, 0.5, 0.5, 0.75, 0.6, 1, 0.5, 0.5)$, $P_{r1} = diag(2/3, 1/3, 2/3)$. The coupling gain $c = 2.2$ satisfies sufficient conditions of Theorems 5 and 6 for the autonomous leader group and the remainder of the graph separately.

Note that this example does not use the all-state direct feedback, rather the controllability properties of single-agent systems are accounted for in the control design. The distributed feedback is designed based on the given drift dynamics' contraction properties and it does not rely simply on overpowering the drift dynamics. The following figures, Figs. 1–4, depict the time-dependence of the first state variable for all 11 agents, with pertaining synchronization errors, including the detailed view of transients for the first 10 s.

7. Conclusions

This paper brings a Lyapunov approach unifying the cooperative regulators and trackers on graphs having a spanning tree. Lyapunov techniques for cooperative stability are here developed

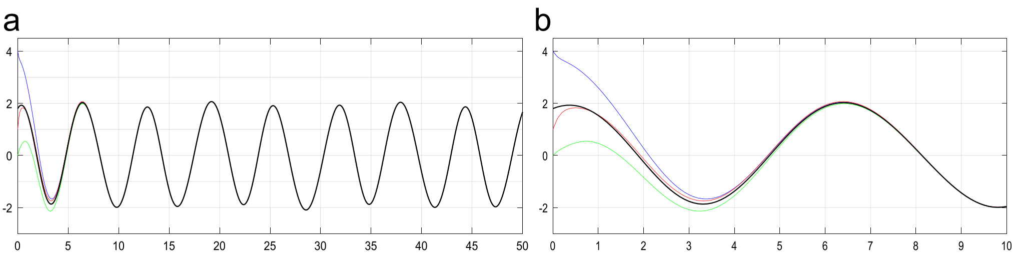

Fig. 1. (a) First state var. of the leader group. (b) First state var. of the leader group—detail.

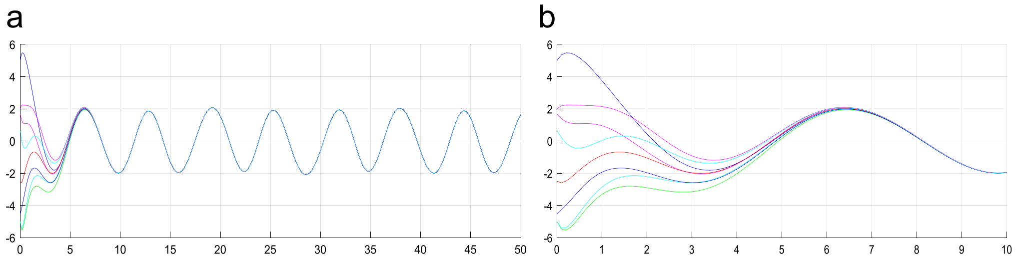

Fig. 2. (a) First state var. remaining agents.(b) First state var. remaining agents—detail.

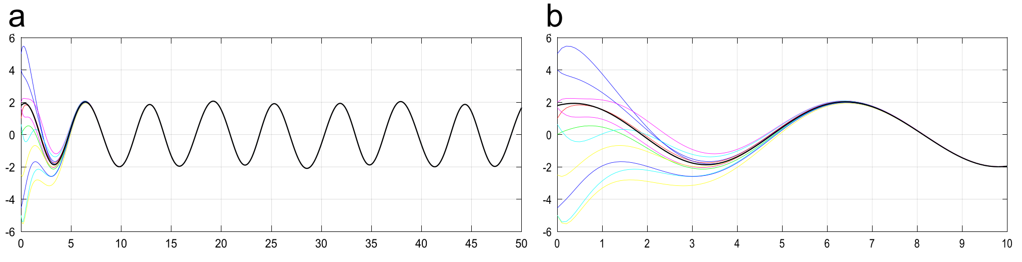

Fig. 3. (a) First state var. entire system. (b) First state var. entire system—detail.

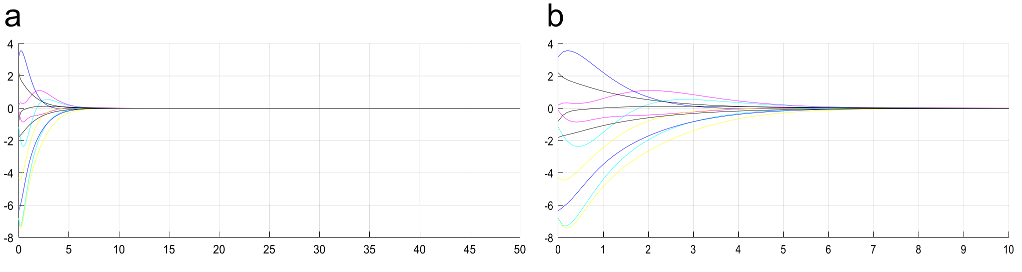

Fig. 4. (a) First state var. synch. error entire system. (b) First state var. synch error entire system—detail.

as a parallel to the versatile synchronizing region approach. Existence of a spanning tree is necessary for the synchronizing region approach which naturally includes all the cases considered. Lyapunov approach presented here, however, requires separate considerations for different types of graphs. Structured quadratic Lyapunov functions are given that guarantee exponential cooperative stability with linear distributed controls for agents whose dynamics belongs to a special class. This class, characterized by incremental properties of the drift dynamics, is termed special QUAD (sQUAD). A sufficient condition is given for a system to be sQUAD, analogous to ARE for LTI systems. This provides a design scheme for affine-in-control systems. Restrictive *a priori* assumptions often found in the literature are removed and the required properties are achieved by design. Future work will apply the structured Lyapunov functions to robust and adaptive distributed control problems on general directed graphs.

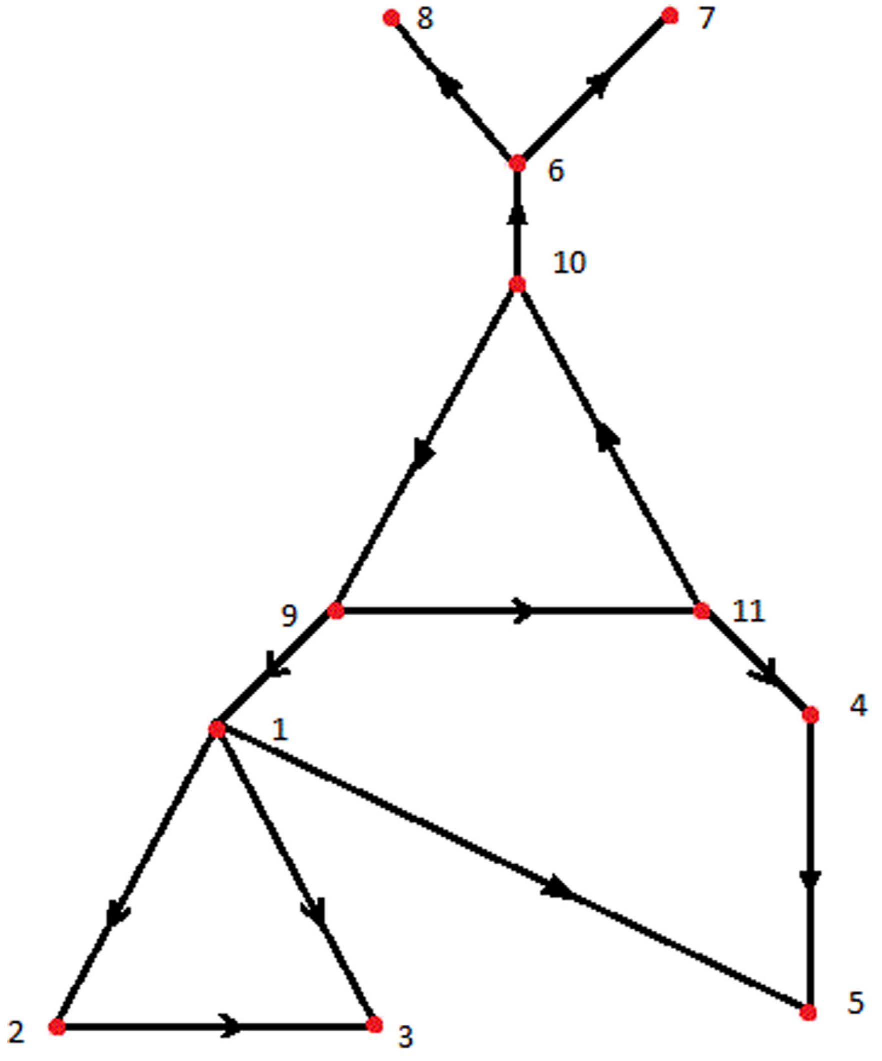

Fig. 5. The interconnecting communication graph.

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