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Probability inequalities for decomposition integrals

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ABSTRACT

Recently, in mathematical economics, Even and Lehrer introduced the decomposition integral (Even and Lehrer, 2014). In this paper, general versions of some well-known probabilistic inequalities for the decomposition integrals and the superdecomposition integrals are discussed that are still open for research. The main results of this paper generalize some previous results for particular integral inequalities obtained by several researchers in generalized probability theory.

1. Introduction and motivation

Optimization theory is a branch of applied mathematics which is used in numerous applications in engineering, economics and statistics [1–7]. Probabilistic inequalities are very important in optimization and approximation theory [8–11]. We recall some important inequalities in probability and measure theory: Hölder's inequality, Minkowski's inequality, Chebyshev's inequality and Jensen's inequality.

Given a probability space $(\Omega, \mathcal{A}, \mathbf{P})$, consider two $\mathcal{A}$-Borel measurable functions X and Y.

Theorem 1.1. *If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then Hölder's inequality*

$$\int_\Omega |XY|\, d\mathbf{P} \leq \left(\int_\Omega |X|^p\, d\mathbf{P}\right)^{\frac{1}{p}} \left(\int_\Omega |Y|^q\, d\mathbf{P}\right)^{\frac{1}{q}}$$

holds.

Theorem 1.2. *If $p \geq 1$, then Minkowski's inequality*

$$\left(\int_\Omega |X+Y|^p\, d\mathbf{P}\right)^{\frac{1}{p}} \leq \left(\int_\Omega |X|^p\, d\mathbf{P}\right)^{\frac{1}{p}} + \left(\int_\Omega |Y|^p\, d\mathbf{P}\right)^{\frac{1}{p}},$$

holds.

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Theorem 1.3. *If X, Y are comonotonic, then Chebyshev's inequality*

$$\int_{\Omega} XYd\mathbf{P} \geq \int_{\Omega} Xd\mathbf{P} \int_{\Omega} Yd\mathbf{P}$$

holds.

Theorem 1.4. *Let I be an open interval and let Ψ be a convex function on I. Let $X : \Omega \to I$ be a random variable. Then the Jensen inequality*

$$\int_{\Omega} \Psi(X)\, d\mathbf{P} \geq \Psi\left(\int_{\Omega} Xd\mathbf{P}\right)$$

holds.

Some new generalizations and refinements of Hölder's inequality were obtained in [12–15]. In 2013, Kochanek and Lewicki [16] characterized L^p-norms on a probabilistic space via a Hölder type inequality. Recently, a Hölder-type inequality on a regular rooted tree was proposed by Falconer [17]. Aldaz [18] presented a stability version of Hölder's inequality. In information theory, a new refinement of the generalized Hölder's inequality was obtained by Tian in [19]. Bourin and Hiai [20] established new extensions of the Minkowski inequality for a large class of operator means. The Pexider type generalization of the Minkowski inequality was proved in [21]. In pseudo analysis, Agahi et al. [22] proved some generalizations of the Hölder's and Minkowski's inequalities for the pseudo-integral. Hölder's and Minkowski's inequalities for Sugeno integral [23–25] with respect to non-additive measures were studied in [15,26,27].

During the last decades, the concept of Choquet integral attracted a great interest of the researchers in many areas of sciences [28–31,11,32–42,23]. In the framework of Choquet integral, several integral inequalities were studied in [43,44,5,45]. For example, Zhao and Zhang [5] proved the Hölder type inequality for Choquet integral based on comonotonicity condition. Let $(\Omega, \mathcal{A})$ be a fixed measurable space. Recall that two functions $X, Y : \Omega \to \mathbb{R}$ are said to be comonotonic if and only if

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geqslant 0$$

for each couple of elements $\omega_1, \omega_2 \in \Omega$. Note that some of the next notions will be properly defined later in Section 2.

Theorem 1.5 ([5,45]). *Let X, Y be two non-negative functions such that $\mathbb{E}_C^{\mu}[XY]$, $\mathbb{E}_C^{\mu}[X^p]$ and $\mathbb{E}_C^{\mu}[Y^q]$ exist for all $p, q > 1$. If X, Y are comonotonic, then*

(i) *Hölder's inequality*

$$\mathbb{E}_C^{\mu}[XY] \leqslant \left(\mathbb{E}_C^{\mu}\left[X^p\right]\right)^{\frac{1}{p}} \left(\mathbb{E}_C^{\mu}\left[Y^q\right]\right)^{\frac{1}{q}}$$

holds where $\frac{1}{p} + \frac{1}{q} = 1, p > 1$.

(ii) *Minkowski's inequality*

$$\left(\mathbb{E}_C^{\mu}\left[(X+Y)^s\right]\right)^{\frac{1}{s}} \leqslant \left(\mathbb{E}_C^{\mu}\left[X^s\right]\right)^{\frac{1}{s}} + \left(\mathbb{E}_C^{\mu}\left[Y^s\right]\right)^{\frac{1}{s}}$$

holds where $s \geq 1$.

Remark 1.6. Let μ be submodular, i.e., $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ *for all* $A, B \in \mathcal{A}$. Then Mesiar et al. [43] and Wang [44] proposed new versions of Hölder's and Minkowski's type inequalities for Choquet integral based on two non-negative functions.

Theorem 1.7 ([46, Theorem 2.2]). *Let $X, Y \geqslant 0$ be comonotonic. Then, the Chebyshev inequality*

$$\|\mu\| \, \mathbb{E}_C^{\mu}[XY] \geq \mathbb{E}_C^{\mu}[X]\, \mathbb{E}_C^{\mu}[Y]$$

holds for any real monotone set function μ.

Theorem 1.8 ([5,43]). *Let υ be a capacity. Let X be a non-negative measurable function. If $\Psi : (0, \infty) \to (0, \infty)$ is a convex non-decreasing function, then Jensen's inequality*

$$\mathbb{E}_C^{\upsilon}[\Psi(X)] \geqslant \Psi\left(\mathbb{E}_C^{\upsilon}[X]\right)$$

holds.

In 2014, Even and Lehrer [30] introduced the concept of decomposition integral which generalizes Choquet integral [41,42,23] and concave integral [35]. Inspired by the idea of decomposition integrals, a new class of integrals based on superdecompositions of integrated functions was introduced by Mesiar et al. [47] in 2015. This paper deals with a review of fundamental concepts on well-known probabilistic inequalities such as Hölder's inequality, Minkowski's inequality, Chebyshev's inequality and Jensen's inequality for the decomposition integrals and the superdecomposition integrals that are still open for research. The proposed inequalities include as special cases the previous probabilistic inequalities for Choquet integral were studied in Zhao and Zhang [5], Mesiar et al. [43], Wang [44], Girotto and Holzer [46], Zhu and Ouyang [45] and others.

The rest of the paper is organized as follows. Some basic well-known notions and definitions that are useful in this paper are given in Section 2. In Section 3, we state the main results of this paper. Finally, some concluding remarks are added.

2. Definitions and notations

In this section, we recall some basic well-known definitions and notations that we will use in the proofs of our results. Let $(\Omega, \mathcal{A})$ be a fixed measurable space.

Definition 2.1 (*[25]*)**.** A set function $\mu : \mathcal{A} \to [0, \infty]$ is called a monotone measure whenever $\mu(\emptyset) = 0$, $\mu(\Omega) > 0$ and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$, moreover, μ is called real if $\|\mu\| = \mu(\Omega) < \infty$ and μ is said to be an additive measure if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$. μ is called a fuzzy measure (or monotone probability or capacity) if $\|\mu\| = 1$. Notice that a capacity with σ-additivity assumption is called a probability measure. A monotone measure μ is also submodular (2-alternating) whenever $\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B)$ for all $A, B \in \mathcal{A}$.

Definition 2.2. A random variable X over N is a function $X : N \to \mathbb{R}$. A subset of N will be called an event. For any event $A \subseteq N$, $\mathbf{1}_A$ denotes the indicator of A, which is the random variable that takes the value 1 over A and the value 0 otherwise.

Definition 2.3 (*[30]*)**.** Let X be a non-negative random variable.

1. A sub-decomposition of X is a finite sum $\sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i}$ such that
   (i) $\sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i} \leq X$; and
   (ii) $\alpha_i \geq 0$ and $A_i \subseteq N$ for every $i = 1, 2, \ldots, k$.
2. Let D be a subset of 2^X. $\sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i}$ is a D-sub-decomposition of X if it is a sub-decomposition of X and $A_i \in D$ for every $i = 1, 2, \ldots, k$.

We say that $\sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i}$ is a decomposition of X if equality replaces inequality in (i). That is, $\sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i}$ is a decomposition of X if it is a sub-decomposition of X, and $\sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i} = X$. A similar definition applies to D-decomposition of X.

For a fixed measurable space $(\Omega, \mathcal{A})$, any finite non-empty subset $D \subseteq \mathcal{A} \setminus \{\varnothing\}$ is called a collection. A system $\mathcal{H}$ of some collections is called a collection system and we denote by $\mathbb{X}$ the set of all collection systems over $(\Omega, \mathcal{A})$. Considering that $(\Omega, \mathcal{A})$ fixed and to shorten the notation, we denote $\mathcal{M}$ as the set of all monotone measures on $(\Omega, \mathcal{A})$ and $\mathcal{F}$ is the set of all bounded measurable functions $X : \Omega \to [0, \infty[$.

Definition 2.4 (*[30]*)**.** The decomposition-integral is defined as follows

$$\mathbb{I}_{\mathcal{H}}[m, X] = \sup\left\{\sum_{i \in I} a_i m(A_i) \mid (A_i)_{i \in I} \in \mathcal{H}, \sum a_i \mathbf{1}_{A_i} \leq X\right\}$$

where all constants a_i, $i \in I$, are non-negative and it is based on a collection system $\mathcal{H}$.

Remark 2.5. (i) Observe that for any collection system $\mathcal{H}$, the corresponding decomposition integral $\mathbb{I}_{\mathcal{H}}$ is monotone and positively homogeneous.

(ii) For a fixed measurable space $(\Omega, \mathcal{A})$, if $\mathcal{H}_1 = \{\{A\} \mid A \in \mathcal{A}\}$, then $\mathbb{I}_{\mathcal{H}_1}$ is a Shilkret integral [48], whereas if $\mathcal{H}_2 = \{\mathcal{B} \mid \mathcal{B}$ is a finite chain in $\mathcal{A}\}$, then $\mathbb{I}_{\mathcal{H}_2}$ is a Choquet expectation (integral) [41,42,23]. Recall that $\mathcal{B}$ is a finite chain in $\mathcal{A}$ if and only if there is an integer k and $\mathcal{B} = \{A_1, A_2, \ldots, A_k\} \subset \mathcal{A}$ that satisfies $A_1 \subset A_2 \subset \cdots \subset A_k$. Considering $\mathcal{H}_3 = \{\mathcal{B} \mid \mathcal{B}$ is a finite subset of $\mathcal{A}\}$, $\mathbb{I}_{\mathcal{H}_3}$ is the concave integral introduced by Lehrer [35].

Definition 2.6 (*[47]*)**.** Let $\mathcal{H} \in \mathbb{X}$ be fixed. Then, the mapping $\mathbb{I}^{\mathcal{H}} : \mathcal{M} \times \mathcal{F} \to [0, \infty]$ given by

$$\mathbb{I}^{\mathcal{H}}[m, X] = \inf\left\{\sum_{A \in \mathcal{C}} a_A m(A) \mid \mathcal{C} \in \mathcal{H}, a_A \geq 0 \text{ for each } A \in \mathcal{C}, \sum_{A \in \mathcal{C}} a_A \mathbf{1}_A \geq X\right\}$$

is called a superdecomposition integral.

Remark 2.7. (i) For a fixed measurable space $(\Omega, \mathcal{A})$, if $\mathcal{H}_1 = \{\{A\} | A \in \mathcal{A}\}$, then

$$\begin{aligned}\mathbb{I}^{\mathcal{H}_1} &= \inf\{a \cdot m(A) | A \in \mathcal{A}, a \cdot \mathbf{1}_A \geq X\} \\ &= \sup\{f(\omega) | \omega \in \Omega\} \cdot m(\{X > 0\}).\end{aligned}$$

If $\mathcal{H}_2 = \{\mathcal{B} | \mathcal{B}$ is a finite chain in $\mathcal{A}\}$, then $\mathbb{I}^{\mathcal{H}_2}$ is a Choquet expectation (integral) [41,42,23] given by

$$\mathbb{I}^{\mathcal{H}_2}[m, X] = \mathbb{E}_C^m[X] = \int_0^\infty m(\{X \geq t\})\, dt.$$

Considering $\mathcal{H}_3 = \{\mathcal{B} | \mathcal{B}$ is a finite subset of $\mathcal{A}\}$, $\mathbb{I}^{\mathcal{H}_3}$ is the convex integral introduced by Mesiar et al. [47].

(ii) For any $\mathcal{H} \in \mathbb{X}$, the corresponding superdecomposition integral $\mathbb{I}^{\mathcal{H}}$ is monotone and positively homogeneous.

3. Main results

In this section, we state the main results of this paper. Our results in this section generalize the previous results obtained by Zhao and Zhang [5], Mesiar et al. [43], Wang [44], Girotto and Holzer [46], Zhu and Ouyang [45].

Theorem 3.1 (*Chebyshev's Inequality*)**.** *Let $m \in \mathcal{M}$ be fixed. If the superdecomposition-integral $\mathbb{I}^{\mathcal{H}}[m, .]$ satisfies the following conditions:*

(i) $\mathbb{I}^{\mathcal{H}}[m, X+Y] = \mathbb{I}^{\mathcal{H}}[m, X] + \mathbb{I}^{\mathcal{H}}[m, Y]$ *for any comonotonic $X, Y \in \mathcal{F}$,*
(ii) $\mathbb{I}^{\mathcal{H}}[m, 1] < \infty$,

then the Chebyshev inequality

$$\mathbb{I}^{\mathcal{H}}[m, XY]\, \mathbb{I}^{\mathcal{H}}[m, 1] \geq \mathbb{I}^{\mathcal{H}}[m, X]\, \mathbb{I}^{\mathcal{H}}[m, Y],$$

holds for any comonotonic random variables $X, Y \in \mathcal{F}$.

Proof. Without loss of generality we assume that $\mathbb{I}^{\mathcal{H}}[m, XY] < \infty$ and $X, Y \neq 0$. Given ω_0, comonotonicity implies that

$$(X - X(\omega_0))(Y - Y(\omega_0)) \geqslant 0,$$

or, equivalently,

$$XY + X(\omega_0) Y(\omega_0) \geqslant Y(\omega_0) X + X(\omega_0) Y.$$

Then, by monotonicity of $\mathbb{I}^{\mathcal{H}}[m, .]$, we have

$$\mathbb{I}^{\mathcal{H}}[m, XY + X(\omega_0) Y(\omega_0)] \geqslant \mathbb{I}^{\mathcal{H}}[m, Y(\omega_0) X + X(\omega_0) Y].$$

Observe that any constant function $c \in \mathcal{F}$ is comonotonic with any random variable $Z \in \mathcal{F}$. Moreover, if $X, Y \in \mathcal{F}$, then, for any positive constants c, d, two random variables cX and cY are also comonotonic. Hence, by condition (i), we get

$$\mathbb{I}^{\mathcal{H}}[m, XY] + \mathbb{I}^{\mathcal{H}}[m, X(\omega_0) Y(\omega_0)] \geqslant \mathbb{I}^{\mathcal{H}}[m, Y(\omega_0) X] + \mathbb{I}^{\mathcal{H}}[m, X(\omega_0) Y].$$

Finally, by positive homogeneity, the inequality

$$\mathbb{I}^{\mathcal{H}}[m, XY] + X(\omega_0) Y(\omega_0)\, \mathbb{I}^{\mathcal{H}}[m, 1] \geqslant Y(\omega_0)\, \mathbb{I}^{\mathcal{H}}[m, X] + X(\omega_0)\, \mathbb{I}^{\mathcal{H}}[m, Y], \tag{3.1}$$

holds for any ω_0, i.e.,

$$\mathbb{I}^{\mathcal{H}}[m, XY] + XY\mathbb{I}^{\mathcal{H}}[m, 1] \geqslant Y\mathbb{I}^{\mathcal{H}}[m, X] + X\mathbb{I}^{\mathcal{H}}[m, Y].$$

Now, we show that $\mathbb{I}^{\mathcal{H}}[m, X]$ and $\mathbb{I}^{\mathcal{H}}[m, Y]$ are both finite. Indeed, assume for instance $\mathbb{I}^{\mathcal{H}}[m, X] = +\infty$, then choose ω_0 such that $Y(\omega_0) > 0$ in order to get, by (3.1) and condition (ii), the contradiction $\mathbb{I}^{\mathcal{H}}[m, XY] \geqslant Y(\omega_0)\, \mathbb{I}^{\mathcal{H}}[m, X] + X(\omega_0)\, \mathbb{I}^{\mathcal{H}}[m, Y] - X(\omega_0) Y(\omega_0)\, \mathbb{I}^{\mathcal{H}}[m, 1] = +\infty$. Consequently, by monotonicity, we have

$$\mathbb{I}^{\mathcal{H}}\left[m, \mathbb{I}^{\mathcal{H}}[m, XY] + XY\mathbb{I}^{\mathcal{H}}[m, 1]\right] \geqslant \mathbb{I}^{\mathcal{H}}\left[m, Y\mathbb{I}^{\mathcal{H}}[m, X] + X\mathbb{I}^{\mathcal{H}}[m, Y]\right]. \tag{3.2}$$

Then condition (i), positive homogeneity and (3.2) imply that

$$\begin{aligned}2\mathbb{I}^{\mathcal{H}}[m, XY]\, \mathbb{I}^{\mathcal{H}}[m, 1] &= \mathbb{I}^{\mathcal{H}}[m, 1]\, \mathbb{I}^{\mathcal{H}}[m, XY] + \mathbb{I}^{\mathcal{H}}[m, XY]\, \mathbb{I}^{\mathcal{H}}[m, 1] \\ &= \mathbb{I}^{\mathcal{H}}\left[m, \mathbb{I}^{\mathcal{H}}[m, XY]\right] + \mathbb{I}^{\mathcal{H}}\left[m, XY\mathbb{I}^{\mathcal{H}}[m, 1]\right] \\ &= \mathbb{I}^{\mathcal{H}}\left[m, \mathbb{I}^{\mathcal{H}}[m, XY] + XY\mathbb{I}^{\mathcal{H}}[m, 1]\right] \geqslant \mathbb{I}^{\mathcal{H}}\left[m, Y\mathbb{I}^{\mathcal{H}}[m, X] + X\mathbb{I}^{\mathcal{H}}[m, Y]\right] \\ &= \mathbb{I}^{\mathcal{H}}\left[m, Y\mathbb{I}^{\mathcal{H}}[m, X]\right] + \mathbb{I}^{\mathcal{H}}\left[m, X\mathbb{I}^{\mathcal{H}}[m, Y]\right] = \mathbb{I}^{\mathcal{H}}[m, X]\, \mathbb{I}^{\mathcal{H}}[m, Y] + \mathbb{I}^{\mathcal{H}}[m, Y]\, \mathbb{I}^{\mathcal{H}}[m, X] \\ &= 2\mathbb{I}^{\mathcal{H}}[m, X]^{\mathcal{H}}[m, Y].\end{aligned}$$

This completes the proof. □

By a similar way, we can obtain Chebyshev's inequality for decomposition-integral.

Theorem 3.2. *Let $m \in \mathcal{M}$ be fixed. If the decomposition-integral $\mathbb{I}_{\mathcal{H}}[m, .]$ satisfies the following conditions:*

(i) $\mathbb{I}_{\mathcal{H}}[m, X+Y] = \mathbb{I}_{\mathcal{H}}[m, X] + \mathbb{I}_{\mathcal{H}}[m, Y]$ *for any comonotonic random variables* $X, Y \in \mathcal{F}$,
(ii) $\mathbb{I}_{\mathcal{H}}[m, 1] < \infty$,

then the Chebyshev inequality

$$\mathbb{I}_{\mathcal{H}}[m, XY]\,\mathbb{I}_{\mathcal{H}}[m, 1] \geq \mathbb{I}_{\mathcal{H}}[m, X]\,\mathbb{I}_{\mathcal{H}}[m, Y],$$

holds for any comonotonic random variables $X, Y \in \mathcal{F}$.

Example 3.3. (i) Since $\mathcal{H}_2 = \{\mathcal{B} | \mathcal{B}$ is a finite chain in $\mathcal{A}\}$, then $\mathbb{I}^{\mathcal{H}_2} = \mathbb{I}_{\mathcal{H}_2}$ is a Choquet integral and we have Chebyshev's inequality for Choquet integral in Theorem 1.7 which was obtained in [46,49].

(ii) Consider $X = \{1, 2\}$, $\mathcal{H} = \{\{1\}, \{1, 2\}\}$, $m(X) = 1$, $m(\{1\}) = a$, $m(\{2\}) = b$ and $a, b \in [0, 1]$. Then, for $X(i) = x_i$, the superdecomposition integral $\mathbb{I}^{\mathcal{H}}[m, .]$ is comonotone additive, and it holds

$$\mathbb{I}^{\mathcal{H}}[m, X] = x_2 + a.\max\{x_1 - x_2, 0\}.$$

Thus $\mathbb{I}^{\mathcal{H}}[m, 1] = 1$, and due to Theorem 3.1, the Chebyshev inequality for this integral holds. Similarly, due to Theorem 3.2, the Chebyshev inequality holds for $\mathbb{I}_{\mathcal{H}}[m, .]$, which is given by

$$\mathbb{I}_{\mathcal{H}}[m, X] = \min\{x_1, x_2\} + a.\max\{x_1 - x_2, 0\}.$$

Theorem 3.4 (*Jensen's Inequality*). *Let $m \in \mathcal{M}$ be fixed and I be an open interval and let Ψ be a twice differentiable function on I satisfying the condition $\Psi''(x) \geq 0$ for $x \in I$. Let $X : \Omega \to I$ be a random variable and $\mathbb{I}_{\mathcal{H}}[\mu, X] \in I$. Then inequality*

$$\mathbb{I}_{\mathcal{H}}\left[\mu, \Psi(X) + \Psi'(\mathbb{I}_{\mathcal{H}}[\mu, X])\,\mathbb{I}_{\mathcal{H}}[\mu, X]\right] \geqslant \mathbb{I}_{\mathcal{H}}\left[\mu, \Psi(\mathbb{I}_{\mathcal{H}}[\mu, X]) + X\Psi'(\mathbb{I}_{\mathcal{H}}[\mu, X])\right] \quad (3.3)$$

holds.

Proof. We use the inequality

$$\Psi(x) \geqslant \Psi(\rho) + \Psi'(\rho)(x - \rho) \quad (3.4)$$

for any $x, \rho \in I$ which follows from convexity of Ψ and Taylor's formula. Then (3.4) implies that

$$\Psi(x) + \rho\Psi'(\rho) \geqslant \Psi(\rho) + \Psi'(\rho)\,x$$

for any $x, \rho \in I$. We set $x = X(\omega)$ and $\mathbb{I}_{\mathcal{H}}[\mu, X] = \rho$ and integrate over the domain Ω. Then

$$\mathbb{I}_{\mathcal{H}}\left[\mu, \Psi(X) + \rho\Psi'(\rho)\right] \geqslant \mathbb{I}_{\mathcal{H}}\left[\mu, \Psi(\rho) + \Psi'(\rho)\,X\right]. \quad \square$$

Corollary 3.5. *Let the capacity υ be fixed and I be an open interval and let Ψ be a twice differentiable function on I satisfying the condition $\Psi''(x) \geq 0$ for $x \in I$. Let $X : \Omega \to I$ be a random variable. If the decomposition-integral $\mathbb{I}_{\mathcal{H}}[\upsilon, .]$ satisfies the following conditions:*

(i) *translation invariant, i.e.,* $\mathbb{I}_{\mathcal{H}}[\upsilon, X + c] = \mathbb{I}_{\mathcal{H}}[\upsilon, X] + c$ *for any* $c \in \mathbb{R}$,
(ii) $\mathbb{I}_{\mathcal{H}}[\upsilon, X] \in I$ *and* $\Psi'(\mathbb{I}_{\mathcal{H}}[\upsilon, X]) \geq 0$ *for capacity* υ,

then inequality

$$\mathbb{I}_{\mathcal{H}}[\upsilon, \Psi(X)] \geqslant \Psi(\mathbb{I}_{\mathcal{H}}[\upsilon, X])$$

holds.

Proof. By condition (i) and (3.3), we have

$$\mathbb{I}_{\mathcal{H}}[\upsilon, \Psi(X)] + \Psi'(\mathbb{I}_{\mathcal{H}}[\upsilon, X])\,\mathbb{I}_{\mathcal{H}}[\upsilon, X] \geqslant \mathbb{I}_{\mathcal{H}}\left[\upsilon, X\Psi'(\mathbb{I}_{\mathcal{H}}[\upsilon, X])\right] + \Psi(\mathbb{I}_{\mathcal{H}}[\upsilon, X]).$$

Now if $\Psi'(\mathbb{I}_{\mathcal{H}}[\upsilon, X]) \geq 0$, then by positive homogeneity of decomposition integral, we have

$$\mathbb{I}_{\mathcal{H}}[\upsilon, \Psi(X)] + \mathbb{I}_{\mathcal{H}}[\upsilon, X]\,\Psi'(\mathbb{I}_{\mathcal{H}}[\upsilon, X]) \geqslant \mathbb{I}_{\mathcal{H}}[\upsilon, X]\,\Psi'(\mathbb{I}_{\mathcal{H}}[\upsilon, X]) + \Psi(\mathbb{I}_{\mathcal{H}}[\upsilon, X]).$$

This completes the proof. $\square$

By a similar way, we can obtain the following theorem for superdecomposition-integral.

Theorem 3.6. *Let $m \in \mathcal{M}$ be fixed and I be an open interval and let Ψ be a twice differentiable function on I satisfying the condition $\Psi''(x) \geq 0$ for $x \in I$. Let $X : \Omega \to I$ be a random variable and $\mathbb{I}^{\mathcal{H}}[\mu, X] \in I$. Then the inequality*

$$\mathbb{I}^{\mathcal{H}}\left[\mu, \Psi(X) + \Psi'\left(\mathbb{I}^{\mathcal{H}}[\mu, X]\right)\mathbb{I}^{\mathcal{H}}[\mu, X]\right] \geqslant \mathbb{I}^{\mathcal{H}}\left[\mu, \Psi\left(\mathbb{I}^{\mathcal{H}}[\mu, X]\right) + X\Psi'\left(\mathbb{I}^{\mathcal{H}}[\mu, X]\right)\right]$$

holds.

Example 3.7. Since $\mathcal{H}_2 = \{\mathcal{B} | \mathcal{B}$ is a finite chain in $\mathcal{A}\}$, then $\mathbb{I}^{\mathcal{H}_2} = \mathbb{I}_{\mathcal{H}_2}$ is a Choquet integral and we have Jensen's inequality for Choquet integral in Theorem 1.8 which was obtained in [5,43].

Theorem 3.8. *Let $m \in \mathcal{M}$ be fixed. If $X, Y \in \mathcal{F}$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have Hölder's inequality for the convex integral*

$$\mathbb{I}^{\mathcal{H}_3}[m, XY] \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^p\right]\right)^{\frac{1}{p}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, Y^q\right]\right)^{\frac{1}{q}}.$$

Proof. Positive homogeneity of $\mathbb{I}^{\mathcal{H}_3}$ implies that

$$\frac{\mathbb{I}^{\mathcal{H}_3}\left[m, X^{\frac{1}{p}} Y^{\frac{1}{q}}\right]}{\left(\mathbb{I}^{\mathcal{H}_3}[m, X]\right)^{\frac{1}{p}} \left(\mathbb{I}^{\mathcal{H}_3}[m, Y]\right)^{\frac{1}{q}}} = \mathbb{I}^{\mathcal{H}_3}\left[m, \left(\frac{X}{\mathbb{I}^{\mathcal{H}_3}[m, X]}\right)^{\frac{1}{p}} \left(\frac{Y}{\mathbb{I}^{\mathcal{H}_3}[m, Y]}\right)^{\frac{1}{q}}\right]. \tag{3.5}$$

So, by geometric inequality and monotonicity of $\mathbb{I}^{\mathcal{H}_3}$, we have

$$\mathbb{I}^{\mathcal{H}_3}\left[m, \left(\frac{X}{\mathbb{I}^{\mathcal{H}_3}[m, X]}\right)^{\frac{1}{p}} \left(\frac{Y}{\mathbb{I}^{\mathcal{H}_3}[m, Y]}\right)^{\frac{1}{q}}\right] \leqslant \mathbb{I}^{\mathcal{H}_3}\left[m, \frac{\frac{1}{p}X}{\mathbb{I}^{\mathcal{H}_3}[m, X]} + \frac{\frac{1}{q}Y}{\mathbb{I}^{\mathcal{H}_3}[m, Y]}\right]. \tag{3.6}$$

Then by convexity of $\mathbb{I}^{\mathcal{H}_3}$, (3.5), (3.6) we have

$$\frac{\mathbb{I}^{\mathcal{H}_3}\left[m, X^{\frac{1}{p}} Y^{\frac{1}{q}}\right]}{\left(\mathbb{I}^{\mathcal{H}_3}[m, X]\right)^{\frac{1}{p}} \left(\mathbb{I}^{\mathcal{H}_3}[m, Y]\right)^{\frac{1}{q}}} \leqslant \frac{1}{p}\mathbb{I}^{\mathcal{H}_3}\left[m, \frac{X}{\mathbb{I}^{\mathcal{H}_3}[m, X]}\right] + \frac{1}{q}\mathbb{I}^{\mathcal{H}_3}\left[m, \frac{Y}{\mathbb{I}^{\mathcal{H}_3}[m, Y]}\right]. \tag{3.7}$$

Again by positive homogeneity and (3.7), we obtain

$$\begin{aligned} \frac{\mathbb{I}^{\mathcal{H}_3}\left[m, X^{\frac{1}{p}} Y^{\frac{1}{q}}\right]}{\left(\mathbb{I}^{\mathcal{H}_3}[m, X]\right)^{\frac{1}{p}} \left(\mathbb{I}^{\mathcal{H}_3}[m, Y]\right)^{\frac{1}{q}}} &\leqslant \frac{1}{p}\frac{\mathbb{I}^{\mathcal{H}_3}[m, X]}{\mathbb{I}^{\mathcal{H}_3}[m, X]} + \frac{1}{q}\frac{\mathbb{I}^{\mathcal{H}_3}[m, Y]}{\mathbb{I}^{\mathcal{H}_3}[m, Y]} \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

i.e.,

$$\mathbb{I}^{\mathcal{H}_3}\left(m, X^{\frac{1}{p}} Y^{\frac{1}{q}}\right) \leqslant \left(\mathbb{I}^{\mathcal{H}_3}(m, X)\right)^{\frac{1}{p}} \left(\mathbb{I}^{\mathcal{H}_3}(m, Y)\right)^{\frac{1}{q}}. \quad \square$$

Corollary 3.9. *Let $m \in \mathcal{M}$ be fixed. If $X, Y \in \mathcal{F}$, then the inequality*

$$\mathbb{I}^{\mathcal{H}_3}[m, XY] \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^2\right]\right)^{\frac{1}{2}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, Y^2\right]\right)^{\frac{1}{2}} \tag{3.8}$$

holds.

Corollary 3.10. *Let $m \in \mathcal{M}$ be fixed. Let $X \in \mathcal{F}$, $p, q \in (0, \infty)$, $t \in (0, 1)$. If $r \in (0, \infty)$ satisfies $\frac{1}{r} = \frac{1-t}{p} + \frac{t}{q}$, then the inequality*

$$\left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^r\right]\right)^{\frac{1}{r}} \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^q\right]\right)^{\frac{t}{q}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^p\right]\right)^{\frac{1-t}{p}}$$

holds.

Proof. Let $p_1 = \frac{p}{r(1-t)}$ and $q_1 = \frac{q}{rt}$. Then $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $q_1 > 0$. So, by Theorem 3.8,

$$\begin{aligned} \mathbb{I}^{\mathcal{H}_3}\left[m, X^r\right] &= \mathbb{I}^{\mathcal{H}_3}\left[m, X^{rt} X^{r(1-t)}\right] \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^{rtq_1}\right]\right)^{\frac{1}{q_1}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^{r(1-t)p_1}\right]\right)^{\frac{1}{p_1}} \\ &= \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^q\right]\right)^{\frac{rt}{q}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^p\right]\right)^{\frac{r(1-t)}{p}}, \end{aligned}$$

i.e.,

$$\left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^r\right]\right)^{\frac{1}{r}} \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^q\right]\right)^{\frac{t}{q}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^p\right]\right)^{\frac{1-t}{p}}.$$

This completes the proof. □

Example 3.11. As an applicable example of the Hölder's inequality in Theorem 3.8, we can present the concept of a convergence in pth mean for convex integral. Let $\{X_n\}$ be a sequence of convex integrable functions on Ω. We say that $\{X_n\}$ converges in pth mean ($p > 0$) to X if

$$\lim_{n\to\infty} \mathbb{I}^{\mathcal{H}_3}\left[m, |X_n - X|^p\right] = 0.$$

Since $0 < r < s$, $Y \equiv 1$, $|X_n - X|^r$ substitute for X and $\frac{s}{r}$ substitute for p, then we have

$$\begin{aligned}\mathbb{I}^{\mathcal{H}_3}\left[m, |X_n - X|^r\right] &\leq \left(\mathbb{I}^{\mathcal{H}_3}\left[m, |X_n - X|^{r\frac{s}{r}}\right]\right)^{\frac{r}{s}} \left(\mathbb{I}^{\mathcal{H}_3}[m, 1]\right)^{\frac{s-r}{s}} \\ &= \left(\mathbb{I}^{\mathcal{H}_3}\left[m, |X_n - X|^s\right]\right)^{\frac{r}{s}} \left(\mathbb{I}^{\mathcal{H}_3}[m, 1]\right)^{\frac{s-r}{s}}. \end{aligned} \tag{3.9}$$

Clearly, if $0 < r < s$, $\{X_n\}$ converges in sth mean, i.e., if $\lim_{n\to\infty} \mathbb{I}^{\mathcal{H}_3}\left[m, |X_n - X|^s\right] = 0$, and $\mathbb{I}^{\mathcal{H}_3}[m, 1]$ is finite, then by (3.9) $\{X_n\}$ converges in rth mean.

Analyzing the proof of Theorem 3.8, we can obtain the following theorems for superdecomposition-integral and decomposition-integral.

Theorem 3.12 (*Hölder's Inequality*)**.** *Let $m \in \mathcal{M}$ be fixed. If $X, Y \in \mathcal{F}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and the superdecomposition-integral $\mathbb{I}^{\mathcal{H}}[m, .]$ satisfies the following condition:*
$\mathbb{I}^{\mathcal{H}}[m, X + Y] \leq \mathbb{I}^{\mathcal{H}}[m, X] + \mathbb{I}^{\mathcal{H}}[m, Y]$ *for any* $X, Y \in \mathcal{F}$,
then the inequality

$$\mathbb{I}^{\mathcal{H}}[m, XY] \leqslant \left(\mathbb{I}^{\mathcal{H}}\left[m, X^p\right]\right)^{\frac{1}{p}} \left(\mathbb{I}^{\mathcal{H}}\left[m, Y^q\right]\right)^{\frac{1}{q}}$$

holds.

Theorem 3.13. *Let $m \in \mathcal{M}$ be fixed. If $X, Y \in \mathcal{F}$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, and the decomposition-integral $\mathbb{I}_{\mathcal{H}}[m, .]$ satisfies the following condition:*
$\mathbb{I}_{\mathcal{H}}[m, X + Y] \leq \mathbb{I}_{\mathcal{H}}[m, X] + \mathbb{I}_{\mathcal{H}}[m, Y]$ *for any* $X, Y \in \mathcal{F}$,
then the inequality

$$\mathbb{I}_{\mathcal{H}}[m, XY] \leqslant \left(\mathbb{I}_{\mathcal{H}}\left[m, X^p\right]\right)^{\frac{1}{p}} \left(\mathbb{I}_{\mathcal{H}}\left[m, Y^q\right]\right)^{\frac{1}{q}}$$

holds.

Example 3.14. Since $\mathcal{H}_2 = \{\mathcal{B} | \mathcal{B}$ is a finite chain in $\mathcal{A}\}$, then $\mathbb{I}^{\mathcal{H}_2} = \mathbb{I}_{\mathcal{H}_2}$ is a Choquet integral.

(I) If X, Y are comonotonic, then we have Hölder's inequality for Choquet integral in Theorem 1.5 which was obtained in [5,45].
(II) Since $m \in \mathcal{M}$ is submodular, then the Choquet integral is subadditive (see Proposition 7.9 in [23]). Then we obtain a new version of Hölder's inequality for Choquet integral which was obtained in [43,44].

Theorem 3.15. *Let $m \in \mathcal{M}$ be fixed. If $X, Y \in \mathcal{F}$, then the inequality*

$$\left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^s\right]\right)^{\frac{1}{s}} + \left(\mathbb{I}^{\mathcal{H}_3}\left[m, Y^s\right]\right)^{\frac{1}{s}} \geqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, (X+Y)^s\right]\right)^{\frac{1}{s}} \tag{3.10}$$

holds any $s \geqslant 1$.

Proof. Without loss of generality, we can assume that $\mathbb{I}^{\mathcal{H}_3}\left[m, (X+Y)^s\right] \neq 0$. Then by monotonicity and convexity of $\mathbb{I}^{\mathcal{H}_3}$, we have

$$\begin{aligned}\mathbb{I}^{\mathcal{H}_3}\left[m, (X+Y)^s\right] &= \mathbb{I}^{\mathcal{H}_3}\left[m, (X+Y)(X+Y)^{s-1}\right] \\ &\leqslant \mathbb{I}^{\mathcal{H}_3}\left[m, (X+Y)(X+Y)^{s-1}\right] \\ &\leqslant \mathbb{I}^{\mathcal{H}_3}\left[m, X(X+Y)^{s-1}\right] + \mathbb{I}^{\mathcal{H}_3}\left[m, Y(X+Y)^{s-1}\right]. \end{aligned} \tag{3.11}$$

By Theorem 3.8, we have

$$\mathbb{I}^{\mathcal{H}_3}\left[m, X\left(X+Y\right)^{s-1}\right] \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^s\right]\right)^{\frac{1}{s}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, \left(X+Y\right)^s\right]\right)^{\frac{s-1}{s}}, \tag{3.12}$$

and

$$\mathbb{I}^{\mathcal{H}_3}\left[m, Y\left(X+Y\right)^{s-1}\right] \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, Y^s\right]\right)^{\frac{1}{s}} \left(\mathbb{I}^{\mathcal{H}_3}\left[m, \left(X+Y\right)^s\right]\right)^{\frac{s-1}{s}}. \tag{3.13}$$

So, (3.11)–(3.13), imply that

$$\mathbb{I}^{\mathcal{H}_3}\left[m, \left(X+Y\right)^s\right] \leqslant \left[\left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^s\right]\right)^{\frac{1}{s}} + \left(\mathbb{I}^{\mathcal{H}_3}\left[m, Y^s\right]\right)^{\frac{1}{s}}\right] \left(\mathbb{I}^{\mathcal{H}_3}\left[m, \left(X+Y\right)^s\right]\right)^{\frac{s-1}{s}},$$

i.e.,

$$\frac{\left(\mathbb{I}^{\mathcal{H}_3}\left[m, \left(X+Y\right)^s\right]\right)}{\left(\mathbb{I}^{\mathcal{H}_3}\left[m, \left(X+Y\right)^s\right]\right)^{1-\frac{1}{s}}} \leqslant \left(\mathbb{I}^{\mathcal{H}_3}\left[m, X^s\right]\right)^{\frac{1}{s}} + \left(\mathbb{I}^{\mathcal{H}_3}\left[m, Y^s\right]\right)^{\frac{1}{s}}.$$

This completes the proof. □

Similarly, we can obtain the following theorems, which present Minkowski's inequality for (super)decomposition-integral.

Theorem 3.16 (*Minkowski's Inequality*). *Let $m \in \mathcal{M}$ be fixed. If $\mathbb{I}^{\mathcal{H}}[m, .]$ satisfies the following condition:*
$\mathbb{I}^{\mathcal{H}}[m, X+Y] \leq \mathbb{I}^{\mathcal{H}}[m, X] + \mathbb{I}^{\mathcal{H}}[m, Y]$ *for any $X, Y \in \mathcal{F}$,*

then the inequality

$$\left(\mathbb{I}^{\mathcal{H}}\left[m, X^s\right]\right)^{\frac{1}{s}} + \left(\mathbb{I}^{\mathcal{H}}\left[m, Y^s\right]\right)^{\frac{1}{s}} \geqslant \left(\mathbb{I}^{\mathcal{H}}\left[m, \left(X+Y\right)^s\right]\right)^{\frac{1}{s}}$$

holds for any $s \geqslant 1$.

Theorem 3.17. *Let $m \in \mathcal{M}$ be fixed. If $\mathbb{I}_{\mathcal{H}}[m, .]$ satisfies the following condition:*
$\mathbb{I}_{\mathcal{H}}[m, X+Y] \leq \mathbb{I}_{\mathcal{H}}[m, X] + \mathbb{I}_{\mathcal{H}}[m, Y]$ *for any $X, Y \in \mathcal{F}$,*

then the inequality

$$\left(\mathbb{I}_{\mathcal{H}}\left[m, X^s\right]\right)^{\frac{1}{s}} + \left(\mathbb{I}_{\mathcal{H}}\left[m, Y^s\right]\right)^{\frac{1}{s}} \geqslant \left(\mathbb{I}_{\mathcal{H}}\left[m, \left(X+Y\right)^s\right]\right)^{\frac{1}{s}}$$

holds for any $s \geqslant 1$.

Example 3.18. When $\mathcal{H}_2 = \{\mathcal{B} | \mathcal{B}$ is a finite chain in $\mathcal{A}\}$, then $\mathbb{I}^{\mathcal{H}_2} = \mathbb{I}_{\mathcal{H}_2}$ is a Choquet integral.

(I) If X, Y are comonotonic, then we have Minkowski's inequality for Choquet integral in Theorem 1.5 which was obtained in [45].

(II) If $m \in \mathcal{M}$ is submodular, then we obtain a new version of Minkowski's inequality for Choquet integral which was obtained in [43,44].

4. Concluding remarks

Probabilistic inequalities are powerful and practical mathematical tools in optimization and approximation theory. In this paper, we have investigated some well-known probabilistic inequalities such as Hölder's inequality, Minkowski's inequality, Chebyshev's inequality and Jensen's inequality for decomposition-integrals and superdecomposition-integrals. Note that the earlier results of Zhao and Zhang [5], Mesiar et al. [43], Wang [44], Girotto and Holzer [46], Zhu and Ouyang [45] are related to the Choquet integral, and their proofs exploit several particular properties of this integral such as the comonotone additivity, for example. We have considered a large class of decomposition (superdecomposition) integrals and thus rather different proof techniques were necessary. For the above mentioned inequalities, we have obtained results valid for all decomposition (super-decomposition) integrals. Having in mind a big potential of these recently introduced integrals in all domains dealing with optimal solutions (decomposition integrals were introduced in [30] in 2014, while superdecomposition integrals in [47] in 2015), we expect several applications of our results just in these domains.

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