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A semiparametric nonlinear quantile regression model for financial returns

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Abstract: Accurately measuring and forecasting value-at-risk (VaR) remains a challenging task at the heart of financial economic theory. Recently, quantile regression models have been used successfully to capture the conditional quantiles of returns and to forecast VaR accurately. In this paper, we further explore nonlinearities in data and propose to couple realized measures with the nonlinear quantile regression framework to explain and forecast the conditional quantiles of financial returns. The nonlinear quantile regression models are implied by the copula specifications and allow us to capture possible nonlinearities, tail dependence, and asymmetries in the conditional quantiles of financial returns. Using high frequency data that covers most liquid US stocks in seven sectors, we provide ample evidence of asymmetric conditional dependence with different levels of dependence, which are characteristic for each industry. The backtesting results of estimated VaR favour our approach.

Keywords: copula quantile regression; realized volatility; value-at-risk.

JEL: C14; C32; C58; F37; G32.

1 Introduction

A number of important financial decisions require proper specification and estimation of the entire financial returns distribution. Models for conditional quantiles help researchers and practitioners to measure and forecast value-at-risk (VaR) (quantiles of the return distribution), although most current approaches rely on unrealistic global distributional assumptions. In turn, many parametric approaches fail in estimation due to a fat tailed, non-elliptical empirical joint distribution of financial returns data in practice. The difficulty further increases if we expect the distributions to change over time, and if we need a proper strategy for capturing the dynamics in quantiles. In this paper, we propose to model the future conditional quantiles of returns using the realized volatility and nonlinear quantile regression (NQR) framework. Our approach is based on parametric copula models that allow the capture of potential nonlinear dynamics while retaining a semiparametric flexibility. Thus, a semiparametric NQR for financial returns should deliver more accurate estimates and forecasts of conditional quantiles that are robust to model misspecification.

Regression quantiles, as introduced in a seminal work by Koenker and Bassett (1978), remain a lively area of research with several recent advances. Koenker (2004) extends the quantile regression to panel data applications and introduces a general approach to estimating quantile regression models for longitudinal data. Xiao (2009) propose a co-integration model with quantile-varying coefficients that are allowed to be affected by the shocks received in each period over the innovation quantile. Chen et al. (2009) introduce nonlinear-in-parameters quantile auto-regression (QAR) models using parametric copulas.

Koenker and Bassett (1978) introduced the regression quantiles more than three decades ago; however, only recently did the financial literature focus more on this. With these advances, quantile regression models

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have important applications in risk management, portfolio optimization, and asset pricing. Engle and Manganelli (2004) introduce the conditional autoregressive value at risk (CAViaR) modelling quantiles of the return distribution directly by regressing on its lagged values. In a recent work, Cappiello et al. (2014) measure co-movements by using regression quantiles. For any set of quantiles, the authors compute the conditional probability that a random variable is lower than a given quantile, when another random variable is also lower than its corresponding quantile.

Fewer contributions explore the relationship between returns and their realized volatility, measured using high frequency data. Andersen et al. (2003) integrate high-frequency data to model and forecast the realized volatility and returns distribution. Cle (2008) find the realized volatility to be useful for computing VaR forecasts. The authors calculate and evaluate quantile forecasts of the daily exchange rate returns of five currencies. Brownlees and Gallo (2010) forecast VaR using different volatility measures. The authors find that the predictive ability of VaR improves considerably when exploiting ultra-high-frequency data volatility measures. By employing two alternative joint specifications of daily returns and realized volatility, Maheu and McCurdy (2011) find that the realized volatility computed at high frequency improves the out-of-sample (OOS) forecasts of the return distribution. Finally, under a semi-parametric quantile regression framework, Žikeš and Baruník (2016) utilize nonparametric measures of the various components of *ex post* variation in asset prices to study the conditional quantiles of daily asset returns and realized volatility and to forecast their future values. We exploit the ideas in a nonlinear semiparametric conditional quantile regression framework to estimate the dependence between returns and the realized volatility at quantiles of interest. Under mild conditions, the model allows for the global misspecification of parametric copulas and marginals.

The contribution of this paper is two-fold. First, we propose to use realized measures in the NQR framework to explain and forecast the conditional quantiles of financial returns. Second, we apply the proposed model to a pool of the most liquid US assets across different industries. The article is structured as follows. Section 2 introduces the copula quantile regression model. Section 3 describes the data under analysis. Section 4 presents an application to real data. Section 5 evaluates the quantile forecast, and Section 6 concludes the article. Notably, the number of results produced by this study is very large, and the results are overlapping; therefore, we relegate our auxiliary results to the online supplementary Appendix that is available in the Supplementary data section published with the article.

2 Nonlinear quantile regression model for financial returns

Žikeš and Baruník (2016) propose a simple linear semiparametric model that successfully captures the conditional quantiles of returns. The approach is based on quantile regressing (log) returns r_{t+1} on its own past realized volatility measured with high frequency data. Žikeš and Baruník (2016) Hence, we assume that the τ-quantile of future returns that are conditional on information set $\mathcal{F}_t$ can be written as a linear function of its past quadratic variation,[1]

$$q_\tau(r_{t+1}|\mathcal{F}_t)=\beta_0(\tau)+\beta_\vartheta(\tau)'\vartheta_t, \tag{1}$$

where ϑ_t is a measure of quadratic variation $QV_t=\int_0^t \sigma_s^2 \mathrm{d}s$ associated with the logarithmic price process $dp_t=\mu_t d_t+\sigma_t dW_t$ that evolves over the time interval $[0, T]$, and $\beta_0(\tau), \beta\vartheta(\tau)$ are vectors of coefficients to be estimated.

Under the assumption of no jumps in the underlying price process, Žikeš and Baruník (2016) establish a connection that exists between the linear quantile regression model (LQR) in Equation 1 and the assumed logarithmic price process. The researchers argue that the conditional quantile $q_\tau(r_{t+1}|\mathcal{F}_t)$ can be obtained from the conditional distribution of r_{t+1}, the given information set $\mathcal{F}_t$ that contains ϑ_t. Thus, the implied conditional quantiles of the returns density $q_\tau(r_{t+1}|\mathcal{F}_t)$ can be approximated conveniently at all quantiles $\tau \in (0, 1)$ by linear

1 Note that Zikeš and Baruník (2016) allow a more general specification that includes various components of volatility and weakly exogenous variables to drive the conditional quantiles in the model.

functions of the current and past values of volatility measures, and Equation 1 can be conveniently used to model and forecast VaR.

A LQR in Equation 1 can be estimated as a solution to the following problem (Koenker and Bassett, 1978):

$$\min_{\beta\in\mathbb{R}^k}\left(\sum_{t\in\mathcal{T}_\tau}\tau|r_{t+1}-\beta_0(\tau)-\beta_\vartheta(\tau)'\vartheta_t|+\sum_{t\in\mathcal{T}_{1-\tau}}(1-\tau)|r_{t+1}-\beta_0(\tau)-\beta_\vartheta(\tau)'\vartheta_t|\right), \tag{2}$$

where $\mathcal{T}_\tau=\{t:r_{t+1}\geq\beta_0(\tau)-\beta\vartheta(\tau)'\vartheta_t\}$ and $\mathcal{T}_{1-\tau}$ is its complement.

In the special case in which $\tau=0.5$, the above quantile regression, delivers the least absolute deviation (LAD) estimate, the LAD is a robust alternative to the classical ordinary least squares (OLS) estimator whenever the errors have a fat-tailed distribution. The problem defined in Equation 2 does not have a closed form-solution; however Portnoy and Koenker (1997) provide a computationally fast algorithm that is also implemented in the `quantreg` package for `R` software.

By quantile regressing the future returns on their past volatility, Žikeš and Baruník (2016) assume that the relationship is linear. In our work, we further extend the researchers successful approach and assume nonlinearities in the relation, which should be explored. To model a nonlinear relationship between future returns and their past volatility, we propose to use NQRs instead. Recall the original problem of explaining the conditional quantiles of future returns r_{t+1} as a function of past volatility ϑ_t. Under a NQR framework, the parameters of quantile regression can be estimated as a solution to the following problem:

$$\min_{\delta}\left(\sum_{t\in\mathcal{T}_p}\tau|r_{t+1}-\boldsymbol{q}(\vartheta_t,\tau;\boldsymbol{\delta})|+\sum_{t\in\mathcal{T}_{1-\tau}}(1-\tau)|r_{t+1}-\boldsymbol{q}(\vartheta_t,\tau;\boldsymbol{\delta})|\right), \tag{3}$$

where $\mathcal{T}_\tau=\{t:r_{t+1}\geq\boldsymbol{q}(\vartheta_t,\tau;\boldsymbol{\delta})\}$ and $\mathcal{T}_{1-\tau}$ is its complement. Koenker and Park (1996) developed an interior point algorithm to compute the quantile regression estimates for problems with nonlinear response functions. The researchers approach to solve the nonlinear problem is to solve a succession of linearized problems, i.e. by splitting the nonlinear problem into a set of linear ones.

In a recent work, Allen et al. (2013) study the inverse volatility-return relationship by examining six volatility-return stock index pairs using linear and NQR models. The researchers find that the relationship is not uniform across the distribution of the volatility-price return pairs. In our work, we utilize a different approach than Allen et al. (2013), and we regress future returns r_{t+1} on their lagged realized volatility computed as the square root of the sum of squared intraday returns

$$\vartheta_t=\sqrt{RV_t}=\sqrt{\sum_{k=1}^{M}(\Delta_k r_t)^2} \tag{4}$$

where $\Delta_k r_t=r_{t-1+\left(\frac{K}{M}\right)}-r_{t-1+\left(\frac{K-1}{M}\right)}$ is the k-th 5-minute intraday return from the sample of $k=1\ldots, M$.

By estimating quantile regressions using small samples, we may encounter a quantile crossing problem caused by estimation error or model misspecification. Recently, Dette and Volgushev (2008) propose a non-parametric estimate of conditional quantiles that avoids quantile crossing. The method uses an initial estimate of the conditional distribution function in the first step and solves the problem of inversion and monotonization with respect to $\tau\in(0, 1)$ simultaneously. Chernozhukov et al. (2009, 2010)) propose a closely related but different method to address the quantile crossing problem.[2] The researchers method consists of sorting the original estimated non-monotone curve into a monotone rearranged curve. This method is a two-step procedure; first, a preliminary (parametric) estimate of the conditional quantile curve is isotonized and inverted. Next, the final non-crossing estimates are constructed by an inversion of the curves that are obtained in the first step. This method is incorporated in the `quantreg` package in `R` software, which is also the one we use in this paper. Here we note that, in our empirical applications reported later in this article, the quantile crossing problem never arises.

2 For a comparison of these approaches, refer to Dette and Volgushev (2008).

2.1 Copulas and nonlinear quantile regression

Having defined the framework for NQR model for financial returns, which we use to model and predict conditional VaR, a function $\boldsymbol{q}(\vartheta_t, \tau; \delta)$ remains to be defined. Here, we employ the approach of Bouyé and Salmon (2009), who consider a NQR model based on copulas.

Copula quantile regression models can be viewed as a special case of the NQR models. Using the properties of conditional probability distribution, the link between copula functions and conditional quantile functions can be established. Consider a random sample of realized volatility $(\vartheta_1, \ldots, \vartheta_T)$ and returns $(r_1, \ldots, r_T)$ from $\mathcal{V}$ and $\mathcal{R}$, respectively. The probability distribution of future return r_{t+1} conditional on the realized volatility ϑ_t can be defined as

$$\begin{aligned}\tau(r_{t+1}|\vartheta_t;\delta) &= \mathbb{P}(\mathcal{R}\le r_{t+1}|\mathcal{V}=\vartheta_t)\\ &= \mathbb{E}(\mathbb{1}\{\mathcal{R}\le r_{t+1}\}|\mathcal{V}=\vartheta_t)\\ &= \lim_{\epsilon\to 0}\mathbb{P}(\mathcal{R}\le r_{t+1}|\vartheta_t\le\mathcal{V}\le\vartheta_t+\epsilon)\\ &= \lim_{\epsilon\to 0}\frac{\boldsymbol{F}(\vartheta_t+\epsilon, r_{t+1};\delta)-\boldsymbol{F}(\vartheta_t, r_{t+1};\delta)}{F_{\mathcal{V}}(\vartheta_t+\epsilon)-F_{\mathcal{V}}(\vartheta_t)}\\ &= \lim_{\epsilon\to 0}\frac{\boldsymbol{C}(F_{\mathcal{V}}(\vartheta_t+\epsilon), F_{\mathcal{R}}(r_{t+1});\delta)-\boldsymbol{C}(F_{\mathcal{V}}(\vartheta_t), F_{\mathcal{R}}(r_{t+1});\delta)}{F_{\mathcal{V}}(\vartheta_t+\epsilon)-F_{\mathcal{V}}(\vartheta_t)}.\end{aligned}$$

Denoted by $C_1(\cdot,\cdot;\delta)$, the partial derivative of the copula function with respect to the first argument, the probability distribution of r_{t+1}, conditional on ϑ_t, can also be written as

$$\tau(r_{t+1}|\vartheta_t;\delta)=\frac{\partial C(u_1, u_2;\delta)}{\partial u_1}=C_1(F_{\mathcal{V}}(\vartheta_t), F_{\mathcal{R}}(r_{t+1});\delta), \tag{5}$$

where $u_1=F_{\mathcal{V}}(\vartheta_t)$ and $u_2=F_{\mathcal{R}}(r_{t+1})$. Refer to the 7 for the proof. In the case that Equation 5 is invertible with respect to r_{t+1}, the relationship between $\mathcal{V}$ and the quantile of $\mathcal{R}$ can be expressed as

$$\mathcal{Q}_{\mathcal{R}|\mathcal{V}}(\tau|\vartheta_t)=\boldsymbol{q}(\vartheta_t, \tau;\delta)=F_{\mathcal{R}}^{-1}(\mathcal{D}(F_{\mathcal{V}}(\vartheta_t), \tau;\delta)), \tag{6}$$

where $\mathcal{D}$ is the partial inverse of C_1 in the second argument and $F_{\mathcal{R}}^{-1}$ is the pseudo-inverse of $F_{\mathcal{R}}$. There is a possibility that the relationship in Equation 5 is not invertible, and if this is the case, numerical methods should be used.[3] We can generate observations on $\mathcal{R}$ given $\mathcal{V}$ by evaluating Equation 6 and replacing τ by independent uniformly distributed draws.

2.2 Copula quantile functions

To complete the model specification, we need to employ specific copula quantile functions. In this paper, we use the two most popular ones, the Normal and the *t* copulas. The bivariate Normal copula function can be written as

3 The Gumbel copula is a typical example in which numerical methods are required. Its τ quantile function is given by the following expression

$$\tau=u_1^{-1}(-\log(u_1))^{\theta-1}e^{-((-\log(u_1))^{\theta}+(-\log(u_2))^{\theta})^{1/\theta}}((-\log(u_1))^{\theta}+(-\log(u_2))^{\theta})^{\frac{1}{\theta}-1}$$

As one can observe, this expression is not invertible. However, the implementation of the numerical methods is time consuming because finding the root of the non-invertible quantile function above needs extra time beyond solving the non-linear quantile regression problem.

$$C_\rho^N(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp\left(\frac{-(k^2 - 2\rho ks + s^2)}{2(1-\rho^2)} \right) dkds,$$

where $\Phi(\cdot)$is the standard Normal distribution, and ρ the linear correlation. The partial derivative with respect to $u_1 = F_{\mathcal{V}}(\vartheta_t)$ is

$$\tau = \Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u_1)}{\sqrt{1-\rho^2}} \right), \tag{7}$$

and the quantile curve implied by the copula function can be obtained by solving Equation 7 for u_2 as

$$u_2 = \Phi(\rho\Phi^{-1}(F_{\mathcal{V}}(\vartheta_t)) + \sqrt{1-\rho^2}\Phi^{-1}(\tau)). \tag{8}$$

Finally, the relationship between ϑ_t and the quantile of r_{t+1} is then

$$\mathcal{Q}_{\mathcal{R}|\mathcal{V}}(\tau|\vartheta_t) = \boldsymbol{q}(\vartheta_t, \tau; \delta) = F_{\mathcal{R}}^{-1}(\Phi(\rho\Phi^{-1}(F_{\mathcal{V}}(\vartheta_t)) + \sqrt{1-\rho^2}\Phi^{-1}(\tau))), \tag{9}$$

where $F_{\mathcal{R}}^{-1}$ is the pseudo-inverse of $F_{\mathcal{R}}$. The distributions of $F_{\mathcal{V}}$ and $F_{\mathcal{R}}$ can be specified either parametrically or non-parametrically. We do not impose any assumption on the distribution of the margins and use the empirical distribution instead. Thus, we avoid the problem of misspecification. If we assume that $F_{\mathcal{R}}$ is known only to a location and scale parameter, the quantile curve will have the following form

$$\mathcal{Q}_{\mathcal{R}|\mathcal{V}}(\tau|\vartheta_t) = \boldsymbol{q}(\vartheta_t, \tau; \delta) = \mu + \sigma F_{\mathcal{R}}^{-1}(\Phi(\rho\Phi^{-1}(F_{\mathcal{V}}(\vartheta_t)) + \sqrt{1-\rho^2}\Phi^{-1}(\tau))). \tag{10}$$

When the margins are estimated non-parametrically, we obtain a semiparametric copula (quantile) model. The properties of this estimator are established by Chen and Fan (2006). The authors also show that the semiparametric conditional quantile estimators are automatically monotonic across quantiles; this is a useful property for conditional VaR models. In this work, for margins of the returns and for realized volatility, we employ the non-parametric empirical distribution F_j introduced by Genest et al. (1995), which consists of modelling the marginal distributions by the (rescaled) empirical distribution as

$$\hat{F}_j(z) = \frac{1}{T+1} \sum_{t=1}^{T} \mathbb{1}\{\hat{z}_{j,t} \le z\}, \ \hat{z}_{j,t} \in \{r_t, \vartheta_t\} \tag{11}$$

where $\mathbb{1}\{.\}$ is an indicator function. The second copula quantile function that we use in our modelling strategy is based on a bivariate t copula

$$C_{\eta,\rho}^t(u_1, u_2) = \int_{-\infty}^{t_\eta^{-1}(u_1)} \int_{-\infty}^{t_\eta^{-1}(u_2)} \frac{\Gamma\left(\frac{\eta+2}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)\pi\eta\sqrt{1-\rho^2}} \left(1 + \frac{k^2 - 2\rho ks + s^2}{\eta(1-\rho^2)}\right)^{-\frac{\eta+2}{2}} dkds,$$

where $\Gamma(\cdot)$ is the Gamma distribution, ρ is the linear correlation and η is the degrees of freedom parameter. As can be observed in the expression below, the partial derivative with respect to u_1 is a bit more complicated

$$\tau = t_{\eta+1}\left(\frac{t_\eta^{-1}(u_2) - \rho t_\eta^{-1}(u_1)}{\sqrt{\frac{(\eta + [t_\eta^{-1}(u_1)]^2)(1-\rho^2)}{\eta+1}}} \right). \tag{12}$$

Following the same steps as previously performed, the quantile curve implied by the t copula is obtained by solving Equation 12 for u_2 as

$$u_2 = t_\eta[t_{\eta+1}^{-1}(\tau)\sqrt{(\eta+1)^{-1}(1-\rho^2)(\eta+[t_\eta^{-1}(F_\mathcal{V}(\vartheta_t))]^2)} + \rho t_\eta^{-1}(F_\mathcal{V}(\vartheta_t))]. \tag{13}$$

Finally, the relationship between ϑ_t and the quantile of r_{t+1} can then be expressed as

$$\begin{aligned} \mathcal{Q}_{\mathcal{R}|\mathcal{V}}(\tau|\vartheta_t) &= \boldsymbol{q}(\vartheta_t, \tau; \delta) \\ &= F_\mathcal{R}^{-1}(t_\eta[t_{\eta+1}^{-1}(\tau)\sqrt{(\eta+1)^{-1}(1-\rho^2)(\eta+[t_\eta^{-1}(F_\mathcal{V}(\vartheta_t))]^2)} + \rho t_\eta^{-1}(F_\mathcal{V}(\vartheta_t))]). \end{aligned} \tag{14}$$

Again, if we assume that $F_\mathcal{R}$ is known only to a location and scale parameter, the quantile curve will have the following form

$$\begin{aligned} \mathcal{Q}_{\mathcal{R}|\mathcal{V}}(\tau|\vartheta_t) &= \boldsymbol{q}(\vartheta_t, \tau; \delta) \\ &= \mu + \sigma F_\mathcal{R}^{-1}(t_\eta[t_{\eta+1}^{-1}(\tau)\sqrt{(\eta+1)^{-1}(1-\rho^2)(\eta+[t_\eta^{-1}(F_\mathcal{V}(\vartheta_t))]^2)} + \rho t_\eta^{-1}(F_\mathcal{V}(\vartheta_t))]). \end{aligned} \tag{15}$$

The theoretical quantile curves of models in Equations 8–9 and 13–14 are plotted in Figure 1, which shows that, for the same correlation parameter and the same margins, different copulas capture different types of dependence (Figure 1C and D). We know that the Normal copula does not have tail dependence for $\rho<1$, whereas the t copula has tail dependence that is symmetric and is estimated as $\tau^L = \tau^U = 2t_{\eta+1}\left(-\sqrt{\frac{(\eta+1)(1-\rho)}{1+\rho}}\right)$.

The tail behaviour of these copulas is depicted by Figure 1A and B.

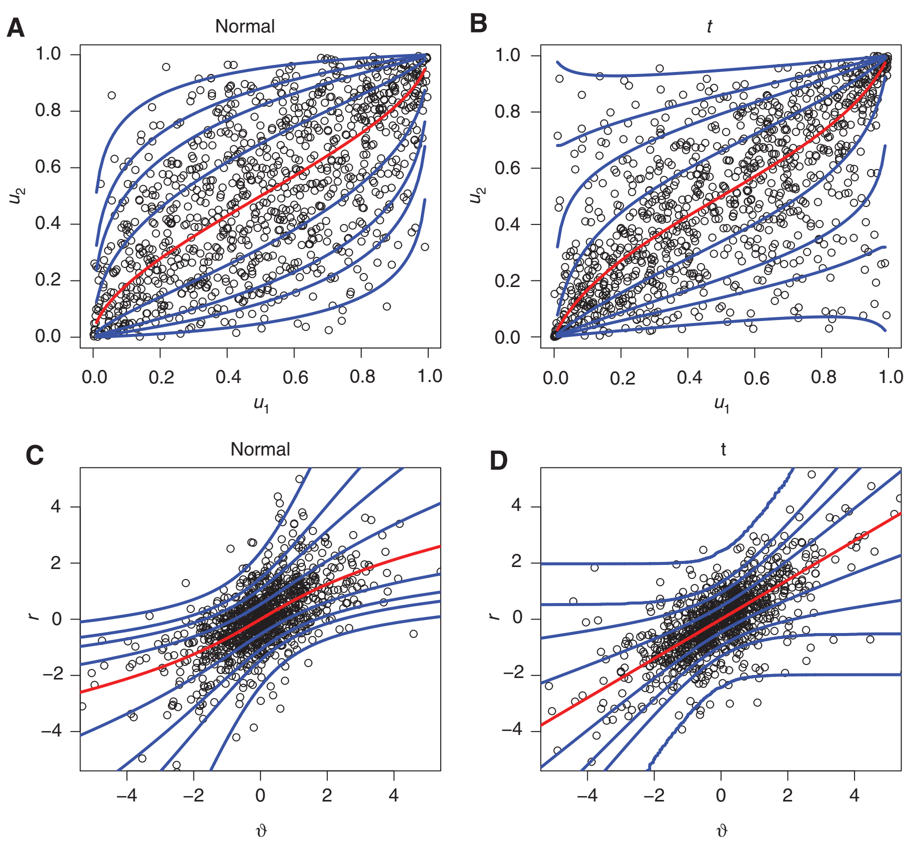

Figure 1: Theoretical quantile curves of the Normal and the t copulas, both with correlation $\rho=0.7$ and for the latter 3 degrees of freedom. The marginal distributions in (C) and (D) are $t_3(\cdot)$, and the quantiles are $\tau \in \{0.01, 0.05, 0.25, 0.5, 0.75, 0.9, 0.95, 0.99\}$ in all cases.

Table 1: Sectors and representative stocks.

Sector	Stocks
Financials	Bank of America Corporation (BAC), Citigroup (C), Wells Fargo & Company (WFC)
Information technology	Apple (AAPL), Intel Corporation (INTC), Microsoft Corporation (MSFT)
Energy	Chevron Corporation (CVX), Schlumberger Limited (SLB), Exxon Mobil Corporation (XOM)
Consumer discretionary	Amazon.com (AMZN), Walt Disney Company (DIS), McDonald's Corp. (MCD)
Consumer staples	Coca-Cola Company (KO), Procter & Gamble Co. (PG), Wal-Mart Stores (WMT)
Telecommunication services	Comcast Corporation (CMCSA), AT&T (T), Verizon Communications (VZ)
Health care	Johnson & Johnson (JNJ), Merck & Co. (MRK), Pfizer (PFE)

3 The data description

We study conditional quantiles of the 21 most liquid US stock returns from the seven main market sectors defined in accordance with the global industry classification standard (GICS).[4] We use three stocks with the highest market capitalization in a sector as representative of the analysed sector. The selected stocks represent approximately half of the total capitalization of the sector. The sectors and representative stocks are listed in Table 1. The data spans from August 2004 to December 2011. The period under study is informative because it covers the recent US recession of Dec. 2007 – June 2009 and 3 years before and after the crisis. The data were obtained from Price-Data.com.[5]

For the computation of realized measures, we restrict the analysis to 5-min returns during the 9:30 a.m. to 4:00 p.m. business hours of the New York Stock Exchange (NYSE). The data are time-synchronized by the same time-stamps. To eliminate potential estimation bias, which could originate from low activity, we eliminate transactions executed on Saturdays and Sundays, US federal holidays, December 24 to 26, and December 31 to January 2. Consequently, our data contains 1835 trading days.

In Table 4, we present descriptive statistics of the returns and realized volatility for the data that constitute our sample. All daily returns series have excess kurtosis and, as usual, the stocks from the Financial sector, on average, have higher volatility than the stocks from the other sectors.

4 Empirical results

4.1 Full sample results

We quantile-regress the returns at time *t+1* conditional on the realized volatility at time *t*, using the full sample of data. In the analysis, we focus on the 1%, 5%, 10%, 25%, 75%, 90% and 95% quantiles because these are most interesting from an economic perspective. We solve the copula quantile regression problem as in Equation 3, in which the quantile curve function $\mathbf{q}(\cdot)$ is in accordance with either Equation 10 in the case of the Normal copula or Equation 15 in the case of the *t* copula. For comparison, we also estimate the LQR as in Equation 1. Given that the parameters of the linear model are not directly comparable to the nonlinear parameters, we do not report them here to save space; however, they are available on request from the authors. Instead, the performance of linear and NQR models is compared via VaR forecast accuracy.

We begin with the empirical estimates, which are synthesized into boxplots for clarity.[6] Figures 2 and 3 plot the parameters ρ, μ, σ for both nonlinear models and the degrees of freedom η for the *t* copula,

4 Morgan Stanley Capital International (MSCI) and Standard & Poor developed the GICS. This standard is a common global classification standard used by the global financial community.
5 http://www.price-data.com/
6 The detailed results of the copula parameters can be found in Tables 1–4 in the supplementary online Appendix.

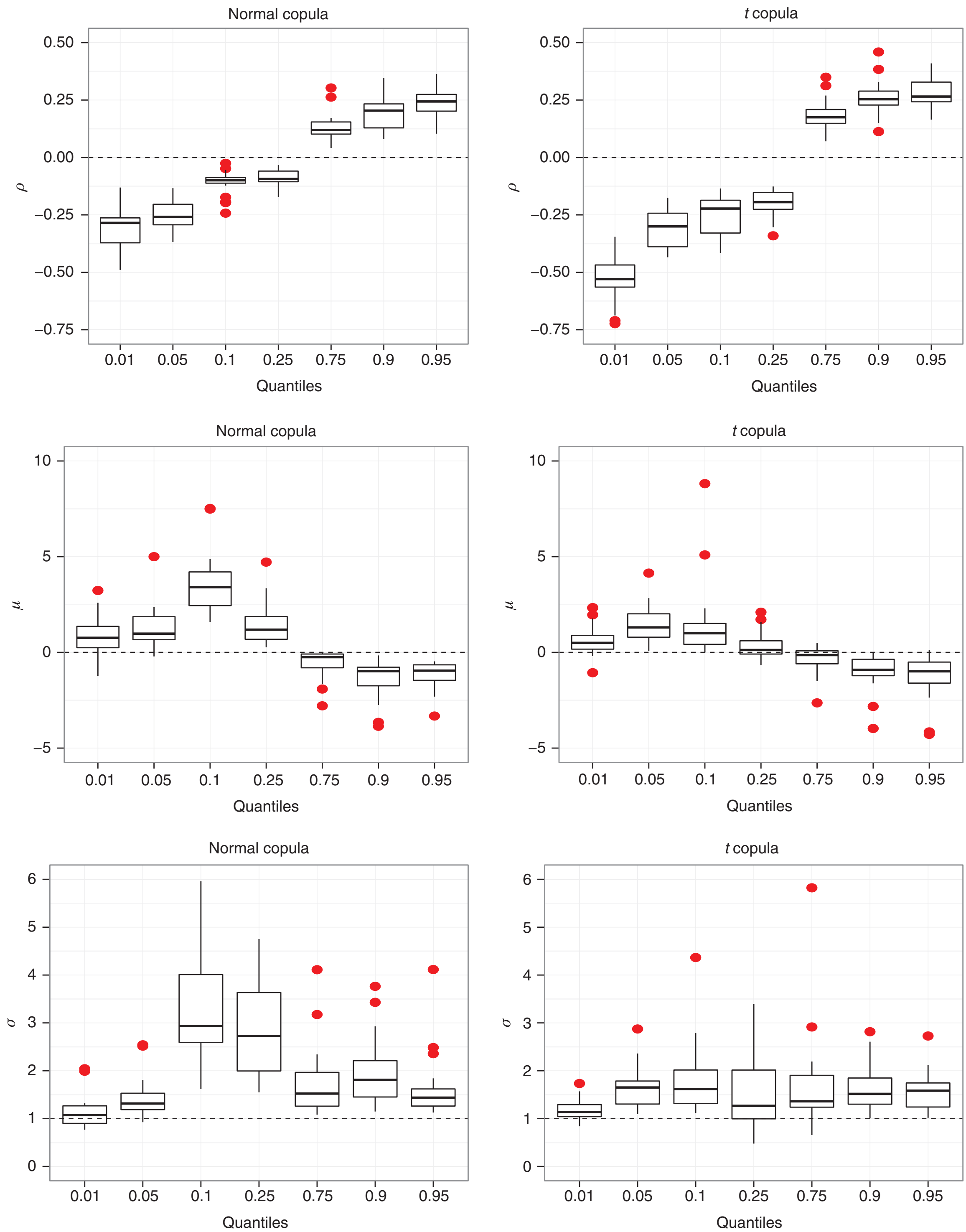

Figure 2: Estimated parameters from the Normal and the *t* copulas using the full sample data. For each quantile level, we summarize the results for 21 assets.

respectively. Examining the estimates of ρ, we notice that the Normal copula, on average, estimates a lower correlation than the *t* counterpart, particularly for lower quantiles that are of the highest interest. In contrast, the *t* copula captures the expected asymmetry with lower (τ=5%) quantiles that have coefficients with much higher magnitudes in comparison to upper (τ=95%) quantiles. Another important finding is that the joint

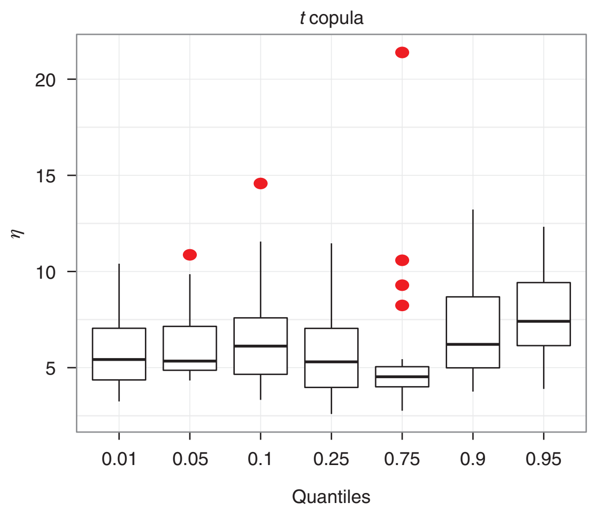

Figure 3: Degrees of freedom for the t copula using the full sample data. For each quantile level, we are summarize the results for 21 assets.

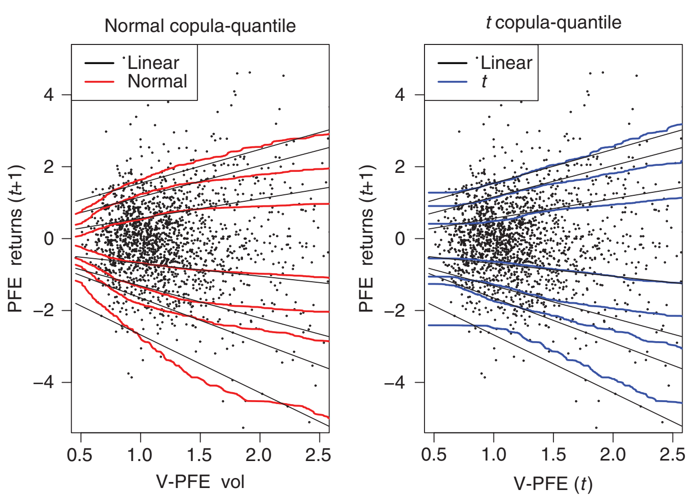

Figure 4: Semiparametric quantile regression of Pfizer returns r_{t+1} on its realized volatility $\sqrt{RV_t}$.

distribution of returns and realized volatility is heavy tailed at all quantiles under analysis. This finding can be observed in Figure 3 in which the average of the degrees of freedom is approximately 5 for all quantiles. The degrees of freedom parameter allows the t copula to capture higher dependence, particularly in the tails of joint distribution. Finally, the location-scale parameters of returns (μ and σ) in general are different than 0 and 1, respectively. Thus, the inclusion of these parameters in the copula quantile curves (Equation 10 and 15) is essential.

Although the parameter estimates from all 21 stock returns used in the analysis are similar, we restrict ourselves to discussing the results for one representative, namely Pfizer. Figure 4 plots the fitted quantile curves for Pfizer using the Normal and the t copulas compared to the fitted curves from the benchmark LQR model. Note that positive (negative) slope curves correspond to a positive (negative) correlation. We notice that there are nonlinear dependencies, particularly for the lower quantiles. Lower quantiles appear to be driven by past volatility to a larger extent than the upper quantiles, which results in asymmetric joint distribution. Figure 4 can be explored in more detail by inspecting the estimated model parameters in Tables 2 and 4 in the supplementary online Appendix. We notice that the estimated correlation coefficients for both nonlinear models have expected signs, i.e. which is negative for lower quantiles and positive for the upper quantiles. In addition, the correlation coefficients are all significantly different from zero. The degrees of freedom parameter for the *t* copula is statistically significant as well for all quantiles under analysis. Regarding the location and scale parameters, with very few exceptions, we confirm the statistical significance as well. The findings from this illustratory stock are uniform across all stocks considered and support our nonlinear model for conditional quantiles of returns.

4.2 Out-of-sample results

The main objective of proposing the nonlinear copula quantile regression for quantiles of the returns distribution is the desire to obtain precise VaR predictions. We split the data to the in-sample 1335 observations that are used for the model fits and the 500 observations for the OOS predictions, which we use later for the evaluation of forecasting models that correspond to the dates from January 2010 until December 2011. Hence,

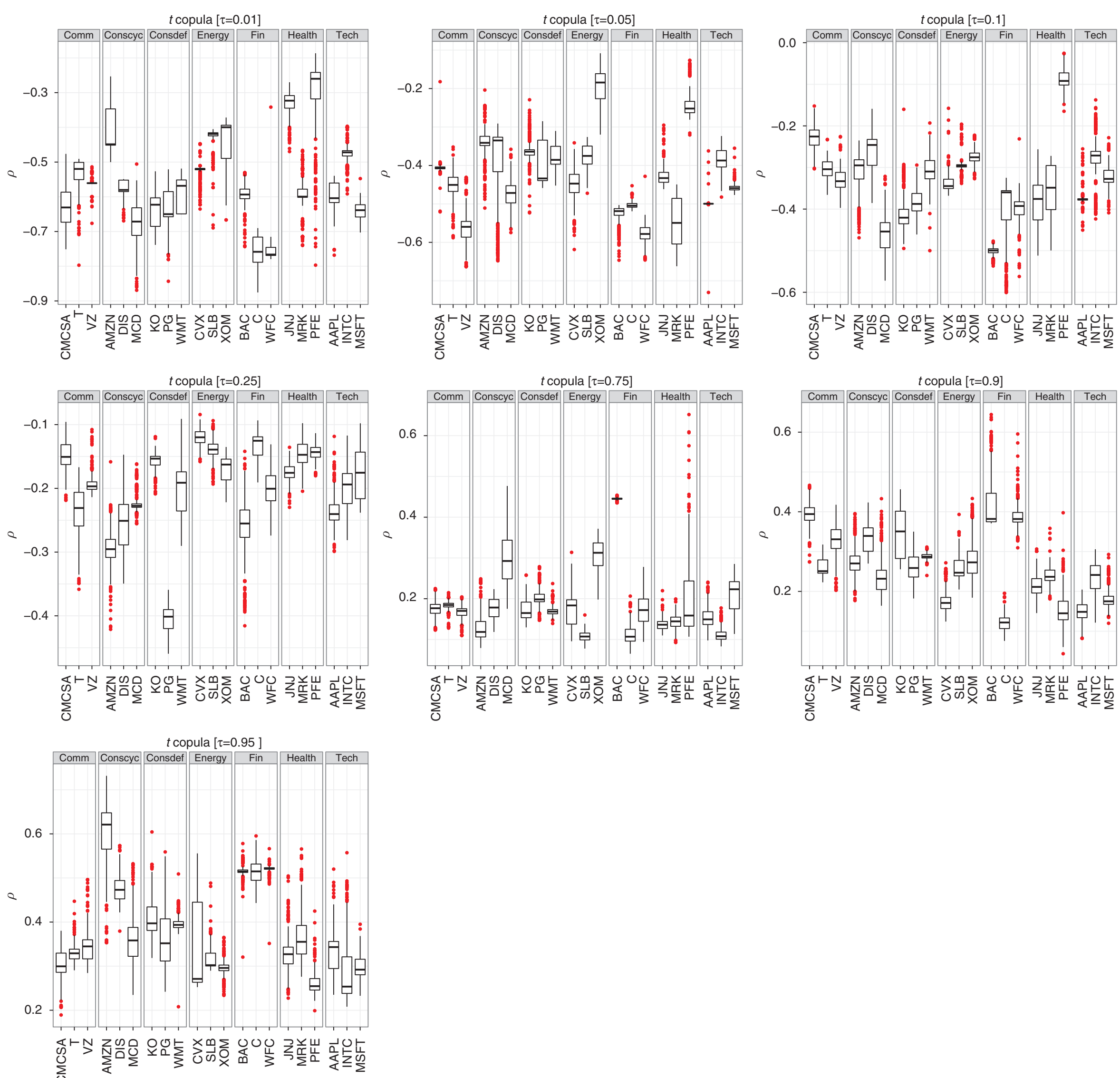

Figure 5: Out-of-sample correlation from the *t* copula. The in-sample period includes 1335 observations, and the out-of-sample period includes 500. We use the one-step-ahead rolling window for a length of approximately 2 years and assume that the degrees-of-freedom are constant throughout the out-of-sample window.

we estimate the models using in-sample data and then use the rolling window[7] to obtain 500 one-step-ahead forecasts of the quantiles of returns (or the VaR). The model parameters for the OOS are represented in Figure 5 for the *t* copula and in Figure 1 of the supplementary online Appendix for the Normal copula. When examining the parameters of the Normal copula, we notice that the estimates have high variance and many outliers. *t* Conversely, the counterpart produces much more stable correlations. The reason for the instability of the results for the Normal copula is that it is not as flexible as *t* and thus cannot cope with large changes in dependence. In addition, for most assets, there is a slight correlation asymmetry when comparing the lower and upper quantiles.

An interesting finding that can be observed from Figure 5 is the large heterogeneity of parameter estimates across industries. The Financial industry tends to have the largest negative correlation. For this industry, the increase in volatility drives (the future) lower quantiles at a higher rate in comparison to other industries. This result is explained by the nature of the Financial industry; it is very sensitive to volatility increases. Conversely, the lower quantiles of the returns of Health and Cyclicals are much less sensitive to increases in volatility. If we observe the volatility impact on upper quantiles close to the right tail of the return distribution, we find that the Financial industry and Consumer Cyclicals (at a higher extent) display the highest correlations.

In accordance with the same approach as above, we obtain the five-steps-ahead predictions for the return quantiles. The estimated model parameters are qualitatively the same. More importantly, we use the obtained forecasts from one and five steps-ahead for comparing model accuracy in the following section. We plot the $VaR_{\tau=5\%}$ forecasts for one step-ahead and five steps-ahead predictions in Figures 2–5 in the supplementary online Appendix. We can observe that all models (the linear, Normal and *t* copulas) provide similar patterns and capture the conditional quantiles well. However, to distinguish which one performs better, we perform statistical testing in the next section.

5 Evaluation of quantile forecasts

We evaluate the *absolute* OOS performance of the various conditional quantile models using a test originally proposed by Engle and Manganelli (2004), who use the *n*-th order autoregression

$$I_t=\omega+\sum_{k=1}^{n}\beta_{1k}I_{t-k}+\sum_{k=1}^{n}\beta_{2k}q^{\tau}_{t-k+1}+u_t, \tag{16}$$

where I_{t+1} is 1 if $y_{t+1}<q_t^{\tau}$, and zero otherwise. Although the hit sequence I_t is a binary sequence, u_t is assumed to follow a logistic distribution, and we can estimate it as a simple logit model and test whether $\mathbb{P}(I_t=1)=q_t^{\tau}$. To obtain the *p*-values, we rely on simulations, as suggested by Berkowitz et al. (2011), and we refer to this test as a DQ test in the results.

The main motivation of the DQ test is to determine whether the conditional quantiles are correctly dynamically specified; hence, it evaluates the absolute performance of the various models. This approach to evaluating the absolute performance of quantile forecasts is only suitable for one-step-ahead forecasts, and, to the best of our knowledge, there is currently no alternative, reliable test for the correct dynamic specification of multi-step conditional quantiles.

To assess the *relative* performance of the models, we evaluate the accuracy of the VaR forecasts statistically by defining the expected loss of the VaR forecast made by a forecaster *m* as

$$\mathcal{L}_{\tau,m}=\mathbb{E}[(\tau-\mathbb{1}\{y_{t,t+1}<q^{\tau,m}_{t,t+1}\})(y_{t,t+1}-q^{\tau,m}_{t,t+1})], \tag{17}$$

which was proposed by Giacomini and Komunjer (2005). The tick loss function penalizes quantile violations more heavily, and the penalization increases with the magnitude of the violation. As argued by Giacomini

7 To ease the computation burden, we estimate the degrees of freedom for the *t* copula only for the in-sample data and then assume it remains constant throughout the OSS because it does not change significantly.

Table 2: Energy: OOS 1-step-ahead VaR evaluation.

CVX	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{\mathcal{C}}_\tau$	0.008	0.048	0.086	0.218	0.742	0.920	0.942
$\hat{\mathcal{L}}$	0.032	0.120	0.200	0.354	0.328	0.182	**0.114**
DQ	0.528	3.097	5.103	7.204	6.358	4.948	5.070
p-values	0.991	0.685	0.403	0.206	0.273	0.422	0.407
NQR–t							
$\hat{\mathcal{C}}_\tau$	0.006	0.030	0.086	0.214	0.762	0.936	0.960
$\hat{\mathcal{L}}$	0.033	0.122	0.198	0.357	0.327	**0.187**	**0.113**
DQ	0.625	8.748	3.705	8.766	9.360	13.662	6.086
p-values	0.987	0.120	0.593	0.119	0.096	0.018	0.298
LQR							
$\hat{\mathcal{C}}_\tau$	0.006	0.034	0.080	0.210	0.752	0.930	0.966
$\hat{\mathcal{L}}$	0.031	0.120	0.197	0.357	0.328	0.181	0.109
DQ	0.625	7.137	4.526	6.363	8.908	11.580	6.641
p-values	0.987	0.211	0.476	0.272	0.113	0.041	0.249
SLB	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{\mathcal{C}}_\tau$	0.004	0.048	0.104	0.246	0.778	0.910	0.960
$\hat{\mathcal{L}}$	0.055	0.189	0.317	0.555	0.537	0.297	0.173
DQ	2.165	3.090	1.714	6.479	7.796	2.743	7.098
p-values	0.826	0.686	0.887	0.262	0.168	0.740	0.213
NQR–t							
$\hat{\mathcal{C}}_\tau$	0.006	0.050	0.104	0.246	0.774	0.918	0.968
$\hat{\mathcal{L}}$	0.052	**0.191**	0.312	0.552	0.537	0.297	**0.175**
DQ	0.625	2.958	5.302	9.207	9.767	7.251	10.555
p-values	0.987	0.707	0.380	0.101	0.082	0.203	0.061
LQR							
$\hat{\mathcal{C}}_\tau$	0.004	0.042	0.104	0.252	0.774	0.912	0.968
$\hat{\mathcal{L}}$	0.051	0.187	0.315	0.552	0.536	0.296	0.174
DQ	2.165	2.551	5.436	9.337	9.156	7.249	10.555
p-values	0.826	0.769	0.365	0.096	0.103	0.203	0.061
XOM	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{\mathcal{C}}_\tau$	0.006	0.042	0.080	0.218	0.754	0.896	0.918
$\hat{\mathcal{L}}$	0.030	0.112	0.185	0.316	0.304	0.171	**0.109**
DQ	0.625	6.284	4.644	8.302	4.540	2.795	17.536
p-values	0.987	0.280	0.461	0.140	0.475	0.732	0.004
NQR–t							
$\hat{\mathcal{C}}_\tau$	0.004	0.034	0.066	0.202	0.772	0.926	0.970
$\hat{\mathcal{L}}$	**0.033**	**0.119**	0.185	**0.321**	**0.310**	0.170	**0.107**
DQ	2.165	6.967	11.610	17.769	2.698	6.640	8.944
p-values	0.826	0.223	0.041	0.003	0.746	0.249	0.111
LQR							
$\hat{\mathcal{C}}_\tau$	0.004	0.036	0.056	0.204	0.752	0.924	0.960
$\hat{\mathcal{L}}$	0.030	0.116	0.184	0.318	0.305	0.168	0.103
DQ	2.165	6.486	17.044	16.395	3.983	10.348	6.086
p-values	0.826	0.262	0.004	0.006	0.552	0.066	0.298

Empirical quantile $\hat{\mathcal{C}}_\tau$, estimated by Giacomini and Komunjer (2005) $\hat{\mathcal{L}}$, logit DQ statistics and its 1000× simulated *p*-values are reported. Moreover, $\hat{\mathcal{L}}$ is tested using Diebold-Mariano statistics with the Newey-West estimator for variance. All models are compared to the linear quantile regression (LQR), whereas models with significantly less accurate forecasts at forecasts at the 95% level are reported in bold, significantly more accurate as underlined. NQR is the nonlinear quantile regression.

Table 3: Energy: 5-days-ahead VaR evaluation.

CVX	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{\mathcal{C}}_\tau$	0.012	0.064	0.102	0.238	0.736	0.894	0.940
$\hat{\mathcal{L}}$	0.069	0.275	0.460	<u>0.797</u>	0.734	0.396	0.233
DM	−0.686	−0.730	−1.314	−2.525	−0.780	0.218	0.788
NQR–t							
$\hat{\mathcal{C}}_\tau$	0.014	0.062	0.102	0.224	0.774	0.922	0.952
$\hat{\mathcal{L}}$	**0.084**	0.288	0.466	<u>0.810</u>	0.748	**0.425**	**0.242**
DM	2.826	1.248	−1.139	−2.035	0.454	1.815	1.923
LQR							
$\hat{\mathcal{C}}_\tau$	0.012	0.056	0.104	0.210	0.726	0.902	0.960
$\hat{\mathcal{L}}$	0.070	0.281	0.476	0.830	0.739	0.394	0.225
SLB	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{\mathcal{C}}_\tau$	0.022	0.068	0.128	0.288	0.780	0.920	0.972
$\hat{\mathcal{L}}$	0.101	0.398	0.676	1.233	1.157	0.640	0.375
DM	−1.100	−0.849	−0.850	0.170	0.861	0.716	1.577
NQR–t							
$\hat{\mathcal{C}}_\tau$	0.018	0.074	0.116	0.274	0.780	0.924	0.948
$\hat{\mathcal{L}}$	0.106	0.404	0.676	1.229	1.156	0.640	0.382
DM	−0.262	−0.646	−1.145	−0.034	0.387	0.772	0.775
LQR							
$\hat{\mathcal{C}}_\tau$	0.018	0.066	0.120	0.282	0.778	0.928	0.972
$\hat{\mathcal{L}}$	0.107	0.411	0.692	1.229	1.155	0.636	0.371
XOM	0.01	0.05	0.1	0.25	0.75	0.9	0.95
NQR–normal							
$\hat{\mathcal{C}}_\tau$	0.014	0.046	0.090	0.238	0.724	0.890	0.932
$\hat{\mathcal{L}}$	<u>0.065</u>	0.250	<u>0.410</u>	0.708	0.688	0.387	**0.239**
DM	−3.030	−1.576	−2.020	−1.340	−0.602	1.520	2.044
NQR–t							
$\hat{\mathcal{C}}_\tau$	0.016	0.048	0.082	0.220	0.736	0.910	0.962
$\hat{\mathcal{L}}$	**0.072**	<u>0.253</u>	<u>0.414</u>	<u>0.705</u>	0.700	**0.392**	**0.233**
DM	1.796	−2.325	−2.014	−2.052	1.079	1.913	1.680
LQR							
$\hat{\mathcal{C}}_\tau$	0.014	0.044	0.078	0.200	0.724	0.906	0.960
$\hat{\mathcal{L}}$	0.070	0.264	0.426	0.725	0.692	0.376	0.225

Empirical quantile $\hat{\mathcal{C}}_\tau$, estimated Giacomini and Komunjer (2005) $\hat{\mathcal{L}}$, $\hat{\mathcal{L}}$ is further tested using Diebold-Mariano statistics with the Newey-West estimator for variance. All models are compared to the linear quantile regression (LQR), whereas models with significantly less accurate forecasts at the 95% level are reported in bold, significantly more accurate as underlined. NQR is nonlinear quantile regression.

and Komunjer (2005), the tick loss is a natural loss function when evaluating conditional quantile forecasts. To compare the forecast accuracy of the two models, we test the null hypothesis that the expected losses for the models are equal, $H_0: d=\mathcal{L}_{\tau,1}-\mathcal{L}_{\tau,2}=0$, against a general alternative. The differences can be tested using Diebold and Mariano (2002) test statistics with Newey-West variance (in case of multi-step-ahead forecasts). Under the null of equal predictive accuracy, the test statistics are distributed as $N(0, 1)$.

5.1 Quantile forecasts results

For brevity, in the body of this paper, we report the forecasting model performance for the Energy sector only representing the results from all other sectors under study. For the remainder, the results are available in

Tables 5–16 in the supplementary online Appendix. Tables 2 and 3 summarize the models performance for the one- and five steps-ahead forecasts. We use different indicators for model comparison, the unconditional coverage $\hat{\mathcal{C}}_\tau$, the tick loss function $\hat{\mathcal{L}}$, DQ test statistics and the simulated p-values.

Examining the unconditional coverage test $\hat{\mathcal{C}}_\tau$, we notice that all models perform similarly for the Energy sector, and we cannot identify a clearly "best performing model." Nevertheless, unconditional coverage is uninformative regarding the dynamic features of the models. One of the main important features of the proposed models is that it models quantiles of the returns distribution conditionally. Under stress events is where most losses occur; hence, conditional coverage tests are of crucial importance. Thus, we focus on the results of the DQ test, which helps to identify whether the models capture the dynamics well. Because we cannot reject most models at the 95% significance level, we conclude that the models are well specified and that they capture the dynamics of quantiles well. For a relative model comparison, we examine the loss function $\hat{\mathcal{L}}$. Particularly for the lower quantiles, the NQR models either are not significantly different than the LQR model or perform worse (the numbers in bold). For the one-step-ahead forecasts of the quantiles, we must conclude that further nonlinearities are not statistically found in the tested data, and the LQRs are not outperformed by nonlinear ones.

The DQ test is not suitable for the evaluation of five-steps-ahead forecasts due to the lack of statistical results regarding their distribution. Hence, for the five-steps-ahead forecasts, we focus on model comparison through the relative performance. We notice an improvement of the performance of the NQR models for the Energy sector e.g. for the lower quantiles of exxon mobil corporation (XOM), both copula quantile models perform very well (the underlined numbers).

Due to the mixed overall results, when we are not able to clearly identify a "best model," we make a summary of the forecasted performance results. For the one-step-ahead forecast, the DQ test rejects the NQR models approximately 12.3% and 20.4% of times for the Normal and the t models, respectively, whereas for LQR, the rejection rate is approximately 13% of times. The Normal copula model performs the best in this category, although it is very close to the LQR. Next, we test the relative model performance, in which the NQR models are compared to the LQR model. Occasionally, the NQR models perform better (with the t copula being the better model); occasionally, it is the LQR copula that performs better. We should note that there are situations in which the NQR models perform worse than LQR; at the same time, all these models are rejected by the DQ test e.g. shown in Table 5 of the supplementary online Appendix for AT&T and quantiles $\tau=\{0.1; 0.9\}$. In such situations, we cannot state which model is the best because none of them pass the DQ.

For the five-steps-ahead forecast, the performance of the NQR models improves significantly, particularly for the t copula. The NQR-t outperforms the LQR model in 13 cases or approximately 9% of the time; however, there are many times in which the tick loss function of the NQR models is lower than the LQR counterpart, although this difference is not statistically significant. In conclusion, the models are well specified, and the NQR models appear to outperform LQR when five-steps-ahead forecasts are considered.

6 Conclusion

This paper proposes to use the NQR with realized measures of volatility to forecast the conditional quantiles of financial asset returns. To make the results robust, we apply this methodology on most liquid US stocks in seven sectors. We argue that using the realized volatility under a copula quantile framework is useful, particularly in the cases in which the quantile dependence is nonlinear. The proposed models capture and forecast the dynamics of quantiles well.

The possible directions for further development would be to study the interdependence between asset returns or to use copula functions that allow for higher dependence in the tails.

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Appendix A Proofs

Probability distribution of r_{t+1} conditional on ϑ_t

$$\tau(r_{t+1}|\vartheta_t)=\frac{\partial C(u_t, v_{t+1})}{\partial u_t}$$

Proof. from

$$\mathbb{P}(\mathcal{R}<r_{t+1}, \mathcal{V}=\vartheta_t)=\frac{\partial F_{\mathcal{V},\mathcal{R}}(\vartheta_t, r_{t+1})}{\partial \vartheta_t}$$

it follows that

$$\begin{aligned}
\mathbb{P}(R<r_{t+1}, \mathcal{V}=\vartheta_t)&=\lim_{\epsilon\to 0}\mathbb{P}(\mathcal{R}\le r_{t+1}|\vartheta_t\le\mathcal{V}\le\vartheta_t+\epsilon)\\
&=\lim_{\epsilon\to 0}\frac{F(\vartheta_t+\epsilon, r_{t+1})-F(\vartheta_t, r_{t+1})}{F_{\mathcal{V}}(\vartheta_t+\epsilon)-F_{\mathcal{V}}(\vartheta_t)}\\
&\approx\frac{(\partial F(\vartheta_t, r_{t+1})/\partial\vartheta_t)\cdot\epsilon}{f_{\mathcal{V}}(\vartheta_t)\cdot\epsilon}\\
&=\frac{1}{f_{\mathcal{V}}(\vartheta_t)}\frac{\partial C(F_{\mathcal{V}}(\vartheta_t), F_{\mathcal{R}}(r_{t+1}))}{\partial\vartheta_t}\\
&=\underbrace{\frac{1}{f_{\mathcal{V}}(\vartheta_t)}}_{*}\frac{\partial C(u_t, v_{t+1})}{\partial u_t}\underbrace{\frac{\partial F_{\mathcal{V}}(\vartheta_t)}{\partial\vartheta_t}}_{*}\\
&=\partial C(u_t, v_{t+1})/\partial u_t
\end{aligned}$$

where $u_t=F_{\mathcal{V}}(\vartheta_t)$, $v_{t+1}=F_{\mathcal{R}}(r_{t+1})$ and * terms cancel out. □

Following the same path it is easy to show that

$$\tau(\vartheta_t|r_{t+1})=\frac{\partial C(u_t, v_{t+1})}{\partial v_{t+1}}$$

Appendix B Tables and Figures

Table 4: Descriptive statistics for daily returns and realized volatility over the sample period extending from August 2004 to December 2011.

Returns

	Information technology			Consumer discretionary			Consumer staples			Telecommunication services		
	AAPL	INTC	MSFT	AMZN	DIS	MCD	KO	PG	WMT	CMCSA	T	VZ
Mean	−0.0003	−0.0001	−0.0001	0.0015	0.0009	0.0004	0.0001	0.0006	−0.0001	0.0002	−0.0001	−0.0004
Std dev	0.0201	0.0164	0.0140	0.0224	0.0156	0.0121	0.0106	0.0099	0.0108	0.0187	0.0132	0.0127
Skewness	−0.3097	0.0641	0.1483	0.3004	0.4682	0.2967	0.0474	−0.0580	0.4404	0.6274	0.5877	0.5984
Kurtosis	3.2914	3.3402	5.8121	4.4135	6.9769	6.0683	8.1274	6.6594	6.5979	18.2322	9.5964	8.3887
Minimum	−0.1223	−0.0907	−0.0755	−0.1313	−0.0909	−0.0799	−0.0717	−0.0660	−0.0653	−0.1416	−0.0629	−0.0760
Maximum	0.1123	0.0880	0.1102	0.1388	0.1185	0.1035	0.0795	0.0776	0.0762	0.2325	0.1242	0.1118

	Financials			Energy			Health care		
	BAC	C	WFC	CVX	SLB	XOM	JNJ	MRK	PFE
Mean	−0.0023	−0.0042	−0.0002	0.0001	−0.0002	0.0005	0.0001	0.0000	−0.0006
Std dev	0.0327	0.0341	0.0272	0.0154	0.0215	0.0147	0.0092	0.0152	0.0133
Skewness	−0.4071	−1.7889	0.2458	0.0847	−0.4012	−0.0108	0.0305	−0.1710	0.1302
Kurtosis	13.6287	20.8588	13.0526	11.9675	5.4278	10.3940	9.6599	6.8905	3.2493
Minimum	−0.2509	−0.3468	−0.2081	−0.1296	−0.1552	−0.1261	−0.0803	−0.1092	−0.0696
Maximum	0.2014	0.1992	0.1933	0.1460	0.1253	0.1189	0.0728	0.0919	0.0714

Realized volatility

	Information technology			Consumer discretionary			Consumer staples			Telecommunication services		
	AAPL	INTC	MSFT	AMZN	DIS	MCD	KO	PG	WMT	CMCSA	T	VZ
Mean	0.0004	0.0003	0.0002	0.0006	0.0003	0.0002	0.0001	0.0001	0.0002	0.0004	0.0002	0.0002
Std dev	0.0008	0.0005	0.0004	0.0009	0.0005	0.0004	0.0003	0.0004	0.0004	0.0007	0.0005	0.0005
Skewness	11.9764	10.7700	7.6919	7.6707	9.8775	25.3799	10.4895	26.0745	20.7189	12.7904	12.0541	15.4769
Kurtosis	209.4541	191.6730	90.6071	80.7537	151.3225	868.6006	175.7496	895.9773	625.6571	243.4969	242.6791	382.1494
Minimum	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Maximum	0.0192	0.0121	0.0070	0.0144	0.0117	0.0157	0.0063	0.0143	0.0126	0.0169	0.0142	0.0148

	Financials			Energy			Health care		
	BAC	C	WFC	CVX	SLB	XOM	JNJ	MRK	PFE
Mean	0.0009	0.0011	0.0007	0.0003	0.0005	0.0002	0.0001	0.0003	0.0002
Std dev	0.0026	0.0040	0.0017	0.0007	0.0009	0.0007	0.0003	0.0007	0.0004
Skewness	7.9607	10.8488	5.7960	17.1272	8.3253	18.3961	18.5604	13.5083	8.8700
Kurtosis	95.3976	166.7985	45.1057	435.1365	111.8150	490.6232	486.2773	263.8237	125.7655
Minimum	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Maximum	0.0489	0.0866	0.0231	0.0207	0.0178	0.0205	0.0090	0.0165	0.0079

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