Optimal Value of Loans via Stochastic Programming

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Abstract. A question of mortgage leads to serious and complicated problems of financial mathematics. On one side is a bank with an aim to have a "good" profit, on the other side is the client trying to invest money safely, with possible "small" risk. Let us suppose that a young married couple is in a position of client. Young people know that an expected and also unexpected unpleasant financial situation can happen. Many unpleasant financial situation can be caused by a random factor. Consequently stochastic methods are suitable to secure against them.

The aim of the suggested model is not only to state a maximal reasonable value of loans, but also to endure unpleasant financial period. To this end we employ stochastic optimization theory. A few suitable models will be introduced. The choice of the model depends on environment of the young people. Models will be with "deterministic" constraints, probability constraints, but also with stochastic dominance constraints. The suggested models will be analyzed both from the numerical point of view and from possible method solution based on data. Except static one–objective problem we suggest also multi–objective models.

Keywords: Loan, debtor, installments, stochastic programming, second order dominance constraints, probability constraints.

JEL classification: C44
AMS classification: 90C15

1 Introduction

Let us construct and analyze a very simple financial model. To this end we consider a situation about a mortgage and instalments supposing a situation of young married couple. Young people wants to get own residence (a flat or a little house). Since they do not posses necessary financial resources, the bank sector offers them a mortgage. Of course bank can employ excellent experts to minimize their risk and maximize profit in dependence of debtor's position. The aim of our approach is to analyze the situation from the second side. In particular our aim is to investigate the possibilities of the debtors not only in dependence of their present–day situation, but also on their future private and subjective decision and on the possible "unpleasant" financial situation. In details, the aim is to suggest a method for a security of a "safe" loan and simultaneously to offer tactics to state a plausible environment for future time. Of course we suppose that our analysis is one of the first contribution to this situation. To this end we start with very standard situation of young people considered already in [6]. The young married couple decides to take loan of the value M. A question is how to choice value of M to be safe for them and simultaneously to secure them their wish of a comfortable flat.

To start with a responsible analyze of their situation we assume that a monthly income of young married people in a start point $t = 0$ is

$$Z_0 = U_0 + V_0, \quad \text{where} \quad U_0 \quad \text{is an income of husband and} \quad V_0 \quad \text{is an income of the wife.}$$

Evidently, this income can be divided into three parts Z_0^1, Z_0^2, Z_0^3, where Z_0^1 denotes means for a basic consumption, Z_0^2 denotes means that can be employed for a repayment of installments and Z_0^3 can be considered as an allocation to saving. Consequently

$$Z_0 = Z_0^1 + Z_0^2 + Z_0^3, \quad Z_0^1,\, Z_0^2 > 0,\, Z_0^3 \geq 0. \tag{1}$$

Supposing the annuity repayments, which is the most standard way of repaying the loan, we denote (as mentioned already above) by a symbol M the value of the loan, by m number of identical installments and by ζ the loan interest rate, then the identical installments $b(M) := b(M, \zeta)$ at time points $t = 1, 2, \ldots, m$ (see, e.g., [7] or [12]) are given by

$$b(M) := b(\zeta) = \begin{array}{ll} \frac{M\zeta}{1-v^m}, & \zeta \neq 0, \quad v = v(\zeta) = (1+\zeta)^{-1}, \\ \frac{1}{m}, & \zeta = 0. \end{array} \tag{2}$$

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It follows from the relations (1), (2) that (in the case when $\zeta \neq 0$) it is desirable (in "static" approach) the following inequality

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} = Z_0^2 \tag{3}$$

to be fulfilled. Of course, this condition (in the extreme case) can be replaced by the inequality

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \le Z_0^2 + Z_0^3. \tag{4}$$

If it is possible to assume that the relations (1), (2) will be fulfilled also in future, then the young people can take the loan equal to the maximal value M for which the inequality (3) (respective (4)) is fulfilled. However mostly it is necessary to assume that the financial situation of young married couple can change. For example: it is reasonable to assume that in some time period, say (m_1, m_2), $0 < m_1 < m_2 < m$ the married couple plan to have a baby. According to this fact and to the social politics of the state the young people can assume the less income in this time. We construct a few mathematical models according to the client possibilities; furthermore we analyze them. To this end the theory of stochastic programming will be employed.

2 Simple Mathematical Models

2.1 Analysis of Situation

To construct mathematical models we suppose that the aim of the young people is to determine maximal "reasonable" safety loan M. To this end let

- $X_t \subset R^n$ ...... nonempty compact sets $t \in \{0, \ldots, m\}$,
- M .... value of loan,
- m .... number of identical installments,
- ζ .... interest rate corresponding to the loan,
- Z_t .... income of young married couple at time point $t \in \{0, 1, \ldots, m\}$,
- U_t .... income of husband at time point $t \in \{0, 1, \ldots, m\}$,
- V_t .... income of wife at time point $t \in \{0, 1, \ldots, m\}$,
- Z_t^1 .... means determined for basic consumption at time point $t \in \{0, 1, \ldots, m\}$,
- Z_t^2 .... means determined for repayment of installment at time point $t \in \{0, 1, \ldots, m\}$,
- Z_t^3 .... allocation (maybe random) for saving in time point $t \in \{0, 1, \ldots, m\}$,
- $\langle m_1, m_2\rangle$ ...... time interval in which income of wife is supposed to be smaller,
- $\xi_{t,j}$, ... random returns at time t and assets j, $t \in \{0, 1, \ldots, m\}$, $j \in \{1, 2\}$,
- $J = \{1, 2\}$ system of assets,
- $x_{t,j}$ ...... decision variables, $t \in \{0, 1, \ldots, m\}$, $j \in \{1, 2\}$,
- $x_t = [x_{t,1}, x_{t,2}]$, $t = 0, 1, \ldots, m$,
- $\xi_t = [\xi_{t,1}, \xi_{t,2}]$, $t = 0, 1, \ldots, m$,
- $g_t = \xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,2}$, $t = 0, \ldots, m$,
- $Y_t = \frac{Z_t^3}{2}\xi_{t,1} + \frac{Z_t^3}{2}\xi_{t,2}$, $t = 0, \ldots, m$,
- F ...... distribution function covering all random values that occur in the corresponding model,
- $\mathcal{Z}$ ...... support corresponding to F.

Remark. We suppose (without loss of generality) that young people can the amount Z_t^3 (at every time point $t \in \langle 0, m\rangle$) invest only in two assets $J = \{1, 2\}$. Of course, in a real–life situation mostly more possibilities exist.

To analyze the situation we suppose that M fulfils the relation

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \le Z_t^2 \quad \text{for} \quad t = 0, \ldots m_1 - 1, m_2 + 1, \ldots, m.$$

Consequently if

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \le Z_t^2 \quad \text{also for every} \quad t \in \langle m_1, m_2\rangle, \tag{5}$$

then everything is OK. (This can happen, if for example one of the following situation happen: M is small in comparison with income of young people; the husband have two positions in the time interval $\langle m_1, m_2\rangle$; the wife can work home; parents helps). But usually the situation is not such pleasant and the following inequality

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} > Z_t^2 \quad \text{can happen for some} \quad t \in \langle m_1, m_2\rangle. \tag{6}$$

Evidently, the trouble starts in the case (6). Young people can protect against them. Especially they can save at the time points $t = 0,\, 1,\, \ldots,\, m_1 - 1$ means Z_t^3 to be fulfilled the inequality

$$\frac{(m_2 - m_1 + 1)M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} \leq (m_2 - m_1 + 1)[Z_0^2 - Z_{m_1}^2] + \sum_{t=0}^{m_1 - 1} Z_t^3, \tag{7}$$

(under the assumption $Z_t^2 = Z_0^2$, $t = 0,\, \ldots,\, m_1 - 1$; $Z_t^2 = Z_{m_1}^2$, $t = m_1,\, \ldots,\, m_2$. If the last inequality (7) is fulfilled, then they endure the time period $\langle m_1,\, m_2 \rangle$ without financial troubles.

2.2 Model Construction

Till now we have assumed that Z_t^3 is deterministic value. It is known that a random component very often exists in salary. Just this component is suitable for saving. We consider in our analysis both cases: deterministic and random. However for simplicity (without loss of generality) we consider only special case $m_1 = 2, m_2 = 4$; m given by the relation (2). First we consider a completely deterministic model:

I. The young people can invest the deterministic amounts $Z_0^3,\, Z_1^3$ into two deterministic assets to obtain

in the first year the value	$c_{0,1}x_{0,1} + c_{0,2}x_{0,2}$
under the assumptions	$x_{0,1} + x_{0,2} \leq Z_0^3, \quad x_{0,1}, x_{0,2} \geq 0, \quad x_0 \in X_0,$
in the second year the value	$c_{1,1}x_{1,1} + c_{1,2}x_{1,2}$
under the assumptions	$x_{1,1} + x_{1,2} \leq Z_1^3, \quad x_{1,1}, x_{1,2} \geq 0, \quad x_1 \in X_1,$

where $c_{i,\,j},\ i = 0,\, 1,\ j = 1,\, 2$ are deterministic constants.
We assume in this model that the profit obtained at time $t = 0$ can not influence the invested amount at time $t = 1$.
Evidently, it is desirable (for young people) fulfilling of the relation

$$\frac{(m_2 - m_1 + 1)M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} \leq (m_2 - m_1 + 1)[Z_0^2 - Z_{m_1}^2] + \sum_{i=0}^{1} [c_{i,1}x_{i,1} + c_{i,2}x_{i,2}]. \tag{8}$$

Consequently, supposing $m_1 = 2,\ m_2 = 4$; $Z_0^2 = Z_1^2$; $Z_{m_1}^2 = Z_t^2, t \in \langle m_1,\, m_2 \rangle$; $c_{i,\,j},\ i,\, j = 1,\, 2$ deterministic, we obtain a deterministic optimization model:

$$\text{Find} \qquad \max M \tag{9}$$

under the constraints

$$\begin{aligned}
x_{0,1} + x_{0,2} &\leq Z_0^3, \quad x_{0,1}, x_{0,2} \geq 0, \quad x_0 \in X_0. \\
x_{1,1} + x_{1,2} &\leq Z_1^3, \quad x_{1,1}, x_{1,2} \geq 0, \quad x_1 \in X_1, \\
\tfrac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} &\leq Z_t^2 \quad \text{for} \quad t = 0, \ldots m_1 - 1,\, m_2 + 1,\, \ldots,\, m, \\
\tfrac{3M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} &\leq 3[Z_0^2 - Z_2^2] + \sum_{i=0}^{1} [c_{i,1}x_{i,1} + c_{i,2}x_{i,2}].
\end{aligned}$$

The problem (9) is a problem of linear programming. Consequently, it can be analyzed and solved employing the theory of linear programming.

II. We consider again $Z_t^3,\, t = 0,\, 1,\, \ldots,\, m$ deterministic. However, in the difference to the case I, young people can invest the value Z_t^3 into two assets with random returns $\xi_{t,1},\, \xi_{t,2},\, t = 0,\, 1$. Consequently it is necessary to determine $x_{0,1},\, x_{0,2},\, x_{1,1},\, x_{1,2}$ fulfilling the relations

$$\begin{aligned}
&x_{0,1} + x_{0,2} \leq Z_0^3, \quad x_{0,1},\, x_{0,2} \geq 0, \\
&x_{1,1} + x_{1,2} \leq Z_1^3, \quad x_{1,1},\, x_{1,2} \geq 0
\end{aligned}$$

to obtain random values

$$g_0 := g_0(x_0,\, \xi_0) = \xi_{0,1}x_{0,1} + \xi_{0,2}x_{0,2},$$

$$g_1 := g_1(x_1,\, \xi_1) = \xi_{1,1}x_{1,1} + \xi_{1,2}x_{1,2}.$$

Evidently, it is possible also to define random values $Y_0,\, Y_1$ by

$$\begin{array}{rcl} Y_0 : Y_0(\xi_0) & = & \frac{Z_0^3}{2}\xi_{0,1} + \frac{Z_0^3}{2}\xi_{0,2}, \\ Y_1 := Y_1(\xi_1) & = & \frac{Z_1^3}{2}\xi_{1,1} + \frac{Z_1^3}{2}\xi_{1,2}. \end{array} \tag{10}$$

Employing the theory of the stochastic dominance [11] it is "reasonable" to determine $x_{0,1},\, x_{0,2},\, x_{1,1},\, x_{1,2}$ such that

$$\begin{array}{lll} & F_{g_0} \succeq_1 F_{Y_0}, & F_{g_1} \succeq_1 F_{Y_1}, \\ \text{or} & F_{g_0} \succeq_2 F_{Y_0}, & F_{g_1} \succeq_2 F_{Y_1}. \end{array} \tag{11}$$

The first relation in (11) is known as a stochastic dominance of the first order; the second relation is known as stochastic dominance of the second order. To define stochastic dominance of the second order it is necessary to assume that the first moments of the random values $g_0(x_0,\, \xi_0),\, g_1(x_1,\, \xi_1),\, Y_0,\, Y_1,$ exist for $x_0 \in X_0,\, x_1 \in X_1$. (More about the definition of the stochastic dominance can be find, e.g., in [11].)

Of course, it is also desirable (for young people) in this case the fulfilling of the relation

$$\frac{3M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} \le 3[Z_0^2 - Z_2^2] + \sum_{i=0}^{1}[\xi_{i,1}x_{i,1} + \xi_{i,2}x_{i,2}]. \tag{12}$$

However, the inequality (12) depends on the random elements $\xi_{i,\,j},\ i = 0,\, 1,\ j = 1,\, 2$. Consequently, the "sense" of this inequality has to be defined. We consider it in probability.

Consequently, we can obtain the following optimization model depending on a probability measure:

$$\text{Find} \qquad \max M \tag{13}$$

under the constraints

$$\begin{array}{ll} x_{0,1} + x_{0,2} & \le Z_0^3, \quad x_{0,1},\, x_{0,2} \ge 0, \\ x_{1,1} + x_{1,2} & \le Z_1^3, \quad x_{1,1}, x_{1,2} \ge 0, \\ \frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} & \le Z_t^2 \quad \text{for} \quad t = 0,\, 1,\, 5,\, \ldots,\, m, \end{array} \tag{14}$$

$$P_F\{\frac{3M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} \le 3[Z_0^2 - Z_{m_1}^2] + \sum_{i=0}^{1}[\xi_{i,1}x_{i,1} + \xi_{i,2}x_{i,2}]\} \ge 1 - \varepsilon, \quad \varepsilon \in (0,\, 1), \tag{15}$$

$$\begin{array}{ll} \mathsf{E}_F(u - g_0(x_0,\, \xi_0))^+ & \le (u - Y_0(\xi_0))^+, \quad u \in R^1,\, x_0 \in X_0, \\ \mathsf{E}_F(u - g_1(x_1,\, \xi_(1)))^+ & \le (u - Y_1(\xi_1))^+, \quad u \in R^1,\, x_1 \in X_1. \end{array} \tag{16}$$

The equivalence of the relation (11) and (16) has been proven by Ruszczynski, see e.g., [11].

The constraints (14) are linear, constraints (16) under general conditions are convex. However (from the numerical point of view) the constraint (15) can be a problem.

III. Deterministic $Z_t^3,\, t = 0,\, 1,\, \ldots,\, m$ can be replaced by random values with probability one to be non negative. We assume that young people can random amounts $Z_0^3,\, Z_1^3$ invest into two assets to obtain:

- in the original year the value $\xi_{0,1}x_{0,1} + \xi_{0,2}x_{0,2}$
  under the assumptions $x_{0,1} + x_{0,2} \le Z_0^3, \quad x_{0,1}, x_{0,2} \ge 0, \quad x_0 \in X_0,$
- in the second year the value $\xi_{1,1}x_{1,1} + \xi_{1,2}x_{1,2}$
  under the assumptions $x_{1,1} + x_{1,2} \le Z_1^3, \quad x_{1,1}, x_{1,2} \ge 0, \quad x_1 \in X_1.$

We assume in this case that the profit obtained in the time $t = 0$ can not influence the invested amount at the time $t = 1$.

Evidently, it is desirable in this case the fulfilling of the relation

$$\frac{3M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} \le 3[Z_0^2 - Z_2^2] + \sum_{t=0}^{1}[\xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,2}] \tag{17}$$

and, simultaneously, the constraints with random factors

$$x_{0,1} + x_{0,2} \le Z_0^3, \quad x_{1,1} + x_{1,2} \le Z_1^3.$$

We consider all these constraints with the random factors in probability. Consequently we obtain stochastic optimization problem:

$$\text{Find} \qquad \max M \tag{18}$$

under the system of constraints

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \le Z_0^2, \quad t = 0, 1, \; 5, ...m, \tag{19}$$

$$P_F\{x_{t,1} + x_{t,2} \le Z_t^3\} \;\; \ge \;\; 1 - \varepsilon_t, \quad \varepsilon_t \in (0, 1), \; x_{t,1}, \, x_{t,2} \, \ge \, 0, \quad t = 0, \, 1, \, 5, ...m, \tag{20}$$

$$\begin{aligned} P_F\{\tfrac{3M[\zeta(1+\zeta)^m}{(1+\zeta)^m-1}\} \;\; \le \;\; & 3[Z_0^2 - Z_2 + \sum_{i=0}^{1}[\xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,\,2}]\} \ge 1 - \varepsilon_0, \\ & \varepsilon_0 \in (0, \, 1). \end{aligned} \tag{21}$$

In this case the constraints (19) is linear, the constraint (20) can be rewritten into linear conditions [4]. Difficulty arises only with the constraint (20); the same as in the problem II.

Remark. In all introduced cases we consider only the problem of choice the value M. However, surely it is very reasonable and suitable to maximize profit obtained by an investment at all time point in the interval $\langle 0, \; m\rangle$ or in the time interval $\langle m_2 + 1, \, m\rangle$. Because the profit is a random value, it is reasonable to look for optimal solution with respect of the mathematical expectation. Evidently in this case we obtain two–objective optimization problem.

3 Conclusion

In the last decades many people try to gain their own residence. Since they do not posses sufficient means, the bank sector offer them the loan. The aim of this contribution is to give a preliminary analysis of their situations and possible responsible behaviour. In the paper a very simple stochastic optimization problems have been introduced. However the model has been constructed to guarantee only very small "risk" for young people. However on the other side only completely deterministic model can be solved by classical numerical methods. The other models obtained do not contain convex constraints.

But replacing theoretical distribution by empirical one we obtain "good" estimates of the original problem. To more details see the literature about empirical estimates e.g.,[1],[2], [4], [5], [8], [9], [10].

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References

[1] Dai, L, Chen, C. H. and Birge,J. R.: Convergence properties of two–stage stochastic programming. *J. Optim. Theory Appl.* **106** (2000), 489–509.

[2] Dvoretzky, A., Kiefer, J. and Wolfowitz, J.: Asymptotic Minimax character of the Sample Distribution Function and the Classical Multinomial Estimate. *Ann. Math. Statist.* **56** (1956), 642–669.

[3] Frank, H. and Glass, A.J.: *Microeconomics and Behavior.* Mc Graw–Hill, New York 1991.

[4] Kaňková, V.: On the stability in stochastic programming: the case of individual probability constraints. *Kybernetika* **33** (1997), 5, 525–546.

[5] Kaňková, V. and Houda, M.: Thin and Heavy Tails in Stochastic Programming. *Kybernetika* **51** (2015), 3, 433–455.

[6] Kaňková, V.: A Note of Optimal Value of Loans. In: *Proceedings of the 34th International Conference Mathematical Methods of Economics 2016* (A. Kocourek and M. Vavroušek, eds.), Technical Univesity Liberec, Liberec 2016, 371–376.

[7] Luderer, B., Nollau, V. and Vetters, K.: *Mathematical Formulas for Economists.* Springer Science & Media, third edition, 2006.

[8] Pflug, G. Ch.: Stochastic Optimization and Statistical Inference. In: *Stochastic Programming, Handbooks in Operations Research and Management Science, Vol. 10* (A. Ruszczynski and A. A. Shapiro, eds.), Elsevier, Amsterdam 2003, pp. 427–480.

[9] Römisch, W.: Stability of Stochastic Programming Problems. In: *Stochastic Programming, Handbooks in Operations Research and Management Science, Vol 10* (A. Ruszczynski and A.A. Shapiro, eds.), Elsevier, Amsterdam 2003, pp. 483–554.

[10] Shapiro, A.: Quantitative stability in stochastic programming. *Math. Program.* **67** (1994), 99–108.

[11] Shapiro, A., Dentcheva, D. and Ruszczynski, A.: *Lectures on Stochastic Programming (Modeling and Theory).* Published by Society for Industrial and Applied Mathematics and Mathematical Programming Society, Philadelphia 2009.

[12] Šmíd, M. and Dufek, J.: Multi–period Factor Model of Loan Portfolio (July 10, 2016). Available at SSRN:http://ssrn.com/abstract=2703884 or http://dx.doi.org/10.2139/ssrn.2703884.