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On the equality of integrals

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ABSTRACT

Considering a finite space X, several necessary conditions and one rather general sufficient condition describing when the Choquet integral coincides with the pan-integral with respect to the standard arithmetic operations are shown. These conditions are characterized by using the minimal atoms of monotone measure. Under the introduced constraints, the calculation of these coinciding two integrals is also given.

1. Introduction

The Choquet integral [4], the pan-integral [32] and the concave integral [14] are three kinds of prominent nonlinear integrals with respect to monotone measure (or capacity), see, for example [3]. All these integrals have numerous application in economy, social sciences, data fusion, multicriteria decision support, etc., see, for example, [8,10,18]. It is well known that for the σ-additive measures all the three types of integrals coincide with the Lebesgue integral (i.e., these three integrals can be seen as particular generalizations of the Lebesgue integral). All these integrals can be seen as particular instances of decomposition integrals [6] (see also [23–25]). However, in general case they are significantly different from each other [22,24,25]. Recall that the concave integral is the greatest decomposition integral, while the pan-integral and the Choquet integral are incomparable, in general [13].

In [14] the relationship between the concave integral and the Choquet integral was discussed, and the concave integral was shown to coincide with the Choquet integral if and only if the underlying monotone measure m is convex (also known as supermodular) (see also [1,16]).

Recently we discussed the relationship between the concave integral and the pan-integral on finite spaces [26]. We introduced the concept of *minimal atom* of a monotone measure. By using the characteristic of minimal atoms we presented a necessary and sufficient condition that the concave integral coincides with the pan-integral with respect to the usual arithmetic operations $+$ and$\cdot$ on finite spaces.

This paper will focus on the relationship between the Choquet integrals and pan-integrals on finite spaces. By means of minimal atoms of a monotone measure we show several necessary conditions and a sufficient condition that the Choquet

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integral coincides with the pan-integral w.r.t. the usual addition + and usual multiplication ·. This characterizes monotone measures for which the related Choquet integrals and pan-integrals coincide. Under the introduced constraints, the calculation of these two coinciding integrals is also given.

Observe that the equality of general pan-integrals and Choquet-like integrals [19] is shortly discussed in Conclusions.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X, and $(X, \mathcal{A})$ denote a measurable space. A set function $m : \mathcal{A} \to [0, +\infty[$ is called a monotone measure [2,13,30], if it satisfies the conditions: (1) $m(\emptyset) = 0$ and $m(X) > 0$; (2) $m(A) \leq m(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

A monotone measure m is said to be *superadditive* if $m(A \cup B) \geq m(A) + m(B)$ for any $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$ [5]; *supermodular* if $m(A \cup B) + m(A \cap B) \geq m(A) + m(B)$ for any $A, B \in \mathcal{A}$ [5].

The concept of a pan-integral was introduced in [32] and it involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see also [20,27,28,30,33]). In this paper we only consider the pan-integrals based on the usual addition + and multiplication ·. We present the following definition.

$\mathcal{F}_+$ denotes the class of all finite nonnegative real-valued measurable functions on $(X, \mathcal{A})$. Let m be a monotone measure and $f \in \mathcal{F}_+$.

The *pan-integral* of f on X with respect to m (based on the usual addition + and usual multiplication ·) is given by

$$\int^{pan} f dm = \sup \Big\{ \sum_{i=1}^{n} \lambda_i m(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \leq f, \\ \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a partition of } X, \lambda_i \geq 0, n \in \mathbb{N} \Big\}.$$

The *concave integral* [14] (see also [15]) of f on X is defined by

$$\int^{cav} f dm = \sup \Big\{ \sum_{i=1}^{n} \lambda_i m(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \leq f, \\ \{A_i\}_{i=1}^{n} \subset \mathcal{A}, \lambda_i \geq 0, n \in \mathbb{N} \Big\}.$$

The *Choquet integral* [4] of f on X with respect to m, is defined by

$$(C) \int f\, dm = \int_0^{\infty} m(\{x : f(x) \geqslant t\})\, dt,$$

where the right side integral is the Riemann integral.

Note that the pan-integral is related to finite partitions of X, the concave integral to any finite set systems of measurable subsets of X. The Choquet integral is based on chains of sets, it can be expressed in the following

$$(C) \int f\, dm = \sup \Big\{ \sum_{i=1}^{n} \lambda_i m(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \leq f, \\ \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a chain}, \lambda_i \geq 0, n \in \mathbb{N} \Big\}.$$

In [26] we have introduced the concept of minimal atom of a monotone measure and by using this concept we have characterized the monotone measures for which the concave integrals coincide with the pan-integrals on finite spaces. We shall see that minimal atoms play an important role also in our discussion. We recall the following definitions. Concerning more details for minimal atoms we refer to [26].

Definition 2.1. [26] Let m be a monotone measure on $(X, \mathcal{A})$. A set $A \in \mathcal{A}$ is called a *minimal atom of m* (or shortly, *m-minimal atom*), if $m(A) > 0$ and for every $B \subset A, B \in \mathcal{A},$ it holds either

(i) $m(B) = 0,$ or
(ii) $A = B.$

Obviously, a minimal atom A of m is a special atom of m (it is also pseudo-atom of m, see [11,17,29,31]). If A is a minimal atom of m, then there is no proper measurable subset B of A such that $m(B) > 0$.

Definition 2.2. [26] A monotone measure m on $(X, \mathcal{A})$ is said to have *the minimal atoms disjointness property*, if every two distinct m-minimal atoms are disjoint, i.e., for every pair of m-minimal atoms A and B, $A \neq B$ implies $A \cap B = \emptyset$.

Definition 2.3. [26] Let X be a finite set. A monotone measure m on $(X, \mathcal{A})$ is said to be *subadditive w.r.t. m-minimal atoms*, if for every set $A \in \mathcal{A}$ with $m(A) > 0$, we have

$$m(A) \leq \sum_{i=1}^{n} m(A_i),$$

where $\{A_i\}_{i=1}^{n}$ is the set of all m-minimal atoms contained in A.

Proposition 2.4. [26] *Let X be a finite set, $\mathcal{A}=2^X$ and m be a monotone measure defined on $(X,\mathcal{A})$. Then every set $E\in\mathcal{A}$ with $m(E)>0$ contains at least one minimal atom of m.*

When X is a finite set and m is a monotone measure on $(X, 2^X)$, it easily follows from the above proposition that each set $E\subset X$ with $m(E)>0$ can be expressed as

$$E=A_1\cup A_2\cup\cdots\cup A_k\cup\widetilde{A}_0, \tag{2.1}$$

where $\{A_1, A_2, \cdots, A_k\}$ is a disjoint system of some m-minimal atoms contained in E, and $m(\widetilde{A}_0)=0, \widetilde{A}_0\cap A_i=\emptyset, i=1,2,\cdots,k$.

We call the expression (2.1) as *the minimal atoms representation of E*, denoted by $E\sim(A_i)_{i=1}^k$.

3. Coincidences of the Choquet and pan-integrals on finite spaces

In the rest of the paper, consider with no loss of generality, $X=\{1,2,\cdots,n\}$ as a fixed finite space for some integer $n\in\mathbb{N}$, and let $\mathcal{M}_n$ be the class of all monotone measures on X, $m\colon 2^X\to[0,\infty[$.

For the convenience of our discussion, we denote $\mathbf{Pan}_m(f)=\int^{pan} f\,dm$, $\mathbf{Ch}_m(f)=(C)\int f\,dm$ and $\mathbf{Cav}_m(f)=\int^{cav} f\,dm$.

Our goal is to investigate monotone measures $m\in\mathcal{M}_n$ such that the related pan and Choquet integrals coincide, i.e., $\mathbf{Pan}_m(f)=\mathbf{Ch}_m(f)$ for each $f\colon X\to[0,\infty[$. Obviously, this happens whenever m is additive, i.e., if there are non-negative constants $a_1, a_2, \cdots, a_n$ such that

$$m(E)=\sum_{i\in E}a_i,\quad \forall E\in 2^X,$$

then

$$\mathbf{Pan}_m(f)=\mathbf{Ch}_m(f)=\sum_{i=1}^n a_i\cdot f(i).$$

Similarly, if m is given, for some set $B\subset X$, $B\neq\emptyset$ and $c>0$, by

$$m(E)=\begin{cases}c & \text{if } B\subset E\\ 0 & \text{else},\end{cases}$$

we have

$$\mathbf{Pan}_m(f)=\mathbf{Ch}_m(f)=c\cdot\min\{f(i)\mid i\in B\}.$$

Lemma 3.1. *Let $m\in\mathcal{M}_n$. Then $\mathbf{Pan}_m\le\mathbf{Ch}_m$ (i.e., for each $f\colon X\to[0,\infty[$, $\mathbf{Pan}_m(f)\le\mathbf{Ch}_m(f)$) if and only if m is superadditive.*

Proof. The “if” part follows directly from Theorem 10.7 in [30]. The “only if” part: Observe that $\mathbf{Ch}_m(\chi_E)=m(E)$ for any $E\subset X$ and, thus for any $A,B\subset X,\ A\cap B=\emptyset$, we have

$$\begin{aligned}m(A\cup B)&=\mathbf{Ch}_m(\chi_{A\cup B})\ge\mathbf{Pan}_m(\chi_{A\cup B})\\&=\sup\Bigg\{\sum_{i=1}^k\lambda_i\cdot m(D_i)\mid(D_i)_{i=1}^k\text{ is a disjoint system,}\\&\quad\lambda_1,\lambda_2,\cdots,\lambda_k\ge 0\text{ and }\sum_{i=1}^k\lambda_i\chi_{D_i}\le\chi_{A\cup B}\Bigg\}\\&\ge m(A)+m(B),\end{aligned}$$

i.e., m is superadditive. □

From the above result, obviously, if $\mathbf{Pan}_m\equiv\mathbf{Ch}_m$, then m is superadditive.

In the following we introduce the concept of *(M)-property* of a monotone measure. We will show that it is a stronger necessary condition for $m\in\mathcal{M}_n$ to satisfy $\mathbf{Pan}_m\equiv\mathbf{Ch}_m$. □

Definition 3.2. Let $m\in\mathcal{M}_n$. If for any $A, B\subset X, A\subset B$, there exists $C\subseteq A$ such that

$$m(C)=m(A)\quad\text{and}\quad m(B)=m(C)+m(B\setminus C), \tag{3.1}$$

then m is called to have *(M)-property*.

The *(M)-property* implies superadditivity. In fact, if m has (M)-property, then for any $A,B\subset X,\ A\cap B=\emptyset$, there is $C\subseteq A$, such that $m(C)=m(A)$ and $m(A\cup B)=m(C)+m((A\cup B)\setminus C)$. Thus, we have $m(A\cup B)=m(A)+m((A\setminus C)\cup B)\ge m(A)+m(B)$, that is, m is superadditive.

The next result shows that *(M)-property* is a necessary condition for $m\in\mathcal{M}_n$ to satisfy $\mathbf{Pan}_m\equiv\mathbf{Ch}_m$.

Lemma 3.3. *Let $m \in \mathcal{M}_n$. If* $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$, *then m has* (M)-property.

Proof. Consider $A \subset B \subset X$. Obviously, (3.1) is valid if $A = \emptyset$ or $A = B$. Suppose $A \neq \emptyset$ and $A \neq B$ and put, for $r \in]0, \infty[$, $f_r = r \cdot \chi_A + \chi_B$. Then

$$\mathbf{Ch}_m(f_r) = r \cdot m(A) + m(B)$$

and

$$\mathbf{Pan}_m(f_r) = \sum_{i=1}^{k} \lambda_i^{(r)} \cdot m(D_i^{(r)})$$

for some disjoint system $(D_i^{(r)})_{i=1}^k$ (supremum is attained due to the finiteness of X). We can split the considered disjoint system $\mathcal{D}^{(r)} = (D_i^{(r)})_{i=1}^k$ into two systems $\mathcal{D}_1^{(r)} = \{D_i^{(r)} \mid i \in \{1, 2, \cdots, k\}, D_i^{(r)} \subset A\}$ and $\mathcal{D}_2^{(r)} = \mathcal{D}^{(r)} \setminus \mathcal{D}_1^{(r)}$. Due to Lemma 3.1, m is superadditive, and thus for $C^{(r)} = \bigcup_{D_i^{(r)} \in \mathcal{D}_1^{(r)}} D_i^{(r)}$ we have $m(C^{(r)}) \geq \sum_{D_i^{(r)} \in \mathcal{D}_1^{(r)}} m(D_i^{(r)})$, and evidently,

$$(r+1) \cdot \chi_{C^{(r)}} \geq \sum_{D_i^{(r)} \in \mathcal{D}_1^{(r)}} \lambda_i^{(r)} \cdot \chi_{D_i^{(r)}}.$$

Similarly, noting that $B \setminus C^{(r)} = \bigcup_{D_i^{(r)} \in \mathcal{D}_2^{(r)}} D_i^{(r)}$, we have

$$m(B \setminus C^{(r)}) \geq \sum_{D_i^{(r)} \in \mathcal{D}_2^{(r)}} m(D_i^{(r)})$$

and

$$\chi_{B \setminus C^{(r)}} \geq \sum_{D_i^{(r)} \in \mathcal{D}_2^{(r)}} \lambda_i^{(r)} \cdot \chi_{D_i^{(r)}}.$$

Consequently,

$$\mathbf{Pan}_m(f_r) = (r+1) \cdot m(C^{(r)}) + m(B \setminus C^{(r)}).$$

There are only finitely many subsets of A, and each $C^{(r)}$, $r \in]0, \infty[$, is a subset of A. Thus there is an $r_0 \in]0, \infty[$ such that $G = \{r \in]0, \infty[\mid C^{(r)} = C^{(r_0)}\}$ is not a singleton. Denote $C = C^{(r_0)}$. For each $r \in G$ it holds

$$\begin{aligned} \mathbf{Pan}_m(f_r) &= (r+1) \cdot m(C^{(r)}) + m(B \setminus C^{(r)}) \\ &= \mathbf{Ch}_m(f_r) = r \cdot m(A) + m(B), \end{aligned}$$

i.e.,

$$r \cdot (m(A) - m(C)) = m(C) + m(B \setminus C) - m(B).$$

Now, it is evident that this equality can hold for each $r \in G$ only if $m(A) = m(C)$ and $m(B) = m(C) + m(B \setminus C)$. □

In the following we use the characteristics of m-minimal atoms to present necessary conditions for $m \in \mathcal{M}_n$ to satisfy $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$. □

Theorem 3.4. *Let $m \in \mathcal{M}_n$. Then* $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$ *only if for any two m-minimal atoms E_1 and E_2, it holds:*

(i) *if $E_1 \cap E_2 = \emptyset$, then $m(E_1 \cup E_2) = m(E_1) + m(E_2)$;*
(ii) *if $E_1 \cap E_2 \neq \emptyset$, then $m(E_1 \cup E_2) = m(E_1) = m(E_2)$. Moreover,*
(iii) *If $E \subset X$ and $m(E) > 0$, then for any minimal atoms representation of E, $E \sim (A_i)_{i=1}^k$, it holds*

$$m(E) = \sum_{i=1}^{k} m(A_i).$$

Proof. Based on Lemma 3.3, and the fact that the only subset C of E_1 such that $m(C) = m(E_1)$ is $C = E_1$ (similarly for E_2), it holds $m(E_1 \cup E_2) = m(E_1) + m(E_2 \setminus E_1) = m(E_2) + m(E_1 \setminus E_2)$. Thus

(i) if $E_1 \cap E_2 = \emptyset$, clearly $m(E_1 \cup E_2) = m(E_1) + m(E_2)$;
(ii) if $E_1 \cap E_2 \neq \emptyset$, then $E_2 \setminus E_1 \neq E_2$ and hence $m(E_1 \cup E_2) = m(E_1)$. Similarly, it holds $m(E_1 \cup E_2) = m(E_2)$.
(iii) For $E \subset X$ with $m(E) > 0$, let $E \sim (A_i)_{i=1}^k$ be minimal atoms representation of E, i.e.,

$$E = A_1 \cup A_2 \cup \cdots \cup A_k \cup \widetilde{A}_0,$$

where $\{A_1, A_2, \cdots, A_k\}$ is a disjoint system of some m-minimal atoms contained in E and $\mu(\widetilde{A}_0) = 0, \widetilde{A}_0 \cap A_i = \emptyset, i = 1, 2, \cdots, k$. It follows from (M)-property that for any $B \subset X$, if A is m-minimal atom contained in B, then $m(B) = m(A) + m(B - A)$. Therefore,

$$m(E) = m\big(A_1 \cup A_2 \cup \cdots \cup A_k \cup \widetilde{A}_0\big)$$

$$= m(A_1) + m\big(A_2 \cup \cdots \cup A_k \cup \widetilde{A}_0\big)$$
$$= m(A_1) + m(A_2) + m\big(A_3 \cup \cdots \cup A_k \cup \widetilde{A}_0\big)$$
$$= \cdots$$
$$= m(A_1) + m(A_2) + \cdots + m(A_k) + m(\widetilde{A}_0)$$
$$= m(A_1) + m(A_2) + \cdots + m(A_k) \quad \square$$

Note 3.5. It is easy to see that the condition (iii) in the above Theorem 3.4 is equivalent to the following condition:
(iii)′ For any $E \subset X$ with $m(E) > 0$,

$$m(E) = \max\left\{\sum_{i=1}^{s} m(C_i) \mid (C_i)_{i=1}^{s} \text{ is a disjoint system of some } m\text{-minimal atoms contained in } E\right\}.$$

Observe that in Theorem 3.4 we only concerned the characteristics of two m-minimal atoms. For 3 different m-minimal atoms such that $E_1 \cap E_2 \neq \emptyset$ and $E_2 \cap E_3 \neq \emptyset$, necessarily $m(E_1) = m(E_2) = m(E_3)$. However, neither $E_1 \cap E_3 \neq \emptyset$ nor $m(E_1 \cup E_2 \cup E_3) = m(E_1)$ should hold.

Example 3.6. Let $X = \{1, 2, 3, 4\}$. The monotone measure m: $2^X \to [0, \infty[$ is defined by

$$m(E) = \begin{cases} 2 & \text{if } E = X \\ 1 & \text{if } |E| = 2 \text{ or } 3 \\ 0 & \text{else,} \end{cases}$$

where $|E|$ stands for the cardinality of E.
Suppose that f is an arbitrary non-negative function on X,

$$f(x) = \begin{cases} a_1 & x = 1 \\ a_2 & x = 2 \\ a_3 & x = 3 \\ a_4 & x = 4. \end{cases}$$

We can assume that $a_1 \geq a_2 \geq a_3 \geq a_4$ without loss of generality. Thus we have $\mathbf{Ch}_m(f) = a_2 + a_4 = \mathbf{Pan}_m(f)$.
On the other hand, {1, 2}, {2, 3} and {3, 4} are 3 different m-minimal atoms. $\{1, 2\} \cap \{2, 3\} \neq \emptyset$, $\{2, 3\} \cap \{3, 4\} \neq \emptyset$, But $\{1, 2\} \cap \{3, 4\} = \emptyset$. Also, $m(\{1, 2\} \cup \{2, 3\} \cup \{3, 4\}) = m(X) = 2 \neq m(\{1, 2\})$ (or $m(\{2, 3\})$, or $m(\{3, 4\})$).

To further investigate the condition for $m \in \mathcal{M}_n$ to satisfy $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$, we need to consider the case of more than two atoms. To this end, we introduce a concept related to m-minimal atoms. We are ready to state a sufficient condition for $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$.

Definition 3.7. Let $m \in \mathcal{M}_n$. We say that m has *minimal atoms partitionable property*, if the following conditions are satisfied: the set $\mathcal{E} = \{E_1, E_2, \cdots, E_k\}$ of all m-minimal atoms can be partitioned into

$$\{\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_p\},$$

where $\mathcal{E}_i = \{E_{i_1}, E_{i_2}, \cdots, E_{i_{k_i}}\}, i = 1, 2, \cdots, p$, are such that

(i) for each i $(i = 1, 2, \cdots, p)$, if $E_{i_j}, E_{i_r} \in \mathcal{E}_i$, then $E_{i_j} \cap E_{i_r} \neq \emptyset$ and $m(E_{i_1}) = m(E_{i_2}) = \cdots = m(E_{i_{k_i}})$, denoted by a_i;
(ii) if $E_{i_j} \in \mathcal{E}_i$ and $E_{t_r} \in \mathcal{E}_t$ for $i \neq t$, then $E_{i_j} \cap E_{t_r} = \emptyset$, and
(iii) for any $E \subset X$ with $\mu(E) > 0$,

$$m(E) = \max\left\{\sum_{l=1}^{s} m(A_l) \mid (A_l)_{l=1}^{s} \text{ is a disjoint system of some } m\text{-minimal atoms contained in } E\right\}.$$

Now, we evaluate the pan-integral and Choquet integral when considering $m \in \mathcal{M}_n$ characterized in Definition 3.7. Observe first that both Choquet integral and pan-integral are positively homogeneous (i.e., for every function f and every non-negative constant $a \geq 0$, the equalities $\mathbf{Ch}_m(a \cdot f) = a \cdot \mathbf{Ch}_m(f)$ and $\mathbf{Pan}_m(a \cdot f) = a \cdot \mathbf{Pan}_m(f)$ hold). Moreover, the Choquet integral is additive in measure (i.e., for any $m_1, m_2 \in \mathcal{M}_n$ and for every function f, $\mathbf{Ch}_{m_1+m_2}(f) = \mathbf{Ch}_{m_1}(f) + \mathbf{Ch}_{m_2}(f)$). Considering the pan-integral, if for any $E \subset X$,

$$m(E) = \sum_{i=1}^{p} m(E \cap G_i)$$

for some disjoint system $\{G_1, G_2, \cdots, G_p\}$, then

$$\mathbf{Pan}_m(f) = \sum_{i=1}^{p} \mathbf{Pan}_m(f \cdot \chi_{G_i}). \tag{3.2}$$

Proposition 3.8. *Let $m \in \mathcal{M}_n$ satisfy the minimal atoms partitionable property (i.e., the conditions (i),(ii) and (iii) introduced in Definition 3.7). Then, for each function f: $X \to [0, \infty[$,*

$$\mathbf{Pan}_m(f) = \sum_{i=1}^{p} a_i \cdot \max\left\{ \min\{f(j) \mid j \in E_t\} \;\mid\; E_t \in \mathcal{E}_i \right\} \tag{3.3}$$

Proof. Denote $G_i = \bigcup_{E_t \in \mathcal{E}_i} E_t,\ i = 1, 2, \cdots, p$. Due to Definition 3.7, $\{G_1, G_2, \cdots, G_p\}$ is a disjoint system of subsets of X. Moreover,

$$\begin{aligned}\mathbf{Pan}_m(f \cdot \chi_{G_i}) &= \max\left\{ \min\{f(j) \mid j \in E_t\} \cdot m(E_t) \;\mid\; E_t \subset G_i \right\} \\ &= a_i \cdot \max\left\{ \min\{f(j) \mid j \in E_t\} \;\mid\; E_t \in \mathcal{E}_i \right\}\end{aligned}$$

Now, the result (3.3) follows from (3.2). □

Proposition 3.9. *Let $m \in \mathcal{M}_n$ satisfy the minimal atoms partitionable property. Then, for each function f: $X \to [0, \infty[$,*

$$\mathbf{Ch}_m(f) = \sum_{i=1}^{p} a_i \cdot \max\left\{ \min\{f(j) \mid j \in E_t\} \;\mid\; E_t \in \mathcal{E}_i \right\} \tag{3.4}$$

Proof. For $i = 1, 2, \cdots, p$, define $m_i \in \mathcal{M}_n$ by

$$m_i(E) = m(E \cap G_i) = \begin{cases} a_i & \text{if } \ E_t \subset E \ \text{ for some } \ E_t \in \mathcal{E}_i, \\ 0 & \text{otherwise,} \end{cases}$$

where G_i was introduced in the proof of Proposition 3.8. Obviously, $m = \sum_{i=1}^{p} m_i$. Moreover, $\frac{m_i}{a_i}$ is a $\{0, 1\}$-valued monotone measure and thus this Choquet integral is a lattice polynomial (see [9]).

$$\begin{aligned}\mathbf{Ch}_{\frac{mi}{a_i}}(f) &= \max\left\{ \min\{f(j) \mid j \in E\} \;\mid\; m_i(E) = a_i \right\} \\ &= \max\left\{ \min\{f(j) \mid j \in E_t\} \;\mid\; E_t \in \mathcal{E}_i \right\}.\end{aligned}$$

Now, the result follows

$$\begin{aligned}\mathbf{Ch}_m(f) &= \sum_{i=1}^{p} \mathbf{Ch}_{m_i}(f) \\ &= \sum_{i=1}^{p} a_i \cdot \mathbf{Ch}_{\frac{mi}{a_i}}(f) \\ &= \sum_{i=1}^{p} a_i \cdot \max\left\{ \min\{f(j) \mid j \in E_t\} \;\mid\; E_t \in \mathcal{E}_i \right\}. \quad \square\end{aligned}$$

Summarizing Propositions 3.8 and 3.9, we obtain a sufficient condition for $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$.

Theorem 3.10. *Let $m \in \mathcal{M}_n$. If m has minimal atoms partitionable property, then $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$, and moreover, for each function f: $X \to [0, \infty[$,*

$$\begin{aligned}\mathbf{Ch}_m(f) &= \mathbf{Pan}_m(f) \\ &= \sum_{i=1}^{p} a_i \cdot \max\left\{ \min\{f(j) \mid j \in E_t\} \;\mid\; E_t \in \mathcal{E}_i \right\}.\end{aligned}$$

The following example illustrates Theorem 3.10.

Example 3.11. Let $X = \{1, 2, 3, 4, 5\}$ and let $m \in \mathcal{M}_5$ be defined as

$$m(E) = \begin{cases} 1 & \text{if } \ E \supset \{1, 2\} \text{ and } E \not\supset \{4, 5\}, \\ 1 & \text{if } \ E \supset \{2, 3\} \text{ and } E \not\supset \{4, 5\}, \\ 2 & \text{if } \ E \supset \{4, 5\} \text{ and } E \not\supset \{1, 2\}, E \not\supset \{2, 3\}, \\ 3 & \text{if } \ E = \{1, 2, 4, 5\} \text{ or } \{2, 3, 4, 5\} \text{ or } X, \\ 0 & \text{otherwise.} \end{cases}$$

Then {1, 2}, {2, 3} and {4, 5} are all m-minimal atoms of m, and m has minimal atoms partitionable property with

$$\mathcal{E}_1 = \{\{1,2\},\{2,3\}\}, \mathcal{E}_2 = \{\{4,5\}\}, p = 2, a_1 = 1, a_2 = 2.$$

Thus, by Theorem 3.10, for each function f: $X \to [0, \infty[$, we have that

$$\begin{aligned}\mathbf{Ch}_m(f) &= \mathbf{Pan}_m(f) \\ &= \sum_{i=1}^{2} a_i \cdot \max\left\{ \min\{f(j) \mid j \in E_t\} \ \mid \ E_t \in \mathcal{E}_i \right\} \\ &= 1 \cdot \max\{\min\{f(1), f(2)\}, \min\{f(2), f(3)\}\} + 2 \cdot \min\{f(4), f(5)\}.\end{aligned}$$

Remark 3.12. The converse of the above theorem may not be true, that is, the minimal atoms partitionable property of m is a sufficient condition for $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$, but it is not necessary. As shown in Example 3.6, introducing a monotone measure m which has not the minimal atoms partitionable property, but still $\mathbf{Pan}_m \equiv \mathbf{Ch}_m$.

4. The equality of the Choquet, pan and concave integrals

Recall that Lehrer in [14] has characterized all monotone measures $m \in \mathcal{M}_n$ for which the Choquet and concave integral coincide.

Proposition 4.1. *[14] Let $m \in \mathcal{M}_n$. Then $\mathbf{Cav}_m \equiv \mathbf{Ch}_m$ if and only if m is supermodular, i.e., for any A, $B \subset X$ it holds*

$$m(A \cup B) + m(A \cap B) \geq m(A) + m(B).$$

Recently, we have characterized in [26] the conditions on $m \in \mathcal{M}_n$ when the concave and pan-integrals coincide.

Proposition 4.2. *Let $m \in \mathcal{M}_n$. Then $\mathbf{Cav}_m \equiv \mathbf{Pan}_m$ if and only if the following two conditions holds:*

(i) *m possesses the m-minimal atoms disjointness property, i.e., any pair of different m-minimal atoms (E_i, E_j) is disjoint;*
(ii) *m is subadditive w.r.t. m-minimal atoms, i.e., for every set $A \in \mathcal{A}$ with $m(A) > 0$, we have*

$$m(A) \leq \sum_{i=1}^{s} m(A_i),$$

where $\{A_i\}_{i=1}^{s}$ is the set of all m-minimal atoms contained in A.

Based on Theorem 3.10, Propositions 4.1 and 4.2, the next result is immediate.

Corollary 4.3. *Let $m \in \mathcal{M}_n$. Then, for any f: $X \to [0, \infty[$,*

$$\mathbf{Ch}_m(f) = \mathbf{Pan}_m(f) = \mathbf{Cav}_m(f)$$

if and only if the system $\mathcal{E} = \{E_1, E_2, \cdots, E_k\}$ of all m-minimal atoms is disjoint, and for any $E \subset X$ with $\mu(E) > 0$,

$$m(E) = \sum_{E_i \subset E} m(E_i), \tag{4.1}$$

and then

$$\begin{aligned}\mathbf{Ch}_m(f) = \mathbf{Pan}_m(f) &= \mathbf{Cav}_m(f) \\ &= \sum_{i=1}^{k} a_i \cdot \min\{f(j) \mid j \in E_i\},\end{aligned}$$

where $a_i = m(E_i), i = 1, 2, \cdots, k$.

The following examples illustrate the validity of Corollary 4.3.

Example 4.4. Let $X = \{1, 2, 3, 4\}$ and let $m \in \mathcal{M}_4$ be given by

$$m(E) = \begin{cases} \frac{1}{3} & \text{if } \ E = \{1,2\}, \{1,2,3\}, \{1,2,4\}, \\ \frac{2}{3} & \text{if } \ E = \{3,4\}, \{1,3,4\}, \{2,3,4\}, \\ 1 & \text{if } \ E = X, \\ 0 & \text{otherwise.} \end{cases}$$

Then m has two minimal atoms, namely, {1, 2} and {3, 4}, and it satisfies the constraints of Corollary 4.3, Therefore, noting that $m(\{1,2\}) = \frac{1}{3}$ and $m(\{3,4\}) = \frac{2}{3}$, we have

$$\begin{aligned}&\mathbf{Ch}_m(f) = \mathbf{Pan}_m(f) = \mathbf{Cav}_m(f) \\ &= \frac{1}{3} \cdot \min\{f(1), f(2)\} + \frac{2}{3} \cdot \min\{f(3), f(4)\}.\end{aligned}$$

Example 4.5. Let $n = 3$, and identify $f\colon X \to [0, \infty[$ by a ternary vector $(x, y, z) \in [0, \infty[^3$.

(1) Define

$$m(E) = \begin{cases} 1 & \text{if } |E| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then m has minimal atoms partitionable property with

$$\mathcal{E}_1 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}, \ \ p = 1 \text{ and } a_1 = 1.$$

Note that due to Theorem 3.10, it holds

$$\begin{aligned} \mathbf{Ch}_m(x, y, z) &= \mathbf{Pan}_m(x, y, z) \\ &= 1 \cdot \max\{\min\{x, y\}, \min\{x, z\}, \min\{y, z\}\} \\ &= med(x, y, z), \end{aligned}$$

i.e., the standard median is recovered. However, m-minimal atoms are not disjoint, thus neither Proposition 4.2 nor Corollary 4.3 can be applied. Indeed, $\mathbf{Cav}_m(1, 1, 1) = \frac{3}{2} > med(1, 1, 1) = 1$.

(2) Define

$$m(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

and $f(j) = 1, \ \forall j$. Then $\mathbf{Cav}_m(f) = \mathbf{Pan}_m(f) = 3$. But $\mathbf{Ch}_m(f) = 1$.

(3) Let $\mathbf{Cav}_m \equiv \mathbf{Ch}_m$. Then m is supermodular. Define

$$m(A) = \begin{cases} |A| & \text{if } |A| < 3, \\ 4 & \text{if } A = X. \end{cases}$$

Let $f(1) = f(2) = 2, \ f(3) = 3$. Then

$$\mathbf{Cav}_m(f) = \mathbf{Ch}_m(f) = 2 \times 4 + 1 \times 2 = 10,$$

but $\mathbf{Pan}_m(f) = 2 \times 4 = 8$.
Observe that m-minimal atoms are the singletons of X and hence they are disjoint. However, neither Proposition 4.2 nor Corollary 4.3 can be applied.

Remark 4.6. Each $m \in \mathcal{M}_n$ characterized by (4.1) can be seen as a multiple of a lower probability in the sense of de Finetti [7], compare also [21]). Then there is another evaluation of the discussed integrals, namely,

$$\begin{aligned} &\mathbf{Ch}_m(f) = \mathbf{Pan}_m(f) = \mathbf{Cav}_m(f) \\ &= \inf\Big\{ \int_X f d\mu \mid \mu \text{ is an additive measure on } X, \text{ such that} \\ &\mu(E_i) = m(E_i), i = 1, 2, \cdots, k \text{ and } \mu(X \setminus \bigcup_{i=1}^{k} E_i) = 0 \Big\}, \end{aligned}$$

where $\int_X f d\mu$ is the standard Lebesgue integral. Observe that this approach is exemplified in Example 4.4, where the monotone measure m is a lower probability in the sense of de Finetti [7] related to a probability measure p defined on an algebra of subsets of X generated by atoms {1, 2} and {3, 4}, where $p(\{1, 2\}) = \frac{1}{3}$ and $p(\{3, 4\}) = \frac{2}{3}$.

5. Conclusions

We have shown several necessary conditions and a sufficient condition for which the Choquet integral coincides with the pan-integral on finite spaces. Such conditions were characterized by minimal atoms of monotone measure (Theorems 3.4 and 3.10). Observe that in multicriteria decision support, as well as in the game theory, the disjointness of considered groups of criteria (of players) is rather often considered, which when evaluating optimal expected value based on a monotone measure yields the pan integral. Our results contribute to the effective computation of pan-integral in particular cases, when it coincides with the related Choquet integral. This is due to the fact that we have several evaluations formulas for the discrete Choquet integral, see, e.g., [8], what is not the case of discrete pan-integrals.

As we have seen, the *minimal atoms partitionable property* is a sufficient condition for $\mathbf{Cav}_m \equiv \mathbf{Pan}_m$, but it is not necessary (Theorem 3.10, Remark 3.12 and Example 3.6). We have also obtained three necessary conditions for $\mathbf{Ch}_m \equiv \mathbf{Pan}_m$ by using the characteristic of minimal atoms of monotone measure in Theorem 3.4 (the conditions (i), (ii) and (iii) in Theorem 3.4). However, we do not know whether this set of conditions is sufficient for $\mathbf{Ch}_m \equiv \mathbf{Pan}_m$.

On the other hand, in [26] we proved a necessary and sufficient condition ensuring that the concave integral coincides with the pan-integral on finite spaces (Proposition 4.2; see also Theorem 4.1 in [26]). Lehrer in [14] has characterized all monotone measures $m \in \mathcal{M}_n$ for which the Choquet and concave integral coincide (Proposition 4.1). These results were summarized in Corollary 4.3 stating a necessary and sufficient condition for the equality $\mathbf{Ch}_m = \mathbf{Pan}_m = \mathbf{Cav}_m$ of the three discussed integrals.

In our further research, we will try to find necessary and sufficient condition characterized by minimal atoms of monotone measure on finite spaces such that the Choquet integral coincides with the Pan-integral.

Observe that for a general pan-integral based on results of Mesiar and Rybárik [20], each pan-integral is either an isomorphic transform of the $(+,\cdot)$-based pan-integral, or it is based on $(\vee, \otimes)$ semiring and then it coincides with the smallest universal integral [12] based on the pseudo-multiplication $\otimes$. Then, in both cases, we have variants of Theorems 3.4 and 3.10, Propositions 4.1 and 4.2, and Corollary 4.3 relating the pan-integrals, Choquet-like integral [19] and pseudo-concave integral [22,23], replacing the standard addition + by a pseudo-addition $\oplus$ whenever + appears in the characterization of the appropriate monotone measures.

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