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Possibility and necessity measures and integral equivalence

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ABSTRACT

Integral equivalence of couples $(\mu, \mathbf{x})$ and $(\mu, \mathbf{y})$, where μ is a possibility (necessity) measure on $[n] = \{1, \dots, n\}$ and $\mathbf{x}, \mathbf{y} \in [0, 1]^n$ is discussed and studied. We characterize the sets $\mathcal{H}(\mu, \mathbf{x})$ of all $\mathbf{y}$ such that the couples $(\mu, \mathbf{x})$ and $(\mu, \mathbf{y})$ are integral equivalent and we add an illustrative example. Subsequently, a new characterization of possibility (necessity) measures is obtained and the coincidence of universal integrals for possibility (necessity) measures and particular vectors from $[0, 1]^n$ is shown, thus generalizing these results introduced by Dubois and Rico for the Choquet and the Sugeno integrals.

1. Introduction

One distinguished class of utility functions exploited in multiple-criteria decision making is formed by universal integrals on $[0, 1]$ introduced by Klement et al. [6]. They calculate a global evaluation of alternatives characterized by score vectors from $[0, 1]^n$, where n is the number of considered criteria. The most applied integrals are the Choquet integral [1] and the Sugeno integral [9]. Among the other universal integrals on $[0, 1]$, recall the Shilkret integral [8], Weber integral [11] and copula-based integrals [7]. All these discrete integrals are based on a normed monotone measure on the space $[n] = \{1, \dots, n\}$ named capacity.

Recently, Dubois and Rico [2] have studied the equality of Choquet (Sugeno) integrals of particular couples of score vectors, considering possibility and necessity measures as underlying capacities. Inspired by their results, we extend their problem to all universal integrals on $[0, 1]$, aiming to characterize, for a fixed possibility (necessity) measure and a fixed score vector $\mathbf{x}$, the class of all score vectors with integral values identical to those related to $\mathbf{x}$, independently of the considered universal integral on $[0, 1]$.

The paper is organized as follows. In the next section, we introduce some necessary preliminaries concerning capacities and universal integrals. In Section 3, we study and discuss the above sketched problem considering possibility measures. In Section 4, necessity measures are considered. Finally, some concluding remarks are added.

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2. Preliminaries

For a fixed $n \geq 2$, we denote $[n] = \{1, \dots, n\}$. For any $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$, we denote by $(\cdot)$ a permutation $(\cdot) : [n] \to [n]$ such that $x_{(1)} \leq \dots \leq x_{(n)}$, and $x_{(0)} = 0$ by convention. More, we will use notation $A_{(i)} = \{(i), \dots, (n)\}$. Though the permutation $(\cdot)$ need not be unique (this happens if there are some ties in the sample $(x_1, \dots, x_n)$), this has no influence on the results presented later. Further, we will use the standard lattice notation $\vee$ for the join (maximum on $[0,1]$) and $\wedge$ for the meet (minimum on $[0,1]$).

Definition 2.1 *([10])*. A monotone set function $\mu : 2^{[n]} \to [0,1]$ is called a capacity whenever it satisfies two boundary conditions $\mu(\emptyset) = 0$ and $\mu([n]) = 1$. A capacity μ is called a possibility (necessity) measure whenever it is maxitive (minitive), i.e., if

$$\mu(A \cup B) = \mu(A) \vee \mu(B) \qquad (\mu(A \cap B) = \mu(A) \wedge \mu(B))$$

for any $A, B \subseteq [n]$. The set of all capacities on $[n]$ will be denoted as $\mathcal{M}_n$.

For any possibility measure Π, the function $\pi : [n] \to [0,1]$, $\pi(i) = \Pi(\{i\})$ is called a possibility distribution (of Π) [12], and for any $A \subseteq [n]$ it holds

$$\Pi(A) = \bigvee_{i \in A} \pi(i),$$

with convention that supremum of the empty set is 0. For any capacity $\mu : 2^{[n]} \to [0,1]$, its dual (conjugate) capacity $\mu^d : 2^{[n]} \to [0,1]$ is given by

$$\mu^d(A) = 1 - \mu([n] \setminus A).$$

Necessity measures are dual to possibility measures, i.e., $N : 2^{[n]} \to [0,1]$ is a necessity measure if and only if its conjugate $N^d = \Pi$ is a possibility measure. Considering the possibility distribution π of Π, it holds

$$N(A) = 1 - \bigvee_{i \notin A} \pi(i).$$

The greatest capacity $\mu^* : 2^{[n]} \to [0,1]$ is given by

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{else} \end{cases},$$

and it is a possibility measure with possibility distribution $\pi^* = 1$.

Its dual μ_*,

$$\mu_*(A) = \begin{cases} 1 & \text{if } A = [n] \\ 0 & \text{else} \end{cases},$$

is a necessity measure, and it is the smallest capacity on $[n]$.

Before introducing the concept of universal integrals on $[0,1]$ we recall the notion of a semicopula.

Definition 2.2 *([3])*. An operation $\otimes : [0,1]^2 \to [0,1]$ is called a semicopula whenever it is increasing in both coordinates and 1 is its neutral element, i.e., $x \otimes 1 = 1 \otimes x = x$ for all $x \in [0,1]$.

The greatest semicopula $\wedge : [0,1]^2 \to [0,1]$ is the standard min operator, $x \wedge y = \min\{x, y\}$. The smallest semicopula is the drastic product T_D,

$$x\, T_D\, y = \begin{cases} x \wedge y & \text{if } x \vee y = 1 \\ 0 & \text{else} \end{cases}.$$

Another distinguished semicopulas are the product T_P, $x\, T_P\, y = x \cdot y$, and the Lukasiewicz t-norm T_L, $x\, T_L\, y = \max\{x + y - 1, 0\}$.

The concept of universal integrals was proposed by Klement et al. [6], and it covers all integrals mentioned in Section 1.

Definition 2.3. Let $\otimes : [0,1]^2 \to [0,1]$ be a fixed semicopula. The mapping $\mathbf{I} : \bigcup_{n \in \mathbb{N}} \mathcal{M}_n \times [0,1]^n \to [0,1]$ is called a ($\otimes$-based) universal integral on $[0,1]$ whenever the next axioms are satisfied:

(U1) $\mathbf{I}$ is increasing in both coordinates, i.e., $\mathbf{I}(\mu_1, \mathbf{x}^{(1)}) \leq \mathbf{I}(\mu_2, \mathbf{x}^{(2)})$ for all $\mu_1, \mu_2 \in \mathcal{M}_n, \mu_1 \leq \mu_2$ and $\mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in [0,1]^n, \mathbf{x}^{(1)} \leq \mathbf{x}^{(2)}$;
(U2) $\mathbf{I}(\mu, c \cdot 1_A) = c \otimes \mu(A)$ for any $\mu \in \mathcal{M}_n, c \in [0,1]$ and $A \subseteq [n]$;
(U3) $\mathbf{I}(\mu_1, \mathbf{x}^{(1)}) = \mathbf{I}(\mu_2, \mathbf{x}^{(2)})$ for any integral equivalent couples
$(\mu_1, \mathbf{x}^{(1)}) \in [0,1]^{n_1}, (\mu_2, \mathbf{x}^{(2)}) \in [0,1]^{n_2}$,
$\mu_1(\{i \in [n_1] | \mathbf{x}_i^{(1)} \geq t\}) = \mu_2(\{j \in [n_2] | \mathbf{x}_j^{(2)} \geq t\})$ for all $t \in [0,1]$.

Note that due to [5, Theorem 3] the axioms (U1) and (U3) can be merged into a unique axiom (U4) equivalent to the simultaneous validity of (U1) and (U3), requiring

(U4) $\mathbf{I}(\mu_1, \mathbf{x}^{(1)}) \leq \mathbf{I}(\mu_2, \mathbf{x}^{(2)})$ whenever for all $t \in [0,1]$ it holds

$$\mu_1(\{i \in [n_1] | \mathbf{x}_i^{(1)} \geq t\}) \leq \mu_2(\{j \in [n_2] | \mathbf{x}_j^{(2)} \geq t\}).$$

Recall that the Choquet and the Shilkret integral are based on the product semicopula and that for $\mu \in \mathcal{M}_n$ and $\mathbf{x} \in [0,1]^n$ they are defined by

$$\mathbf{Ch}(\mu, \mathbf{x}) = \sum_{i=1}^{n} (x_{(i)} - x_{(i-1)})\, \mu(A_{(i)}) \qquad \text{and}$$

$$\mathbf{Sh}(\mu, \mathbf{x}) = \bigvee_{i=1}^{n} x_{(i)}\, \mu(A_{(i)}),$$

respectively. Among some other universal integrals based on the product we recall integrals $\mathbf{I}_{T_P}^{(k)}$, $k \in \mathbb{N}$, which are the only integrals being simultaneously universal and decomposition integrals [4],

$$\mathbf{I}_{T_P}^{(k)}(\mu, \mathbf{x}) = \sup\left\{ \sum_{i=1}^{k} a_i \mu(A_i) \,\Big|\, (A_i)_{i=1}^{k} \text{ is a chain, } (a_i)_{i=1}^{k} \in [0,1]^k, \sum_{i=1}^{k} a_i 1_{A_i} \leq \mathbf{x} \right\}.$$

Note that $\mathbf{Sh} = \mathbf{I}_{T_P}^{(1)} \leq \mathbf{I}_{T_P}^{(2)} \leq \cdots \leq \mathbf{I}_{T_P}^{(k)} \leq \ldots$ and that $\mathbf{Ch} = \bigvee_{k \in \mathbb{N}} \mathbf{I}_{T_P}^{(k)}$. In fact, on $\mathcal{M}_n \times [0,1]^n$, the integrals $\mathbf{Ch}$ and $\mathbf{I}_{T_P}^{(n)}$ coincide.

Sugeno integral $\mathbf{Su}$ is related to the greatest semicopula $\wedge$ and it is given by

$$\mathbf{Su}(\mu, \mathbf{x}) = \bigvee_{i=1}^{n} (x_{(i)} \wedge \mu(A_{(i)})).$$

Coming back to the axiom (U3), denote by $h_{\mu,\mathbf{x}} : [0,1] \to [0,1]$ a function given by

$$h_{\mu,\mathbf{x}}(t) = \mu(\{i \in [n] | x_i \geq t\}).$$

Due to the increasingness of μ, the function $h_{\mu,\mathbf{x}}$ is decreasing and $h_{\mu,\mathbf{x}}(0) = 1$. In probability theory, $h_{\mu,\mathbf{x}}$ can be seen as a survival function of random vector $\mathbf{x}$ (considering the probability space $([n], 2^{[n]}, \mu)$), and with a slight abuse of terminology, we will call $h_{\mu,\mathbf{x}}$ a survival function for any couple $(\mu, \mathbf{x}) \in \mathcal{M}_n \times [0,1]^n$, $n \in \mathbb{N}$. Recall that, in probability theory, the survival function $S : \mathbb{R} \to [0,1]$ of a random variable X (on a probability space $(\Omega, \mathcal{A}, P)$) is given by $S(x) = P(X \geq x)$. In our context, for the extremal capacities μ^* and μ_* it holds

$$h_{\mu^*,\mathbf{x}}(t) = \begin{cases} 1 & \text{if } t \leq x_{(n)} \\ 0 & \text{otherwise} \end{cases}, \quad \text{and } h_{\mu_*,\mathbf{x}}(t) = \begin{cases} 1 & \text{if } t \leq x_{(1)} \\ 0 & \text{otherwise} \end{cases}.$$

Due to the axiom (U3), the coincidence of survival functions $h_{\mu_1,\mathbf{x}^{(2)}} = h_{\mu_2,\mathbf{x}^{(2)}}$ ensures the coincidence $\mathbf{I}(\mu_1, \mathbf{x}^{(1)}) = \mathbf{I}(\mu_2, \mathbf{x}^{(2)})$ for any universal integral $\mathbf{I}$ acting on $[0,1]$. So, for example, if the considered capacity $\mu \in \mathcal{M}_n$ is symmetric, i.e., $\mu(A)$ depends on cardinality $|A|$ of the set $A \subseteq [n]$ only, then for any permutation $\sigma : [n] \to [n]$, any $\mathbf{x} \in [0,1]^n$ and any universal integral $\mathbf{I}$ it holds $\mathbf{I}(\mu, \mathbf{x}) = \mathbf{I}(\mu, \mathbf{x}_\sigma)$, where $\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Inspired by results of Dubois and Rico [2], we focus on possibility and necessity measures, and in particular on the equality of the related survival functions. This equality ensures the equality of related universal integrals, including the equality of Choquet and Sugeno integrals studied in [2].

3. Possibility measures and equality of survival functions

For a fixed $n \geq 2$, let $\Pi \in \mathcal{M}_n$ be a possibility measure, and let $\pi : [n] \to [0,1]$ be the related possibility distribution. For a fixed score vector $\mathbf{x} \in [0,1]^n$, we describe the corresponding survival function.

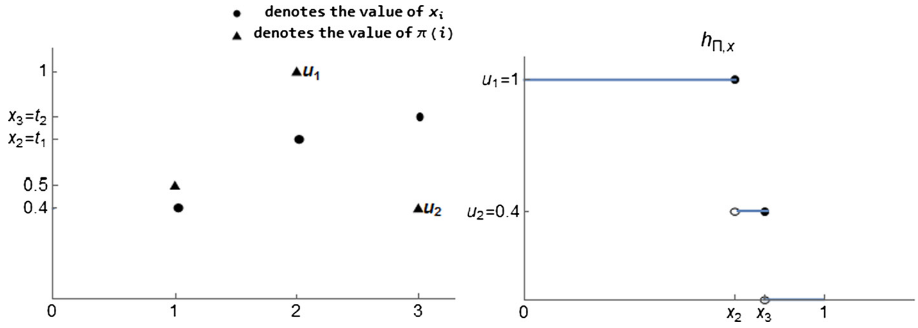

Fig. 1. Illustration of Example 3.1, construction of $h_{\Pi,\mathbf{x}}$.

Proposition 3.1. *The survival function* $h_{\Pi,\mathbf{x}} : [0,1] \to [0,1]$ *is given by*

$$h_{\Pi,\mathbf{x}}(t) = \bigvee_{x_i \geq t} \pi(i) = \Pi(A_{(i_{\mathbf{x},t})}), \text{ where } i_{\mathbf{x},t} = \min\{j \in [n] | x_{(j)} \geq t\}. \tag{1}$$

The proof of Proposition 3.1 follows from the previous definitions. For a better clarification of survival function $h_{\Pi,\mathbf{x}}$, observe first that

$$h_{\Pi,\mathbf{x}}(t) = 1 = u_1 \quad \text{for any} \quad t \in [0, t_1],\ t_1 = \bigvee_{\pi(i)=1} x_i.$$

Recall that $u_1 = \Pi([n])$.

For $u_2 = \Pi([n] \setminus \{i \in [n] | x_i \leq t_1\}) = \Pi(\{i | x_i > t_1\}) = \bigvee_{x_i > t_1} \pi(i)$, we have

- either $u_2 = 0$ and then $h_{\Pi,\mathbf{x}}(t) = 0$ for any $t \in]t_1, 1]$,
- or $u_2 > 0$ then $h_{\Pi,\mathbf{x}}(t) = u_2$ for any $t \in]t_1, t_2]$, where $t_2 = \bigvee_{\pi(i)=u_2} x_i$.

By induction, we have

- $h_{\Pi,\mathbf{x}}(t) = u_k > 0$ for any $t \in]t_{k-1}, t_k]$ for $k = 1, \ldots, r$, and
- $h_{\Pi,\mathbf{x}}(t) = 0$ for any $t \in]t_r, 1]$,

where $t_0 = 0$, $u_k = \bigvee_{x_i > t_{k-1}} \pi(i)$, and ◇ either $u_k = 0$ and then $r = k - 1$,
◇ or $u_k > 0$ and then $t_k = \bigvee_{\pi(i)=u_k} x_i$.

Observe that if, by chance, an interval $]t, t]$ is considered, then the corresponding claim is always valid due to the fact that $]t, t] = \emptyset$.

Example 3.1. Consider a possibility distribution $\pi : [3] \to [0, 1]$ given by

$$\pi(1) = 0.5,\ \pi(2) = 1 \text{ and } \pi(3) = 0.4.$$

Then for any $\mathbf{x} \in [0,1]^3$ such that $x_1 < x_2 < x_3$ we have $t_1 = x_2$, i.e., $h_{\Pi,\mathbf{x}}(t) = 1$ for any $t \in [0, x_2]$. Next, $u_2 = \Pi(\{3\}) = \pi(3) = 0.4$ and $t_2 = x_3$, i.e., $h_{\Pi,\mathbf{x}}(t) = 0.4$ for any $t \in]x_2, x_3]$. As $u_3 = 0$, we can summarize

$$h_{\Pi,\mathbf{x}}(t) = \begin{cases} 1 & \text{if } t \in [0, x_2] \\ 0.4 & \text{if } t \in]x_2, x_3] \\ 0 & \text{if } t \in]x_3, 1]. \end{cases}$$

For illustration, see Fig. 1.

Similarly, if $x_1 > x_2 > x_3$, it holds

$$h_{\Pi,\mathbf{x}}(t) = \begin{cases} 1 & \text{if } t \in [0, x_2] \\ 0.5 & \text{if } t \in]x_2, x_1] \\ 0 & \text{if } t \in]x_1, 1]. \end{cases} \quad \square$$

As already mentioned, for the greatest capacity $\mu^* \in \mathcal{M}_n$ it holds

$$h_{\mu^*,\mathbf{x}}(t) = \begin{cases} 1 & \text{if } t \in \left[0, \bigvee_{i=1}^{n} x_i\right] \\ 0 & \text{else.} \end{cases}$$

Obviously, $h_{\mu^*,\mathbf{x}} = h_{\mu^*,\mathbf{y}}$ if and only if $\bigvee_{i=1}^{n} x_i = \bigvee_{i=1}^{n} y_i$. Then the greatest score vector $\mathbf{x}^* \in [0,1]^n$ such that $h_{\mu^*,\mathbf{x}} = h_{\mu^*,\mathbf{x}^*}$ is the constant vector

$$\mathbf{x}^* = \left(\bigvee_{i=1}^{n} x_i, \ldots, \bigvee_{i=1}^{n} x_i\right).$$

On the other hand, there is no smallest vector $\mathbf{x}_*$ such that $h_{\mu^*,\mathbf{x}} = h_{\mu^*,\mathbf{x}_*}$ whenever $\mathbf{x} \neq (0,\ldots,0)$. However then, there are n minimal vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ such that $h_{\mu^*,\mathbf{x}} = h_{\mu^*,\mathbf{x}_1} = \cdots = h_{\mu^*,\mathbf{x}_n}$, where

$$\mathbf{x}_1 = \left(\bigvee_{i=1}^{n} x_i, 0, \ldots, 0\right), \ldots, \mathbf{x}_n = \left(0, \ldots, 0, \bigvee_{i=1}^{n} x_i\right).$$

Definition 3.1. Let $\mu \in \mathcal{M}_n$ and $\mathbf{x} \in [0,1]^n$ be fixed. Any score vector $\mathbf{y} \in [0,1]^n$ such that $h_{\mu,\mathbf{x}} = h_{\mu,\mathbf{y}}$ is called μ-integral equivalent to $\mathbf{x}$.

The set of all score vectors μ-integral equivalent to $\mathbf{x}$ is denoted as $\mathcal{H}(\mu,\mathbf{x})$.

The main aim of this section is the description of the set $\mathcal{H}(\Pi,\mathbf{x})$, where Π is a possibility measure on $[n]$, and $\mathbf{x}$ is a score vector from $[0,1]^n$. Note that for any possibility measure $\Pi \in \mathcal{M}_n$, such that $\Pi(A) > 0$ whenever $A \neq \emptyset$, $\mathcal{H}(\Pi,(0,\ldots,0)) = \{(0,\ldots,0)\}$.

Considering again the greatest capacity $\mu^* \in \mathcal{M}_n$, we see that

$$\mathcal{H}(\mu^*,\mathbf{x}) = \left\{\mathbf{y} \in [0,1]^n \,\middle|\, \bigvee_{i=1}^{n} y_i = \bigvee_{i=1}^{n} x_i\right\}.$$

Then, if $\mathbf{y},\mathbf{z} \in \mathcal{H}(\mu^*,\mathbf{x})$ also $\mathbf{y} \vee \mathbf{z} \in \mathcal{H}(\mu^*,\mathbf{x})$, i.e., $\mathcal{H}(\mu^*,\mathbf{x})$ is an upper semi-lattice with top element x^*. Moreover, if $\mathbf{x} \neq (0,\ldots,0)$, then $\mathcal{H}(\mu^*,\mathbf{x})$ is not a convex subset of $[0,1]^n$, though it can be represented as a union of $2^n - 2$ convex subsets of $[0,1]^n$. Denoting $\bigvee_{i=1}^{n} x_i = c > 0$, the mentioned convex classes are related to proper subsets $A \subset [n]$,

$$\mathcal{H}_A(\mu^*,\mathbf{x}) = \{\mathbf{y} \in [0,c]^n \mid y_i = c \text{ for any } i \in A\}.$$

For any capacity $\mu \in \mathcal{M}_n$, we have an equivalence relation $\sim_\mu$ on $[0,1]^n$ defined by $\mathbf{x} \sim_\mu \mathbf{y}$ if and only if $h_{\mu,\mathbf{x}} = h_{\mu,\mathbf{y}}$. Then it is obvious that $\mathcal{H}(\mu,\mathbf{x})$ is an equivalence class of $\sim_\mu$ containing the vector $\mathbf{x}$.

Theorem 3.1. *Under the previous notation,* $\mathbf{y} \in \mathcal{H}(\Pi,\mathbf{x})$ *if and only if*

$$\bigvee_{\pi(i)=u_j} y_i = t_j \text{ and } y_i \le t_j \text{ whenever } \pi(i) > u_{j+1} \text{ for all } j = 1,\ldots,r.$$

The set $\mathcal{H}(\Pi,\mathbf{x})$ *is an upper semi-lattice with the top element* $\mathbf{x}^\Pi$ *given by*

$$x_i^\Pi = \begin{cases} t_j & \text{whenever } u_j \ge \pi(i) > u_{j+1}, \qquad j = 1,\ldots,r \\ 1 & \text{whenever } \pi(i) = 0 \end{cases},$$

for $i = 1,\ldots,n$.

Proof. The proof follows from Proposition 3.1. Indeed, based on (1) we see that $\mathbf{y} \in \mathcal{H}(\Pi,\mathbf{x})$ if and only if, for any $t \in [0,1]$, it holds $\bigvee_{y_j \ge t} \pi(j) = \bigvee_{x_i \ge t} \pi(i)$. In particular, we see that $\bigvee_{\pi(i)=1} y_i = t_1$ (recall that $u_1 = 1$), and thus $h_{\Pi,\mathbf{y}}(t) = 1$ for any $t \in [0,t_1]$. Moreover, $\bigvee_{y_i > t_1} \pi(i) = u_2$, and $\bigvee_{\pi(i)=u_2} y_i = t_2$, hence $h_{\Pi,\mathbf{y}}(t) = u_2$ for any $t \in]t_1,t_2]$. The proof of equality $h_{\Pi,\mathbf{y}} = h_{\Pi,\mathbf{x}}$ follows by induction.

For any $\mathbf{y}, \mathbf{z} \in [0,1]^n$,

$$h_{\Pi,\mathbf{y}\vee\mathbf{z}}(t) = \Pi(\{i \in [n] | y_i \vee z_i \geq t\}) = \Pi(\{i \in [n] | y_i \geq t\} \cup \{i \in [n] | z_i \geq t\}) =$$
$$= \Pi(\{i \in [n] | y_i \geq t\}) \vee \Pi(\{i \in [n] | z_i \geq t\}) = h_{\Pi,\mathbf{y}}(t) \vee h_{\Pi,\mathbf{z}}(t).$$

Then, for any $\mathbf{y}, \mathbf{z} \in \mathcal{H}_{\Pi,\mathbf{x}}$, $h_{\Pi,\mathbf{y}\vee\mathbf{z}} = h_{\Pi,\mathbf{y}} \vee h_{\Pi,\mathbf{z}}$, which implies $\mathbf{y} \vee \mathbf{z} \in \mathcal{H}_{\Pi,\mathbf{x}}$, i.e., $\mathcal{H}_{\Pi,\mathbf{x}}$ is an upper semi-lattice. The description of its top element $\mathbf{x}^{\Pi}$ follows from the first part of this theorem. □

Theorem 3.1 was already exemplified considering the greatest possibility measure μ^*. As another extremal case, consider the Dirac measure δ_k, $k \in [n]$, $\delta_k(A) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{otherwise} \end{cases}$. The measure δ_k is one of n minimal possibility measures on $[n]$, and its possibility distribution is just the characteristic function $\mathbf{1}_{\delta_k,\mathbf{x}}(t) = \begin{cases} 1 & \text{if } t \in [0, x_k] \\ 0 & \text{otherwise} \end{cases}$, and $\mathcal{H}(\delta_k, \mathbf{x}) = \{\mathbf{y} \in [0,1]^n |\, y_k = x_k\}$ with top element $\mathbf{x}^{\delta_k} = (1, \ldots, 1, x_k, 1, \ldots, 1)$.

Example 3.2. Continuing in Example 3.1, for any $\mathbf{x} \in [0,1]^3$ such that $x_1 < x_2 < x_3$ it holds

$$\mathcal{H}(\Pi, \mathbf{x}) = \{\mathbf{y} \in [0,1]^3 |\, y_1 \leq x_2, y_2 = x_2, y_3 = x_3\}.$$

Note that the permutation σ related to Π is given by $\sigma = (2, 1, 3)$ and $\mathbf{x}^{\sigma,+} = (x_2, x_2, x_3)$ is the top element of $\mathcal{H}(\Pi, \mathbf{x})$. □

Dubois and Rico [2] have introduced two special members of the class $\mathcal{H}(\Pi, \mathbf{x})$. If Π is a possibility measure related to a possibility distribution $\pi : [n] \to [0,1]$ which is decreasing, i.e., $1 = \pi(1) \geq \pi(2) \geq \cdots \geq \pi(n)$, then their score vector $\mathbf{x}^+ \in [0,1]^n$ given by

$$x_i^+ = \bigvee_{\pi(j) \geq \pi(i)} x_j, \qquad i \in [n], \tag{2}$$

satisfies the constraints given in Theorem 3.1 and thus $\mathbf{x}^+ \in \mathcal{H}(\Pi, \mathbf{x})$, and, moreover $\mathbf{x}^+$ is the top element of $\mathcal{H}(\Pi, \mathbf{x})$, $\mathbf{x}^+ = \mathbf{x}^{\Pi}$ whenever $\pi(n) > 0$.

Observe that if $\pi(n) = 0$ then the value x_n has no influence on the survival function $h_{\Pi,\mathbf{x}}$, and then, if $x_n^+ \neq 1$, clearly $\mathbf{x}^+$ cannot be a top element of $\mathcal{H}(\Pi, \mathbf{x})$.

Note that if $1 = \pi(1) > \pi(2) > \cdots > \pi(n)$ then

$$x_i^+ = \bigvee_{j=1}^{i} x_j.$$

If there are some ties, $\pi(j) = \pi(j+1)$, then

$$x_i^+ = \bigvee_{j=1}^{k_i} x_j \qquad \text{for all } i \in [n],$$

where $k_i = \max\{j \in [n] | \pi(j) = \pi(i)\}$.

To see the fact that, if $\pi(n) > 0$, $\mathbf{x}^+$ is the top element of $\mathcal{H}(\Pi, \mathbf{x})$ (i.e., $\mathbf{x}^+ \in \mathcal{H}(\Pi, \mathbf{x})$, and for any $\mathbf{y} \in \mathcal{H}(\Pi, \mathbf{x})$ it holds $\mathbf{y} \leq \mathbf{x}^+$), suppose that there is $\mathbf{y} \in \mathcal{H}(\Pi, \mathbf{x})$ not satisfying $\mathbf{y} \leq \mathbf{x}^+$. This means that $y_i > x_i^+$ for some $i \in [n]$.

- Suppose first $y_1 > x_1^+$. Then $h_{\Pi,\mathbf{y}}(y_1) = 1$ but $h_{\Pi,\mathbf{x}}(t) < 1$ if $t > x_1^+$. This means that $h_{\Pi,\mathbf{x}}(y_1) < 1$ and thus $\mathbf{y} \notin \mathcal{H}(\Pi, \mathbf{x})$.
- For the rest of the proof, suppose $y_1 \leq x_1^+, \ldots, y_r \leq x_r^+$ and $y_{r+1} > x_{r+1}^+$ for some $r \in \{1, \ldots, n-1\}$. Then $h_{\Pi,\mathbf{y}}(y_{r+1}) = \pi(r+1)$ but

$$h_{\Pi,\mathbf{x}}(y_{r+1}) = \max\{\pi(j) |\, j > r+1, x_j \geq y_{r+1}\}$$
$$\leq \max\{\pi(j) |\, j > r+1, x_j > x_{r+1}^+\} < \pi(r+1)$$

(with convention $\max \emptyset = 0$), and thus also now $\mathbf{y} \notin \mathcal{H}(\Pi, \mathbf{x})$.

For a general possibility measure Π, based on a permutation $\sigma : [n] \to [n]$ such that $1 = \pi(\sigma(1)) \geq \pi(\sigma(2)) \geq \cdots \geq \pi(\sigma(n))$, they have introduced a score vector $\mathbf{x}^{\sigma,+} \in \mathcal{H}(\Pi, \mathbf{x})$, where

$$x_i^{\sigma,+} = \bigvee_{j=1}^{i} x_{\sigma(j)} \qquad i \in [n]. \tag{3}$$

Observe that σ is not unique whenever the range of π consists from less than n elements, i.e., if $\pi(i) = \pi(j)$ for some $i \neq j$. Then also the score vector $\mathbf{x}^{\sigma,+}$ need not be uniquely determined by Π, and hence it need not be the top element of $\mathcal{H}(\Pi, \mathbf{x})$. As an example, consider again the greatest possibility measure μ^*. Then any permutation σ satisfies $\pi(\sigma(1)) \geq \pi(\sigma(2)) \geq \cdots \geq \pi(\sigma(n))$. For $n = 3$ and $\mathbf{x} = \left(\frac{1}{3}, \frac{2}{3}, 1\right)$, it holds:

$$\mathbf{x}^{(1\,2\,3),+} = \left(\frac{1}{3}, \frac{2}{3}, 1\right) = \mathbf{x},$$
$$\mathbf{x}^{(1\,3\,2),+} = \left(\frac{1}{3}, 1, 1\right),$$
$$\mathbf{x}^{(2\,1\,3),+} = \left(\frac{2}{3}, \frac{2}{3}, 1\right),$$
$$\mathbf{x}^{(2\,3\,1),+} = \left(\frac{2}{3}, 1, 1\right),$$
$$\mathbf{x}^{(3\,1\,2),+} = (1, 1, 1) = \mathbf{x}^{\mu^*},$$
$$\mathbf{x}^{(3\,2\,1),+} = (1, 1, 1) = \mathbf{x}^{\mu^*}.$$

Coming back to $\mathbf{x}^+$ given by (2), note that this formula can be applied for any possibility measure Π with a possibility distribution π. We have the next corollary of Theorem 3.1 and whose particular case when $1 = \pi(1) \geq \cdots \geq \pi(n)$ was already discussed above.

Corollary 3.1. *For $\mathbf{x} \in [0, 1]^n$, define $\mathbf{x}^+$ by (2), i.e., $x_i^+ = \bigvee_{\pi(j) \geq \pi(i)} x_j$. Then $\mathbf{x}^+ \in \mathcal{H}(\Pi, \mathbf{x})$. Moreover, $\mathbf{x}^+ = \mathbf{x}^{\Pi}$ is the top element of $\mathcal{H}(\Pi, \mathbf{x})$ whenever $\pi(i) > 0$ for any $i \in [n]$.*

Example 3.3. For $k = 1$, consider the Dirac measure δ_1 and the score vector $\mathbf{x} = (a, 0, \ldots, 0)$, $a \in [0, 1]$. Then

$$\mathbf{x}^+ = \mathbf{x} = (a, 0, \ldots, 0) \text{ but } \mathbf{x}^{\delta_1} = (a, 1, \ldots, 1).$$

Observe that $\pi(n) = 0$ in this case and that $\mathbf{x}^+$ is not a top element of $\mathcal{H}(\delta_1, \mathbf{x})$. On the other hand, considering the greatest possibility measure μ^* we have

$$\mathbf{x}^+ = \mathbf{x}^{\mu^*} = (a, \ldots, a).$$

Recall that for μ^* it holds $\pi^*(1) = \cdots = \pi^*(n) = 1 > 0$, while for δ_1 we have $\pi(1) = 1$ but $\pi(2) = \cdots = \pi(n) = 0$. □

The next characterization of possibility measures is a modification of Theorem 1 (related to the Choquet integral) and Theorem 6 (related to the Sugeno integral) from Dubois and Rico paper [2], where no integral is applied, and thus a strengthening of the results of [2].

Theorem 3.2. *Let $\mu \in \mathcal{M}_n$ be a capacity. Then the following are equivalent:*

(i) μ is a possibility measure;
(ii) there is a permutation $\sigma : [n] \to [n]$ such that for any $A \subseteq [n]$, $(\mathbf{1}_A)^{\sigma,+} \in \mathcal{H}(\mu, \mathbf{1}_A)$, where $\mathbf{1}_A$ is the characteristic function of A, $\mathbf{1}_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$, and $(\mathbf{1}_A)^{\sigma,+}$ is given by (3), i.e.,

$$(\mathbf{1}_A)^{\sigma,+}(i) = \bigvee_{j=1}^{i} \mathbf{1}_A(\sigma(j)).$$

Proof. The implication $(i) \Rightarrow (ii)$ follows from Corollary 3.1.

Suppose that (ii) is valid and denote $\pi(i) = \mu(\{i\})$. Observe that, for any $A \subseteq [n]$, $h_{\mu,\mathbf{1}_A}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \mu(A) & \text{else} \end{cases}$. Fix $k \in [n]$ and put $A_k = \{k\}$. Then

$$(\mathbf{1}_{A_k})^{\sigma,+} = \mathbf{1}_{B_k}, \text{ where } B_k = \{i \in [n] | \sigma^{-1}(k) \leq i\},$$

and hence

$$\pi(k) = \mu(A_k) = \mu(B_k) = \mu(\{i \in [n] | \sigma^{-1}(k) \leq i\}).$$

For an arbitrary $A \subseteq [n]$, $|A| > 1$, it holds

$$(\mathbf{1}_A)^{\sigma,+} = \mathbf{1}_B, \text{ where } B = \left\{ i \in [n] \,\middle|\, \bigvee_{k \in A} \sigma^{-1}(k) \le i \right\}.$$

Let $k_0 \in A$ be such that $\sigma^{-1}(k_0) = \bigvee_{k \in A} \sigma^{-1}(k)$. Then $\pi(k_0) = \bigvee_{k \in A} \pi(k)$ and $B = B_{k_0}$. It follows that

$$\mu(A) = \mu(B) = \mu(B_{k_0}) = \pi(k_0) = \bigvee_{k \in A} \pi(k),$$

i.e., μ is a possibility measure and π is the corresponding possibility distribution. □

Example 3.4.

(i) Consider the possibility measure Π on [3] from Example 3.1. For the permutation $\sigma = \sigma^{-1} = (2\ 1\ 3)$ we have:

$$\begin{array}{lllllllll} \mathbf{1}_A & (0,0,0), & (1,0,0), & (0,1,0), & (0,0,1), & (1,1,0), & (1,0,1), & (0,1,1), & (1,1,1) \\ \mathbf{1}_A{}^{\sigma,+} & (0,0,0), & (0,1,1), & (1,1,1), & (0,0,1), & (1,1,1), & (0,1,1), & (1,1,1), & (1,1,1) \end{array}.$$

In all 8 cases it holds $h_{\Pi,\mathbf{1}_A} = h_{\Pi,(\mathbf{1}_A)^{\sigma,+}}$. Note that if $\mathbf{1}_A = (e_1, e_2, e_3) \in \{0,1\}^3$ then

$$\mathbf{1}_A^{\sigma,+} = (e_2, e_1 \vee e_2, e_1 \vee e_2 \vee e_3).$$

(ii) For $n = 3$, define a capacity $\mu \in \mathcal{M}_3$, $\mu(A) = \begin{cases} 1 & \text{if } \operatorname{card} A > 1 \\ 0 & \text{otherwise} \end{cases}$. For $A = \{1\}$, $h_{\mu,\mathbf{1}_A}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$.

On the other hand, for any ternary permutation σ such that $\sigma(3) = 3$,

$$h_{\mu,(\mathbf{1}_A)^{\sigma,+}}(t) = 1 \quad \text{for all} \quad t \in [0,1].$$

Similarly, if $\sigma(3) \ne 3$, the property (ii) in Theorem 3.2 is violated. Obviously, the considered capacity μ is not a possibility measure. □

4. Necessity measures and equality of survival functions

For any capacity $\mu \in \mathcal{M}_n$, for its dual capacity $\mu^d \in \mathcal{M}_n$ it holds, for any $\mathbf{x} \in [0,1]^n$,

$$h_{\mu^d,\mathbf{x}}(t) = \mu^d(\{i \in [n] |\, x_i \ge t\}) = 1 - \mu(\{j \in [n] |\, x_j < t\}).$$

Thus $h_{\mu^d,\mathbf{x}} = h_{\mu^d,\mathbf{y}}$ for some $\mathbf{y} \in [0,1]^n$ if and only if

$$\mu(\{j \in [n] |\, x_j < t\}) = \mu(\{i \in [n] |\, y_i < t\}) \text{ for any } t \in [0,1],$$

or, equivalently,

$$\mu(\{j \in [n] |\, 1 - x_j > u\}) = \mu(\{i \in [n] |\, 1 - y_i > u\}) \text{ for any } u \in [0,1].$$

On the other hand, we have, for any $t \in]0,1]$,

$$h_{\mu,1-\mathbf{x}}(t) = \mu(\{i \,|1 - x_i \ge t\}) = \inf\{\mu(\{i \,|1 - x_i > u\}) |\, u \in [0,t[\,\}.$$

These facts prove the next result.

Proposition 4.1. *For any capacity $\mu \in \mathcal{M}_n$ and score vector $\mathbf{x} \in [0,1]^n$, it holds the next equality*

$$1 - \mathcal{H}(\mu,\mathbf{x}) = \mathcal{H}(\mu^d, 1 - \mathbf{x}), \tag{4}$$

where $1 - \mathcal{H}(\mu,\mathbf{x}) = \{1 - \mathbf{y} |\, \mathbf{y} \in \mathcal{H}(\mu,\mathbf{x})\}$.

The equality (4) allows to rewrite all results introduced in Section 3 for possibility measures for the case when necessity measures are considered.

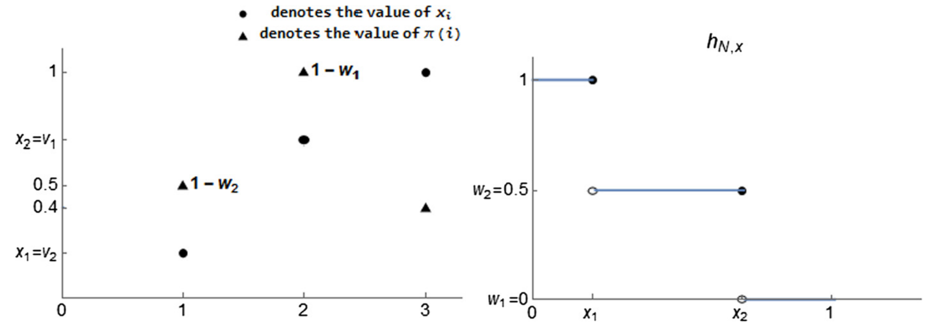

Fig. 2. Illustration of constructing $h_{N,\mathbf{x}}$.

Let $N \in \mathcal{M}_n$ be a necessity measure related to a possibility distribution π, and let $\mathbf{x} \in [0,1]^n$. Then the next results hold.

Theorem 4.1. *The survival function $h_{N,\mathbf{x}} : [0,1] \to [0,1]$ is given by*

$$h_{N,\mathbf{x}}(t) = 1 - \bigvee_{x_i < t} \pi(i) \tag{5}$$

and then $h_{N,\mathbf{x}}(t) = 0 = w_1$ for any $t \in]v_1, v_0]$, where $v_0 = 1$, $v_1 = \bigwedge_{\pi(i)=1-w_1} x_i = \bigwedge_{\pi(i)=1} x_i$. Note that if $v_1 = 1$ then $h_{N,\mathbf{x}} > 0$ for all $t \in [0,1]$.

Next, put $w_2 = 1 - \bigvee_{x_i < t_1} \pi(i)$ and $v_2 = \bigwedge_{\pi(i)=1-w_2} x_i$. Then

- *either $w_2 = 1$ and then $h_{N,\mathbf{x}}(t) = 1$ for any $t \in [0, v_1]$,*
- *or $w_2 < 1$ then $h_{N,\mathbf{x}}(t) = w_2$ for any $t \in]v_2, v_1]$.*

By induction, we have

- *$h_{N,\mathbf{x}}(t) = w_k$ for any $t \in]v_k, v_{k-1}]$ for $k = 1, \ldots, r$, and*
- *$h_{N,\mathbf{x}}(t) = 1$ for any $t \in [0, v_r]$,*

where $w_k = 1 - \bigvee_{x_i < v_{k-1}} \pi(i)$, and $\diamond$ *either $w_k = 1$, and then $r = k-1$,*
$\diamond$ *or $w_k < 1$, and then $v_k = \bigwedge_{\pi(i)=1-w_k} x_i$.*

The construction of $h_{N,\mathbf{x}}$ (based on $\pi, \mathbf{x}$ from Example 3.1) is illustrated in Fig. 2.

Theorem 4.2. *Under the notation of Theorem 4.1, $\mathbf{y} \in \mathcal{H}(N, \mathbf{x})$ if and only if $\bigwedge_{\pi(i)=1-w_j} y_i = v_j$ and $y_i \geq v_j$ whenever $\pi(i) > 1 - w_{j+1}$ for all $j = 1, \ldots, r$. The set $\mathcal{H}(N, \mathbf{x})$ is a lower semi-lattice with the bottom element $\mathbf{x}_N$ given by*

$$(x_N)_i = \begin{cases} w_j & \text{whenever } 1 - w_j \geq \pi(i) > 1 - w_{j+1}, \qquad j = 1, \ldots, r \\ 0 & \text{whenever } \pi(i) = 0, \end{cases}$$

for $i = 1, \ldots, n$.

Example 4.1.

(i) Dirac measures are simultaneously possibility and necessity measures. Thus, for any $k \in [n]$ and $\mathbf{x} \in [0,1]^n$, $\mathcal{H}(\delta_k, \mathbf{x})$ is a lattice with the top element $\mathbf{x}^{\delta_k} = (1, \ldots, x_k, 1, \ldots, 1)$ and the bottom element $\mathbf{x}_{\delta_k} = (0, \ldots, x_k, 0, \ldots, 0)$. Note that

$$\mathcal{H}(\delta_k, \mathbf{x}) = \{\mathbf{y} \in [0,1]^n \mid y_k = x_k\},$$

and it is a convex set.

(ii) The smallest capacity $\mu_* \in \mathcal{M}_n$ is a necessity measure and

$$\mathcal{H}(\mu_*, \mathbf{x}) = \left\{ \mathbf{y} \in [0,1]^n \,\middle|\, \bigwedge_{i=1}^n y_i = \bigwedge_{i=1}^n x_i \right\}.$$

Obviously, its bottom element is $\mathbf{x}_{\mu_*} = \left(\bigwedge_{i=1}^{n} x_i, \ldots, \bigwedge_{i=1}^{n} x_i\right)$, but it has no top element whenever $\mathbf{x} \neq (1, \ldots, 1)$. On the other hand, the lower semi-lattice $\mathcal{H}(\mu_*, \mathbf{x})$ has n maximal elements whenever $\mathbf{x} \neq (1, \ldots, 1)$. □

Due to Theorem 4.2, we see that for any score vector $\mathbf{x} \in [0, 1]^n$ and any necessity measure N related to a possibility distribution π, the vector $\mathbf{x}^-$ introduced by Dubois and Rico [2] by

$$x_i^- = \bigwedge_{\pi(j) \geq \pi(i)} x_j, \qquad i \in [n], \tag{6}$$

belongs to the class $\mathcal{H}(N, \mathbf{x})$.

Similarly, if $\sigma : [n] \to [n]$ is a permutation such that $\pi(\sigma(1)) \geq \pi(\sigma(2)) \geq \cdots \geq \pi(\sigma(n))$, then the vector $\mathbf{x}^{\sigma,-}$ given by

$$x_i^{\sigma,-} = \bigwedge_{j=1}^{i} x_{\sigma(j)}, \qquad i \in [n], \tag{7}$$

belongs to $\mathcal{H}(N, \mathbf{x})$.

Based on Corollary 3.1, we have also the next result.

Theorem 4.3. *Let N be a necessity measure related to a possibility distribution π such that $\pi(i) > 0$ for all $i \in [n]$. Then, for any score vector $\mathbf{x} \in [0, 1]^n$, the vector $\mathbf{x}^-$ given by (6) is the bottom element of $\mathcal{H}(N, \mathbf{x})$, i.e., $\mathbf{x}^- = \mathbf{x}_N$.*

Finally, based on Theorem 3.2, we introduce a new characterization of necessity measures.

Theorem 4.4. *Let $\mu \in \mathcal{M}_n$ be a capacity. Then the following are equivalent:*

(i) μ is a necessity measure;
(ii) there is a permutation $\sigma : [n] \to [n]$ such that for any $A \subseteq [n]$, $(\mathbf{1}_A)^{\sigma,-} \in \mathcal{H}(\mu, \mathbf{1}_A)$, where $(\mathbf{1}_A)^{\sigma,-}$ is given by (7), i.e., for $i \in [n]$

$$(\mathbf{1}_A)^{\sigma,-}(i) = \bigwedge_{j=1}^{i} \mathbf{1}_A(\sigma(j)).$$

5. Concluding remarks

We have discussed the equality of survival functions $h_{\mu,\mathbf{x}} = h_{\mu,\mathbf{y}}$ for particular capacities μ, namely for possibility and necessity measures related to a possibility distribution π. Our results generalize some results of Dubois and Rico [2] focused on the study of the equality of Choquet integrals $\mathbf{Ch}(\mu, \mathbf{x}) = \mathbf{Ch}(\mu, \mathbf{y})$ and of Sugeno integrals $\mathbf{Su}(\mu, \mathbf{x}) = \mathbf{Su}(\mu, \mathbf{y})$. Our approach, based on the equality of survival functions, ensures the equality $\mathbf{I}(\mu, \mathbf{x}) = \mathbf{I}(\mu, \mathbf{y})$ for an arbitrary universal integral $\mathbf{I}$ acting on $[0, 1]$, compare [6]. As a by-product, we have obtained an alternative characterization of possibility and necessity measures. We expect the generalization of our results to domains where the interval $[0, 1]$ is replaced by some other ordered structure, for example a bounded distributive lattice L. In particular, when L is a finite bounded chain, then possibility and necessity measure can be easily introduced and studied by means of approaches exploited in this paper, and several results introduced here can be just copied. Moreover, we aim to study some other particular classes of capacities from $\mathcal{M}_n$ in a way similar to the presented study of possibility and necessity measures.

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