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On the equivalence of the Choquet, pan- and concave integrals on finite spaces

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ABSTRACT

In this paper we introduce the concept of *maximal cluster of minimal atoms* on monotone measure spaces and by means of this new concept we continue to investigate the relation between the Choquet integral and the pan-integral on finite spaces. It is proved that the (M)*-property* of a monotone measure is a sufficient condition that the Choquet integral coincides with the pan-integral based on the usual addition $+$ and multiplication $\cdot$. Thus, combining our recent results, we provide a necessary and sufficient condition that the Choquet integral is equivalent to the pan-integral on finite spaces. Meanwhile, we also use the characteristics of minimal atoms of monotone measure to present another necessary and sufficient condition that these two kinds of integrals are equivalent on finite spaces. The relationships among the Choquet integral, the pan-integral and the concave integral are summarized.

1. Introduction

In nonlinear integral theory, related to the standard arithmetical operations on reals, there are three kinds of important integrals, the Choquet integral [3], the pan-integral [25] and the concave integral [7,8]. It is well known that all the three types of integrals are particular generalizations of the Lebesgue integral (for a σ-additive measure, these integrals coincide with the Lebesgue integral). All these integrals are particular instances of decomposition integrals [6,12,14]. However, in general case they are significantly different from each other [11], for instance, the pan-integral and the Choquet integral are incomparable [12].

Recently we discussed the relationship between the concave integral and the pan-integral on finite spaces [16]. We introduced the concept of *minimal atom* of a monotone measure. By using the characteristics of

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minimal atoms we presented a set of necessary and sufficient conditions that the concave integral coincides with the pan-integral based on the usual addition + and multiplication · on finite spaces.

In a recent paper [10], we investigated the relationship between the Choquet integral and the pan-integral on finite spaces. By means of minimal atoms we described several necessary conditions and a sufficient condition that the Choquet integral and the pan-integral are equivalent. It was shown that a necessary condition that the Choquet integral coincides with the pan-integral based on the usual addition + and multiplication · is that the involved monotone measure has the so-called (M)*-property*.

In this paper we continue to explore the relations between the Choquet integral and the pan-integral. We will introduce the concepts of *a cluster of minimal atoms* and *a maximal cluster of minimal atoms* on a monotone measure space. By means of *maximal clusters of minimal atoms*, we prove that the (M)-property is not only necessary, but also sufficient for the equivalence of the Choquet integral and the pan-integral based on the usual addition + and multiplication · on finite spaces. Meanwhile, we also use the characteristics of minimal atoms to present another necessary and sufficient condition that these two kinds of integrals are equivalent on finite spaces.

Note that Lehrer and Teper [8] (see also [1,7]) discussed the relation between the concave integral and the Choquet integral and showed that the concave integral coincides with the Choquet integral if and only if the underlying monotone measure is convex (also known as supermodular). Thus, the coincidences among these three types of nonlinear integrals, the Choquet integral, the concave integral and the pan-integral with respect to the standard arithmetic operations, are completely characterized on finite spaces.

2. The Choquet, concave and pan-integrals

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X, and $(X, \mathcal{A})$ denote a measurable space. $\mathcal{F}_+$ denotes the class of all finite nonnegative real-valued measurable functions on $(X, \mathcal{A})$.

A set function $\mu\colon \mathcal{A} \to [0, +\infty[$ is called a monotone measure [2] if it satisfies the conditions: (1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$; (2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

In this paper we always suppose that μ is a monotone measure defined on $(X, \mathcal{A})$.

We recall three types of nonlinear integrals. Let $f \in \mathcal{F}_+$, and let χ_A denote the characteristic function of $A \in \mathcal{A}$.

The *Choquet integral* [3,4] of f on X with respect to μ, is defined by

$$\int^{Cho} f \, d\mu = \int_0^{\infty} \mu(\{x : f(x) \geqslant t\}) \, dt,$$

where the right-hand side integral is the improper Riemann integral.

The *pan-integral* [23,25] of f on X with respect to μ (based on the usual addition + and usual multiplication ·) is given by

$$\int^{pan} f d\mu = \sup\left\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \le f, \ \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a partition of } X, \ \lambda_i \ge 0, \ n \in \mathbb{N} \right\}.$$

The *concave integral* [7] (see also [8]) of f on X is defined by

$$\int^{cav} f d\mu = \sup\left\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \le f, \ \{A_i\}_{i=1}^{n} \subset \mathcal{A}, \ \lambda_i \ge 0, \ n \in \mathbb{N} \right\}.$$

Note that the pan-integral is related to finite partitions of X, the concave integral to any finite set systems of measurable subsets of X. The Choquet integral is based on chains of sets, it can be expressed as the following form:

$$\int^{Cho} f\,d\mu = \sup\left\{\sum_{i=1}^{n}\lambda_i\mu(A_i) : \sum_{i=1}^{n}\lambda_i\chi_{A_i} \le f,\ \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a chain},\ \lambda_i \ge 0,\ n \in \mathbb{N}\right\}.$$

When $X = \{x_1, \cdots, x_n\}$ is a finite set then the Choquet integral has a simple expression:

$$\int^{Cho} f d\mu = \sum_{i=1}^{n}\left(f(x_{(i)}) - f(x_{(i+1)})\right)\mu(B_i),$$

where $(x_{(1)}, \cdots, x_{(n)})$ is a permutation of $(x_1, \cdots, x_n)$ such that $f(x_{(1)}) \ge \cdots \ge f(x_{(n)}) \ge 0$ and $B_i = \{x_{(1)}, \cdots, x_{(i)}\}$ and by convention $f(x_{(n+1)}) = 0$.

3. Minimal atoms and cluster of minimal atoms of a monotone measure

In [10,16] we introduced the concept of minimal atom of a monotone measure. By means of the concept we described the relations between the pan-integral and the concave integral, and the Choquet integral and the pan-integral. In these discussions the minimal atoms played important roles. We will see, they still play an key role in our further discussion. We recall the following definition.

Definition 3.1. [16] Let μ be a monotone measure on $\mathcal{A}$. A set $A \in \mathcal{A}$ is called a *minimal atom* of μ if $\mu(A) > 0$ and for every $B \subset A$ holds either

(i) $\mu(B) = 0$, or
(ii) $A = B$.

Note. The concept of atom in classical measure theory was generalized in nonadditive measure theory, see [5,22] and further discussed, see [9,18,19,24]. It is easy to see that, a minimal atom A of μ is a special atom of μ (it is also a pseudo-atom of μ, see [24]).

Proposition 3.2. *[16] Let X be a finite space. Then every set $A \in \mathcal{A}$ with $\mu(A) > 0$ contains at least one minimal atom of μ.*

In [10] we introduced the concept of (M)*-property* of a monotone measure.

Definition 3.3. [10] Let μ be a monotone measure on $(X, \mathcal{A})$. If for any $A, B \in \mathcal{A}, A \subset B$, there exists $C \in A \cap \mathcal{A}$ such that

$$\mu(C) = \mu(A) \text{ and } \mu(B) = \mu(C) + \mu(B \setminus C),$$

then μ is called to have (M)-property.

If X is a finite set, then the (M)-property is equivalent to a simpler expression by minimal atoms described. We can easily obtain the following result.

Proposition 3.4. *Let X be a finite set and μ be a monotone measure defined on $(X, \mathcal{A})$. Then the following two statements are equivalent:*

(i) *μ has (M)-property.*
(ii) *For any $B \in \mathcal{A}$ with $\mu(B) > 0$ and any minimal atom $A \subset B$, it holds that*

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

In the rest of paper, we always suppose that X is a finite set and $\mathcal{A}$ is an algebra over X.

To characterize the equivalence of the Choquet and pan-integral, we introduce the following concepts.

Definition 3.5. (i) Let $A \in \mathcal{A}$ and $A_1, \cdots, A_s$ be some minimal atoms contained in A. If for any $1 \le i, j \le s$, there exist minimal atoms $\tilde{A}_1, \cdots, \tilde{A}_k \subset \cup_{i=1}^s A_i$ such that $\tilde{A}_1 = A_i$, $\tilde{A}_k = A_j$ and $\tilde{A}_l \cap \tilde{A}_{l+1} \neq \emptyset, 1 \le l \le k-1$, then we call the set $E = \cup_{i=1}^s A_i$ *a cluster of minimal atoms* contained in A. If $A = X$ then we simply call E a cluster of minimal atoms.

(ii) Let $E \subset A$ be a cluster of minimal atoms contained in A. If for any minimal atom $B \subset A \setminus E$, we have that $B \cap E = \emptyset$, then we call E *a maximal cluster of minimal atoms* contained in A. If $A = X$ then we simply call E a maximal cluster of minimal atoms.

Due to Proposition 3.2, when X is a finite space, for any $A \in \mathcal{A}$ with $\mu(A) > 0$ contains at least one minimal atom of μ. It is easy to see that each minimal atom C of μ contained in A can be expanded into a cluster of minimal atoms contained in A. Thus A also has at least one maximal cluster of minimal atoms.

Example 3.6. Let $X = \{x_1, \cdots, x_5\}$ and the monotone measure $\mu\colon \mathcal{P}(X) \to [0, \infty]$ be defined as

$$\mu(A) = \begin{cases} 1 & \text{if } x_5 \in A \text{ and } |A| < 3 \text{ or,} \\ & \quad A \subsetneqq \{x_1, x_2, x_3, x_4\} \text{ and } |A| \ge 2 \\ 2 & \text{if } A = \{x_1, x_2, x_3, x_4\} \text{ or,} \\ & \quad x_5 \in A \text{ and } 3 \le |A| \le 4 \\ 3 & A = X \\ 0 & \text{else.} \end{cases}$$

Apparently, $\{x_5\}$ is a minimal atom and each set A contained in $\{x_1, x_2, x_3, x_4\}$ and such that $|A| = 2$ is also a minimal atom. The space X contains two maximal clusters of minimal atoms, namely, $\{x_1, x_2, x_3, x_4\}$ and $\{x_5\}$.

Note 3.7. It is easy to see that the monotone measure μ in Example 3.6 possesses (M)-property.

The following result shows that a monotone measure with (M)-property has nice nature. Its proof is essentially the same as which of Theorem 3.4 in [10].

Proposition 3.8. *Let E be a maximal cluster of minimal atoms. If μ has (M)-property, then for arbitrary minimal atoms $A_s, A_t \subset E$, $\mu(A_s) = \mu(A_t)$. Moreover,*

$$E = \Big(\bigcup_{i=1}^{l} A_i\Big) \bigcup E_0 \tag{3.1}$$

for some l, where $A_i, i = 1, \cdots, l$ are pairwise disjoint minimal atoms and $\mu(E_0) = 0$, and thus $\mu(E) = l\mu(A_1)$.

Note 3.9. (i) Notice that the representation (3.1) may not be unique, but the number l is unique (it is determined by E and thus we will denote it by l_E). For example, let $X = \{x_1, x_2, x_3\}$ and $\mu(A) = 1$ if $|A| \geq 2$ and 0 otherwise. Then X itself is a maximal cluster of minimal atoms and it has three ways of representation

$$X = \{x_1, x_2\} \cup \{x_3\} = \{x_1, x_3\} \cup \{x_2\} \cup \{x_2, x_3\} \cup \{x_1\},$$

but $l_X = 1$ is fixed. Also Example 3.6, where for the maximal cluster $E = \{x_1, x_2, x_3, x_4\}$ we have 3 different decomposition $\{x_i, x_j\} \cup \{x_r, x_k\}$ with $\{i, j, r, k\} = \{1, 2, 3, 4\}$, then $\mu(E) = 2 \cdot \mu(\{x_1, x_2\}) = 2$.

(ii) Let $E_1, \cdots, E_k$ be all of the maximal clusters of minimal atoms. Then X can obviously be represented as $X = \cup_{i=1}^k E_i \cup X_0$, where X_0 is a set of measure zero (otherwise, either X contains another maximal cluster of minimal atoms or at least one of E_i is not a maximal cluster of minimal atoms). By using (3.1), we have

$$X = \left(\bigcup_{i=1}^{k} \bigcup_{j=1}^{l_{E_i}} A_j^{(i)} \right) \bigcup \tilde{X}_0,$$

where $\tilde{X}_0 = \left(X_0 \bigcup \left(\bigcup_{i=1}^k E_0^{(i)} \right) \right)$ is also a set of measure zero. In fact, if $\mu(\tilde{X}_0) > 0$ then it contains at least one minimal atom A. Noting that each $E_0^{(i)}$ as well as X_0 is of measure zero, there exists some i such that both $A \cap E_0^{(i)} \neq \emptyset$ and which imply that E_i is not a maximal cluster of minimal atoms.

Theorem 3.10. *Let $E_1, \cdots, E_k$ be all of the maximal clusters of minimal atoms. If μ has (M)-property then for any $B \in \mathcal{A}$ we have that*

$$\mu(B) = \sum_{i=1}^{k} \mu(B \cap E_i). \tag{3.2}$$

Proof. Suppose that $B \cap E_i$ contains m_i pairwise disjoint minimal atoms $A_1^{(i)}, \cdots, A_{m_i}^{(i)}$. Then, similar to the above proposition, $B \cap E_i$ can be represented as

$$B \cap E_i = \left(\bigcup_{j=1}^{m_i} A_j^{(i)} \right) \bigcup \tilde{E}_0^{(i)},$$

where $\tilde{E}_0^{(i)}$ is a set with measure zero. Then, by using the (M)-property repeatedly,

$$\mu(B \cap E_i) = \sum_{j=1}^{m_i} \mu(A_j^{(i)}) + \mu(\tilde{E}_0^{(i)}) = \sum_{j=1}^{m_i} \mu(A_j^{(i)}).$$

Noting that $X = \cup_{i=1}^k E_i \cup X_0$, we have

$$\begin{aligned}
\mu(B) &= \mu\left(\left(\bigcup_{i=1}^{k} (B \cap E_i) \right) \bigcup (B \cap X_0) \right) \\
&= \sum_{j=1}^{m_1} \mu(A_j^{(1)}) + \mu\left(\left(\bigcup_{i=2}^{k} (B \cap E_i) \right) \bigcup \left(\tilde{E}_0^{(1)} \bigcup \left(B \cap X_0 \right) \right) \right) \\
&= \cdots \\
&= \sum_{i=1}^{k} \sum_{j=1}^{m_i} \mu(A_j^{(i)}) + \mu\left(\left(\bigcup_{i=1}^{k} \tilde{E}_0^{(i)} \right) \bigcup \left(B \cap X_0 \right) \right).
\end{aligned}$$

Applying the same argument of Note 3.9(ii), we get

$$\mu\left(\Big(\bigcup_{i=1}^{k}\tilde{E}_0^{(i)}\Big)\bigcup\Big(B\cap X_0\Big)\right)=0,$$

and thus

$$\mu(B)=\sum_{i=1}^{k}\sum_{j=1}^{m_i}\mu(A_j^{(i)})=\sum_{i=1}^{k}\mu(B\cap E_i).$$

Specifically, we have that $\mu(X)=\sum_{i=1}^{k}\mu(E_i)$. □

4. Coincidence of the Choquet integral and the pan-integral

In [10] we have proved that a necessary condition that the Choquet integral coincides with the pan-integral based on the usual addition + and multiplication · on finite space is that the involved monotone measure has (M)-property. In this section we will show that the (M)-property is not only necessary, but also sufficient. To prove our main result, we need some subsequent lemmas. As their proofs are straightforward, these proofs are omitted.

Lemma 4.1. *Let $E_1,\cdots,E_k$ be all of the maximal clusters of minimal atoms. If μ has (M)-property, then for any $f\in\mathcal{F}_+$ it holds*

$$\overset{Cho}{\int} f d\mu=\sum_{i=1}^{k}\overset{Cho}{\int_{E_i}} f d\mu. \tag{4.1}$$

Lemma 4.2. *Under the assumptions of Lemma 4.1, we have that*

$$\overset{pan}{\int} f d\mu=\sum_{i=1}^{k}\overset{pan}{\int_{E_i}} f d\mu. \tag{4.2}$$

Note 4.3. By Lemmas 4.1, 4.2, to reach our goal of this section, it suffices to prove that the Choquet integral coincides with the pan-integral on each maximal cluster of minimal atoms.

Lemma 4.4. *For each minimal atom A and $f\in\mathcal{F}_+$, the following equality holds,*

$$\overset{Cho}{\int_{A}} f d\mu=\overset{pan}{\int_{A}} f d\mu. \tag{4.3}$$

We also need the following result.

Lemma 4.5. *Let μ be a monotone measure defined over $(X,\mathcal{A})$ and E be a maximal cluster of minimal atoms. If μ has (M)-property, then for $f\in\mathcal{F}_+$, there exist pairwise disjoint minimal atoms $A_1,\cdots,A_{l_E}\subset E$ (depending on f) such that*

$$\overset{Cho}{\int_{E}} f d\mu=\sum_{i=1}^{l_E}\overset{Cho}{\int_{A_i}} f d\mu. \tag{4.4}$$

Proof. We can assume $\mathcal{A} = \mathcal{P}(X)$ without loss of generality (for a general algebra $\mathcal{A}$, it is not difficult to prove the conclusion still holds).

Suppose $E = \{x_1, \cdots, x_n\}$ and f is fixed. Without loss of generality, we can assume that $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n)$, then

$$\int_E^{Cho} f d\mu = \sum_{i=1}^{n} (f(x_i) - f(x_{i+1}))\, \mu(B_i),$$

where $B_i = \{x_1, \cdots, x_i\}$ and $f(x_{n+1}) \triangleq 0$.

By the definition of maximal cluster of minimal atoms, each $x \in E$ is contained in at least one minimal atom. Suppose i_1 be such that $\mu(B_{i_1-1}) = 0$ and $\mu(B_{i_1}) > 0$, then B_{i_1} contains one minimal atom, denoted by A_1 (note that $x_{i_1} \in A_1$). Again, let $i_2 > i_1$ be such that $\mu(B_{i_2-1}) = \mu(A_1)$ and $\mu(B_{i_2}) > \mu(A_1)$, then by (M)-property, $\mu(B_{i_2} \setminus A_1) = \mu(B_{i_2}) - \mu(A_1) > 0$. Thus, $B_{i_2} \setminus A_1$ contains one minimal atom, denoted by A_2 (again we have $x_{i_2} \in A_2$), and by Proposition 3.8 $\mu(A_2) = \mu(A_1)$ and thus $\mu(B_{i_2}) = 2\mu(A_1)$.

Generally, let i_j be such that $\mu(B_{i_j-1}) = (j-1)\mu(A_1)$ and $\mu(B_{i_j}) = j\mu(A_1)$, then $B_{i_j} \setminus \left(\bigcup_{i=1}^{j-1} A_i\right)$ contains one minimal atom, denoted by A_j ($x_{i_j} \in A_j$). We repeat this procedure unless we reach B_n.

Now the Choquet integral of f can be computed in the following way

$$\begin{aligned}
\int_E^{Cho} f d\mu &= \sum_{i=1}^{i_1-1} (f(x_i) - f(x_{i+1}))\, \mu(B_i) + \sum_{i=i_1}^{i_2-1} (f(x_i) - f(x_{i+1}))\, \mu(B_i) + \\
&\quad \cdots + \sum_{i=i_{l_E-1}}^{i_{l_E}-1} (f(x_i) - f(x_{i+1}))\, \mu(B_i) + \sum_{i=i_{l_E}}^{n} (f(x_i) - f(x_{i+1}))\, \mu(B_i) \\
&= \sum_{i=1}^{i_1-1} (f(x_i) - f(x_{i+1})) \cdot 0 + \sum_{i=i_1}^{i_2-1} (f(x_i) - f(x_{i+1}))\, \mu(A_1) + \cdots \\
&\quad + \sum_{i=i_{l_E-1}}^{i_{l_E}-1} (f(x_i) - f(x_{i+1}))\, (\mu(A_1) + \cdots + \mu(A_{l_E-1})) \\
&\quad + \sum_{i=i_{l_E}}^{n} (f(x_i) - f(x_{i+1}))\, (\mu(A_1) + \cdots + \mu(A_{l_E})) \\
&= \sum_{j=1}^{l_E} f(x_{i_j})\mu(A_j) = \sum_{j=1}^{l_E} \min\{f(x) : x \in A_j\}\mu(A_j) \\
&= \sum_{i=1}^{l_E} \int_{A_i}^{Cho} f d\mu. \qquad \square
\end{aligned}$$

The following theorem is our main result.

Theorem 4.6. *Let $(X, \mathcal{A})$ be a finite space and μ be a monotone measure defined on $(X, \mathcal{A})$. The following three statements are equivalent:*

(i) for each $f \in \mathcal{F}_+$, it holds

$$\int^{Cho} f d\mu = \int^{pan} f d\mu; \tag{4.5}$$

(ii) μ has (M)-property;
(iii) for any $B \in \mathcal{A}$ with $\mu(B) > 0$ and any minimal atom $A \subset B$, it holds that

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

Proof. By Proposition 3.4, it remains to prove the equivalence of (i) and (ii). For (i) implies (ii), see Lemma 3.3 in [10]. Now we show (ii) implies (i).

Let $E_1, \cdots, E_k$ be all of the maximal clusters of minimal atoms and f be an arbitrary but fixed function. For each E_i, by Lemma 4.5, we can find a disjoint minimal atoms system $A_1^{(i)}, \cdots, A_{l_{E_i}}^{(i)}$ such that

$$\int_{E_i}^{Cho} f d\mu = \sum_{j=1}^{l_{E_i}} \int_{A_j^{(i)}}^{Cho} f d\mu.$$

Then, combining above Lemmas 4.1, 4.2, 4.4, and Lemma 2 in [15], we have

$$\begin{aligned}
\int^{Cho} f d\mu &= \sum_{i=1}^{k} \int_{E_i}^{Cho} f d\mu = \sum_{i=1}^{k} \sum_{j=1}^{l_{E_i}} \int_{A_j^{(i)}}^{Cho} f d\mu \\
&= \sum_{i=1}^{k} \sum_{j=1}^{l_{E_i}} \int_{A_j^{(i)}}^{pan} f d\mu = \sum_{i=1}^{k} \sum_{j=1}^{l_{E_i}} \int^{pan} f \cdot \chi_{A_j^{(i)}} d\mu \\
&\leq \sum_{i=1}^{k} \int^{pan} \sum_{j=1}^{l_{E_i}} f \cdot \chi_{A_j^{(i)}} d\mu \\
&\leq \sum_{i=1}^{k} \int^{pan} f \cdot \chi_{E_i} d\mu = \int^{pan} f d\mu.
\end{aligned}$$

On the other hand, (M)-property implies the superadditivity of μ [10], and for a superadditive measure μ, the Choquet integral is greater than or equal to the pan-integral [23], thus $(ii) \Rightarrow (i)$ follows. □

Example 4.7. The monotone measure μ in Example 3.6 has (M)-property. For any nonnegative function f, we have

$$\int^{Cho} f d\mu = \int^{pan} f d\mu = m + M + f(x_5),$$

where $m = \min_{1 \leq i \leq 4} f(x_i)$ and $M = \max_{x \in \{x_1, \cdots, x_4\} \setminus \{x^*\}} f(x)$, x^* is one of the points such that $f(x^*) = \max_{1 \leq i \leq 4} f(x_i)$.

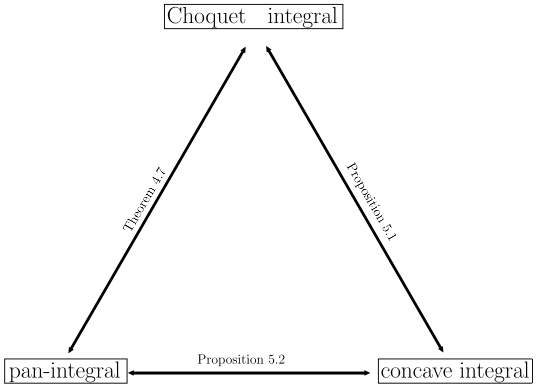

Fig. 5.1. Relationships among the three types of integrals.

5. Summarization of the relationships among the Choquet, concave and pan-integrals

Recall that Lehrer in [7] has characterized all monotone measures μ for which the Choquet and concave integrals coincide.

Proposition 5.1. *[7] The Choquet integral coincides with the concave integral, that is,* $\int^{Cho} f d\mu = \int^{cav} f d\mu$ *holds for any* f *if and only if* μ *is supermodular, i.e., for any* $A, B \subset X$ *it holds*

$$\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

Recently, we have characterized in [16] the conditions on μ when the concave and pan-integrals coincide with each other.

Proposition 5.2. *[16] The equality*

$$\int^{pan} f d\mu = \int^{cav} f d\mu$$

holds for each f *if and only if the following two conditions hold:*

(i) μ *possesses the minimal atoms disjointness property, i.e., any pair of different minimal atoms* (E_i, E_j) *is disjoint;*
(ii) μ *is subadditive w.r.t. minimal atoms, i.e., for every set* $A \in \mathcal{A}$ *with* $\mu(A) > 0$*, we have*

$$\mu(A) \leq \sum_{i=1}^{s} \mu(A_i),$$

where $\{A_i\}_{i=1}^{s}$ *is the set of all minimal atoms contained in* A.

The following Fig. 5.1 depicts the relationships among the Choquet integral, the concave integral and the pan-integral.

Based on Theorem 4.6, Propositions 5.1 and 5.2, the next result is immediate.

Theorem 5.3. *The equalities*

$$\int^{Cho} f d\mu = \int^{pan} f d\mu = \int^{cav} f d\mu$$

hold for each f if and only if the following two conditions hold:

(i) μ possesses the minimal atoms disjointness property;
(ii) μ has (M)-property.

Obviously, if μ satisfies the conditions (i) and (ii) of the above theorem then for any $E \in \mathcal{F}$ it holds $\mu(E) = \sum_{A_i \subset E} \mu(A_i)$ and

$$\int_E^{Cho} f d\mu = \int_E^{pan} f d\mu = \int_E^{cav} f d\mu = \sum_i \min_{x \in A_i} f(x) \cdot \mu(A_i),$$

where A_i are all minimal atoms contained in E. It is worth noting that a monotone measure μ satisfies (i) and (ii) need not be additive. For example, let X be a finite set with more than one point and the monotone measure μ be defined as $\mu(A) = 1$ if $A = X$ and 0 otherwise. Then μ satisfies the conditions of Theorem 5.3 and μ is not additive. For any nonnegative measurable function f, all the three types of integrals are equal to $\min_{x \in X} f(x)$. Similarly, consider a proper subset A of X and define a measure μ_A by

$$\mu_A(B) = \begin{cases} 1 & \text{if } \ A \subset B, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., μ_A is a so called unanimity game known from the game theory. Then μ_A satisfies all constraints of Theorem 5.3, and for any nonnegative measurable function f, all the three types of integrals are equal to $\min\{f(x) : x \in A\}$.

When we characterize the relationship of the Choquet, concave and pan-integrals, the following three structural characteristics are important:

(i) μ possesses the minimal atoms disjointness property;
(ii) μ has (M)-property;
(iii) μ is supermodular.

In the following, we discuss the relationship of (i), (ii) and (iii). From Theorem 5.3 and Proposition 5.1, the following results is straightforward.

Proposition 5.4. *If both (i) and (ii) hold, then (iii) is also true.*

Proposition 5.5. *If both (ii) and (iii) hold, then (i) is also true.*

The following example shows that $(i) + (iii) \not\Rightarrow (ii)$.

Example 5.6. Let $X = \{x_1, \cdots, x_4\}$ and $\mathcal{F} = \mathcal{P}(X)$. The monotone measure μ is defined in the following way:

$$\mu(A) = \begin{cases} 3 & \text{if } A = X \\ 1 & \text{if } \{x_1, x_2\} \subset A \neq X \text{ or, } \{x_3, x_4\} \subset A \neq X \\ 0 & \text{else.} \end{cases}$$

Then μ is supermodular and satisfies the minimal atoms disjoint property, but μ has not (M)-property. In fact, let $A = \{x_1, x_2\}$ and $B = X$, then for any $C \subset A$ with $\mu(C) = \mu(A)$ we have $C = A$ and

$$\mu(B) = 3 > \mu(A) + \mu(B \setminus A) = \mu(C) + \mu(B \setminus C).$$

6. Concluding remarks

In this paper, we have shown that a sufficient condition that the Choquet integral coincides with the $(+,\cdot)$-based pan-integral on finite spaces is that the involved monotone measure has (M)-property (Theorem 4.6). In [10] we also proved that the (M)-property is also necessary for the equivalence of these two kinds of integrals on finite spaces (Lemma 3.3 in [10]). Thus we have obtained an essential result: a necessary and sufficient condition that the Choquet integral coincides with the $(+,\cdot)$-based pan-integral on finite spaces is that the involved monotone measure possesses (M)-property (Theorem 4.6). We also give a simpler equivalent expression of (M)-property by minimal atoms described (Proposition 3.4, Theorem 4.6). Therefore, the equivalence of the Choquet integral and the pan-integral based on the usual addition + and multiplication · on finite spaces are characterized by the minimal atoms of monotone measures.

It is worth noting that we introduced the concept of *a maximal cluster of minimal atoms* on a monotone measure space (Definition 3.5) and it possesses some interesting characteristics (Proposition 3.8, Theorem 3.10). By means of characteristics of *maximal clusters of minimal atoms*, we obtained our results. As we have seen (Lemma 4.1, 4.2, 4.4 and 4.5), the concepts of *minimal atom* of monotone measure and *maximal cluster of minimal atoms* played very important roles in our discussions.

Notice that due to the results of equivalence of the concave integral and pan-integral on finite space we have obtained in [16] (Proposition 5.2, see also Theorem 4.1 in [16]), and of the equivalence between the concave integral and the Choquet integral in [8] (Proposition 5.1, see also Proposition 2 in [8]), then the coincidence among these three types of nonlinear integrals, the Choquet integral, the concave integral and the pan-integral with respect to the standard arithmetic operations, is completely solved on finite spaces.

We point out that our results were obtained only on finite spaces. It is important to investigate the relationships of these three types of integrals on infinite spaces. For subadditive monotone measure spaces we have got some results ([17]). On the other hand, the concept of a pan-integral was introduced in [25] and it involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see also [13,19,21,23,26]). In this paper we only consider the pan-integrals based on the usual addition + and multiplication ·. Concerning the Choquet integral, it was generalized in [13] into Choquet-like integrals, where the standard arithmetic operations + and · on reals, as well as the lattice operations max and min on reals were replaced by a pseudo-addition $\oplus$ and a pseudo-multiplication $\otimes$. Similarly, the concave integral of Lehrer, which is based on + and ·, was generalized into pseudo-concave integral [11] based on pan-operations $\oplus$ and $\otimes$. Certainly, all our results are valid for these integrals based on g-addition and g-multiplication [19], which can be seen as an isomorphic transformation of the standard arithmetical operations + and ·. In the specific case when we consider idempotent operations, i.e., when $\oplus = \vee$ and $\otimes = \wedge$, then all three kinds of integrals coincide (for any measure m) and they are just the Sugeno integral [20]. For general pan-operations $\oplus$ and $\otimes$ (they were completely characterized in [13]), the issue of coincidences of the three mentioned types of integrals is an interesting topic for the further investigation.

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