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Towards Implementable Prescriptive Decision Making

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Abstract

The need for inspecting (ir)rationality in decision making (DM) — the observed discrepancy between real and prescriptive DMs — stems from omnipresence of DM in individuals' and society life. Active approaches try to diminish this discrepancy either by changing behaviour of participants (DM subjects) or modifying prescriptive theories as done in this text. It provides a core of unified merging methodology of probabilities serving for knowledge fusion and information sharing exploited in cooperative DM. Specifically, it unifies merging methodologies supporting a flat cooperation of interacting self-interested DM participants. They act without a facilitator and they are unwilling to spare a non-negligible deliberation effort on merging. They are supposed to solve their DM tasks via the fully probabilistic design (FPD) of decision strategies. This option is motivated by the fact that FPD is axiomatically justified and extends standard Bayesian DM.

Merging is a supporting DM task and is also solved via FPD. The proposed merging formulation tries to be as general as possible without entering into technicalities of measure theory. The results generalise and unify earlier work and open a pathway to systematic solutions of specific, less general, problems.

Keywords: fully probabilistic design, distributed decision making, cooperation

1. Introduction

Decision making (DM), seen as a purposeful choice among available options, covers a substantial portion of human activities as well as of institutions and devices created by people. DM almost always runs under uncertainty and with incomplete knowledge. This makes DM, by its nature optimising, extremely demanding on cognitive resources of any participant — the DM subject. Among a range of DM methodologies those based on Savage's DM concept, Savage (1954), seem to be the most promising and they are permanently generalised and refined. The fully probabilistic design of decision strategies (FPD), Kárný (1996); Kárný and Guy (2006); Kárný and Kroupa (2012), on which this paper relies, is of a Savage's type. The space limitations prevent us to discuss very rich related work even immediate predecessors as Kárný et al. (2009); Sečkárová (2013). The next characterisation of the used FPD is an exception enforced by its weak penetration to DM communities.

FPD, as all Bayesian solution, models probabilistically the closed-loop behaviour — the collection of all considered and opted variables within the DM task. Unlike its predecessors, FPD probabilistically describes the DM preferences through an ideal closed-loop model. FPD selects the optimal strategy — the sequence of randomised decision rules mapping the knowledge on actions — minimising the KLD divergence, Kullback and Leibler (1951), of the closed-loop model to its ideal counterpart.

DM is always performed with limited resources — the time-span devoted to the solved DM task, the extent of the manageable knowledge, physical and computational resources. This naturally led to a division of DM tasks. In it, the involved participants (individuals or groups formed both by people, technical and organisational tools they use) solve smaller DM sub-tasks and select actions — irreversibly implemented decisions. This allows groups of participants to function but at substantial costs expended on cooperation — goods and knowledge exchanging, sharing, making concessions with respect to participant's aims, etc.

Governance of the cooperation process by a participant at a higher hierarchical level (facilitator, coordinator, institutional or legal authority, etc.) is a DM on its own. It can make the multi-participants' DM efficient but with increasing complexity of the addressed DM tasks demands on governance resources quickly increase and its efficiency strongly drops. Due to this, the need arises for distributed DM (almost) without a facilitator — a sort of democratic scenario arises, Kárný and Guy (2004).

The democratic scenario essentially lets individual DM subjects to act in a selfish way. The selfish (self-centered) participant cares about her own DM tasks only. She possibly cooperates with her neighbours — the participants with which she interacts and about which she is able and willing to care. The DM aims, however, persist: each participant tries to optimally reach her "personal" DM objectives under the given circumstances.

Note that the term selfish lacks here a moral dimension. For instance, it can be perceived as a quite positive if the care about societal welfare is adopted by the participant as her personal aim. Also note that real limited cognitive resources of any participant imply that the number of her neighbours will remain relatively small.

This paper aims to equip any selfish participant having limited cognitive resources with a tool supporting her DM by taking into account influence of neighbours. In the quest for an applicable prescriptive DM methodology, the constructed "advisor": i) is to be impersonal and implementable as a feasible algorithm, which adds as little as possible (ideally none) additional options (parameters, tuning knobs); ii) must not require from the supported participants more than what they handle anyway; iii) must offer its support in a way understandable to individual participants: as a by-product this guarantees privacy of the respective participants; iv) must let the supported participants decide whether they accept the advise or not.

The proposed solution assumes that all involved participants use FPD as the tool for solving their DM tasks. This is the most general prescriptive DM methodology, which is feasible for realistic but sufficiently small DM tasks. It also covers Bayesian participants.

Section 2 prepares the problem formulation and solution presented in Section 3. The solution depends on unwanted options (parameters), which are unambiguously specified in Section 4. The solution operates on joint probabilities acting on the same collection of variables. Section 5 describes the way how to cope with this restrictive condition. Section 6 adds remarks on conversions of non-probabilistic elements into the merged probabilities, recommends when to accept the gained advices and outlines open problems.

2. Preliminaries

We use the following notions and conventions.

DM subjects: A participant $\mathcal{P}$ belongs to a group of her neighbours $(\mathcal{P}_p)_{p\in p^\star}$, $|p^\star| < \infty$. The term advisor $\mathcal{A}$ refers to the cooperation-supporting algorithm serving to this group.

Sets, mappings, finite collections: Sets of entities $a, X, \mathsf{R}, \dots$ are denoted $a^\star, X^\star, \mathsf{R}^\star, \dots$ $|X^\star|$ means cardinality of the set $X^\star$. Sets are subsets of separable spaces, typically, finite sets of integers, finite-dimensional real spaces or sets of probability densities (pd, Radon-Nikodým derivatives with respect to counting or Lebesgue's measure — both denoted $\mathrm{d}\bullet$). Pds and other mappings are distinguished by $\mathsf{san\ serif\ fonts}$. A finite collection of entities, say real scalars $(\lambda_p)_{p\in p^\star}$, is often referred as $\lambda = (\lambda_p)_{p\in p^\star}$. Inequalities like $\lambda > 0$ are then understood component-wise.

Behaviour, ignorance, action, knowledge: Each supported participant $\mathcal{P}_p$, $p \in p^\star$, operates on a specific (closed-loop) *behaviour* $b_p \in b_p^\star$, which is adopted name for all variables that $\mathcal{P}_p$ considers, opts or knows. The opted action splits the behaviour as follows[1]

$$\begin{aligned} b_p &= (g_p, a_p, k_p) = \text{(ignorance, action, knowledge) behaviour parts} \\ &= \text{(considered but inaccessible, opted by } \mathcal{P}_p\text{, used for the action choice) parts.} \end{aligned} \tag{1}$$

The performed minimisation of any *expected* loss over randomised decision rules reduces to the minimisation of the expected loss conditioned on the knowledge available for the action choice, Berger (1985). This allows us to simplify presentation by (mostly) not spelling explicitly the used knowledge in conditions of the involved pds. This formally reduces behaviours to

$$\begin{aligned} b_p &= (g_p, a_p) = \text{(ignorance, action) behaviour parts} \\ &= \text{(considered but inaccessible, opted by } \mathcal{P}_p\text{) parts}, \ \ p \in p^\star. \end{aligned} \tag{2}$$

Simplifying assumptions: The following concessions from full generality will be made.

- The cardinalities of the behaviour sets of all participants are finite

  $$b_p \in b_p^\star = \{1, \dots, |b_p^\star|\}, \ 2 \le |b_p^\star| < \infty, \quad p \in p^\star. \tag{3}$$

  The theory of the numerical representation of DM preferences, Debreu (1954), implies that the considered, numerically representable, behaviour spaces have to be separable. Then, the considered finite sets of behaviours can be seen as images of finite projections (discretisations) of the underlying separable spaces of infinite cardinalities.

- Static DMs are considered. Each participant selects a single action, i.e. she selects and applies a single randomised decision rule described by a pd $\mathsf{r}(a_p) = \mathsf{r}(a_p|k_p)$, $p \in p^\star$.

1. While terms "knowledge" and "action" (an irreversibly adopted decision) are common and well-accepted, the unusual term "ignorance" (linguistically opposite to knowledge) is often felt as inappropriate. We ask patient reader to take it as technical term describing the part of the closed-loop behaviour differing from the action and knowledge.

FPD: Each participant $\mathcal{P}_p$, $p \in p^\star$, within a group of neighbours deals with her closed-loop model, see e.g. Kárný and Guy (2006), which is a joint pd,

$$\underbrace{\mathsf{c}_{\mathsf{r}_p}(b_p)}_{closed-loop\,model} = \underbrace{\mathsf{m}(g_p|a_p)}_{environment\,model} \times \underbrace{\mathsf{r}(a_p)}_{decision\,rule}, \quad b_p \in b_p^\star. \tag{4}$$

The factorisation (4) is implied by the chain rule for pds. The first factor on the right-hand side relates action to ignorance, i.e. to the considered but (yet) unknown reaction of the participant's environment. This motivates its interpretation. The second factor already has been recognised as the model of the decision rule.

$\mathcal{P}_p$ applying FPD possesses a preference-expressing ideal counterpart $\mathsf{c}_{Ip}(b_p)$ of $\mathsf{c}_{\mathsf{r}_p}(b_p)$, which determines her FPD-optimal decision rule $\mathsf{r}_{Op} = \mathsf{r}_{Op}(a_p)$

$$\mathsf{r}_{Op} \in \mathrm{Arg}\min_{\mathsf{r}_p \in \mathsf{r}_p^\star} \int_{b_p^\star} \mathsf{c}_{\mathsf{r}_p}(b_p) \ln\left(\frac{\mathsf{c}_{\mathsf{r}_p}(b_p)}{\mathsf{c}_{Ip}(b_p)}\right) \mathrm{d}b_p = \mathrm{Arg}\min_{\mathsf{r}_p \in \mathsf{r}_p^\star} \mathsf{KL}(\mathsf{c}_{\mathsf{r}_p}||\mathsf{c}_{Ip}),\ p \in p^\star, \tag{5}$$

i.e. the optimal decision rule r_{Op} minimizes the KLD $\mathsf{KL}(\mathsf{c}_{\mathsf{r}_p}||\mathsf{c}_{Ip})$ of $\mathsf{c}_{\mathsf{r}_p}$ from c_{Ip}.

3. Merging Problem Formulation and Solution

We assume that a participant seeks for support and her abilities delimit a group of neighbours. This defines a group of supported participants. Any participant can be a member of many groups, each with its advisor. Groups act in an asynchronous way and advices are offered for exploitation when created. The following design concerns a fixed group with a fixed knowledge processed by the group members having a single fixed advisor $\mathcal{A}$.

The required support: Each group member $\mathcal{P}_p$, $p \in p^\star$, provides the advisor $\mathcal{A}$ her closed-loop model with the aim obtaining an advice about a non-void factor of her closed-loop model (4), which she is willing to change according to the $\mathcal{A}$ advice.

The group behaviour, $\mathcal{A}$ action: The group behaviour[2] $B \in B^\star$ has the structure

$$\begin{aligned} B &= (G, A) = (\text{ignorance}, \text{action}) \text{ of the group behaviour parts} \\ G &= \text{the group ignorance consists of behaviours (2) of all participants} \\ G^\star &= \cup_{p \in p^\star} b_p^\star \\ A &= \mathsf{A}(G) = \text{the } \mathcal{A} \text{ action is a pd on } G^\star \text{ merging pds } \mathsf{c}_\mathsf{r} = (\mathsf{c}_{\mathsf{r}_p})_{p \in p^\star} \\ \mathsf{A}^\star &= \text{all pds with the domain } G^\star \text{ of a finite cardinality} \\ K &= \text{the } \mathcal{A} \text{ knowledge consists of the pds } \mathsf{c}_\mathsf{r} \text{ and possibly } \mathsf{c}_I. \end{aligned} \tag{6}$$

Verbally, the advisor action A is the pd $\mathsf{A}(G) = \mathsf{A}(G|K)$ modelling the group ignorance G formed by $(g_p, a_p)_{p \in p^\star}$ — the advice is offered to any participant $\mathcal{P}_p$ before she makes her action a_p. Knowledge K is insufficient for a unique specification of the pd $\mathsf{A} \in \mathsf{A}^\star$, and a randomised decision rule $\mathsf{R}(\mathsf{A}) = \mathsf{R}(\mathsf{A}|K)$ is to be designed. This is the factor of the group closed-loop model C_R

$$\mathsf{C}_\mathsf{R}(B) = \mathsf{C}_\mathsf{R}(G|A)\mathsf{C}_\mathsf{R}(A) = \mathsf{A}(G)\mathsf{R}(\mathsf{A}). \tag{7}$$

2. The mathematical entities related to the supported participants are marked by small letters while the group quantities, handled by $\mathcal{A}$, are denoted by their capital counterparts.

The first equality in (7) uses the definition of the group behaviour $B = (G, A) \in (G^\star, A^\star)$ (6) and the chain rule for pds. The second equality in (7) is implied by the definition of the advisor action $A = \mathsf{A}$ and of the decision rule R. The knowledge implicit in conditions of processed pds (cf. simplified version (2) of (1) is uninfluenced by the optimised rule R.

The desired merger: Let $\mathsf{e}_p(G)$ be a model of the group behaviour $G \in G^\star$ from the $\mathcal{P}_p$ point of view. Section 5 discusses the construction of these pds from the processed pds $\mathsf{c}_\mathsf{r}(G) = (\mathsf{c}_{\mathsf{r}_p}(b_p))_{p \in p^\star}$. It contains factors, which the cooperating participants are ready to replace according to the $\mathcal{A}$ advice. Of course, $\mathcal{P}_p$ may be inclined to accept the replacement if the *merger* — the advisor action — $A = \mathsf{A}$ is not too far from e_p. The approximation quality *from the participant view point* should be measured by the KLD, Bernardo (1979); Kárný and Guy (2012),

$$\mathsf{KL}(\mathsf{e}_p||\mathsf{A}) = \int_{G^\star} \mathsf{e}_p(G) \ln\left(\frac{\mathsf{e}_p(G)}{\mathsf{A}(G)}\right) \mathrm{d}G,\ p \in p^\star. \tag{8}$$

The decision rule R *is to provide mergers* $\mathsf{A} \in \mathsf{A}^\star$*, which make the KLD (8) small* $\forall p \in p^\star$*.*

The supporting DM task: The choice of R is the DM task, which is here formulated as FPD of the advisor decision rule entering the group closed-loop model (7). The ideal group closed-loop model described by a pd $\mathsf{C}_I(B)$ quantifies the $\mathcal{A}$ DM aim

$$\mathsf{C}_I(B) = \mathsf{C}_I(G|A)\mathsf{C}_I(A) = \mathsf{A}(G)\mathsf{R}_I(\mathsf{A}),\ G \in G^\star,\ \mathsf{A} \in \mathsf{A}^\star. \tag{9}$$

The first equality in (9) follows the definition of the group behaviour (6) and the chain rule for pds. The second one expresses the wish to get such a merger A, which describes well the group ignorance $G \in G^\star$. The ideal decision rule $\mathsf{R}_I(\mathsf{A})$ is one of the design options (tuning knob) through which the merging aim is fed into the solved group DM task.

For options (7), (9), the optimal decision rule $\mathsf{R}_\mathsf{O} = \mathsf{R}_\mathsf{O}(\mathsf{A}),\ \mathsf{A} \in \mathsf{A}^\star$, is

$$\mathsf{R}_\mathsf{O} \in \mathrm{Arg}\min_{\mathsf{R} \in \mathsf{R}^\star} \mathsf{KL}(\mathsf{C}_\mathsf{R}||\mathsf{C}_I) = \mathrm{Arg}\min_{\mathsf{R} \in \mathsf{R}^\star} \mathsf{KL}(\mathsf{R}||\mathsf{R}_I), \tag{10}$$

where the equality follows from cancelling the common factor A and integrating out the group ignorance $G \in G^\star$.

The set of admissible DM rules: The optimisation task (10) is determined by the set $\mathsf{R}^\star$ of admissible DM rules R and by the ideal decision rule R_I. They are gradually selected with the selection finalised in Section 4.

The KLD $\mathsf{KL}(\mathsf{R}||\mathsf{R}_I)$ is finite iff the support of R

$$\mathrm{supp}[\mathsf{R}] = \{\mathsf{A} \in \mathsf{A}^\star : \mathsf{R}(\mathsf{A}) > 0\} \tag{11}$$

is included in the support of R_I. Thus, meaningful choices for $\mathsf{R}^\star$ are

$$\emptyset \neq \mathsf{R}^\star \subset \{\mathsf{R} : \mathrm{supp}[\mathsf{R}] = \mathrm{supp}[\mathsf{R}_I]\}. \tag{12}$$

A specific choice of the set $\mathsf{R}^\star$ should primarily reflect the already formulated wish to deal with such Rs, which generate mergers $\mathsf{A} \in \mathsf{A}^\star$, which make KLDs (8) small. This singles out the rules R, for which the expectations (with respect to $\mathsf{R}(\mathsf{A})$) of divergences

$\mathsf{KL}(\mathsf{e}_p||\mathsf{A})$, $p \in p^\star$, (8) are finite and small. This wish and non-negativity of the inspected KLDs delimit

$$\begin{aligned} \mathsf{R}^\star &= \left\{ \mathsf{R} : \int_{\mathsf{A}^\star} \mathsf{R}(\mathsf{A}) \mathsf{KL}(\mathsf{e}_p||\mathsf{A}) \mathrm{d}\mathsf{A} \right. \\ &= \left. \int_{\mathsf{A}^\star} \mathsf{R}(\mathsf{A}) \left[\int_{G^\star} \mathsf{e}_p(G) \ln\left(\frac{\mathsf{e}_p(G)}{\mathsf{A}(G)}\right) \mathrm{d}G \right] \mathrm{d}\mathsf{A} \leq \phi_p < \infty,\ p \in p^\star \right\}, \end{aligned} \tag{13}$$

where the positive scalars $\phi = (\phi_p)_{p\in p^\star}$ should be chosen as small as possible. This intuitive wish is given the precise meaning in Section 4. The limits ϕ serve us a technical tool for determining the structure of the FPD-optimal decision rule $\mathsf{R_O}$ (10). Section 4 removes these undesirable tuning knobs of the advisor $\mathcal{A}$.

The next proposition determines the FPD-optimal merger selected from (13).

Proposition 1 (The Form of the FPD-Optimal Merger) *Let us consider a fixed ideal decision rule* R_I *and a non-empty set* $\mathsf{R}^\star$ *(13) of optional* $\mathcal{A}$ *decision rules. Then, the FPD-optimal decision rule* $\mathsf{R}_O(\mathsf{A})$ *solving (10) on this set is proportional to*

$$\begin{aligned} & \mathsf{R}_I(\mathsf{A}) \exp\left[\sum_{p\in p^\star} \lambda_p \int_{G^\star} \mathsf{e}_p(G) \ln\left(\mathsf{A}(G)\right) \mathrm{d}G \right] \\ = \ & \mathsf{R}_I(\mathsf{A}) \prod_{G\in G^\star} \left(\mathsf{A}(G)\right)^{\nu_\lambda(G)}, \quad \nu_\lambda(G) = \sum_{p\in p^\star} \lambda_p \mathsf{e}_p(G), \end{aligned} \tag{14}$$

where the non-negative Kuhn-Tucker multipliers $\lambda = (\lambda_p)_{p\in p^\star}$ *are chosen so that inequalities in (13) are satisfied.*

Proof We use the Kuhn-Tucker functional, Kuhn and Tucker (1951), respecting the constraints (13) on the optimised rules $\mathsf{R} \in \mathsf{R}^\star$ and rearrange it into the KLD of the optimised R to $\mathsf{R_O}$ (10), which is minimised by equating these arguments. The factor independent of A cancels and the final form of (14) uses simple operations relying on $|G^\star| < \infty$. □

4. Choice of the Tuning Knobs

The usefulness of the solution described by Proposition 1 is strongly influenced by the optional tuning knobs in them. They are gradually and unambiguously specified here.

The replacement of ϕ by the choice λ: A good advisor $\mathcal{A}$ should make the approximation of all processed pds $\mathsf{e} = (\mathsf{e}_p)_{p\in p^\star}$ in (13) as tight as possible. It primarily means that all inequalities are to be active and thus $\lambda > 0$. In this case, the choice of $\phi > 0$ becomes formally equivalent to the choice of $\lambda > 0$.

A desired *impartial* $\mathcal{A}$ must not prefer any $\mathcal{P}_p$. This implies the *basic requirement* on possible $\lambda > 0$, cf. Sečkárová (2015),

$$\int_{\mathsf{A}^\star} \mathsf{R_O}(\mathsf{A}) \mathsf{KL}(\mathsf{e}_p||\mathsf{A}) \mathrm{d}\mathsf{A} = \Phi,\ \forall p \in p^\star, \tag{15}$$

where the finite constant Φ is common for all $p \in p^\star$.

This specify $|p^\star| - 1$ conditions on $|p^\star|$ Kuhn-Tucker multipliers $\lambda > 0$. The quest for *tightness* of the approximation (8) implies that the advisor $\mathcal{A}$, which uses the ideal decision rule R_I leading to a Φ in (15), is preferable against the advisor $\tilde{\mathcal{A}}$ with $\tilde{\mathsf{R}}_I$ leading to a $\tilde{\Phi} \geq \Phi$. This provides the needed $|p^\star|$-th condition for an unambiguous choice of $\lambda > 0$.

The ideal decision rule R_I: The ideal decision rule $\mathsf{R}_I(\mathsf{A})$ is chosen as a finite mixture of Dirichlet pds $\mathsf{D}(\cdot|\cdot)$, which can arbitrarily-well approximate any $\mathsf{R}_I(\mathsf{A})$, Antoniak (1974),

$$\begin{aligned}
\mathsf{R}_I(\mathsf{A}) &= \sum_{k\in k^\star} \alpha_k \mathsf{D}(\mathsf{A}|\nu_{Ik}),\ k^\star = \{1,\ldots,|k^\star|\},\ |k^\star| < \infty, \quad \text{where} \qquad (16)\\
\alpha \in \alpha^\star &= \Big\{\alpha_k \geq 0,\ \sum_{k\in k^\star} \alpha_k = 1\Big\},\ \nu_{Ik} = (\nu_{Ik}(G))_{G\in G^\star} > 0,\ k \in k^\star,\\
\mathsf{D}(\mathsf{A}|\nu) &= \frac{\prod_{G\in G^\star} \mathsf{A}(G)^{\nu(G)-1}}{\mathsf{B}(\nu)},\ \mathsf{B}(\nu) = \frac{\prod_{G\in G^\star}\Gamma(\nu(G))}{\Gamma\left(\sum_{G\in G^\star}\nu(G)\right)},\ \nu = (\nu(G))_{G\in G^\star},
\end{aligned}$$

where the gamma function $\Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)\mathrm{d}t$, for $x > 0$.

The ideal decision rule (16) leads to an optimal rule (14) with the Dirichlet mixture form

$$\mathsf{R}_\mathsf{O}(\mathsf{A}) = \sum_{k\in k^\star} \alpha_k \mathsf{D}(\mathsf{A}|\nu_{Ik} + \nu_\lambda). \qquad (17)$$

The specific choice of α, $|k^\star|$ and ν_I follows from the required impartiality of the advisor $\mathcal{A}$ and from the following simple uncontroversial requirement: An impartial $\mathcal{A}$ chooses its ideal decision rule beforehand for all possible $\mathsf{e} = (\mathsf{e}_p)_{p\in p^\star}$. It means that it has to process well even the special case $\mathsf{e}_1 = \ldots = \mathsf{e}_{|p^\star|} \neq \nu_{Ik}$, $k \in k^\star$. In this case, the expected values

$$\int_{\mathsf{A}(G)^\star} \mathsf{A}(G)\mathsf{R}_\mathsf{O}(\mathsf{A}(G))\mathrm{d}\mathsf{A}(G) = \mathsf{e}_1(G) = \ldots = \mathsf{e}_{|p^\star|}(G), \quad \forall G \in G^\star, \qquad (18)$$

represent the only meaningful option. The optimal rule (17) has the expected values

$$\int_{\mathsf{A}(G)^\star} \mathsf{A}(G)\mathsf{R}_\mathsf{O}(\mathsf{A}(G))\mathrm{d}\mathsf{A}(G) = \sum_{k\in k^\star} \alpha_k \frac{\nu_{Ik}(G) + \nu_\lambda(G)}{\sum_{G\in G^\star}(\nu_{Ik}(G) + \nu_\lambda(G))}, \quad \forall G \in G^\star, \qquad (19)$$

for which the equality (18) is reachable iff $(\nu_{Ik}(G))_{k\in k^\star, G\in G^\star} = 0$. Thus, only identical improper components $\mathsf{D}(\mathsf{A}|\nu_{Ik} = 0)$ in (16) meet (18). For them, the mixture (16) reduces to the single-component improper ideal decision rule $\mathsf{R}_I(\mathsf{A}) = \mathsf{D}(\mathsf{A}|\nu_I = 0)$. This ideal decision rule always gives the proper optimal rule R_O (14) for the considered $\lambda > 0$, which makes $\nu_\lambda > 0$. Indeed, in the linear combination ν_λ (14) at least one e_p assigns a positive value to any $G \in G^\star$ as $G^\star$ is delimited by this requirement.

The choice of Φ (15): The above considerations uniquely specified the improper ideal decision rule R_I and the form of the optimal rule R_O

$$\mathsf{R}_I(\mathsf{A}) \;=\; \mathsf{D}(\mathsf{A}|\nu_I = 0) \propto \prod_{G\in G^\star} \mathsf{A}(G)^{-1} \overset{(17)}{\Rightarrow} \mathsf{R}_\mathsf{O}(\mathsf{A}) = \mathsf{D}(\mathsf{A}|\nu_\lambda). \qquad (20)$$

It also provided $|p^\star| - 1$ conditions (15) for $|p^\star|$ Kuhn-Tucker multipliers $\lambda > 0$. Thus, it remains to specify the constant Φ in (15).

For a given $\nu_\lambda > 0$ (14), and arbitrary $p \in p^\star$, the left-hand side of (15) reads

$$\begin{aligned} \Phi_p &= \int_{\mathsf{A}^\star} \mathsf{R}_{\mathsf{O}}(\mathsf{A}) \mathsf{KL}(\mathsf{e}_p||\mathsf{A}) \mathrm{d}\mathsf{A} \qquad (21) \\ &= -\sum_{G \in G^\star} \mathsf{e}_p(G) \int_{\mathsf{A}^\star} \mathsf{D}(\mathsf{A}||\nu_\lambda) \ln(\mathsf{A}(G)) \mathrm{d}\mathsf{A} - \underbrace{\sum_{G \in G^\star} \mathsf{e}_p(G) \ln(\mathsf{e}_p^{-1}(G))}_{\mathsf{H}(\mathsf{e}_p)} \\ &= -\sum_{G \in G^\star} \mathsf{e}_p(G) \left[\Psi(\nu_\lambda(G)) - \Psi\Big(\sum_{\tilde{G} \in G^\star} \nu_\lambda(\tilde{G}) \Big) \right] - \mathsf{H}(\mathsf{e}_p), \end{aligned}$$

where Ψ is the digamma function, the derivative of logarithm of the gamma function, Abramowitz and Stegun (1972). The formula for the expectation of $\ln(\mathsf{A}(G))$ with respect to Dirichlet pd is, e.g., in Kárný et al. (2006). The λ-independent summand $\mathsf{H}(\mathsf{e}_p)$ is the entropy of the processed pd e_p. For a simple presentation, let us denote[3]

$$\begin{aligned} \mu &= \sum_{p \in p^\star} \lambda_p, \quad \zeta_p = \frac{\lambda_p}{\sum_{p \in p^\star} \lambda_p}, \quad \mathsf{e}_\zeta(G) = \sum_{p \in p^\star} \zeta_p \mathsf{e}_p(G) \overbrace{\Rightarrow}^{(14),(21)} \qquad (22) \\ \Phi_p &= -\int_{G^\star} \mathsf{e}_p(G) \Psi(\mu \mathsf{e}_\zeta(G)) \mathrm{d}G + \Psi(\mu) - \mathsf{H}(\mathsf{e}_p). \end{aligned}$$

There, the pd e_ζ on $G^\star$ is the mixture of the pds supplied by the respective participants. Given (22), the choice of $\lambda > 0$ is equivalent to the choice of the free $\mathsf{p} - 1$ positive weighting probabilities $\zeta = (\zeta_p)_{p \in p^\star}$ and the scaling factor $\mu > 0$. For a given pd $\zeta > 0$, determined so that all $\Phi_p = \Phi$, the choice of Φ is equivalent to the choice of $\mu > 0$.

The value Φ_p is a decreasing function of μ and its absolutely smallest value $-\mathsf{H}(\mathsf{e}_p)$ is reached for $\mu = \infty$. Thus, μ is found by solving (15) with the biggest $\mu < \infty$, for which its solution exist. With the introduced notations and performed evaluations, (15) gets the form of $|p^\star| - 1$ equations for the probabilistic weights $\zeta > 0$

$$\int_{G^\star} (\mathsf{e}_p(G) - \mathsf{e}_1(G)) \Psi(\mu \mathsf{e}_\zeta(G)) \mathrm{d}G = \mathsf{H}(\mathsf{e}_1) - \mathsf{H}(\mathsf{e}_p), \ \forall p \in p^\star, \qquad (23)$$

and the scalar $\mu > 0$ is selected according to the above dictum.

Remark An analysis is needed for whether a solution of (23) exists, i.e. whether $\mathsf{R}^\star \neq \emptyset$. Also uniqueness is to be inspected and an efficient algorithm for its construction designed. All these important tasks are out of the paper scope.

5. Extension of Processed PDs c_r to PDs e Acting on Group Behaviour

The proposed merging works with the collection of pds $\mathsf{e} = (\mathsf{e}_p)_{p \in p^\star}$ defined over the group ignorance $G \in G^\star$, but participants $\mathcal{P}_p$, $p \in p^\star$, provide their closed-loop models $\mathsf{c}_{\mathsf{r}_p}$ assigning

3. The integral notation used hereafter underlines the conjecture that the *results* hold for $|B^\star| = \infty$.

probabilities to $b_p^\star$ only. The needed extension $\mathsf{c_r} = (\mathsf{c}_{\mathsf{r}_p})_{p\in p^\star} \to \mathsf{e} = (\mathsf{e}_p)_{p\in p^\star}$ is to be done the advisor $\mathcal{A}$. Altogether, $\mathcal{A}$ has to select a decision rule $\mathsf{R}_E : \mathsf{c_r} \to (\mathsf{A}, \mathsf{E})$, generating the action pair consisting of the derived merger A and the pd $\mathsf{E} \in \mathsf{E}^\star$ on the possible extensions $\mathsf{e} = (\mathsf{e}(G)_p)_{p\in p^\star, G\in G^\star}$ of $\mathsf{c_r} = ((\mathsf{c}_{\mathsf{r}_p}(b_p))_{b_p\in b_p^\star})_{p\in p^\star}$. The subscript $_E$ stresses that the discussed decision rule extends that treated in previous sections.

The randomised rule $\mathsf{R}_E(\mathsf{A}, \mathsf{E})$ describes action pairs $(\mathsf{A}(G|\mathsf{E}), \mathsf{E}(\mathsf{e}))$ occurring in the chain-rule factorisation

$$\mathsf{R}_E(\mathsf{A}, \mathsf{E}) = \mathsf{R}_E(\mathsf{A}|\mathsf{E})\mathsf{R}_E(\mathsf{E}). \tag{24}$$

The (extended) group behaviour is $B = (G, (\mathsf{A}, \mathsf{E})) = (\text{group ignorance}, (\text{action pair}))$ parts and the closed-loop model

$$\mathsf{C}_{\mathsf{R}_E}(B) = \mathsf{C}_{\mathsf{R}_E}(G|\mathsf{A}, \mathsf{E})\mathsf{C}_{\mathsf{R}_E}(\mathsf{A}|\mathsf{E})\mathsf{C}_{\mathsf{R}_E}(\mathsf{E}) = \mathsf{A}(G|\mathsf{E})\mathsf{R}_E(\mathsf{A}|\mathsf{E})\mathsf{R}_E(\mathsf{E}), \tag{25}$$

where the plain chain rule gives the first equality and the second one directly follows from the definitions of individual factors. The ideal closed-loop model is

$$\mathsf{C}_{IE}(B) = \mathsf{A}(G|\mathsf{E})\mathsf{R}_{IE}(\mathsf{A}|\mathsf{E})\mathsf{R}_{IE}(\mathsf{E}), \tag{26}$$

where the chain rule is used and the wish that A should describe well group ignorance is again applied. With this, the FPD-optimal extended decision rule $\mathsf{R}_{\mathrm{O}E}(\mathsf{A}, \mathsf{E}) = \mathsf{R}_{\mathrm{O}E}(\mathsf{A}|\mathsf{E})\mathsf{R}_{\mathrm{O}E}(\mathsf{E})$ is the minimising argument in

$$\begin{aligned} &\min_{\mathsf{R}_E\in\mathsf{R}_E^\star} \mathsf{KL}(\mathsf{C}_{\mathsf{R}_E}||\mathsf{C}_{IE}) = \min_{\mathsf{R}_E\in\mathsf{R}_E^\star} \mathsf{KL}(\mathsf{R}_E||\mathsf{R}_{IE}) \\ = &\min_{\mathsf{R}_E(\mathsf{E})} \int_{\mathsf{E}^\star} \mathsf{R}_E(\mathsf{E}) \left[\ln\left(\frac{\mathsf{R}_E(\mathsf{E})}{\mathsf{R}_{EI}(\mathsf{E})}\right) + \underbrace{\min_{\mathsf{R}_E(\mathsf{A}|\mathsf{E})} \int_{\mathsf{A}^\star} \mathsf{R}_E(\mathsf{A}|\mathsf{E}) \ln\left(\frac{\mathsf{R}_E(\mathsf{A}|\mathsf{E})}{\mathsf{R}_{IE}(\mathsf{A}|\mathsf{E})}\right) \mathrm{d}\mathsf{A}}_{\Phi(\mathsf{E})} \right] \mathrm{d}\mathsf{E}, \end{aligned} \tag{27}$$

where again $\mathsf{A}(G|\mathsf{E})$ has cancelled and G integrated out.

As demonstrated below, the treatment of the second summand in (27) reduces to that specified and optimised in previous sections. Thus, the ideal pd $\mathsf{R}_{IE}(\mathsf{E})$ determining the first summand only needs to be chosen. Its support has to allow only the extensions, which preserve the closed-loop models $\mathsf{c_r}$ provided by individual participants, i.e.

$$\mathrm{supp}[\mathsf{R}_E(\mathsf{E})] \quad = \quad \{\mathsf{E} : \mathrm{supp}[\mathsf{E}] = \{\mathsf{e} = (\mathsf{e}_p)_{p\in p^\star} : \mathsf{e}_p(G) = \mathsf{e}_p(G_{-p}|b_p)\mathsf{c_r}(b_p)\}\}, \tag{28}$$

where G_{-p} complements b_p to G, i.e. $G = (G_{-p}, b_p),\ \forall p \in p^\star$.

The following adopted leave-to-the-fate option, Kárný et al. (2006),

$$\mathsf{R}_{IE}(\mathsf{E}) = \mathsf{R}_E(\mathsf{E}) \text{ on } \mathrm{supp}[\mathsf{R}_E(\mathsf{E})] \tag{29}$$

respects that no other requirements exist with respect to $\mathsf{R}_E(\mathsf{E})$. Under (29), the minimised functional (27) is linear in the optimised $\mathsf{R}_E(\mathsf{E})$ and minimum is reached by the deterministic rule $\mathsf{R}_{\mathrm{O}E}(\mathsf{E})$ concentrated on a minimiser $\mathsf{E_O}$ of $\Phi(\mathsf{E})$ defined in (27).

For a fixed E concentrated on a point e in supp[E] (28), the second minimisation in (27), defining $\Phi(\mathsf{E})$, coincides with the optimisation in Section 3, Proposition 1. Also, the choice of the tuning knobs, Section 4, is the same when taking into account the correspondence

$$\mathsf{R}(\mathsf{A}) \leftrightarrow \mathsf{R}_E(\mathsf{A}|\mathsf{E}),\ \mathsf{R}_I(\mathsf{A}) \leftrightarrow \mathsf{R}_{IE}(\mathsf{A}|\mathsf{E}). \tag{30}$$

This uniquely determines $\mathsf{R}_{IE}(\mathsf{A}|\mathsf{E}) = \mathsf{D}(\mathsf{A}|\nu_I = 0)$ and the optimal $\mathsf{R}_{OE}(\mathsf{A}|\mathsf{E}) = \mathsf{D}(\mathsf{A}|\nu_\lambda)$ and the function $\Phi(\mathsf{E})$ to be minimised with respect to free factors $(\mathsf{e}(G_{-p}|b_p))_{p\in p^\star}$ coincides with the common value Φ (15).

A general pd E in the support of $\mathsf{R}_{IE}(\mathsf{E})$ (28) is a convex combination of pds E concentrated on pds $\mathsf{e} \in \mathsf{e}^\star = \text{supp}[\mathsf{E}]$. The complete analogy of reasoning made in Section 4 recommends the ideal decision rule $\mathsf{R}_{IE}(\mathsf{A}|\mathsf{E}) = \mathsf{D}(\mathsf{A}|\nu_I = 0)$. The set $\mathsf{e}^\star$ defined by constraints on possible e (28) is the convex set. Thus, the optimisation over $\mathsf{E} \in \mathsf{E}^\star$ reduces to optimisation over those concentrated on points $\mathsf{e} = (\mathsf{e}(G_{-p}|b_p)\mathsf{c}_{\mathsf{rp}}(b_p))_{p\in p^\star, G\in G^\star}$. This finalises the solution of the general merging case summarised now for reference purposes.

Proposition 2 (General Merging) *Let the closed-loop models of neighbours*[4] $\mathsf{c}_{\mathsf{r}}(G) = (\mathsf{c}_{\mathsf{r}_p}(b_p))_{p\in p^\star}$ *be given and the group ignorance* $G \in G^\star$ *be the concatenation of all variables occurring in* b_p, $p \in p^\star$, *while for each* $G \in G^\star$ *at least one* $\mathsf{c}_{\mathsf{r}_p}(b_p)$ *is positive. Then, the FPD-optimal (extended) advising rule of an impartial advisor* $\mathcal{A}$ *respecting (18) is described as follows. Among pds of the form* $\mathsf{e}_p(G) = \mathsf{e}(G_{-p}|b_p)\mathsf{c}_{\mathsf{r}_p}(b_p)$, $p \in p^\star$, $G = (G_{-p}, b_p)$, *find*

$$\min_{(\mathsf{e}(G_{-p}|b_p))_{p\in p^\star}} \Phi(\mathsf{e}) = \tag{31}$$

$$\min_{(\mathsf{e}(G_{-p}|b_p))_{p\in p^\star},\, \mu>0,\, \zeta\in\zeta^\star} \int_{G^\star} \mathsf{e}_1(G)\Psi(\mu\mathsf{e}_\zeta(G))\mathrm{d}G + \Psi(\mu) - \mathsf{H}(\mathsf{e}_1)$$

with the optimised $\mu > 0$ *and the probabilistic weights* $\zeta > 0$ *entering*

$$\mathsf{e}_\zeta(G) = \sum_{p\in p^\star} \zeta_p \mathsf{e}(G_{-p}|b_p)\mathsf{c}_{\mathsf{r}_p}(b_p) \text{ and}$$

$$\int_{G^\star} (\mathsf{e}_p(G_{-p}|b_p)\mathsf{c}_{\mathsf{r}_p}(b_p) - \mathsf{e}_1(G_{-1}|b_1)\mathsf{c}_{\mathsf{r}1}(b_1))\Psi(\mu\mathsf{e}_\zeta(G))\mathrm{d}G = \mathsf{H}(\mathsf{e}_1) - \mathsf{H}(\mathsf{e}_p),\ \forall p \in p^\star,$$

with $\Psi(\mu)$ *being digamma function and the entropy* $\mathsf{H}(\mathsf{e}_p)$ *defined*

$$\mathsf{H}(\mathsf{e}_p) = -\int_{G^\star} \mathsf{e}_p(G_{-p}|b_p)\mathsf{c}_{\mathsf{r}_p}(b_p)\ln(\mathsf{e}_p(G_{-p}|b_p)\mathsf{c}_{\mathsf{r}_p}(b_p))\mathrm{d}G. \tag{32}$$

The minimiser $(\mathsf{e}_O, \mu_O, \zeta_O)$ *determines the optimal advisory rule with* $\mathsf{R}_{OE}(\mathsf{E})$ *concentrated on* e_O *and* $\mathsf{R}_{OE}(\mathsf{A}|\mathsf{E}) = \mathsf{R}_{OE}(\mathsf{A}|\mathsf{e}_O) = \mathsf{D}(\mathsf{A}|\mu_O\mathsf{e}_{O\zeta_O})$, *with* $\mathsf{e}_{O\zeta_O} = \sum_{p\in p^\star} \zeta_{Op}\mathsf{e}_{op}(G_{-p}|b_b)\mathsf{c}_{\mathsf{r}_p}(b_p)$.

6. Concluding Remarks

The contribution to a prescriptive DM theory, which respects constraints of real DM participants in knowledge sharing, is the main message brought by this paper. The proposed merging of probabilistic knowledge of neighbours is based on a systematic use of FPD and the impartiality requirement suppressing undesirable tuning knobs. Limited space prevents us in describing how to support participants in transforming their non-probabilistic knowledge into probabilities (crisp values as measures concentrated on them, marginal pds and deterministic relations, Sečkárová (2015), extended by minimum cross-entropy principle, Shore and Johnson (1980), re-interpretation of fuzzy rules as conditional pds, etc.)

The paper described how to construct advices but does not guide when accept them. It has, however, a relatively clear conceptual solution. After merging, the advisor $\mathcal{A}$ generates

4. The same procedure applies even when factors of $\mathsf{c}_{\mathsf{r}_p}(b_b)$ are communicated to $\mathcal{A}$ only.

an advice $\mathsf{A_O}$ either via sampling from $\mathsf{R}_{OE}(\mathsf{A}|\mathsf{e_O})$, Proposition 2, or by taking $\mathsf{A_O}$ as the expected value of the optimal advising rule. Then the advice $\mathsf{c}_{\mathcal{A}p}$ is simply the marginal pd of $\mathsf{A_O}(G),\ G \in G^\star$

$$\mathsf{c}_{\mathcal{A}p}(b_p) = \int_{G^\star_{-p}} \mathsf{A_O}(G_{-p}, b_p)\mathrm{d}G_{-p}. \tag{33}$$

It operates on variables known to the participant $\mathcal{P}_p$ and as such it is understandable to her. If the participant replaces $\mathsf{c}_{\mathsf{r}_p}$ or its factor by the corresponding factor of the advice $\mathsf{A_O}$ she gets the modified closed-loop model $\mathsf{c}_{\mathcal{A}p}$. Naturally, she takes it as a good advice iff

$$\mathsf{D}(\mathsf{c}_{\mathcal{A}p}||\mathsf{c}_{Ip}) \leq \mathsf{D}(\mathsf{c}_{\mathsf{r}_p}||\mathsf{c}_{Ip}). \tag{34}$$

Thus, $\mathcal{P}_p$ evaluates the quality of the advice according to her original selfish aim. The improvement (34) leading to advice acceptance is possible due to the fact that the advisor $\mathcal{A}$ operates on a wider knowledge and may respect supportive as well as competitive tendencies in the group interactions.

Specialisations to subclasses of our general solution, like Quinn et al. (2016), theoretical analysis of the proposed solution and its steps as well as the conversion of the conceptual solution into a practical tool are obvious directions to be addressed.

Good news are that preliminary brute-force numerical experiments (made without extensions only) indicate desirable properties of the merger: i) the average is the optimal merger iff the merged pds have identical entropy but its weight $\mu < |p^\star|$; ii) the weight ζ_p assigned to e_p increases with entropy $\mathsf{H}(\mathsf{e}_p)$, which makes the merger robust; iii) the Bayes' rule is the optimal merger if the merged pds are concentrated on crisp values.

It remains to discuss the concessions made, see Section 2. The obtained results are conjectured to be amendable to unbounded refinements of the discretisation mappings. This makes the assumption about finite cardinality of the behaviours'sets (3) non-restrictive. Also, dynamic DM is expected to be solvable in the exactly the same way as the considered static case. Note that whenever an external decision layer provides relative importance degrees of the respective participants within a group then the advisor can respect them by making the value Φ (15) the participant-specific.

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