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Generalized expectation with general kernels on g-semirings and its applications

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Abstract The connection between probability and g-integral is investigated. The purposes of this paper are mainly to introduce the concept g-expectation with general kernels on a g-semiring, and then extend the Jensen type inequality in general form, thus refining the previous results in probability and measure theory.

Keywords Probability theory · g-expectation · Fractional integral · Jensen's inequality

Mathematics Subject Classification 60E15 · 28C99 · 60A99

1 Introduction

The theory of pseudo-analysis, as a generalization of the classical analysis, has obtained a growing interest in many areas such as probability and statistics, measure theory, partial differential equations, optimization, control theory, decision making, knowledge based systems [3,8,9,17,22–24,27,28,32,34,35]. For example, in 2013, Bede and O'Regan [3] proposed the theory of pseudo-linear operators which advances the theory of aggregation operators in knowledge based systems. Some other applications of pseudo-analysis can be found in [8,9,17,22–24,27,28,32,34,35].

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Inequalities are powerful tools in many areas of mathematics, especially in information sciences, engineering, probability theory and economics. For example, in the expression of the relationship between convergence concepts in probability theory, we always use probabilistic inequalities. Therefore, inequalities should be analyzed and considered when using the theory of pseudo-analysis. Recently, there were obtained generalizations of inequalities based on the theory of pseudo-analysis [1,2,10,29–31]. The classical Jensen inequality [16] is one of the most important inequalities for convex functions in mathematics, especially in mathematical engineering, information sciences, probability theory and stochastic processes.

Theorem 1.1 *Let $I \subset \mathbb{R}$ be a real interval and $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $X : \Omega \to I$ be a μ-integrable function. Then the classical Jensen inequality*

$$\frac{1}{\mu(\Omega)} \int_{\Omega} \Psi \circ X d\mu \geqslant \Psi \left(\frac{1}{\mu(\Omega)} \int_{\Omega} X d\mu \right)$$

holds for every convex function Ψ on I.

In particular, if $\mu = \mathbf{P}$, $\mathbf{P}$ is the probability measure, then the classical Jensen inequality

$$\mathbb{E}[\Psi(X)] \geq \Psi(\mathbb{E}[X]) \tag{1.1}$$

holds.

The study of this inequality is important in many fields, such as pseudo-analysis [29, 31], probability theory and statistics [18,19], generalized measure theory [30], multivariate analysis [14], stochastic processes [7], Markov diffusion processes [15], information theory [11] and etc. In 2010, generalizations of the Jensen integral inequality for pseudo-integral on two cases of the real semiring with pseudo-operations was proposed by Pap and Štajner in [29]. In probability theory and statistics, Jensen's inequality for medians and for multivariate medians was proposed by Merkle [18,19]. In 2014, Terán [33] extended Jensen's inequality to metric spaces endowed with a convex combination operation. He also proposed some applications of this inequality for both random elements and random sets. In multivariate analysis, a refined Jensen's inequality in Hilbert spaces and empirical approximations were proved by [14]. In generalized measure context, Román-Flores et al. [30] studied the Jensen type inequality for Sugeno integral. Some new generalizations of Jensen type inequality for generalized Sugeno integral can be found as the result of Kaluszka et al. [10] in 2014. In matrix-valued measures, Farenick and Zhou obtained a Jensen's inequality relative to matrix-valued measures [6]. In Markov diffusion process, Lerner [15] proved a Jensen's inequality for the entropy functional of a Markov diffusion process.

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two finite measure spaces. Define two operators

$$A_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1) := \frac{1}{K_{\mathbf{id}}^{\Omega_2}(\omega_1)} \int_{\Omega_2} k(\omega_1, \omega_2) X(\omega_2) d\mu_2(\omega_2), \tag{1.2}$$

$$\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1) := \int_{\Omega_2} k(\omega_1, \omega_2) X(\omega_2) d\mu_2(\omega_2), \tag{1.3}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}^+$ is measurable and non-negative kernel, X is measurable function on Ω_2,

$$K_{\mathbf{id}}^{\Omega_2}(\omega_1) := \int_{\Omega_2} k(\omega_1, \omega_2) d\mu_2(\omega_2), \quad \omega_1 \in \Omega_1.$$

In the following theorems, we propose general versions of Jensen inequality in probability and measure theory.

Theorem 1.2 *Let* $(\Omega_1, \mathcal{F}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{F}_2, \mu_2)$ *be two finite measure spaces and* I *be a finite or infinite open interval and let* Ψ *be a differentiable function on a finite or infinite open interval* I *containing zero. Let* $X : \Omega_2 \to I$ *be a random variable such that* $A_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1) \in I$ *for each fixed* $\omega_1 \in \Omega_1$*. If* Ψ *is convex, then inequality*

$$A_{\mathbf{id}}^{k,\Omega_2}[\Psi(X)](\omega_1) \geqslant \Psi\left[A_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1)\right] \frac{K_{\mathbf{id}}^{G}(\omega_1)}{K_{\mathbf{id}}^{\Omega_2}(\omega_1)} + \frac{K_{\mathbf{id}}^{G^c}(\omega_1)}{K_{\mathbf{id}}^{\Omega_2}(\omega_1)} \left(A_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1)\Psi'\left(A_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1)\right) + \Psi(0)\right)$$

holds for each fixed $\omega_1 \in \Omega_1$ *where* $G = \{\omega_2 \in \Omega_2 : X(\omega_2) \neq 0\}$ *and* $A_{\mathbf{id}}^{k,\Omega_2}$ *is defined by (1.2).*
In particular, taking $k(\omega_1, \omega_2) \equiv 1$*, we get the inequality*

$$\int_{\Omega_2} \Psi(X)\, d\mu_2 \geqslant \Psi\left(\frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} X d\mu_2\right) \mu_2(\{\omega_2 : X(\omega_2) \neq 0\}) + \left[\left(\frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} X d\mu_2\right) \Psi'\left(\frac{1}{\mu_2(\Omega)} \int_{\Omega_2} X d\mu_2\right) + \Psi(0)\right] \mu_2(\{\omega_2 : X(\omega_2) = 0\}). \tag{1.4}$$

Proof See Appendix A, proof of Theorem 1.2.

Remark 1.3 (I) If $\mu_2(\{\omega_2 : X(\omega_2) = 0\}) = 0$, then (1.4) reduces to Jensen's inequality

$$\frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} \Psi(X)\, d\mu_2 \geqslant \Psi\left(\frac{1}{\mu_2(\Omega_2)} \int_{\Omega_2} X d\mu_2\right).$$

(II) If $\mu_2 = \mathbf{P}$, $\mathbf{P}$ is the probability measure, and $k(\omega_1, \omega_2) \equiv 1$ in Theorem 1.2, then we get

$$\mathbb{E}[\Psi(X)] \geq \Psi(\mathbb{E}[X])\, \mathbf{P}(X \neq 0) + \big(\mathbb{E}[X]\, \Psi'(\mathbb{E}[X]) + \Psi(0)\big)\, \mathbf{P}(X = 0), \tag{1.5}$$

and if $\mathbf{P}(X = 0) = 0$, we get the classical Jensen inequality. If $\mathbf{P}(X = 0) > 0$ and

$$\Psi(0) \geq \Psi(\mathbb{E}[X]) - \mathbb{E}[X]\, \Psi'(\mathbb{E}[X])$$

for every convex and differentiable function Ψ, then inequality (1.5) is a refinement of the classical Jensen inequality.

In a similar way we can prove the following theorem.

Theorem 1.4 *Let* $(\Omega_1, \mathcal{F}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{F}_2, \mu_2)$ *be two finite measure spaces and* I *be a finite or infinite open interval and let* Ψ *be a differentiable function on a finite or infinite open* *interval I containing zero. Let $X : \Omega_2 \to I$ be a random variable such that $\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1) \in I$ for each fixed $\omega_1 \in \Omega_1$. If Ψ is convex, then inequality*

$$\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}(\Psi(X)) \geqslant \Psi\left(\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}[X]\right)\mathfrak{A}_{\mathbf{id}}^{k,G}[1] + \Psi'\left(\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}[X]\right)\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}[X]\left(1-\mathfrak{A}_{\mathbf{id}}^{k,G}[1]\right) + \Psi(0)\,\mathfrak{A}_{\mathbf{id}}^{k,G^c}[1].$$

holds for each fixed $\omega_1 \in \Omega_1$ where $G = \{\omega_2 \in \Omega_2 : X(\omega_2) \neq 0\}$ and $\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}$ is defined by (1.3).

Proof See Appendix A, proof of Theorem 1.4.

The purposes of this paper are mainly to introduce the concept of pseudo-expectation with general kernels, and then extend the Jensen type inequality in general form, thus generalizing and improving the previous results in literature [29].

The rest of the paper is organized as follows. Some notions and definitions that are useful in this paper are given in Sect. 2. In this section, we also introduce the concept of pseudo-expectation with general kernels in Definition 2.4. In next section, we establish some refinements of Jensen's inequality in general form. Finally, some concluding remarks are given.

2 Pseudo-expectation with general kernels

In this section, we first recall some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals [1,2,23,29]. Then we introduce the concept of pseudo-expectation with general kernels in Definition 2.4.

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 2.1 A binary operation $\oplus$ on $[a, b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to $\preceq$), continuous, associative, and with a zero (neutral) element different from b and denoted by $\mathbf{0}$. Let $[a, b]_+ = \{x \mid x \in [a, b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a, b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and with a unit element $\mathbf{1} \in [a, b]_+$, i.e., for each $x \in [a, b]$, $\mathbf{1} \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a, b], \oplus, \odot)$ is a *semiring* (see [13]).

Let Ω be a non-empty set. Let $\mathcal{A}$ be a σ-algebra of subsets of a set Ω.

Definition 2.2 [27] A set function $m : \mathcal{A} \to [a, b]_+$ (or semiclosed interval) is a $\oplus$-measure if there holds:

(i) $m(\phi) = \mathbf{0}$ (if $\oplus$ is not idempotent);
(ii) m is σ-$\oplus$-(decomposable) measure, i.e.

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$$

holds for any sequence $\{A_i\}_{i\in N}$ of pairwise disjoint sets from $\mathcal{A}$. If $\oplus$ is idempotent operation condition (i) can be left out and sets from sequence $\{A_i\}$ do not have to be pairwise disjoint.

We consider an important case of pseudo-integrals.

Definition 2.3 First case of pseudo-integrals is when pseudo-operations are generated by a monotone bijection $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}\left(g(x) + g(y)\right), \quad x \odot y = g^{-1}\left(g(x)\, g(y)\right).$$

Then the pseudo-integral for a function $X : \Omega \to [a, b]$ reduces to the g-integral [23,26],

$$\mathfrak{A}^{\Omega}_{\oplus,\odot}[X] := \int_{\Omega}^{\oplus} X \odot dm = g^{-1}\left(\int_{\Omega} (g \circ X)\, d(g \circ m)\right),$$

where the integral applied on the right side is the standard Lebesgue integral. In special case, when $m = g^{-1} \circ \mu$, μ is the standard Lebesgue measure, then we obtain

$$\int_{\Omega}^{\oplus} X \odot dm = g^{-1}\left(\int_{\Omega} g(X(\omega))\, d\mu(\omega)\right).$$

When $m = g^{-1} \circ \mathbf{P}$, $\mathbf{P}$ is the probability measure, then

$$\mathbb{E}^{\Omega}_{\oplus}[X] := g^{-1}\left(\int_{\Omega} (g \circ X)\, d\mathbf{P}\right) = g^{-1}\left(\mathbb{E}[g(X)]\right).$$

More on this structure as well as on corresponding measures and integrals can be found in [23,26].

Definition 2.4 Let a generator g be the same as in Definition 2.3. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces and $X : \Omega_2 \to [a, b]$ be a measurable function. Then for any σ-$\oplus$-measure μ_2 and for each fixed $\omega_1 \in \Omega_1$, we define an operator $\mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}$,

$$\begin{aligned}\mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[X](\omega_1) &:= \int_{\Omega_2}^{\oplus} \left(k(\omega_1, \omega_2) \odot X(\omega_2)\right) \odot d\mu_2(\omega_2) \\ &= g^{-1}\left(\int_{\Omega_2} g\left(k(\omega_1, \omega_2) \odot X(\omega_2)\right) d\left(g \circ \mu_2(\omega_2)\right)\right),\end{aligned}$$

by using Definition 2.3, where $k : \Omega_1 \times \Omega_2 \to [a, b]$ is measurable kernel. In particular, when $\mu_2 = g^{-1} \circ \mathbf{P}$, $\mathbf{P}$ is the probability measure, then we define

$$\mathbb{E}^{k,\Omega_2}_{\oplus,\odot}[X](\omega_1) := \mathbb{E}^{\Omega_2}_{\oplus}\left[k(\omega_1, \omega_2) \odot X(\omega_2)\right].$$

In Definition 2.4, if $g = \mathbf{id}$ (i.e., $g(x) = x$ for all x), then we define $\mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[.] = \mathfrak{A}^{k,\Omega_2}_{\mathbf{id}}[.]$.

3 Main results: Some refinements of Jensen's inequality

In this section, we establish some refinements of Jensen's inequality in general form.

Theorem 3.1 *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces and $X : \Omega_2 \to [a, b]$ be a measurable function, $\Phi : [a, b] \to [a, b]$ be a convex and nonincreasing function and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a convex and increasing function such that $g(X(\omega_2)) \in (a, b)$ for any $\omega_2 \in \Omega_2$ and $g \circ \Phi \circ g^{-1}$ is a differentiable function. If $\mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[X](\omega_1) \in (a, b)$ for each fixed $\omega_1 \in \Omega_1$, then for any σ-$\oplus$-measure μ_2, we have*

$$
\begin{aligned}
&\mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[\Phi(X)](\omega_1) \oplus \left(\beta \odot \mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[X](\omega_1) \odot \mathfrak{A}^{k,G}_{\oplus,\odot}\left[g^{-1}(1)\right](\omega_1)\right) \\
&\quad \geqslant \left[\Phi\left(\mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[X](\omega_1)\right) \odot \left(\mathfrak{A}^{k,G}_{\oplus,\odot}\left[g^{-1}(1)\right](\omega_1)\right)\right] \\
&\quad\quad \oplus \left(\left[\beta \odot \mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[X](\omega_1)\right] \oplus \left[\Phi\left(g^{-1}(0)\right) \odot \mathfrak{A}^{k,G^c}_{\oplus,\odot}\left[g^{-1}(1)\right](\omega_1)\right]\right),
\end{aligned}
$$

where $G = \{\omega_2 \in \Omega_2 : (g \circ X)(\omega_2) \neq 0\}$ and $\beta = g^{-1}[(g \circ \Phi \circ g^{-1})'(g \circ \mathfrak{A}^{k,\Omega_2}_{\oplus,\odot}[X](\omega_1))]$.

Proof Let $\Psi = g \circ \Phi \circ g^{-1}$. It is easy to see that Ψ is a convex function. Apply Theorem 1.4 with Ψ and replace $X(\omega_2)$, $k(\omega_1, \omega_2)$ and $\mu_2(\omega_2)$ by $g \circ X(\omega_2)$, $g \circ k(\omega_1, \omega_2)$ and $g \circ \mu_2(\omega_2)$, respectively. Then

$$
\begin{aligned}
&\int_{\Omega_2} g \circ k(\omega_1, \omega_2)\, g \circ \Phi \circ X(\omega_2)\, d(g \circ \mu_2(\omega_2)) \\
&\quad \geqslant g \circ \Phi \circ g^{-1}\left(\int_{\Omega_2} g \circ k(\omega_1, \omega_2)\, g \circ X(\omega_2)\, d(g \circ \mu_2(\omega_2))\right) \\
&\quad\quad \times \int_{G} g \circ k(\omega_1, \omega_2)\, d(g \circ \mu_2(\omega_2)) \\
&\quad\quad + \Psi'\left(\int_{\Omega_2} g \circ k(\omega_1, \omega_2)\, g \circ X(\omega_2)\, d(g \circ \mu_2(\omega_2))\right) \\
&\quad\quad \times \int_{\Omega_2} g \circ k(\omega_1, \omega_2)\, g \circ X(\omega_2)\, d(g \circ \mu_2(\omega_2)) \\
&\quad\quad - \Psi'\left(\int_{\Omega_2} g \circ k(\omega_1, \omega_2)\, g \circ X(\omega_2)\, d(g \circ \mu_2(\omega_2))\right) \\
&\quad\quad \times \int_{\Omega_2} g \circ k(\omega_1, \omega_2)\, g \circ X(\omega_2)\, d(g \circ \mu_2(\omega_2)) \\
&\quad\quad \times \int_{G} g \circ k(\omega_1, \omega_2)\, d(g \circ \mu_2(\omega_2)) + \Psi(0) \int_{G^c} g \circ k(\omega_1, \omega_2)\, d(g \circ \mu_2(\omega_2)).
\end{aligned}
$$

So,

$$\begin{aligned}
&\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot\Phi\left(X\left(\omega_2\right)\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\\
&\quad\geqslant g\circ\Phi\circ g^{-1}\left(\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\right)\\
&\qquad\times\int\limits_{G} g\circ k\left(\omega_1,\omega_2\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\\
&\qquad+\Psi'\left(\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\right)\\
&\qquad\times\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\\
&\qquad-\Psi'\left(\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\right)\\
&\qquad\times\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\\
&\qquad\times\int\limits_{G} g\circ k\left(\omega_1,\omega_2\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)+\Psi\left(0\right)\int\limits_{G^c} g\circ k\left(\omega_1,\omega_2\right)d\left(g\circ\mu_2\left(\omega_2\right)\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot\Phi\left(X\left(\omega_2\right)\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\\
&\qquad+\Psi'\left(\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\right)\\
&\qquad\times\int\limits_{\Omega_2} g\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\\
&\qquad\times\int\limits_{G} g\circ k\left(\omega_1,\omega_2\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\\
&\quad\geqslant g\circ\Phi\circ g^{-1}\left(\int\limits_{\Omega_2} g\circ\left(k\left(\omega_1,\omega_2\right)\odot X\left(\omega_2\right)\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)\right)\\
&\qquad\times\int\limits_{G} g\circ k\left(\omega_1,\omega_2\right)d\left(g\circ\mu_2\left(\omega_2\right)\right)
\end{aligned}$$

$$+ \Psi' \left(\int_{\Omega_2} g\left(k\left(\omega_1, \omega_2\right) \odot X\left(\omega_2\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right)$$
$$\times \int_{\Omega_2} g\left(k\left(\omega_1, \omega_2\right) \odot X\left(\omega_2\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right)$$
$$+\Psi(0) \int_{G^c} g \circ k\left(\omega_1, \omega_2\right) d\left(g \circ \mu_2\left(\omega_2\right)\right).$$

Since g is an increasing function, its inverse g^{-1} is also an increasing function and we have

$$g^{-1} \left(\int_{\Omega_2} g\left(k\left(\omega_1, \omega_2\right) \odot \Phi\left(X\left(\omega_2\right)\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right)$$
$$\oplus \left[g^{-1} \left(\Psi' \left(\int_{\Omega_2} g\left(k\left(\omega_1, \omega_2\right) \odot X\left(\omega_2\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right) \right) \right.$$
$$\odot g^{-1} \left(\int_{\Omega_2} g\left(k\left(\omega_1, \omega_2\right) \odot X\left(\omega_2\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right)$$
$$\left. \odot g^{-1} \left(\int_{G} g \circ k\left(\omega_1, \omega_2\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right) \right]$$
$$\geqslant \left[\Phi \circ g^{-1} \left(\int_{\Omega_2} g \circ \left(k\left(\omega_1, \omega_2\right) \odot X\left(\omega_2\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right) \right.$$
$$\left. \odot g^{-1} \left(\int_{G} g \circ k\left(\omega_1, \omega_2\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right) \right]$$
$$\oplus \left[g^{-1} \left(\Psi' \left(\int_{\Omega_2} g\left(k\left(\omega_1, \omega_2\right) \odot X\left(\omega_2\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right) \right) \right.$$
$$\left. \odot g^{-1} \left(\int_{\Omega_2} g\left(k\left(\omega_1, \omega_2\right) \odot X\left(\omega_2\right)\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right) \right]$$
$$\oplus \left[g^{-1}\left(\Psi(0)\right) \odot g^{-1} \left(\int_{G^c} g \circ k\left(\omega_1, \omega_2\right) d\left(g \circ \mu_2\left(\omega_2\right)\right) \right) \right].$$

So,

$$\mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}\left[\Phi(X)\right]\left(\omega_1\right)$$
$$\oplus \left[g^{-1} \left(\Psi' \left(g \left(\mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X]\left(\omega_1\right) \right) \right) \right) \odot \mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X]\left(\omega_1\right) \odot \mathfrak{A}_{\oplus,\odot}^{k,G}\left[g^{-1}(1)\right]\left(\omega_1\right) \right]$$

$$\geqslant \left[\Phi\left(\mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1)\right) \odot \left(\mathfrak{A}_{\oplus,\odot}^{k,G}\left[g^{-1}(1)\right](\omega_1)\right)\right]$$
$$\oplus \left[g^{-1}\left(\Psi'\left(g\left(\mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1)\right)\right)\right) \odot \mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1)\right]$$
$$\oplus \left[g^{-1}\left(g \circ \Phi \circ g^{-1}(0)\right) \odot \mathfrak{A}_{\oplus,\odot}^{k,G^c}\left[g^{-1}(1)\right](\omega_1)\right].$$

This completes the proof. □

If $k(\omega_1, \omega_2) \equiv g^{-1}(1)$ in Theorem 3.1, then we have the following corollary.

Corollary 3.2 *For a given measurable space $(\Omega_2, \mathcal{F}_2)$, let $X : \Omega_2 \to [a, b]$ be a measurable function, $\Phi : [a, b] \to [a, b]$ be a convex and nonincreasing function and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a convex and increasing function such that $g(X(\omega_2)) \in (a, b)$ for any $\omega_2 \in \Omega_2$ and $g \circ \Phi \circ g^{-1}$ is a differentiable function. If $\mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X] \in (a, b)$, then for any σ-$\oplus$-measure μ_2, we have*

$$\mathfrak{A}_{\oplus,\odot}^{\Omega_2}[\Phi(X)] \oplus \left(\beta \odot \mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X] \odot \mu(G)\right)$$
$$\geqslant \left[\Phi\left(\mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X]\right) \odot \mu(G)\right] \oplus \left(\left[\beta \odot \mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X]\right] \oplus \left[\Phi\left(g^{-1}(0)\right) \odot \mu\left(G^c\right)\right]\right),$$

where $G = \{\omega_2 \in \Omega_2 : (g \circ X)(\omega_2) \neq 0\}$ and $\beta = g^{-1}\left(\left[\left(g \circ \Phi \circ g^{-1}\right)'\left(g \circ \mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X]\right)\right]\right)$.

Remark 3.3 When $\mu_2 = g^{-1} \circ \mathbf{P}$, $\mathbf{P}$ is the probability measure, then we obtain

$$\mathbb{E}_{\oplus}^{\Omega_2}[\Phi(X)] \oplus \left(\beta \odot \mathbb{E}_{\oplus}^{\Omega_2}[X] \odot g^{-1}(\mathbf{P}(g(X) \neq 0))\right)$$
$$\geqslant \left[\Phi\left(\mathbb{E}_{\oplus}^{\Omega_2}[X]\right) \odot g^{-1}(\mathbf{P}(g(X) \neq 0))\right]$$
$$\oplus \left[\beta \odot \mathbb{E}_{\oplus}^{\Omega_2}[X]\right] \oplus \left[\Phi\left(g^{-1}(0)\right) \odot g^{-1}(\mathbf{P}(g(X) = 0))\right].$$

In particular, taking $\mathbf{P}\left(X = g^{-1}(0)\right) = 0$, we get the Jensen inequality

$$\mathbb{E}_{\oplus}^{\Omega_2}[\Phi(X)] \geq \Phi\left(\mathbb{E}_{\oplus}^{\Omega_2}[X]\right). \tag{3.1}$$

Example 3.4 Let $g(x) = x^\gamma$, $\gamma \in [1, \infty)$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\gamma]{x^\gamma + y^\gamma}$ and $x \odot y = xy$. Then (3.1) reduces to the following inequality

$$\left(\mathbb{E}\left[(\Phi(X))^\gamma\right]\right)^{\frac{1}{\gamma}} \geq \Phi\left(\left(\mathbb{E}\left[X^\gamma\right]\right)^{\frac{1}{\gamma}}\right).$$

Theorem 3.5 *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces and $X : \Omega_2 \to [a, b]$ be a measurable function, $\Phi : [a, b] \to [a, b]$ be a concave and non-decreasing function and let a generator $g : [a, b] \to [0, \infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a convex and decreasing function such that $g(X(\omega_2)) \in (a, b)$ for any $\omega_2 \in \Omega_2$ and $g \circ \Phi \circ g^{-1}$ is a differentiable function. If $\mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1) \in (a, b)$ for each fixed $\omega_1 \in \Omega_1$, then for any σ-$\oplus$-measure μ_2, we have*

$$\mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[\Phi(X)](\omega_1) \oplus \left(\beta \odot \mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1) \odot \mathfrak{A}_{\oplus,\odot}^{k,G}\left[g^{-1}(1)\right](\omega_1)\right)$$
$$\leqslant \left[\Phi\left(\mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1)\right) \odot \left(\mathfrak{A}_{\oplus,\odot}^{k,G}\left[g^{-1}(1)\right](\omega_1)\right)\right]$$
$$\oplus \left(\left[\beta \odot \mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1)\right] \oplus \left[\Phi\left(g^{-1}(0)\right) \odot \mathfrak{A}_{\oplus,\odot}^{k,G^c}\left[g^{-1}(1)\right](\omega_1)\right]\right),$$

where $G = \{\omega_2 \in \Omega_2 : (g \circ X)(\omega_2) \neq 0\}$ *and* $\beta = g^{-1}([(g \circ \Phi \circ g^{-1})'(g \circ \mathfrak{A}_{\oplus,\odot}^{k,\Omega_2}[X](\omega_1))])$.

Proof Since g is a decreasing and convex function, g^{-1} is also decreasing but concave function. Let $\Psi = g \circ \Phi \circ g^{-1}$. It is easy to see that Ψ is a convex function. Similarly as in the proof of Theorem 3.1, we obtain the desired inequality. □

Corollary 3.6 *For a given measurable space* $(\Omega_2, \mathcal{F}_2)$, *let* $X : \Omega_2 \to [a, b]$ *be a measurable function,* $\Phi : [a, b] \to [a, b]$ *be a concave and non-decreasing function and let a generator* $g : [a, b] \to [0, \infty]$ *of the pseudo-addition* $\oplus$ *and the pseudo-multiplication* $\odot$ *be a convex and decreasing function such that* $g(X(\omega_2)) \in (a, b)$ *for any* $\omega_2 \in \Omega_2$ *and* $g \circ \Phi \circ g^{-1}$ *is a differentiable function. If* $\mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X] \in (a, b)$, *then for any* σ-$\oplus$ *-measure* μ_2, *we have*

$$\mathfrak{A}_{\oplus,\odot}^{\Omega_2}[\Phi(X)] \oplus \left(\beta \odot \mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X] \odot \mu(G)\right)$$
$$\leqslant \left[\Phi\left(\mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X]\right) \odot \mu(G)\right] \oplus \left(\left[\beta \odot \mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X]\right] \oplus \left[\Phi\left(g^{-1}(0)\right) \odot \mu\left(G^c\right)\right]\right),$$

where $G = \{\omega_2 \in \Omega_2 : (g \circ X)(\omega_2) \neq 0\}$ *and* $\beta = g^{-1}([(g \circ \Phi \circ g^{-1})'(g \circ \mathfrak{A}_{\oplus,\odot}^{\Omega_2}[X])])$.

Remark 3.7 When $\mu_2 = g^{-1} \circ \mathbf{P}$, $\mathbf{P}$ is the probability measure, then we obtain

$$\mathbb{E}_{\oplus}^{\Omega_2}[\Phi(X)] \oplus \left(\beta \odot \mathbb{E}_{\oplus}^{\Omega_2}[X] \odot g^{-1}(\mathbf{P}(g(X) \neq 0))\right)$$
$$\leqslant \left[\Phi\left(\mathbb{E}_{\oplus}^{\Omega_2}[X]\right) \odot g^{-1}(\mathbf{P}(g(X) \neq 0))\right]$$
$$\oplus \left(\left[\beta \odot \mathbb{E}_{\oplus}^{\Omega_2}[X]\right] \oplus \left[\Phi\left(g^{-1}(0)\right) \odot g^{-1}(\mathbf{P}(g(X) = 0))\right]\right),$$

where $\beta = g^{-1}\left(\left(g \circ \Phi \circ g^{-1}\right)'\left(g \circ \mathbb{E}_{\oplus}^{\Omega_2}[X]\right)\right)$.
In particular, taking $\mathbf{P}\left(X = g^{-1}(0)\right) = 0$, we get the Jensen inequality

$$\mathbb{E}_{\oplus}^{\Omega_2}[\Phi(X)] \leqslant \Phi\left(\mathbb{E}_{\oplus}^{\Omega_2}[X]\right).$$

4 Concluding remarks

We have introduced and discussed the concept of pseudo-expectation with general kernels and then have established some refinements of Jensen's inequality in general form. This inequality includes pseudo-integral, expectation, convolution integral, fractional integral, as special cases. As we have seen,

- for $k(\omega_1, \omega_2) \equiv \mathbf{1}$ and $\mu_2 = g^{-1} \circ \mathbf{P}$, $\mathbf{P}$ is the probability measure, in Corollary 3.2, we get the refined Jensen's inequality for g -expectation. In particular, taking $[a, b] = [0, \infty]$, $g = \mathbf{id}$ and $\mathbf{P}(X = 0) = 0$, we get the Jensen's inequality (1.1).
- For $k(\omega_1, \omega_2) \equiv \mathbf{1}$, $\mu_2 = g^{-1} \circ \mu$, μ is the standard Lebesgue measure, $\Omega_2 = [0, 1]$, in Corollary 3.2, we have the Jensen type for pseudo-integral obtained by Pap and Štrboja [29].
- For $k(\omega_1, \omega_2) = k(\omega_1 - \omega_2)$ and $\mu_2 = g^{-1} \circ \mu$, μ is the standard Lebesgue measure, $\Omega_2 = [0, \omega_1]$ for each fixed ω_1, in Theorem 3.1, we get the Jensen's inequality for g-convolution integral.

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Appendix

Proof of Theorem 1.2 We use the inequality

$$\Psi(x) \geqslant \Psi(\rho) + \Psi'(\rho)(x-\rho) \tag{5.1}$$

for any $x, \rho \in I$ which follows from convexity of Ψ. Multiplying both sides of (5.1) by $k(\omega_1, \omega_2)$, we have

$$k(\omega_1,\omega_2)\Psi(x) \geqslant k(\omega_1,\omega_2)\Psi(\rho) + k(\omega_1,\omega_2)\Psi'(\rho)(x-\rho). \tag{5.2}$$

We set $x = X(\omega_2)$ and $A_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1) = \frac{1}{K_{\mathbf{id}}^{\Omega_2}(\omega_1)} \int_{\Omega_2} (k(\omega_1,\omega_2) X(\omega_2))\, d\mu_2(\omega_2) = \rho$ and integrate over the domain $G = \{\omega_2 \in \Omega_2 : X(\omega_2) \neq 0\}$. Then

$$\begin{aligned}
\int_G k(\omega_1,\omega_2)\Psi[X(\omega_2)]\,d\mu_2(\omega_2) &\geqslant \Psi(\rho)\int_G k(\omega_1,\omega_2)\,d\mu_2(\omega_2)\\
&\quad + \Psi'(\rho)\left(\int_G k(\omega_1,\omega_2)X(\omega_2)\,d\mu_2(\omega_2) - \rho\int_G k(\omega_1,\omega_2)\,d\mu_2(\omega_2)\right)\\
&= \Psi(\rho)\int_G k(\omega_1,\omega_2)\,d\mu_2(\omega_2) + \rho\Psi'(\rho)\\
&\quad \times \left(\int_{\Omega_2} k(\omega_1,\omega_2)\,d\mu_2(\omega_2) - \int_G k(\omega_1,\omega_2)\,d\mu_2(\omega_2)\right),
\end{aligned}$$

which gets the desired inequality

$$\begin{aligned}
&\int_{\Omega_2} k(\omega_1,\omega_2)\Psi[X(\omega_2)]\,d\mu_2(\omega_2)\\
&\quad \geqslant \Psi(\rho)\int_G k(\omega_1,\omega_2)\,d\mu_2(\omega_2) + \left(\rho\Psi'(\rho) + \Psi(0)\right)\left(\int_{G^c} k(\omega_1,\omega_2)\,d\mu_2(\omega_2)\right).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.4 **.** Using (5.2), set $x = X(\omega_2)$ and

$$\mathfrak{A}_{\mathbf{id}}^{k,\Omega_2}[X](\omega_1) = \int_{\Omega_2} k(\omega_1,\omega_2)X(\omega_2)\,d\mu_2(\omega_2) = \rho$$

and integrate over the domain $G = \{\omega_2 \in \Omega_2 : X(\omega_2) \neq 0\}$. Then

$$\begin{aligned}
&\int_G k(\omega_1, \omega_2)\, \Psi(X(\omega_2))\, d\mu_2(\omega_2) \geqslant \Psi(\rho) \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) \\
&\quad + \Psi'(\rho) \left(\int_G k(\omega_1, \omega_2)\, X(\omega_2)\, d\mu_2(\omega_2) - \rho \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) \right) \\
&= \Psi(\rho) \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) \\
&\quad + \Psi'(\rho) \left(\int_{\Omega_2} k(\omega_1, \omega_2)\, X(\omega_2)\, d\mu_2(\omega_2) - \rho \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) \right) \\
&= \Psi(\rho) \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) + \rho \Psi'(\rho) \left(1 - \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) \right).
\end{aligned}$$

So,

$$\begin{aligned}
&\int_{\Omega_2} k(\omega_1, \omega_2)\, \Psi(X(\omega_2))\, d\mu_2(\omega_2) \geqslant \Psi(\rho) \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) \\
&\quad + \Psi'(\rho) \int_{\Omega_2} k(\omega_1, \omega_2)\, X(\omega_2)\, d\mu_2(\omega_2) - \rho \Psi'(\rho) \int_G k(\omega_1, \omega_2)\, d\mu_2(\omega_2) \\
&\quad + \Psi(0) \int_{G^c} k(\omega_1, \omega_2)\, d\mu_2(\omega_2).
\end{aligned}$$

This completes the proof. □

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