KYBERNETIKA — VOLUME *53* (2017), NUMBER 6, PAGES 1026–1046

STABILITY, EMPIRICAL ESTIMATES AND SCENARIO GENERATION IN STOCHASTIC OPTIMIZATION – APPLICATIONS IN FINANCE

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Economic and financial processes are mostly simultaneously influenced by a random factor and a decision parameter. While the random factor can be hardly influenced, the decision parameter can be usually determined by a deterministic optimization problem depending on a corresponding probability measure. However, in applications the "underlying" probability measure is often a little different, replaced by empirical one determined on the base of data or even (for numerical reason) replaced by simpler (mostly discrete) one. Consequently, real one and approximate one correspond to applications. In the paper we try to investigate their relationship. To this end we employ the results on stability based on the Wasserstein metric and $\mathcal{L}_1$ norm, their applications to empirical estimates and scenario generation. Moreover, we apply the achieved new results to simple financial applications. The corresponding model will a problem of stochastic programming.

Keywords: stochastic programming problems, probability constraints, stochastic dominance, stability, Wasserstein metric, $\mathcal{L}_1$ norm, Lipschitz property, empirical estimates, scenario, error approximation, financial applications, loan, debtor, installments, mortgage, bank

Classification: 90C15

1. INTRODUCTION

Let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi\,(:= \xi(\omega) = (\xi_1(\omega), \dots, \xi_s(\omega))$ an s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F_\xi(z), z \in \mathbb{R}^s)$ the distribution function of ξ; P_F, Z_F the probability measure and a support corresponding to F. Let, moreover, $g_0(:= g_0(x, z))$ be a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^s$; $X_F \subset X \subset \mathbb{R}^n$ a nonempty set generally depending on F, $X \subset \mathbb{R}^n$ a nonempty "deterministic" set. If E_F denotes the operator of mathematical expectation corresponding to F and if for $x \in X$ there exists a finite $\mathsf{E}_F g_0(x, \xi)$, then one-stage (static) "classical" stochastic optimization problem can be introduced in the form:

Find

$$\varphi(F, X_F) = \inf\{\mathsf{E}_F g_0(x, \xi) | x \in X_F\}. \tag{1.1}$$

DOI: 10.14736/kyb-2017-6-1026

To our purpose we recall only special cases of X_F. We consider the case $X_F = X$ ("deterministic" constraints); the case when there exist continuous functions $\bar{g}_i (:= \bar{g}_i(x), x \in \mathbb{R}^n)$, $i = 1, \ldots, s$ and $\bar{g}_i(x, z)$, $i = 1, \ldots, l$ defined on $\mathbb{R}^n \times \mathbb{R}^s$ such that the following constraints sets can be defined:

- $$\begin{aligned} X_F(:= X_F(\alpha)) &= \bigcap_{i=1}^{s} \{x \in X : P_F[\omega : \bar{g}_i(x) \le \xi_i] \ge \alpha_i\}, \\ &\quad \alpha_i \in (0, 1),\ i = 1, \ldots, s, \quad \alpha = (\alpha_1, \ldots, \alpha_s) \end{aligned} \tag{1.2}$$

  known, from the stochastic programming literature, as the problem with individual probability constraints,

- $$X_F(: X_F(\bar{\alpha})) = \{x \in X : P_F[\omega : \bar{g}_i(x, \xi) \le 0,\ i = 1, \ldots, l] \ge \bar{\alpha}\},\ \bar{\alpha} \in (0, 1), \tag{1.3}$$

  known as joint probability constraints.

To define the last constraints sets X_F let $g(:= g(x, z))$ be a function defined on $\mathbb{R}^n \times \mathbb{R}^s$, $Y(:= Y(z)) := Y(\xi)$ a random value with the distribution function F_Y, such that for every $x \in X$ there exist finite $\mathsf{E}_F g(x, \xi)$, $\mathsf{E}_F Y(\xi)$. Furthermore, let

$$F^2_{g(x, \xi)}(u) = \int_{-\infty}^{u} F_{g(x, \xi)}(y)\,\mathrm{d}y, \quad F^2_{Y(\xi)}(u) = \int_{-\infty}^{u} F_{Y(\xi)}(y)\,\mathrm{d}y, \quad u \in \mathbb{R}^1.$$

- $$X_F = \{x \in X : F_{g(x, \xi)}(u) \le F_{Y(\xi)}(u) \quad \text{for every} \quad u \in \mathbb{R}^1\} \tag{1.4}$$

  is known as first order stochastic dominance constraints,

- $$X_F = \{x \in X : F^2_{g(x, \xi)}(u) \le F^2_{Y(\xi)}(u) \quad \text{for every} \quad u \in \mathbb{R}^1\} \tag{1.5}$$

  as second order stochastic dominance constraints. (For more information about stochastic dominance see, e. g., [26].)

Evidently, just introduced problems are often rather complicated as from the theoretical so from the numerical point of view. Moreover, employing them to applications, other difficulties can appear. We recall some essential of them:

1. the "underlying" distribution function F can be a little changed;

2. the probability measure is unknown and the problem has to be solved on the data base. It means that the underlying probability measure P_F has to be replaced by its statistical estimates, mostly by empirical distribution function;

3. the distribution function F corresponds to real situation, however, the optimization problem (1.1) is (from the numerical point of view) very complicated. Consequently, the "underlying" distribution function has to be approximated by simpler one (usually the continuous function is replaced by discrete one).

Evidently, in all these cases, two optimization problems (real and approximate) correspond to real applications. The relationship between these problems has been investigated in the stochastic programming literature many times, however mostly, in the case when the distribution F has been replaced by empirical one (see, e. g., [1, 2, 7, 8, 11, 19, 21, 22, 24, 25, 26, 30]). The aim of this paper is, first, to recall some of these results and furthermore to employ them to scenario generation. Moreover we try to employ these results to a simple financial problem.

The paper is organized as follows. First, we recall some definitions and auxiliary assertions (section 2). Section 3 is devoted to suitable results on the stability based on the Wasserstein metric corresponding to $\mathcal{L}_1$ norm and their applications to empirical estimates. Applications to scenario generation can be found in Section 4. Section 5 is devoted to an analysis of a simple financial model. The paper is ended by a short Conclusion (section 6).

2. SOME DEFINITION AND AUXILIARY ASSERTION

To recall suitable stability assertions we first recall some definitions. To this end, if $\mathcal{P}(\mathbb{R}^s)$ denotes the set of all (Borel) probability measures on $\mathbb{R}^s$, then we can define the system $\mathcal{M}_1^1(\mathbb{R}^s)$ by the relation:

$$\mathcal{M}_1^1(\mathbb{R}^s) := \left\{ \nu \in \mathcal{P}(\mathbb{R}^s) : \int_{\mathbb{R}^s} \|z\|_1 \mathrm{d}\nu(z) < \infty \right\}, \ \|\cdot\|_1 := \|\cdot\|_1^s \text{ denotes } \mathcal{L}_1 \text{ norm in } \mathbb{R}^s. \tag{2.6}$$

Evidently, if $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$ (and the corresponding problems (1.1) are well defined), then employing the triangular inequality we can obtain

$$|\varphi(F, X_F) - \varphi(G, X_G)| \le |\varphi(F, X_F) - \varphi(F, X_G)| + |\varphi(F, X_G) - \varphi(G, X_G)|. \tag{2.7}$$

Defining quantil's vector $k_F(\alpha)$ and the set $\bar{X}(v)$, $v \in \mathbb{R}^s$ by the relations

$$\begin{aligned} k_F(\alpha) &= (k_{F_1}(\alpha_1), \ldots, k_{F_s}(\alpha_s)), \ \alpha = (\alpha_1, \ldots, \alpha_s), \\ k_{F_i}(\alpha_i) &= \sup\{z_i | \, P_{F_i}\{\omega | \, z_i \le \xi_i(\omega)\} \ge \alpha_i\}, \ \alpha_i \in (0, 1), \ i = 1, \ldots, s, \end{aligned} \tag{2.8}$$

$$\bar{X}(v) = \bigcap_{i=1}^{s} \{x \in X | \, \bar{g}_i(x) \le v_i\}, \ v = (v_1, \ldots, v_s), \ v \in \mathbb{R}^s, \tag{2.9}$$

we can recall the following auxiliary assertions.

Lemma 2.1. (Kaňková [10]) Let $\bar{g}_i(x)$, $i = 1, \ldots, s$ be continuous functions defined on $\mathbb{R}^n$, P_{F_i}, $i = 1, \ldots, s$ be absolutely continuous w. r. t. the Lebesgue measure on $\mathbb{R}^1$. Let, moreover, $X_F(\alpha)$ be defined by (1.2), then

$$X_F(\alpha) = \bar{X}(k_F(\alpha)), \quad \alpha = (\alpha_1, \ldots, \alpha_s), \ \alpha_i \in (0, 1), i = 1, \ldots, s.$$

(F_i, $i = 1, \ldots, s$ denote one–dimensional marginal distribution functions corresponding to F.)

Lemma 2.2. Let $g(x, z)$, $Y(z)$ be for every $x \in X$ a Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $x \in X$. Let, moreover, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$. If X_F is defined by the relation (1.5), then

1. $$X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+, \quad u \in \mathbb{R}^1\},$$

2. $$(u - g(x, z))^+, \ (u - Y(z))^+, \quad u \in \mathbb{R}^1, x \in \mathbb{R}^n$$

   are Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $u \in \mathbb{R}^1, x \in \mathbb{R}^n$.

Proof. The first assertion follows from the relation (4.7) in [26], second assertion can be found in [14]. □

To recall assertions we introduce the system of assumptions.

A.0 $g_0(x, z)$ is for $x \in X$ a Lipschitz function of $z \in \mathbb{R}^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x,

A.1 $g_0(x, z)$ is either a uniformly continuous function on $X \times \mathbb{R}^s$, or there exists $\varepsilon > 0$ such that $g_0(x, z)$ is a function convex on X^ε and bounded on $X^\varepsilon \times \mathbb{R}^s$, where X^ε denotes the $\varepsilon-$ neighbourhood of the set X,

A.2
- $\{\xi^i\}_{i=1}^{\infty}$ is a sequence of independent random vectors corresponding to F,
- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$, $N = 1, 2, \ldots,$

A.3 P_{F_i}, $i = 1, \ldots, s$ are absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^1$ (we denote by f_i, $i = 1, \ldots, s$ the probability densities corresponding to F_i),

A.4 there exist constants $\vartheta_i > 0$, $\delta_i > 0$ and δ_i- neighbourhood $U_i^{\delta_i}(k_{F_i}(\alpha_i))$ of $k_{F_i}(\alpha_i)$ such that $f_i(z_i) > \vartheta_i$ for $z_i \in U_i^{\delta_i}(k_{F_i}(\alpha_i))$, $\alpha_i \in (0, 1)$, $i = 1, \ldots, s$,

A.5 $\mathsf{E}_F g_0(x, \xi)$ is a Lipschitz function on X with the Lipschitz constant $\bar{L}$.

2.1. Stability

First, we recall (for us) an important stability assertion.

Proposition 2.3. (Kaňková and Houda [11]) Let $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$. If Assumption A.0 is fulfilled, then

$$|\mathsf{E}_F g_0(x, \xi) - \mathsf{E}_G g_0(x, \xi)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i \quad \text{for all} \quad x \in X. \qquad (2.10)$$

If, moreover, X is a compact set, A.1 is fulfilled, then also

$$|\inf_{x\in X}\mathsf{E}_F g_0(x,\,\xi) - \inf_{x\in X}\mathsf{E}_G g_0(x,\,\xi)| \le L\sum_{i=1}^{s}\int_{-\infty}^{+\infty}|F_i(z_i)-G_i(z_i)|\,\mathrm{d}z_i. \tag{2.11}$$

If the assumptions of Lemma 2.1 are fulfilled, X is a compact set, then X_F, X_G defined by (1.2) are also compact sets. Assumptions under which it is possible to find out $\bar{C}>0$ such that

$$\Delta[X_F(\alpha),\,X_G(\alpha)] = \Delta[\bar{X}(k_F(\alpha)),\,\bar{X}(k_G(\alpha))] \le \bar{C}\sum_{i=1}^{s}|k_{F_i}(\alpha_i) - k_{G_i}(\alpha_i)|$$

are introduced in [10]. ($\Delta[\cdot,\,\cdot]=\Delta_n[\cdot,\,\cdot]$ denotes the Hausdorff distance in the space of nonempty closed subsets of $\mathbb{R}^n$; for the definition of the Hausdorff distance see, e.g., [20] or [23].)

Lemma 2.4. Let X be a nonempty compact set, $\alpha=(\alpha_1,\,\ldots,\,\alpha_s)$, $\alpha_i\in(0,\,1)$, $i=1,\,\ldots,\,s$, Assumptions A.1, A.3 be fulfilled. If

1. $g_0(x,\,z)$ is a Lipschitz function on X with the Lipschitz constant $\bar{L}$ not depending on $z\in Z_F\bigcup Z_G$,
2. $P_F,\,P_G\in\mathcal{M}_1^1(\mathbb{R}^s)$ and moreover $\bar{X}(k_F(\alpha))$, $\bar{X}(k_G(\alpha))$ are nonempty sets,
3. there exists a constant $\bar{C}$ such that
$$\Delta[\bar{X}(v^1),\,\bar{X}(v^2)] \le \bar{C}\|v^1-v^2\|_2 \quad \text{for every} \quad v^1,\,v^2\in Z_F\bigcup Z_G,$$

then

$$|\inf_{x\in\bar{X}(k_F(\alpha))}\mathsf{E}_F g_0(x,\,\xi) - \inf_{x\in\bar{X}(k_G(\alpha))}\mathsf{E}_F g_0(x,\,\xi)| \le \bar{L}\bar{C}\|k_F(\alpha)-k_G(\alpha)\|_2. \tag{2.12}$$

($\|\cdot\|_2 := \|\cdot\|_2^s$ denotes the Euclidean norm in $\mathbb{R}^s$.)

Proof. First, it follows from the Assumption 1 that $\mathsf{E}_F g_0(x,\,\xi)$ is a Lipschitz function on X with the Lipschitz constant $\bar{L}$ (Assumption A.5). Consequently, it is easy to see that the assertion of Lemma 2.4 follows from Proposition 1 in [10]. □

Furthermore, evidently, if moreover Assumption A.4 is fulfilled and if

$$|G_i(z_i)-F_i(z_i)| \le \frac{1}{2}\vartheta_i\delta_i \quad \text{for} \quad z_i\in U_i^{\delta_i}(k_{F_i}(\alpha_i)), \quad i=1,\,\ldots,\,s,$$

then

$$G_i(k_{F_i}(\alpha_i)-\delta_i) \le F_i(k_{F_i}(\alpha_i)) - \frac{1}{2}\vartheta_i\delta_i, \quad G_i(k_{F_i}(\alpha_i)+\delta_i) \ge F_i(k_{F_i}(\alpha_i)) + \frac{1}{2}\vartheta_i\delta_i.$$

Consequently $k_{G_i}(\alpha_i)\in\langle k_{F_i}(\alpha_i)-\delta_i,\,k_{F_i}(\alpha_i)+\delta_i\rangle$ and

$$|\inf_{x\in\bar{X}(k_F(\alpha))}\mathsf{E}_F g_0(x,\,\xi) - \inf_{x\in\bar{X}(k_G(\alpha))}\mathsf{E}_F g_0(x,\,\xi)| \le \bar{L}\bar{C}\sum_{i=1}^{s}\delta_i. \tag{2.13}$$

Summarizing now the relation (2.7), Proposition 2.3 and Lemma 2.4 we can obtain.

Proposition 2.5. Let X be nonempty compact set, $\alpha = (\alpha_1, \ldots, \alpha_s)$, $\alpha_i \in (0, 1)$, $i = 1, \ldots, s$, $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, Assumptions A.0, A.1, A.3 be fulfilled. If, moreover,

1. $g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant $\bar{L}$ not depending on z,
2. $\bar{X}(k_F(\alpha))$, $\bar{X}(k_G(\alpha))$ are nonempty sets,
3. there exists a constant $\bar{C}$ such that

$$\Delta[\bar{X}(v^1), \bar{X}(v^2)] \le \bar{C}\|v^1 - v^2\|_2 \quad \text{for every} \quad v^1, v^2 \in Z_F \bigcup Z_G,$$

then

$$\begin{aligned} &\left| \inf_{x \in \bar{X}(k_F(\alpha))} \mathsf{E}_F g_0(x, \xi) - \inf_{x \in \bar{X}(k_G(\alpha))} \mathsf{E}_G g_0(x, \xi) \right| \le \\ &L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i + \bar{L}\bar{C}\|k_F(\alpha) - k_G(\alpha)\|_2. \end{aligned} \tag{2.14}$$

Analyzing further the case of constraints set X_F introduced by (1.5) we first define for $\varepsilon \in \mathbb{R}^1$ the sets

$$X_F^{\varepsilon} = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \le \varepsilon, \quad u \in \mathbb{R}^1\}, \quad \varepsilon \in \mathbb{R}^1; \tag{2.15}$$

evidently $X_F^0 = X_F$.

If the assumptions of Lemma 2.2 are fulfilled, P_F, $P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, $u \in \mathbb{R}^1$, $x \in X$, then

$$\begin{aligned} |\mathsf{E}_F(u - g(x,\xi))^+ - \mathsf{E}_G(u - g(x,\xi))^+| &\le L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i, \\ |\mathsf{E}_F(u - Y(\xi))^+ - \mathsf{E}_G(u - Y(\xi))^+| &\le L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i. \end{aligned} \tag{2.16}$$

Consequently

$$\begin{aligned} &|\mathsf{E}_F(u - g(x,\xi))^+ + \mathsf{E}_G(u - g(x,\xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ - \mathsf{E}_G(u - Y(\xi))^+| \\ &\qquad \le 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i, \end{aligned}$$

$$\begin{aligned} x \in X_F &\Rightarrow |\mathsf{E}_G(u - g(x,\xi))^+ - \mathsf{E}_G(u - Y(\xi)^+| \le 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i, \\ x \in X_G &\Rightarrow |\mathsf{E}_F(u - g(x,\xi))^+ - \mathsf{E}_F(u - Y(\xi)^+| \le 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i \end{aligned}$$

or equivalently

$$x \in X_F \Longrightarrow x \in X_G^{\varepsilon}, \quad x \in X_G \Longrightarrow x \in X_F^{\varepsilon} \quad \text{with} \quad \varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i.$$

More generally

$$X_G^{\delta-\varepsilon} \subset X_F^{\delta} \subset X_G^{\delta+\varepsilon} \quad \text{for} \quad \delta \in \mathbb{R}^1. \tag{2.17}$$

Lemma 2.6. Let X be a nonempty compact set, $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, Assumption A.1 be fulfilled. Let, moreover, $g(x, z)$ be for every $x \in X$ a Lipschitz function of $z \in Z_F \bigcup Z_G$ with the Lipschitz constant L_g not depending on $x \in X$. If

1. $g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant $\bar{L}$ not depending on z (Assumption 5),

2. X_F, X_G defined by (1.5) are nonempty compact sets,

3. there exists a constant $D > 0$ such that

$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \le D\varepsilon \quad \text{for every} \quad \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon \rangle,$$

   with $\varepsilon = 2L_g \sum\limits_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i$,

then

$$\left| \inf_{x \in X_F} \mathsf{E}_F g_0(x, \xi) - \inf_{x \in X_G} \mathsf{E}_F g_0(x, \xi) \right| \le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i. \tag{2.18}$$

P r o o f. Since the Hausdorff distance is a metric in the space of compact subsets of $\mathbb{R}^n$, (see, e. g., [23]) we can obtain that

$$\Delta[X_F, X_G] \le \Delta[X_F, X_F^{-\varepsilon}] + \Delta[X_F^{-\varepsilon}, X_G].$$

Further employing (2. 17) and assumption 3 of Lemma 2.6 we can obtain

$$\Delta[X_F, X_G] \le 2D\varepsilon.$$

The assertion of Lemma 2.6 now already follows from Proposition 1 in [10]. □

Employing the relation (2.7), Proposition 2.3 and Lemma 2.6 we can obtain.

Proposition 2.7. Let X be a compact set, $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, Assumptions A.0, A.1 and A.3 be fulfilled. Let, moreover, $g(x, z)$ be for every $x \in X$ a Lipschitz function of $z \in Z_F \bigcup Z_G$ with the Lipschitz constant L_g not depending on $x \in X$. If

1. $g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant L not depending on $z \in Z_F \bigcup Z_G$,

2. X_F, X_G defined by (1.5) are nonempty compact sets,

3. there exists a constant $D > 0$ such that

$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \leq D\varepsilon \quad \text{for every} \quad \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon \rangle,$$

with $\varepsilon = 2L_g \sum\limits_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i$,

then there exists a constant $D' > 0$ such that

$$\left| \inf_{x \in X_F} \mathsf{E}_F g_0(x, \xi) - \inf_{x \in X_G} \mathsf{E}_G g_0(x, \xi) \right| \leq D' \sum_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i. \tag{2.19}$$

Remark 2.8. The constant D' in Proposition 2.7 can be estimated by the value $L + 3D\bar{L}L_g$.

2.2. Empirical estimates

Since in applications very often P_F has to be replaced by empirical P_{F^N}, the solution of (1.1) has to be (mostly) sought w.r.t. an "empirical problem":

Find

$$\varphi(F^N, X_{F^N}) = \inf\{\mathsf{E}_{F^N} g_0(x, \xi) | x \in X_{F^N}\}, \tag{2.20}$$

where F^N denotes an empirical distribution function determined by random sample $\{\xi^i\}_{i=1}^N$ (not necessarily independent) corresponding to F. If we denote by symbols $\mathcal{X}(F, X_F)$ and $\mathcal{X}(F^N, X_{F^N})$ the optimal solution sets of (1.1) and (2.20), then under rather general assumptions, $\varphi(F^N, X_{F^N})$, $\mathcal{X}(F^N, X_{F^N})$ are "good" statistical estimates of $\varphi(F, X_F)$, $\mathcal{X}(F, X_F)$. The investigation of these estimates started in 1974 (see [30]) and it was followed by [7, 8] and many others. (Some of them were mentioned already in the Introduction, see also the subsection STABILITY.)

Results on the statistical estimates (in the stochastic programming literature) are often based on the large deviations (started by employing the inequality published in [4]), on the stability assertions corresponding to different distances in the spaces of the probability measures, see, e. g., [6, 8, 9, 19, 21, 22]. In this paper we focus mostly on the case when the empirical estimates are based on the stability results and $\mathcal{L}_1$ norm and the results corresponding to quantils (see subsection STABILITY). Replacing G by F^N (in the stability results) we can investigate the relationship between $\varphi(F^N, X_{F^N})$,

$\mathcal{X}(F^N, X_{F^N})$ and $\varphi(F, X_F)$, $\mathcal{X}(F, X_F)$. We focus on the investigation of relationship between $\varphi(F, X_F)$ and $\varphi(F^N, X_{F^N})$. To this end it is evidently suitable to investigate

$$\int_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| \,\mathrm{d}z_i, \quad |k_{F_i}(\alpha_i) - k_{F_i^N}(\alpha_i)|, \, \alpha_i \in (0, 1), \, i = 1, \ldots, s.$$

Lemma 2.9. (Shorack and Welner [29]) Let $s = 1$, $P_F \in \mathcal{M}_1^1(\mathbb{R}^1)$. Let, moreover, the assumption A.2 be fulfilled, then

$$P\left\{\omega : \int_{-\infty}^{\infty} |F(z) - F^N(z)| \,\mathrm{d}z \underset{N\to\infty}{\longrightarrow} 0\right\} = 1.$$

Evidently, the results of Lemma 2.9 hold (for one–dimensional random value) when the finite first moment exists. The case of convergence rate is more complicated. To investigate it we recall the following auxiliary assertion.

Lemma 2.10. (Houda and Kaňková [5]) Let $s = 1, r > 0, t > 0$, Assumptions A.2, A.3 be fulfilled. Let, moreover, ξ be a random variable such that $\mathsf{E}_F|\xi|^r < \infty$. If constants $\beta, \gamma > 0$ fulfil the inequalities

$$0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1 - r)\gamma < 0,$$

then

$$P\left\{\omega : N^{\beta} \int_{-\infty}^{\infty} |F(z) - F^N(z)| > t\right\} \underset{N\to\infty}{\longrightarrow} 0. \tag{2.21}$$

Evidently, the convergence rate $\beta := \beta(r)$ (introduced in Lemma 2.10) depends on the existence of finite absolutely moments. Unfortunately, we cannot obtain (by this approach) any results in the case when there exist only $\mathsf{E}_F|\xi|^r < \infty$ for $r < 2$. Some weaker assertions (for this case) can be found in [5].

Lemma 2.11. (Kaňková [13]) Let $s = 1$, $\alpha \in (0, 1)$. If Assumptions A.2, A.3 and A.4 are fulfilled, $0 < t' < \delta$, then

$$P\{\omega : |k_F(\alpha) - k_{F^N}(\alpha)| > t'\} \le 2\exp\{-2N(\vartheta t')^2\}, \quad N \in \mathcal{N}.$$

($\mathcal{N}$ denotes the set of natural numbers, $\delta = \delta_1$, $\vartheta = \vartheta_1$ defined by A.4.)

Employing Lemma 2.10, Lemma 2.11, Proposition 2.3, Proposition 2.5 and the properties of the exponentional function we obtain.

Proposition 2.12. Let X be a compact set, Assumptions A.0, A.1, A.2, A.3 and A.4 be fulfilled, $\alpha = (\alpha_1, \ldots, \alpha_s)$, $\alpha_i \in (0, 1)$ and $t > 0$. Let moreover X_F be defined by the relation (1.2). If

1. for every $v \in Z_F$, $\bar{X}(v)$ are nonempty sets and moreover there exists a constant $\bar{C}$ such that

$$\Delta[\bar{X}(v^1),\, \bar{X}(v^2)] \le \bar{C}\|v^1 - v^2\|_2, \quad v^1, v^2 \in Z_F,$$

2. $g_0(x,\, z)$ is for every $z \in Z_F$ a Lipschitz function of $x \in X$ with the Lipschitz constant $\bar{L}$ not depending on $z \in Z_F$ (Assumption A.5),

3. for all components ξ_i, $i = 1, \ldots, s$ of the vector ξ and $r > 0$ there exist finite $\mathsf{E}_F|\xi_i|^r$. If constants $\beta,\, \gamma > 0$ fulfil the inequalities

$$0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1-r)\gamma < 0,$$

then

$$P\left\{\omega :\, N^{\beta}|\inf_{\bar{X}(k_F(\alpha))} \mathsf{E}_F g_0(x,\, \xi) - \inf_{\bar{X}(k_{F^N}(\alpha))} \mathsf{E}_{F^N} g_0(x,\, \xi)| > t\right\} \xrightarrow[N\to\infty]{} 0. \tag{2.22}$$

Considering X_F defined by (1.5) we can obtain.

Proposition 2.13. (Kaňková and Houda [14]) Let X be a compact set, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, Assumptions A.0, A.1, A.2 and A.3 be fulfilled, X_F be defined by the relation (1.5). Let, moreover, $g(x,\, z)$ be for every $x \in X$ a Lipschitz function of $z \in Z_F$ with the Lipschitz constant not depending on $x \in X$. If

1. $g_0(x,\, z)$ is a Lipschitz function on X with the Lipschitz constant L not depending on $z \in Z_F$,

2. X_F defined by (1.5) is a nonempty compact set,

3. there exists $\varepsilon_0 > 0$ such that X_F^{ε} are nonempty compact sets for every $\varepsilon \in \langle -\varepsilon_0,\, \varepsilon_0\rangle$ and, moreover, there exists a constant $\hat{C} > 0$ such that

$$\Delta[x_F^{\varepsilon},\, X_F^{\varepsilon'}] \le \hat{C}|\varepsilon - \varepsilon'| \quad \text{for} \quad \varepsilon,\, \varepsilon' \in \langle -\varepsilon_0,\, \varepsilon_0\rangle,$$

4. for all components ξ_i, $i = 1, \ldots,\ , s$ of the vector ξ and $r > 0$ there exist finite $\mathsf{E}_F|\xi_i|^r$. If constants $\beta,\, \gamma > 0$ fulfil the inequalities

$$0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1-r)\gamma < 0,$$

then

$$P\left\{\omega :\, N^{\beta}|\inf_{X_F} \mathsf{E}_F g_0(x,\, \xi) - \inf_{X_{F^N}} \mathsf{E}_{F^N} g_0(x,\, \xi)| > t\right\} \xrightarrow[N\to\infty]{} 0. \tag{2.23}$$

3. SCENARIO GENERATION

Evidently, it can be very complicated (from the numerical point of view) to solve the stochastic programming problems; especially, when the "underlying" probability measure belongs to a continuous type. Consequently the "underlying" continuous probability measure is often replaced by discrete one with finite number of atoms (scenaria). We employ the results on the stability (subsection 2.1) to suggest one of possibilities to scenario generation. First, we consider the case of static (one–stage) stochastic programming problems, further we try to generalize this approach to special case of multistage stochastic programming problem.

3.1. One–stage case

First, we shall deal with the case of static stochastic programs. In particular, the aim of this subsection is to deal with the above mentioned discrete approximation in the case when the approximation error can be estimated by the sum of one–dimensional Wasserstein distances. To this end we employ the stability results. Namely considering Problem (1.1) with $X_F = X$ or with X_F fulfilling the definition (1.5) and assuming that X is a compact set, A.0, A.1 are fulfilled and $P_F,\, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$ we can see that

$$|\varphi(F,\, X_F) - \varphi(G,\, X_G)|$$

can be bounded by the value

$$L' \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|\, \mathrm{d}z_i, \quad L' \quad \text{"suitable"constant.}$$

Consequently, if we suppose that F is "underlying original" distribution function and G approximate one, then evidently we can construct G with given approximation error. To this end we employ the approach of [15]. If A.3 is fulfilled, $P_{F_i} \in \mathcal{M}_1^1(\mathbb{R}^1)$, $i = 1, \ldots, s$, then for given $M_i > 0$, $i = 1, \ldots, s$ there exist natural numbers m_i, $i = 1, \ldots, s$, points $z_{i,\,j},\, \in \bar{\mathbb{R}}^1$, $j = 0, 1, \ldots, m_i$ and one–dimensional discrete distribution functions G_i such that

$$-\infty = z_{i,\,0} < z_{i,\,1} < z_{i,\,2} < \ldots < z_{i,\,m_{i-1}} < z_{i,\,m_i} = \infty,$$

and, simultaneously,

$$(L/s) \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)|\, \mathrm{d}z_i \le M_i,\; i = 1, \ldots, s.$$

($\bar{\mathbb{R}}^1$ denotes the extended real line.)

Furthermore, it follows from the last relations that there exists s–dimensional distribution function G with marginals G_i, $i = 1, \ldots, s$ such that

$$L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|\, \mathrm{d}z_i \le \sum_{i-1}^{s} M_i. \tag{3.24}$$

Employing the relation (3.24), Proposition 2.3 and Proposition 2.7, we can see that the following Proposition is valid.

Proposition 3.1. Let Assumption A.0, A.1, A.3 be fulfilled, $P_{F_i} \in \mathcal{M}_1^1(\mathbb{R}^1)$. Let, moreover, $M,\, \bar{M} > 0$, X be a compact set. If

1. $X_F = X$, then there exists a discrete distribution function G with discrete marginals G_i, $i = 1, \ldots, s$ and finite number of atoms such that

$$|\varphi(F,\, X) - \varphi(G,\, X)| \le M, \tag{3.25}$$

2.
   - X_F fulfils the relation (1.5) and the assumption 2 of Lemma 2.12 holds,
   - $g(x,\, z)$ is for every $x \in X$ a Lipschitz function of $z \in Z_F$ with the Lipschitz constant L_g not depending on $x \in X$,
   - there exists ε_0, $\hat{D} > 0$ such that

$$\Delta[X_F^{\varepsilon},\, X_F^{\varepsilon'}] \le \hat{D}|\varepsilon - \varepsilon'| \quad \text{for} \quad \varepsilon,\, \varepsilon' \in \langle -3\varepsilon_0, 3\varepsilon_0 \rangle,$$

   then there exists a discrete distribution function $\bar{G}$ with finite number of atoms such that

$$|\varphi(F,\, X_F) - \varphi(\bar{G},\, X_{\bar{G}})| \le \bar{M}. \tag{3.26}$$

Remark 3.2.

- According to Proposition 2.7 it is suitable the following relation

$$\varepsilon_0 \ge 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - \bar{G}_i(z_i)|\, \mathrm{d}z_i,$$

  be fulfilled.

- In the case when X_F fulfils the relation (1.2), then it is necessary to consider the case 1, to assume A.4, A.5, $L\bar{C}\|k_F(\alpha) - k_G(\alpha)\|_2$ be sufficiently small, employ the relation (2.7) and Proposition 2.5.

3.2. Multistage case

Till now we have considered problems with respect to one time point. However economic activities are mostly developing in time and moreover it is reasonable to determine a decision (in given time point) as a function of a random sequence realization and decision to this time. Multistage stochastic programming problems pose to such situation. In this paper we consider only special case of the multistage programs. To this end we recall $T+1-$ stage stochastic programs by the following way:

Find

$$\varphi_{\mathcal{F}}(T) = \inf \{\mathsf{E}_{F^{\xi^0}} g^0_{\mathcal{F}}(x^0,\, \xi^0)|\; x^0 \in \mathcal{K}^0\}, \tag{3.27}$$

where the function $g^0_{\mathcal{F}}(x^0,\, z^0)$ is defined recursively

$$\begin{aligned} g^k_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) &= \inf\{\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}\, g^{k+1}_{\mathcal{F}}(\bar{x}^{k+1},\, \bar{\xi}^{k+1})\,|x^{k+1} \in \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k)\},\\ k &= 0,\, 1,\, \ldots,\, T-1,\\ g^T_{\mathcal{F}}(\bar{x}^T,\, \bar{z}^T) &:= g^T_0(\bar{x}^T,\, \bar{z}^T), \quad \mathcal{K}_0 := X^0. \end{aligned} \tag{3.28}$$

$\xi^j := \xi^j(\omega)$, $j = 0,\, 1,\, \ldots,\, T$ denotes an s–dimensional random vector defined on a probability space $(\Omega,\, \mathcal{S},\, P)$; $F^{\xi^j}(z^j)$, $z^j \in \mathbb{R}^s$, $j = 0,\, 1\, \ldots,\, T$ the distribution function of ξ^j and $F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})$, $z^k \in \mathbb{R}^s$, $\bar{z}^{k-1} \in \mathbb{R}^{(k-1)s}$, $k = 1,\, \ldots,\, T$ the conditional distribution function (ξ^k conditioned by $\bar{\xi}^{k-1}$); $P_{F^{\xi^j}}$, $P_{F^{\xi^{k+1}|\bar{\xi}^k}}$, $j = 0,\, 1,\, \ldots,\, T$, $k = 0,\, 1,\, \ldots,\, T-1$ the corresponding probability measures; $Z^j := Z_{F^{\xi^j}} \subset \mathbb{R}^s$, $j = 0,\, 1,\, \ldots,\, T$ the support of the probability measure $P_{F^{\xi^j}}$. Furthermore, the symbol $g^T_0 := g^T_0(\bar{x}^T,\, \bar{z}^T)$ denotes a continuous function defined on $\mathbb{R}^{n(T+1)} \times \mathbb{R}^{s(T+1)}$; $X^k \subset \mathbb{R}^n$, $k = 0,\, 1,\, \ldots,\, T$ is a nonempty compact set; the symbol $\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k,\, \bar{z}^k) := \mathcal{K}^{k+1}_{F^{\xi^{k+1}|\bar{\xi}^k}}(\bar{x}^k,\, \bar{z}^k)$, $k = 0,\, 1,\, \ldots$ $\ldots, T-1$ denotes a measurable multifunction defined on $\mathbb{R}^{n(k+1)} \times \mathbb{R}^{s(k+1)}$ with "values" subsets of $\mathbb{R}^n$. $\bar{\xi}^k (:= \bar{\xi}^k(\omega)) = [\xi^0,\, \ldots,\, \xi^k]$; $\bar{z}^k = [z^0,\, \ldots,\, z^k]$, $z^j \in \mathbb{R}^s$; $\bar{x}^k = [x^0,\, \ldots,\, x^k]$, $x^j \in \mathbb{R}^n$; $\bar{X}^k = X^0 \times X^1 \ldots \times X^k$; $\bar{Z}^k := \bar{Z}^k_{\mathcal{F}} = Z_{F^{\xi^0}} \times Z_{F^{\xi^1}} \ldots \times Z_{F^{\xi^k}}$, $j = 0,\, 1,\, \ldots, k$, $k = 0,\, 1,\, \ldots,\, M$. Symbols $\mathsf{E}_{F^{\xi^0}}$, $\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}$, $k = 0,\, 1,\, \ldots,\, T-1$ denote the operators of mathematical expectation corresponding to F^{ξ^0}, $F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}$, $k = 0,\, \ldots,\, T-1$.

In the multistage case, we restrict to the case when the following assumption is fulfilled:

C.1 Random sequence $\{\xi^k\}_{k=-\infty}^{\infty}$ follows (generally) nonlinear autoregressive sequence

$$\xi^k = H(\xi^{k-1},\, \nu^k),$$

where ξ^0, ν^k, $k = 1,\, 2,\, \ldots$ are stochastically independent s–dimensional random vectors defined on $(\Omega,\, \mathcal{S},\, P)$ and, moreover, ν^k, $k = 1,\, \ldots$ identically distributed. $H = (H_1,\, \ldots,\, H_s)$ is a Lipschitz vector function defined on $\mathbb{R}^s$. We denote the distribution function corresponding to $\nu^1 = (\nu^1_1,\, \ldots,\, \nu^1_s)$ by the symbol F^ν and suppose the realization ξ^0 to be known.

Evidently, the multistage stochastic programming problem (3.27), (3.28) depends essentially on a system of (generally) conditional distribution functions

$$\mathcal{F} = \{F^{\xi^0}(z^0),\, F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\, k = 1,\, \ldots,\, T\}. \tag{3.29}$$

Consequently, if we replace $\mathcal{F}$ by another system $\mathcal{G}$

$$\mathcal{G} = \{G^{\xi^0}(z^0),\, G^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1}),\, k = 1,\, \ldots,\, T\}, \tag{3.30}$$

we obtain another multistage stochastic programming problem with the optimal value denoted $\varphi_{\mathcal{G}}(T)$.

Under Assumption C.1 the system $\mathcal{F}$ is determined by F^{ξ^0} and F^{ν}. Consequently, if we replace these two probability distribution functions by another G^{ξ^0} and G^{ν}, we obtain another system $\mathcal{G}$.

Considering, furthermore, the constraint sets $\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k)$, $k = 0, \ldots, T-1$ not depending on the probability measure, then the assumptions under which

$$|\varphi_{\mathcal{F}}(M) - \varphi_{\mathcal{G}}(M)| \le \sum_{i=1}^{s} C^i_W \int_{R^1} |F^{\nu}_i(z_i) - G^{\nu}_i(z_i)| \, \mathrm{d}z_i, \quad C^i_W > 0, \, i = 1, \ldots, s$$

can be found in [12]. Consequently, if we define discrete distributions G^{ξ_0}, G^{ν} determined by the approach of Proposition 3.1 (the case 1), then we have an approximating system $\mathcal{G}$ given by discrete mostly conditional distributional functions. Furthermore, it follows from results of the above mentioned work that this approach can be generalized to the case when constraints sets are given by the individual probability constraints (for more details see, e. g., [12]).

Remark 3.3. In this subsection we have denoted only to deterministic scenario generation. However random scenario we can obtain by random sample. The corresponding property of such approximation are given in the subsection "Empirical Estimation".

4. SIMPLE FINANCIAL PROBLEM

We analyze very simple financial problem from the point of view of the stochastic programming theory. The aim of this example is not only to demonstrate a utilizing the former theory but also to present a life situation when this theory can not be completely employed. To this end we consider a situation about a mortgage and its instalments. Let us start with a common situation. People try to gain (in the last decades) own residence (a flat or little house). Since young people do not posses necessary financial resources, the bank sector offers them a mortgage. Of course banks can employ excellent experts to minimize their risk and maximize profit in dependence of debtor's position. The aim of our approach is to analyze the situation from the second side. In particular, our aim is to investigate the possibilities of the debtors not only in dependence on their present–day situation, but also on their future private and subjective decisions and on possible "unpleasant" events. In details the aim is to suggest a method for a recognition of a "safe" loan and simultaneously to offer tactics to state a plausible environment for future time. Of course we suppose that our analysis is first contribution to this situation. The stochastic programming theory will be employed to it. Let us start with simple standard situation. A young married couple wants to gain own flat. They already obtained an offer from banks determined by their present–day situation. However they have subjective plans. According to this fact we try to analyze their possibilities. To this end let us assume that (in the start time) their monthly income is

$$Z_0 = U_0 + V_0, \quad \text{where } U_0 \text{ is an income of husband and } V_0 \text{ is an income of wife.}$$

Evidently, this income can be divided into three parts Z_0^1, Z_0^2, Z_0^3, where Z_0^1 denotes means for a basic consumption, Z_0^2 denotes means that can be employed for a repayment of installments and Z_0^3 can be considered as an allocation to saving. Consequently

$$Z_0 = Z_0^1 + Z_0^2 + Z_0^3, \quad Z_0^1,\, Z_0^2 > 0,\, Z_0^3 \geq 0. \tag{4.31}$$

Given the annuity repayments, which is the most standard way of repaying the loan; if we denote by a symbol M the value of the loan, by m number of identical installments and by ζ the loan interest rate, then the identical installments $b(M) := b(\zeta)$ in time points $t = 1,\, 2,\, \ldots,\, m$ (see, e. g., [17] or [28]) are given by

$$b(M) := b(\zeta) = \begin{array}{ll} \frac{M\zeta}{1-v^m}, & \zeta \neq 0, \quad v = v(\zeta) = (1+\zeta)^{-1}, \\ \frac{1}{m}, & \zeta = 0. \end{array} \tag{4.32}$$

It follows from the relations (4.31), (4.32) that (in the case when $\zeta \neq 0$) it is desirable (in "static" approach) the following inequality

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \leq Z_0^2 \tag{4.33}$$

to be fulfilled. Of course, this condition (in the extreme case) can be replaced by the inequality

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \leq Z_0^2 + Z_0^3. \tag{4.34}$$

If it is possible to assume that the relations (4.31), (4.33) will be fulfilled also in future, then the young people can take the loan equal to the maximal value M for which the inequality (4.33) (respectively (4.34)) is fulfilled. However mostly it is necessary to assume that the financial situation of young married couple can change. For example: it is reasonable to assume that in some time period, say $(m_1,\, m_2)$, $0 < m_1 < m_2 \leq m$ the married couple plan to have a baby. According to this fact and to the social politics of a state the young people can assume the less income in this time, approximately equal to

$$Z_1 = U_0 + V_1 = Z_0^1 + Z_1^2 + Z_1^3, \quad Z_1^2,\, Z_1^3 \geq 0, Z_1^2 \leq Z_0^2,$$

where V_1 is the supposed income of wife in the time interval $\langle m_1,\, m_2 \rangle$; Z_1^2 denotes the means, that can be employed for a repayment of installments (of course $Z_1^2 \leq Z_0^2$) and Z_1^3 saved amount in every year of this time interval (of course mostly $0 \leq Z_1^3 \leq Z_0^3$). Evidently without financial reserve the inequalities

$$Z_0^1 + Z_0^2 \leq U_0 + V_1$$

need to be fulfilled. Consequently, if

$$U_0 + V_1 < Z_0^1 + Z_0^2,$$

then a very serious trouble could arise. However, if the young couple saved every time point $t \in \{0,\, \ldots,\, m_1 - 1\}$ the amount Z_0^3 and if the inequality

$$\frac{(m_2 - m_1)M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} \leq (m_2 - m_1)[Z_0^2 - Z_1^2] + (m_1 - 1)Z_0^3 \tag{4.35}$$

is fulfilled, then they endure the time period $\langle m_1, m_2\rangle$ without financial troubles.

To construct the relation (4.35), it has been assumed that the amount Z_0^3 is deterministic, the same in every time point $t \in \{0, \ldots, m_1 - 1\}$ and that this amount can not be changed. However this situation can be a little different. To explain a new approach we suppose $m_1 = 2, \quad m_2 - m_1 = 2$; it means $m_1 = 2$, $m_2 = 4$ and m is determined by the relation (4.32) (consequently dependent on M). Furthermore we denote Z_t^2, $t = 0, \ldots m$ the means that can be employed for a repayment of installments, Z_t^3 an allocation for a saving ($0 \le Z_t^2 \le Z_0^2$, $t = m_1, \ldots, m_2$, $Z_t^3 \ge 0$, $t = 0, \ldots, m$).

We consider two special cases.

D 1. The deterministic value Z_0^3 (in the relation (4.31)) can be replaced by random values Z_t^3; Z_t^3, $t \in \{0, 1, \ldots, m\}$ with probability one non negative. Consequently the deterministic income $Z_0 = Z_0^1 + Z_0^2 + Z_0^3$ is replaced by random $Z_t = Z_0^1 + Z_t^2 + Z_t^3$ in all points $t = 0, 1, \ldots, m$. We assume that young people can these random amount (in time point $t = 0$, 1) invest (for example) into two assets to obtain:

- in the original year the value $\quad \xi_{0,1}x_{0,1} + \xi_{0,2}x_{0,2}$
under the assumptions $\quad x_{0,1} + x_{0,2} \le Z_0^3, \quad x_{0,1}, x_{0,2} \ge 0,$

- in the second year the value $\quad \xi_{1,1}x_{1,1} + \xi_{1,2}x_{1,2}$
under the assumptions $\quad x_{1,1} + x_{1,2} \le Z_1^3, \quad x_{1,1}, x_{1,2} \ge 0$

(under the assumptions that the profit obtained in the time $t = 0$ can not influence the invested amount in the time $t = 1$). Evidently, it is desirable (for young people under the assumption $Z_2^2 = Z_3^2 = Z_4^2$) the fulfilling of the relation

$$\frac{(m_2 - m_1)M[\zeta(1+\zeta)^m]}{(1+\zeta)^m - 1} \le 3[Z_0^2 - Z_2^2] + \sum_{t=0}^{1}[\xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,2}], \qquad (4.36)$$

and of course the maximization of a possible profit.

Z_0^3, Z_1^3, $\xi_{0,1}$, $\xi_{0,2}$, $\xi_{1,2}$, $\xi_{1,2}$ are generally supposed to be random variables with "positive support". Consequently, it is necessary to "specify" the sense of relations in D.1. In details, it is necessary to "specify" when the operator of mathematical expectation, probability constraints, risk constraints or stochastic dominance constraints are employed in the optimization problems. We set (for simplicity) to this case a very simple stochastic optimization problem.

$$\text{Find} \qquad \max M \qquad (4.37)$$

under the system of constraints

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \le Z_0^2, \qquad (4.38)$$

$$P_F\{x_{t,1} + x_{t,2} \le Z_t^3\} \quad \ge \quad 1 - \varepsilon_t, \quad \varepsilon_t \in (0,1),\ x_{t,1},\, x_{t,2} \ge 0, \quad t = 0,\, 1, \tag{4.39}$$

$$\begin{aligned} P_F\{\tfrac{3M[\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \quad \le \quad & \sum_{t=2}^{4}[Z_0^2 - Z_t^2] + \sum_{t=0}^{1}[\xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,2}]\} \ge 1 - \varepsilon_0, \\ & \varepsilon_0 \in (0,\, 1). \end{aligned} \tag{4.40}$$

Evidently, in this case it is reasonable to add to an objective function (4.37) the second one

$$\mathsf{E}_F \sum_{t=m_1}^{m} [\xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,2}] \tag{4.41}$$

with the corresponding constraints

$$\begin{aligned} P_F\{x_{t,1} + x_{t,2} \le \max(0, Z_t^3)\} \quad & \ge \quad 1 - \varepsilon_t,\ \varepsilon_t \in (0,\, 1),\ x_{t,\,1},\, x_{t,\,2} \ge 0, \\ & \qquad t = 2,\, 3,\, 4, \\ P_F\{x_{t,1} + x_{t,2} \le Z_t^3\} \quad & \ge \quad 1 - \varepsilon_t,\ \varepsilon_t \in (0,\, 1),\ x_{t,\,1},\, x_{t,\,2} \ge 0, \\ & \qquad t = 5,\, \ldots,\, m. \end{aligned} \tag{4.42}$$

$\xi_{t,1}$, $\xi_{t,\,2}$, $t = m_1, \ldots, m$ random value. Consequently, we have constructed two objective stochastic programming problem with objective (4.37) and (4.41) and constraints (4.38), (4.39), (4.40) and (4.42). Analyzing this model we can see

- constraints (4.38) are linear deterministic,
- constraints (4.39) and (4.42) (according to Lemma 2.1) can be expressed in equivalent form of linear inequalities,
- (4.37) and (4.41) are linear objective functions. Employing a convex combination we obtain only one linear objective.

Evidently, mentioned objective and constraints are suitable for discrete approximation. However, it remains to deal with constraints set (4.40). This condition can be rewritten in the equivalent form

$$\begin{aligned} P_F\Big\{\sum_{t=2}^{4}[Z_t^2 - Z_0^2] \quad \le \quad & \sum_{t=0}^{1}[\xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,2}] - \frac{3M[\zeta(1+\zeta)^m}{(1+\zeta)^m - 1}\Big\} \ge 1 - \varepsilon_0, \\ & \varepsilon_0 \in (0,\, 1). \end{aligned} \tag{4.43}$$

There the function in the probability is not simultaneously, in decision parameter and random factor, convex. Consequently, the corresponding constraint is not generally convex. It is question if it is not reasonable to employ Markowitz approach (for more details see, e. g. [3]) and to replace (4.43). by

$$\mathsf{E}_F \sum_{t=0}^{1} [\xi_{t,1}x_{i,1} + \xi_{t,2}x_{t,2}] - KV \left[\sum_{i=0}^{1} [\xi_{t,1}x_{t,1} + \xi_{t,2}x_{t,2} \right] \geq \sum_{t=2}^{4} [Z_t^2 - Z_0^2], \qquad (4.44)$$

where V denotes a symbol for the Variation, $K > 0$ i s a suitable constant. How to take the constant K is however evidently beyond the scope of this paper.

D.2 $Z_t^3(1)$, $t = 0, 1, \ldots, m$ have a deterministic character. Let us assume that these amounts can be invested into two assets (portfolio) with random returns $\bar{\xi}_{t,1}, \bar{\xi}_{t,2}$. Mathematically saying, it is desirable to determine $x_{0,1}, x_{0,2}, x_{1,1}, x_{1,2}$ fulfilling the relations

$$\begin{aligned} & & & x_{t,1} + x_{t,2} \leq Z_t^3, \quad x_{t,1}, x_{t,2} \geq 0, \\ & \text{to obtain random values} & & \hat{g}_t = \bar{\xi}_{t,1}x_{t,1} + \bar{\xi}_{t,2}x_{t,2}, \\ & & & t = 0, 1, \ldots, m. \end{aligned}$$

Evidently, it is possible also to define random values Y_t, by the following relation

$$Y_t = \frac{1}{2}\bar{\xi}_{t,1} + \frac{1}{2}\bar{\xi}_{t,2}, \quad t = 0, 1, \ldots, m. \qquad (4.45)$$

$\hat{g}_t, Y_t$ are random values "depending" on Z_t^3.

Employing the theory of the stochastic dominance [26] it is "reasonable" to determine $x_{t,1}, x_{t,2}$ such that

$$F_{\hat{g}_t} \succeq_1 F_{Y_t}, \quad \text{or} \quad F_{\hat{g}_t} \succeq_2 F_{Y_t}, \quad t = 0, 1, \ldots, m. \qquad (4.46)$$

Evidently, in this case we can construct the following optimization problem.

$$\text{Find} \qquad \max M \qquad (4.47)$$

under the system of constraints

$$\frac{M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1} \leq Z_0^2, \qquad (4.48)$$

$$F_{\hat{g}_t} \succeq_2 F_{Y_t}, \quad t = 0, 1 \qquad (4.49)$$

$$\begin{aligned} P_F\{\tfrac{3M\zeta(1+\zeta)^m}{(1+\zeta)^m - 1}\} \quad \leq \quad & \sum_{t=2}^{4}[Z_0^2 - Z_t^2] + \sum_{t=0}^{1}[\bar{\xi}_{t,1}x_{t,1} + \bar{\xi}_{t,2}x_{t,2}]\} \geq 1 - \varepsilon_0, \\ & \varepsilon_0 \in (0, 1). \end{aligned} \qquad (4.50)$$

It is reasonable to add to an objective function (4.47) the second one

$$\mathsf{E}_F \sum_{t=5}^{m} [\bar{\xi}_{t,1} x_{t,1} + \bar{\xi}_{t,2} x_{t,2}] \tag{4.51}$$

with the corresponding constraints

$$F_{\hat{g}_t} \succeq_2 F_{Y_t}, \quad t = 2, \ldots, m. \tag{4.52}$$

Analyzing the approach D.2 we can see that in the constraints (4.49) and (4.52) the "underlying" distribution function can be replaced by discrete one. However, the constraint (4.50)) appears. This constraint is not convex, and evidently it has to be replaced by another. However for philosophy of the problem this condition is very important.

Remark 4.1. We have tried (in the last example) to analyze a situation of young married couple and their problem with mortgage. We have included in the model their private plans (to have baby), however we have neglected many troubles and situations that can happen (e. g. illness, a loss of employment). But we also omitted a possibility to gain "better" career or only increasing salary. Every of these possibilities are waiting to be included.

5. CONCLUSION

The aim of the paper is to summarize a possibility to employ the results on the stability (based on the Wassersten metric and $\mathcal{L}_1$ norm) to some other parts of stochastic programming. In particular, stochastic programming problems with "deterministic" constraints, individual probability constraints and stochastic dominance constraints are considered. To introduce the survey of possible applications, first, the corresponding stability results are recalled. Further, the stability results are employed to empirical estimates in the stochastic programming and scenario generation. The paper is finished with a simple financial problem.

Evidently, by this approach a stochastic dependence between components of the random element is neglected. However, all formulas are simple and, moreover, they are acceptable from the numerical point of views. The idea to reduce s–dimensional case to one–dimensional case is credited to G. Pflug [19] (see also [27]).

A very simple example is presented at the end of the paper. Two reasons exist for this. First, to show that it is possible to analyze not only the optimize behaviour banks but also debtors. Moreover, the theory of stochastic programming can be employed to it. However to present a suitable numerical solution is beyond of the scop of this paper. Maybe, it appears later including more complicated model.

ACKNOWLEDGEMENT

This work was supported by the Czech Science Foundation under Grants 15-10331S.

(Received February 13, 2017)

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