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RESEARCH REPORT

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Multi-period Factor Model of a Loan Portfolio[1]

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Abstract

We construct a general dynamic model of losses of a large loan portfolio, secured by collaterals. In the model, the wealth of a debtor and the price of the corresponding collateral depend each on two factors: a common one, having a general distribution, and an individual one, following an AR(1) process. The default of a loan happens if the wealth stops to be sufficient for repaying the loan.

We show that the mapping transforming the common factors into the probability of default (PD) and the loss given default (LGD) is one-to-one twice continuously differentiable.

As the transformation is not analytically tractable, we propose a numerical technique for its computation and demonstrate its accuracy by a numerical study.

We show that the results given by our multi-period model may differ significantly from those resulting from single-period models, and demonstrate that our model naturally replicates the empirically observed decrease of PDs within a portfolio in time.

In addition, we give a formula for the overall loss of the portfolio and, as an example of its application, we formulate a simple optimal scoring decision problem and discuss its solution.

Keywords: Credit Risk, Structural Factor Models, Loan Portfolio Management

1 Introduction

At present, factor models form one of the main branches in credit risk modeling. Pioneering work on this topic was done by [24], [5] or [22], recent development may be found e.g. in [14], [23], [3] or [16]. For a survey of present state of the art, see e.g., [2]. It is also worth noting that many migration-rating models, which form another strong branch of credit risk modeling, may be reformulated as factor ones (see [9]).

In a typical factor model, the (conditional) probability of default (PD) is dependent on one or more real-valued factors. Particular functions used to express this dependence include logit [14], probit [23], and e^{-x} [3].

In the case of structural models - a special subgroup of the factor models - the shape of the transforming function emerges implicitly from the distribution of the debtors' wealth. In particular, given the usual assumption that the (log) wealth is equal the sum of a common and an individual factor,

$$PD = q(-Y^\star)$$

where $Y^\star$ is the difference between the common factor and the minimal wealth needed to repay the loan, and q is the cumulative distribution function of the individual factor; thus, structural models with normal individual factors thus coincide with general factor models having the probit transformation function.

If a collateral is incorporated in the model and its price is equal to the sum of another common and another individual factor, then the expected LGD may be computed by an integration over a censored distribution of the individual factor. When, specially, the individual factor is normal independent of the individual factor underlying the defaults, a closed form formula for LGD exists (for details, see e.g. [22] or Eq. (17) of [6]).

With a time series of factors and a structural model at hand, it seems natural to model the dynamics of the PD and LGD by repeated transformations of the common factors according to the model, as e.g. [21], [8] or [7] do. There is, however, a drawback to this approach because, by the repeated usage of the single-period model, it is implicitly assumed that the debtors' wealth "starts from scratch" at each period, which would be justifiable only if the loans took exactly one period; in practice, however, the duration of the loans counts in years, so the past of the wealth should be taken into account.

To be specific, there are at least three sources of the time dependence of the wealth within a loan portfolio:

(i) the time dependence of the common factors,

(ii) the time dependence of the individual factors,

(iii) the periodic “cutting-off the poorest” from the portfolio by the defaults.

By its nature, the “single-period” approach is able to accomodate only (i). However, as it is demonstrated in our paper, (ii) and (iii) may influence the losses of a portfolio significantly (see Table 1); thus, they have to be taken into account, too.

The model proposed in the present paper, despite it is nothing more than a natural dynamic generalization of single-period structural models, is able to handle all the three effects. Simultaneously, it is general enough: we do not put any specific restrictions on the distribution of the process of the common factors. The individual factors, in line with empirical evidence (see, e.g., [20] or [10]), are assumed to be AR(1) withe arbitrary “reasonable” distributions of their residua, which may also be stochastically dependent.

We prove that, similarly to existing structural model, a one-to-one mapping transforming the common factors into PDs and LGDs exists and, moreover, is twice continuously differentiable all in the common factors and in the parameters of the model, which are the volatilities of individual factors, their AR coefficients and the loan interest rate. Consequently, we show that the overall percentage loss of the portfolio, seen as a function of the common factors and the parameters, is twice continuously differentiable, too.

As both the transformation and the overall loss are analytically intractable, we further propose a numerical technique for the their computation. We also present a simple numerical study demonstrating efficiency of the technique.

Is it was mentioned above, we also demonstrate that the PDs resulting from our model may significantly differ from those resulting from the repeated usage of a one-period model. We also show that, given a positive AR parameter, our model replicates the empirically observed decrease of the PDs in time within the portfolio (see [1]).

Our model is widely applicable. Not only can it may be combined with any appropriate (macroeconomic) model of the factors' evolution to describe the dynamics of PDs and LGDs, but it may also serve in concrete portfolio management, namely for stress testing (via factors perturbations) or for portfolio optimization. In all these cases, the theoretical properties proved in our paper, especially the differentiability, can help a great deal, either in numerical optimization or in statistical estimation (for the role of the differentiability in the statistical estimation, see [12] and/or [19], Chp. 2.3.).

As an example of the application of our model, we formulate a single optimal scoring model with the debtors' minimal wealth as the decision variable.

The paper is organized as follows: after this Introduction, the model is formulated (Section 2). Then, formulas for the PD and the mean charge-off, from which the LGD is subsequently computed, are given (Section 3). Next, the bijectiveness of the transformation, its differentiability and the differentiability of the overall loss are proved (Section 4). Consequently, the numerical technique for the computation of the mapping and its inverse is described (Section 5). Further (Section 7), the accuracy or our technique, the difference to the single-period models, and the decrease of the PDs in time are demonstrated. Finally, the optimal scoring problem is presented (Section 7) and the paper is concluded (Section 8). Some auxiliary mathematical material and proofs are contained in the Appendix.

2 Setting

We consider a portfolio of N loans such that

- the amounts of loans are identical for all the debtors, equal to one without loss of generality,
- all the loans have the same duration m and the same interest rate ϵ,
- each loan is arranged at time 0 and is amortized annuity way, i.e., by identical installments

$$b \quad = \quad b(\epsilon) = \begin{cases} \frac{\epsilon}{1-v^m} & \epsilon \neq 0 \\ \frac{1}{m} & \epsilon = 0 \end{cases}, \qquad v = v(\epsilon) = (1+\epsilon)^{-1},$$

payed at each of the times $1, 2, \ldots, m$ (see [17], p. 39. for the formula determining the repayment given annuity amortization).

Assume further that nominal free wealth A_t^i of the i-th debtor at time t fulfills

$$A_t^i = \exp\left\{Y_t + Z_t^i\right\}, \qquad t \geq 1,$$

where

- Y is a stochastic processes (common factor),
- Z^i (individual factor) is a stochastic process such that
  - $Z_1^i = \sigma_1 U_1^i$, $\sigma_1 > 0$, where U_1^i is a centered standardized[2] random variable,
  - $Z_t^i = \phi Z_{t-1}^i + \sigma U_t^i$, $t > 1$, for some constants $\phi \in \mathbb{R}$, $\sigma > 0$, where $U_2^i, U_3^i, \ldots$ are identically distributed centered standardized.

Further, assume that the i-th loan is secured by a collateral with price P^i fulfilling

$$P_0^i = 1$$

(i.e., is equal to the size of the loan) and

$$P_t^i = \exp\left\{I_t + E_t^i\right\}, \qquad t > 0,$$

where

- I is a stochastic processes (another common factor),
- E^i (individual factor) is a stochastic process fulfilling $E_t^i = \psi E_{t-1}^i + \rho V_t^i$, $t > 0$, for some constants $\psi \in \mathbb{R}$ and $\rho > 0$ where $E_0^i \equiv 0$ and $V_1^i, V_2^i, \ldots$ are identically distributed centered standardized.

Finally, for any t and i, denote $\Xi_t^i = (U_t^i, V_t^i)$ and assume that

- $\Xi_1^1, \Xi_1^2, \ldots, \Xi_1^N, \Xi_2^1, \Xi_2^2, \ldots$ are mutually independent, independent of Y, I.
- $\Xi_1^1, \Xi_1^2, \ldots, \Xi_1^N$ are identically distributed with a (joint) c.d.f. W_1, strictly increasing in both its arguments, having continuous uniformly bounded first- and second-order derivatives and, moreover,

$$|u| w_1^1(u) \text{ and } |v| w_1^2(v) \text{ are bounded} \tag{1}$$

where $w_1^1(u) = \frac{\partial}{\partial u} W_1(u, \infty)$ and $w_1^2(v) = \frac{\partial}{\partial v} W_1(\infty, v)$ are densities of U_1, V_1, respectively.

[2] i.e., having zero mean and unit second moment

- $\Xi_2^1, \Xi_2^2, \dots, \Xi_2^N, \Xi_3^1, \dots$ are identically distributed with a (joint) c.d.f. W, strictly increasing in both its arguments, having continuous uniformly bounded first- and second-order derivatives and, moreover,
$$|u|w^1(u) \text{ and } |v|w^2(v) \text{ are bounded} \tag{2}$$
where $w^1(u) = \frac{\partial}{\partial u} W(u, \infty)$ and $w^2(v) = \frac{\partial}{\partial v} W(\infty, v)$ are densities of U_2, V_2, respectively.

Remark 1. *Our assumptions are met if distributions of Ξ_1^1 and Ξ_1^2 are non-degenerated joint normal.*

Proof. The bounded differentiability may be easily verified using well known formulas for the densities. The monotonicity of the c.d.f.'s follows from the positiveness of their densities. The boundedness of $|u|w_1^1(u)$ follows from its positivity and the monotonicity of its tails. □

Remark 2. *Let $m = 1$.*
(i) If $Y_1 \sim \mathcal{N}(0, \varsigma^2)$, $Z_1 \sim \mathcal{N}(0, \sigma^2)$, $\varsigma^2 + \sigma^2 = 1$ then our setting replicates the Vasicek Model [24].
(ii) If Y_1, Z_1, are as in (i), $I_1 = \alpha + \gamma Y_1$ for some $\alpha, \gamma > 0$, $E_1 \sim \mathcal{N}(0, \varrho^2)$, $0 \le \varrho \le 1$, $E_1 \perp\!\!\!\perp Z_1$, then we are getting the model by [5] up to taking logarithms of the collateral price.
(iii) If Y_1, I_1, Z_1, E_1 are as in (ii) with (E_1, Z_1) joint normal correlated then we have the model by [22].
(iv) If Y_1, I_1, Z_1, E_1 are general with $E_1 \perp\!\!\!\perp Z_1$ then we get [6].

Proceeding with definitions, denote
$$B_\tau^i = \mathbf{1}\left[A_\tau^i < \tau b\right] = \mathbf{1}\left[Z_\tau^i < -Y_\tau^\star\right], \qquad Y_\tau^\star = Y_\tau - \log \tau - \log b, \tag{3}$$
the variable indicating insufficiency of the i-th debtor's wealth to cover the (accumulated) installments at τ[3] and denote
$$S_\tau^i = \mathbf{1}[B_1^i = 0, B_2^i = 0, \dots, B_\tau^i = 0]$$
an indicator of "survival" of the i-th debt up to time τ ($S_0 \equiv 1$ by definition).

We say that the i-th debtor *defaults* at t if
$$Q_t^i = 1$$
where
$$Q_t^i = \mathbf{1}[B_t^i = 1, S_{t-1}^i = 1].$$
The *loss given default* (LGD) of the i-th loan at time t is given
$$G_t^i = \frac{\max(0, h_t - P_t^i)}{h_t}$$
where
$$h_t = h_t(\epsilon) = \begin{cases} b \sum_{\tau=t}^m v^{m-\tau+1} = bv\frac{1-v^{m-t+1}}{1-v} & \epsilon \ne 0 \\ \frac{m-t+1}{m} & \epsilon = 0 \end{cases}$$
is the principal outstanding at t (see [17] for a corresponding formula).

[3] Alternatively, additional obligations $\gamma_\tau > 0$, $1 \le \tau \le m$, of the debtor may be considered changing our definitions to
$$B_\tau^i = \mathbf{1}\left[A_\tau^i < \tau b + \sum_{\nu=1}^{\tau} \gamma_\tau\right], \qquad Y_\tau^\star = Y_\tau - \log\left(\tau b + \sum_{\nu=1}^{\tau} \gamma_\tau\right)$$
without affecting our further results. Moreover, both the factors transformation and the overall loss would be differentiable in all $\gamma_1, \dots, \gamma_m$ thanks to their differentiability in factor Y, proved later.

The *charge-off* (percentage loss) of the i-th loan is then given as

$$L_t^i = Q_t^i G_t^i \tag{4}$$

Further, denote

$$N_t = \sum_{i=1}^{N} S_{t-1}^i \qquad t > 0,$$

the number of the debts having survived until t and define

$$Q_{N,t} = \frac{\sum_{1\leq i\leq N} Q_t^i}{N_t}, \qquad t > 0,$$

the *default rate*,

$$L_{N,t} = \frac{\sum_{1\leq i\leq N} L_t^i}{N_t}$$

the *charge-off rate* and

$$G_{N,t} = \frac{\sum_{1\leq i\leq N, Q_t^i=1} G_t^i}{\sum_{1\leq i\leq N} Q_t^i} = \frac{L_{N,t}}{Q_{N,t}}, \qquad t > 0,$$

the *average loss given default*.

Finally, denote

$$L_N = \frac{1}{N}\sum_{t=1}^{m} \beta^{t-1} \sum_{1\leq i\leq N} L_t^i, \qquad \beta \in (0,1],$$

the *overall relative discounted loss* of the portfolio.

Finally, denote

$$Q_t = \lim_{N\to\infty} Q_{N,t}, \qquad G_t = \lim_{N\to\infty} G_{N,t}, \qquad t \geq 1,$$
$$L = \lim_{N\to\infty} L_N$$

the asymptotic versions of the default rate, loss given default and the overall loss (i.e., those given a hypothetical infinite size of the portfolio).

We hold on the established practice and call Q_t *probability of default* (PD), even though this name is inaccurate because Q_t is a random variable (equal, by the way, to the conditional probability of default given the vector of factors up to t, see Proposition 1 (i)). Variables G_t and L, we call simply LGD, *overall loss*, respectively.

The main goal of the present paper is to examine properties and computability of the mapping Φ_t transforming the factors up to time t into the PDs and LGDs up to time t, i.e.

$$\Phi_t(Y_1, I_1, \dots Y_t, I_t; \theta) = (Q_1, G_1, \dots, Q_t, G_t), \tag{5}$$

its inversion, and the mapping Λ transforming the common factors into the overall loss, i.e. fulfilling

$$L = \Lambda(Y_1, I_1, \dots, Y_m, I_m; \theta) \tag{6}$$

in dependence on parameter vector

$$\theta = (\epsilon, \sigma_1, \sigma, \rho, \phi, \psi)$$

taking values in space

$$\Theta = (-1,\infty) \times (0,\infty)^3 \times \mathbb{R}^2.$$

3 Probability of Default and Mean Charge-off

In the present Section, mappings transforming the common factors into the probability of default, the mean charge-off rate, respectively, are studied under temporary assumption that

$$Y \equiv y,\ I \equiv \iota \text{ for some deterministic } y \in \mathbb{R}^{\mathbb{N}}, \iota \in \mathbb{R}^{\mathbb{N}} \tag{7}$$

(meanwhile, variables Ξ are kept stochastic). Readers, interested only in our main results, may skip this Section without loss of understanding; further text, however, references several formulas from the present Section. For notational simplicity, we write Ξ_τ instead of Ξ^1_τ, U_τ instead of U^1_τ, etc., throughout the Section.

We start with some properties of the conditional distribution of (Z_t, E_t) given survival of the debt until $t-1$:

Lemma 1. *For any $t \geq 1$, denote*

$$\xi_t = (y_t, \dots, y_1, \theta).$$

and put

$$F_t(z, e; \xi_{t-1}) = \begin{cases} W_1\left(\frac{z}{\sigma_1}, \frac{e}{\rho}\right) & t = 1, \\ \frac{A_t}{C_t} & t > 1, \end{cases} \tag{8}$$

$$\begin{aligned} A_t &= A_t(z, e; \xi_{t-1}) \\ &= \int_{\{r \geq -y^\star_{i-1}, s \in \mathbb{R}\}} W\left(\frac{z - \phi r}{\sigma}, \frac{e - \psi s}{\rho}\right) dF_{t-1}(r, s; \xi_{t-2}), \\ C_t &= C_t(\xi_{t-1}) = 1 - F_{t-1}(-y^\star_{t-1}, \infty; \xi_{t-2}), \\ y^\star_t &= y^\star_t(y_t, \epsilon) = y_t - \log t - \log b(\epsilon). \end{aligned} \tag{9}$$

Let $t \geq 1$ and let ξ_{t-1} be feasible, i.e. $\xi_{t-1} \in \mathbb{R}^{t-1} \times \Theta$. Then

(i) *for each $z, e \in \mathbb{R}$,*

$$\mathbb{P}[Z_t \leq z, E_t \leq e | S_{t-1} = 1] = F_t(z, e; \xi_{t-1}),$$

(ii) *F_t is strictly increasing in both z and e,*

(iii) *F_t is continuously differentiable in (z,e,ξ_{t-1}),*

(iv) *for each symbol $s \in \{z, e, y_{t-1}, \dots, y_1, \epsilon, \sigma_1, \sigma, \phi, \rho, \psi\}$ there exists continuous $\alpha^{(s)}_t(\xi) \in \mathbb{R}^+$ such that*

$$\left|\frac{\partial}{\partial s} F_t(z, e; \xi_{t-1})\right| \leq \alpha^{(s)}_{t-1}(\xi_{t-1}).$$

If, in addition,

$$u^2 \left|\frac{\partial}{\partial u} w^1_1(u)\right|,\ v^2 \left|\frac{\partial}{\partial v} w^2_1(v)\right| \text{ and } |uv|\, w_1(u, v) \text{ are bounded, the first two being integrable} \tag{10}$$

$$u^2 \left|\frac{\partial}{\partial u} w^1(u)\right|,\ v^2 \left|\frac{\partial}{\partial v} w^2(v)\right| \text{ and } |uv|\, w(u, v) \text{ are bounded, the first two being integrable,} \tag{11}$$

$$w^{1|2}_1(u|v) \leq \omega^{1|2}_1, \qquad w^{1|2}(u|v) \leq \omega^{1|2}, \qquad w^{2|1}(v|u) \leq \omega^{2|1}, \qquad u, v \in \mathbb{R}, \tag{12}$$

where $w_1 = \frac{\partial}{\partial u \partial v} W_1$ and $w = \frac{\partial}{\partial u \partial v} W$ are densities of (U_1, V_1), (U_2, V_2), respectively, w^i and w^i_1 denote the i-th coordinates of w_1, w, respectively, $\omega^{1|2}_1$, $\omega^{1|2}$ and $\omega^{2|1}$ are finite constants, and where $w^{1|2}_1$, $w^{1|2}$ and $w^{2|1}$ are the conditional densities of $U_1|V_1$,$U_2|V_2$, $V_2|U_2$, respectively, then

(v) *F_t is twice continuously differentiable in (z, e, ξ_{t-1}),*

(vi) *for any symbols $r, s \in \{z, e, y_{t-1}, \dots, y_1, \epsilon, \sigma_1, \sigma, \phi, \rho, \psi\}$ there exists a continuous $\alpha^{(r,s)}(\xi) \in \mathbb{R}^+$ such that*

$$\left| \frac{\partial}{\partial r \partial s} F_t(z, e; \xi_{t-1}) \right| \leq \alpha^{(r,s)}(\xi_{t-1}), \qquad z, e \in \mathbb{R}.$$

Proof. See Appendix B. □

Remark 3. *Conditions (10), (11) and (12) are met if distributions of Ξ_1^1 and Ξ_1^2 are non-degenerated joint normal.*

Proof. Denote f the standard normal density. Using the fact that f is decreasing on $\mathbb{R}^+$ and the By Parts Formula gradually, we get $\int_{\mathbb{R}^+} u^2 |\frac{\partial}{\partial u} w_1^1(u)| du = -\int_{\mathbb{R}^+} u^2 \frac{\partial}{\partial u} f(u) du = 2 \int u f(u) du = \mathbb{E}|U_1|$, and, symmetrically, $\int_{\mathbb{R}^-} u^2 |w_1^1(u)| du = \mathbb{E}|U_1|$, which imply the first formula in (10) (the boundedness of function $u^2 |\frac{\partial}{\partial u} w_1^1(u)|$ may be proved by showing that its tails are monotone beyond certain threshold).

Further, as $U_1 | V_1 \sim \mathcal{N}(\varrho V_1, 1 - \varrho^2)$ where ϱ is the correlation coefficient between U_1 and V_1 (see [4], Proposition 3.13), we have that

$$w_1^{1|2}(u|v) = cf\left(c(u + \varrho v)\right) \leq cf(0), \qquad c = \frac{1}{\sqrt{1 - \varrho^2}}$$

proving the first formula of (12). As, further,

$$\begin{aligned}
\max_{u,v} |uv| w_1(u,v) &= \max_{u,v} |uv| cf\left(c(u + \varrho v)\right) f(v) \\
&\overset{x = c(u+\varrho v)}{=} \max_{v,x} \left| \frac{x}{c} - \varrho v \right| |v| cf(x) f(v) \\
&\leq \max_{v,x} |xv| f(x) f(v) + c\varrho \max_{v,x} v^2 f(x) f(v) \\
&= \max_x |x| f(x) \max_v |v| f(v) + c\varrho \max_x f(x) \max_v v^2 f(v) < \infty
\end{aligned}$$

(the finiteness follows from the fact that the tails of the maximized functions are vanishing at ∞), i.e. the last formula of (1) is proved. The proofs of the rest of (10), (11) and (12) are analogous. □

The following two Propositions describe properties of the probability of default, the expected charge-off rate, respectively, of a single debt in dependence on ξ_t.

Proposition 1. *Let $t \geq 1$ and let ξ_{t-1} be feasible. Put*

$$q_t = q_t(\xi_t) = F_t(-y_t^\star, \infty; \xi_{t-1}).$$

Then

(i) $\mathbb{P}[Q_t = 1 | S_{t-1} = 1] = q_t$,

(ii) *q_t is strictly decreasing in y_t,*

(iii) *q_t is continuously differentiable in ξ_t,*

(iv) *$q_t(\bullet, \xi_{t-1})$ is a bijection between $\mathbb{R}$ and (0,1),*

(v) *$q_t^{-1}(z, \xi_{t-1})$, where the inversion is meant with respect to the first argument of q, is continuously differentiable in (z, ξ_{t-1}),*

(vi) *if (10), (11) and (12) hold then q_t and q_t^{-1} are twice continuously differentiable in ξ_t, (z, ξ_{t-1}),* respectively.

Proof. (i) follows from (3) and from the fact that $Q_t = B_t$ on $[S_{t-1} = 1]$, implying that $\mathbb{P}[Q_t = 1|S_{t-1} = 1] = \mathbb{P}[B_t = 1|S_{t-1} = 1]$. (ii) and (iii) are implied by Lemma 1 (ii), (iii), respectively, and by the fact that $b(\bullet)$ is continuously differentiable.[4](iv) follows from the continuity of $F_t(\bullet, \infty)$, its strict monotonicity and the fact that it is a c.d.f., i.e., its limits in $-\infty$ and $+\infty$ are zero, one, respectively. It remains to prove (v) and (vi): To this end, let $\hat{z} \in (0, 1)$ be a constant and let

$$\hat{y} = q_t^{-1}(\hat{z}, \xi).$$

Define function ϕ by

$$\phi(y, z, \xi) = q_t(y, \xi) - z.$$

As

$$\phi(\hat{y}, \hat{z}, \xi) = 0$$

and as $\frac{\partial}{\partial y}\phi(\hat{y}, \hat{z}, \xi_{t-1}) < 0$ by (ii) and (iii), it follows from the Implicit Function Theorem that there exists a neighborhood N of $(\hat{z}, \xi_{t-1})$ and a continuously differentiable function υ, uniquely defined on N, such that

$$\phi(\upsilon(z, \xi), z, \xi) = 0, \qquad (z, \xi) \in N. \tag{13}$$

However, as (13) also uniquely defines q_t^{-1}, necessarily $\upsilon(z, \xi) = q_t^{-1}(z, \xi)$ for any $(z, \xi) \in N$, i.e..(v) is proved.

Finally, if the assumptions of (vi) hold true, then, by Lemma 1 (v) and thanks to continuous second order differentiability of b (which can be proved analogously as in Footnote 4), $q_t(\xi_t)$ is twice continuously differentiable. Moreover, by the Implicit Function Theorem, υ is twice continuously differentiable, i.e. (vi) is proved. □

Proposition 2. *Let $t \geq 1$ and let ξ_t be feasible. Put*

$$\lambda_t = \lambda_t(\iota_t, \xi_t) = \lambda_t^\star(\iota_t - \log h_t, \xi_t)$$

where

$$\begin{aligned}
\lambda_t^\star(s, \xi_t) &= H_t(-s; \xi_t) - \exp(s) \int_{-\infty}^{-s} \exp(x) dH_t(x; \xi_t) && (14)\\
&= \exp(s) \int_{-\infty}^{-s} H_t(x; \xi_t) \exp(x) dx, && (15)
\end{aligned}$$

$$H_t(e; \xi_t) = F_t(-y_t^\star, e; \xi_{t-1}).$$

Then

(i) $\mathbb{E}[L_t|S_{t-1} = 1] = \lambda_t$,[5]

(ii) *λ_t is strictly decreasing in ι_t and y_t,*

(iii) *λ_t is continuously differentiable in (ι_t, ξ_t),*

(iv) *$\lambda_t(\bullet, \xi_t)$ is a bijection between $\mathbb{R}$ and $(0, q_t)$,*

(v) *$\lambda_t^{-1}(x, \xi_t)$, where the inversion is with respect to the first argument of λ_t, is continuously differentiable in (x, ξ_t),*

[4]The differentiability in zero, which is the only problematic point, follows from Lemma 5.
[5]Rigorously: $\mathbb{E}[L_t|S_{t-1}] = \lambda_t$ on $[S_{t-1} = 1]$.

(vi) *if (10), (11) and (12) hold then both* λ_t *and* λ_t^{-1} *are twice continuously differentiable in* (ι_t, ξ_t), (x, ξ_t), *respectively.*

Proof. Ad (i). Using probability calculus, we gradually get that, on $[S_{t-1} = 1]$,

$$
\begin{aligned}
\mathbb{E}[L_t|S_{t-1}] &= h_t^{-1}\mathbb{E}(D_t \max(0, h_t - P_t)|S_{t-1}) \\
&= h_t^{-1} \int (h_t - p)\mathbf{1}[h_t - p > 0]\mathbf{1}[z < -y_{t-1}^\star] d\mathbb{P}_{P_t, Z_t|S_{t-1}}(p, z) \\
&= h_t^{-1} \int (h_t - \exp\{\iota_t + e\})\mathbf{1}[z < -y_{t-1}^\star]\mathbf{1}[e < \log h_t - \iota_t] dF_t(e, z) \\
&= h_t^{-1} \Big[h_t \int \mathbf{1}[z < -y_{t-1}^\star]\mathbf{1}[e < \log h_t - \iota_t] dF_t(e, z) \\
&\qquad - \exp\{\iota_t\} \int \mathbf{1}[z < -y_{t-1}^\star]\mathbf{1}[e < \log h_t - \iota_t] \exp\{e\} dF_t(e, z) \Big] \\
&= F_t(-y_{t-1}^\star, \log h_t - \iota_t) - \exp\{\iota_t - \log h_t\} \int \mathbf{1}[z < -y_{t-1}^\star]\mathbf{1}[e < \log h_t - \iota_t] \exp\{e\} dF_t(e, z) \\
&= H_t(\iota_t - \log h_t) - \exp\{\iota_t - \log h_t\} \int \mathbf{1}[e < \log h_t - \iota_t] \exp\{e\} dH_t(e)
\end{aligned}
$$

which proves (i) in its variant (14) (at the last equality, we have used the fact that, for any positive measurable f, $\int f(e)\mathbf{1}[z \leq -y_{t-1}^\star] dF_t(z, e) = \int f(e) dH_t(e)$ which may be easily verified e.g. by an approximation by simple functions). Finally, put $\eta_t(e; \xi_t) = \frac{\partial}{\partial e} F_t(-y_t^\star, e; \xi_{t-1})$ (the existence of the derivative is guaranteed by Lemma 1 (iii)) and note that η_t is a density corresponding to H_t. Using the By Parts formula, we get

$$
\int_{-\infty}^{-s} \exp(x) dH_t(x) = \int_{-\infty}^{-s} \exp(x)\eta_t(x) dx = H_t(-s)\exp\{-s\} - \int_{-\infty}^{-s} H_t(x)\exp(x) dx,
$$

which, plugged into (14), gives (15).
Ad (ii). By differentiating (14) with $\eta_t(x)dx$ in place of $dH_t(x)$ we get, using the Leibnitz Rule, that

$$
\begin{aligned}
\frac{\partial}{\partial s}\lambda_t^\star(s) &= -\eta_t(-s) - \exp(s)\left(\int_{-\infty}^{-s} \exp(x)\eta_t(x) dx - \exp(-s)\eta_t(-s) \right) \\
&= -\exp(s) \int_{-\infty}^{-s} \exp(x)\eta_t(x) dx < 0 \qquad (16)
\end{aligned}
$$

which proves the monotonicity of $\lambda_t^\star$ in s, implying the monotonicity of λ_t in ι_t. The monotonicity in y_t follows from that of H_t, which is inherited from F_t, and the strict positivity of the integrand in (15).
Ad (iii). The continuous differentiability of λ_t in ι_t follows from that of $\lambda_t^\star$ in s (see (16) and note that the continuity of the derivative follows from absolute continuity of the Lebesgue measure). The differentiability in ξ_t^i (the i-th coordinate of ξ_t) and the continuity of the derivative follows from Lemma 4. In particular, by differentiating (15) we get

$$
\frac{\partial}{\partial \xi_t^i}\lambda^\star(s, \xi_t) = \exp(s) \int_{-\infty}^{-s} \frac{\partial}{\partial \xi_t^i} F_t(-y_t^\star, x; \xi_t)\exp(x) dx;
$$

with the integrable upper bound being equal to $\exp(x)\alpha_t^{(z)}$ (if we differentiate according to y_t) or $\exp(x)\max_{|\xi - \xi_t| \leq \epsilon} \alpha_t^{(i)}(\xi)$ (in the remaining cases). The continuity of the derivative follows from the well known theorem guaranteeing continuity of a Lebesgue integral dependent on a parameter

([18], Theorem 9.1.). (iii) now follows from the continuous differentiability of h_t.[6]
Ad (iv). As $\lambda_t^\star(s) \geq 0$ by (14) and $\lambda_t^\star(s) \leq H_t(-s)$ by (15), necessarily

$$\lim_{\iota_t \to \infty} \lambda_t(\iota_t, \xi_t) = \lim_{s \to \infty} \lambda_t^\star(s, \xi_t) = 0. \tag{17}$$

Further, we have

$$\iota_t \to -\infty \Longrightarrow P_t \to 0 \text{ a. s. } \Longrightarrow G_t \to 1 \text{ a. s.}$$
$$\Longrightarrow \mathbb{E}(L_t|S_{t-1} = 1) \to \mathbb{E}(D_t|S_{t-1} = 1) \Rightarrow \lambda_t(\iota_t, \xi_t) \to q_t$$

(the expectations converge because L_t and D_t are uniformly bounded). Therefore and thanks to (17), point (ii) of the present Proposition and the continuity of λ_t (following from its differentiability in ι_t) suffices for (iv).
Ad (v). The assertion may be proved analogously to (v) of Proposition 1.
Ad (vi). For any $1 \leq i, j \leq \dim(\xi_t)$,

$$\frac{\partial}{\partial \xi^i \partial \xi^j} \lambda_t^\star(s, \xi_t) = \exp(s) \int_{-\infty}^{-s} \frac{\partial}{\partial \xi^i \partial \xi^j} F_t(-y_t^\star, x; \xi_t) \exp(x) dx$$

(an integrable upper bound here is $\exp(x) \max_{|\xi - \xi_t| \leq \epsilon} \alpha_t^{(i,j)}(\xi)$). Further, by differentiation of (16),

$$\frac{\partial}{\partial s \partial s} \lambda_t^\star = -\exp(s) \left[\int_{-\infty}^{-s} \exp(x) \eta_t(x) dx - \exp(-s) \eta_t(-s) \right]$$
$$= \eta_t(-s) - \exp(s) \int_{-\infty}^{-s} \exp(x) \eta_t(x) dx$$

(the derivative is continuous because η_t is continuous by Lemma 1 (iii)) and, by differentiation of (16) again,

$$\frac{\partial}{\partial s \partial \xi_t^i} \lambda_t^\star = -\exp(s) \int_{-\infty}^{-s} \exp(x) \frac{\partial}{\partial x \partial \xi_t^i} F_t(-y_t^\star, x) dx$$

(an integrable upper bound being $\exp(x) \max_{|\xi - \xi_t| \leq \epsilon} \alpha_t^{(z,j)}(\xi)$ here). As $\frac{\partial}{\partial s \partial \xi_t^i} \lambda_t^\star = \frac{\partial}{\partial \xi_t^i \partial s} \lambda_t^\star$ thanks to the continuity of the l.h.s., we have proved that $\lambda_t^\star$ is twice continuously differentiable in (s, ξ_t) which, together with the second continuous differentiability of h_t[7] proves the continuous differentiability of λ_t in (ι_t, ξ_t) which itself suffices for (vi) (for details, the the analogous proof of Proposition 1 (vi)). □

Next Corollary discusses two important special cases

Corollary 1. *(i) If $U \perp\!\!\!\perp V$ then*

$$\lambda_t = q_t \gamma_t, \qquad \gamma_t = \gamma_t(\iota_t, \theta) = \gamma_t^\star(\iota_t - \log h_t, \theta)$$

where

$$\gamma_t^\star(s, \theta) = R_t(-s) - \exp(s) \int_{-\infty}^{-s} \exp(x) dR_t(x) = \exp(s) \int_{-\infty}^{-s} R_t(x) \exp(x) dx$$
$$R_t(e) = R_t(e; \theta) = \mathbb{P}[E_t \leq e]. \tag{18}$$

[6] The differentiability of h_t follows similarly as that of b, see Footnote 4.
[7] Proved analogously to Footnote 4

(ii) If, $U \perp\!\!\!\perp V$ and $V_1 \sim \mathcal{N}(0,1)$ then

$$\gamma_t^\star(s,\theta) = \varphi(-\frac{s}{\rho_t}) - \exp\left\{s + \frac{\rho_t^2}{2}\right\}\varphi\left(\frac{-s-\rho_t^2}{\rho_t}\right), \qquad \rho_t^2 = \begin{cases} \rho^2 \cdot \frac{1-\psi^{2t+2}}{1-\psi^2} & \psi \neq 1 \\ \rho^2 t & \psi = 1 \end{cases} \tag{19}$$

where φ is the standard normal c.d.f.

Proof. (i) follows from the fact that

$$U \perp\!\!\!\perp V \Rightarrow E \perp\!\!\!\perp Z \Rightarrow E_t \perp\!\!\!\perp Z_t, S_{t-1}$$

(in the last implication, we have used the fact that S_{t-1} is a function of $Z_1, Z_2, \ldots, Z_{t-1}$) implying

$$\begin{aligned} H_t(e) &= \mathbb{P}[Z_t \leq -y_{t-1}^\star, E_t \leq e | S_{t-1} = 1] \\ &= \mathbb{P}[E_t \leq e | Z_t \leq -y_{t-1}^\star, S_{t-1} = 1]\mathbb{P}[Z_t \leq -y_{t-1}^\star | S_t = 1] = R_t(e) q_t \end{aligned}$$

which, plugged into (14) and (15), gives (i).
Ad (ii). By an easy calculation we get that $E_t \sim \mathcal{N}(0, \rho_t^2)$, i.e.,

$$R_t(e) = \varphi\left(\frac{e}{\rho_t}\right), \qquad dR_t(e) = \frac{1}{\rho_t\sqrt{2\pi}} \exp\left\{-\frac{e^2}{2\rho_t^2}\right\} dx,$$

and, consequently,

$$\begin{aligned} \int_{-\infty}^{-s} \exp(x) dR_t(x) &= \int_{-\infty}^{-s} \frac{1}{\rho_t\sqrt{2\pi}} \exp\left(x - \frac{x^2}{2\rho_t^2}\right) dx \\ &= \int_{-\infty}^{-s} \frac{1}{\rho_t\sqrt{2\pi}} \exp\left(-\frac{(x-\rho_t^2)^2}{2\rho_t^2} + \frac{\rho_t^2}{2}\right) dx \\ &= \exp\left\{\frac{\rho_t^2}{2}\right\} \mathbb{P}[\mathcal{N}(\rho_t^2, \rho_t^2) \leq -s] = \exp\left\{\frac{\rho_t^2}{2}\right\} \varphi\left(\frac{-s-\rho_t^2}{\rho_t}\right) \end{aligned}$$

which, together with (i), gives (ii). □

Now we may prove the required properties of a mapping, which will later be shown to fulfill (5).

4 Main Results

We start the present Section by the construction of the mapping transforming the factors to the rates.

Theorem 1. *Let $t \geq 1$.*

(i)

$$(Q_1, G_1, \ldots, Q_t, G_t) = \Phi_t(Y_1, I_1, \ldots Y_t, I_t; \theta) \tag{20}$$

where

$$\begin{aligned} &\Phi_t(y_1, \iota_1, \ldots, y_t, \iota_t; \theta) \\ &\quad = (q_1(y_1,\theta), g_1(\iota_1, y_1, \theta), \ldots, q_t(y_t, \ldots, y_1, \theta), g_t(\iota_t, y_t, \ldots, y_1, \theta)) \end{aligned} \tag{21}$$

and

$$g_\tau(\iota_\tau, y_\tau, \ldots, y_1, \theta) = \frac{\lambda_\tau(\iota_\tau, y_\tau, \ldots, y_1, \theta)}{q_\tau(y_\tau, \ldots, y_1, \theta)}, \qquad 1 \leq \tau \leq t.$$

Moreover, Φ_t is one-to-one between $\mathbb{R}^{2\times t}$ and $(0,1)^{2\times t}$ and is continuously differentiable in $(y_1, \iota_1, \ldots, y_t, \iota_t, \theta)$ (see Propositions 1 and 2 for definitions of q_τ, λ_τ, respectively).

(ii) *The inversion of Φ_t is given by*

$$\Psi_t(z_1, x_1, \dots, z_t, x_t; \theta) \\ = (\tilde{y}_1(z_1;\theta), \tilde{\iota}_\tau(z_1, x_1;\theta), \dots, \tilde{y}_t(z_t, \dots, z_1;\theta), \tilde{\iota}_t(x_t, z_t, \dots, z_1;\theta)) \quad (22)$$

where

$$\tilde{y}_1(z_1;\theta) = q_1^{-1}(z_1, \theta), \qquad \tilde{\iota}_1(z_1, x_1;\theta) = \lambda_1^{-1}(x_1 z_1, \tilde{y}_1(z_1;\theta), \theta), \qquad z_1, x_1 \in (0,1),$$

and, for any $\tau > 1$,

$$\tilde{y}_\tau(z_\tau, \dots z_1;\theta) = q_\tau^{-1}(z_\tau, \tilde{y}_{\tau-1}(z_{\tau-1}, \dots, z_1;\theta), \dots, \tilde{y}_1(z_1;\theta);\theta), \\ \tilde{\iota}_\tau(x_\tau, z_\tau, \dots z_1;\theta) = \lambda_\tau^{-1}(x_\tau z_\tau, \tilde{y}_\tau(z_1, \dots, z_\tau;\theta), \dots, \tilde{y}_1(z_1;\theta);\theta), \\ z_1, z_2, \dots, z_\tau, x_\tau \in (0,1) \quad (23)$$

(the inversions of q_τ and λ_τ are meant with respect to their first arguments). Moreover, Ψ_t is continuously differentiable in $(z_1, x_1, \dots, z_t, x_t, \theta)$.

(iii) *If (10), (11) and (12) hold true then Φ_t and Ψ_t are twice continuously differentiable in $(y_1, \iota_1, \dots, y_t, \iota_t, \theta)$, $(z_1, x_1, \dots, z_t, x_t, \theta)$, respectively.*

Proof. We start with the proof of (ii). If $t > 1$, then the assertion may be easily verified by Proposition 1 (iv) and Proposition 2 (iv). Let $t > 1$ and assume (ii) to hold for $t-1$. Let $\nu_t = (z_1, x_1, \dots, z_t, x_t) \in (0,1)^{2\times t}$. Then

$$\Phi_t(\Psi_t(\nu_t)) = \Phi_t(\Psi_{t-1}(\nu_{t-1}), q_t^{-1}(z_t, \tilde{y}_{t-1}, \dots, \tilde{y}_1;\theta), \lambda_t^{-1}(z_t x_t, \tilde{y}_t, \dots, \tilde{y}_1;\theta)) \\ = \Big(\Phi_{t-1}(\Psi_{t-1}(\nu_{t-1})), q_t(q_t^{-1}(z_t, \tilde{y}_{t-1}, \dots, \tilde{y}_1;\theta), \tilde{y}_{t-1}, \dots, \tilde{y}_1;\theta), \\ \frac{\lambda_t(\lambda_t^{-1}(z_t x_t, \tilde{y}_t, \dots, \tilde{y}_1;\theta), \tilde{y}_t, \dots, \tilde{y}_1;\theta)}{q_t(q_t^{-1}(z_t, \tilde{y}_{t-1}, \dots, \tilde{y}_1;\theta), \tilde{y}_{t-1}, \dots, \tilde{y}_1;\theta)}\Big) \\ = (z_1, x_1, \dots, z_t, x_t) = \nu_t. \quad (24)$$

Similarly we could show that $\Phi_t(\Psi_t(\upsilon_t)) = \upsilon_t$ for any $\upsilon_t \in \mathbb{R}^{2\times t}$ which, together with (24), would prove that Ψ_t is an inverse of Φ_t. The differentiability of Ψ_t is easy to verify using (v)'s of Propositions 1, 2, respectively, and the Chain Rule for Derivatives.

Ad (i). Let $1 \le \tau \le t$. First, assume (7) (i.e. that Y and I are deterministic). Then, by the Strong Law of Large Numbers ([15] Theorem 4.23), $\frac{\sum_{1\le i\le N} Q_\tau^i}{N} \to \mathbb{P}[Q_\tau^1 = 1]$ and $\frac{N_{\tau-1}}{N} \to \mathbb{P}[S_{\tau-1}^1 = 1]$ almost sure, hence in probability, so, by Lemma 3 (see Appendix),

$$Q_\tau = \frac{p\lim_N \frac{\sum_{1\le i\le N} Q_\tau^i}{N}}{p\lim_N \frac{N_{\tau-1}}{N}} = \frac{\mathbb{P}[Q_\tau^1 = 1]}{\mathbb{P}[S_{\tau-1}^1 = 1]} \\ = \frac{\mathbb{P}[Q_\tau^1 = 1 | S_{\tau-1}^1 = 1]\mathbb{P}[S_{\tau-1}^1 = 1] + \mathbb{P}[Q_\tau^1 = 1 | S_{\tau-1}^1 = 0]\mathbb{P}[S_{\tau-1}^1 = 0]}{\mathbb{P}[S_{\tau-1}^1 = 1]} \\ = \mathbb{P}[Q_\tau^1 = 1 | S_{\tau-1}^1 = 1] = q_\tau(Y_\tau, \dots, Y_1, \theta) \quad (25)$$

by Proposition 1 (i) (we have used the fact that $Q_\tau^1 = 0$ on $[S_{\tau-1}^1 = 0]$ implying $\mathbb{P}[Q_\tau^1 = 1 | S_{\tau-1}^1 =$

$0] = 0)$ and, by analogous arguments,

$$G_\tau = \frac{p\lim_N L_\tau^N}{p\lim_N Q_\tau^N} = \frac{\mathbb{E}[L_\tau^1]}{\mathbb{P}[Q_\tau^1 = 1]} =$$
$$= \frac{\mathbb{E}[L_\tau^1|S_{\tau-1}^1 = 1]\mathbb{P}[S_{\tau-1}^1 = 1] + \mathbb{E}[L_\tau^1|S_{\tau-1}^1 = 0]\mathbb{P}[S_{\tau-1}^1 = 0]}{\mathbb{P}[Q_\tau^1 = 1|S_{\tau-1}^1 = 1]\mathbb{P}[S_{\tau-1}^1 = 1]}$$
$$= \frac{\mathbb{E}[L_\tau^1|S_{\tau-1}^1 = 1]}{\mathbb{P}[Q_\tau^1 = 1|S_{\tau-1}^1 = 1]} = \frac{\lambda_\tau(I_\tau, Y_\tau, \dots, Y_1, \theta)}{q_\tau(Y_\tau, \dots, Y_1, \theta)} \tag{26}$$

by Proposition 1 (i) and Proposition 2 (i) (we have used fact that $\mathbb{E}[L_\tau^1|S_{\tau-1}^1 = 0] = 0$ which is true because $S_{\tau-1}^1 = 0 \Rightarrow Q_{\tau-1}^1 = 0 \Rightarrow L_\tau^1 = 0$).

Now, stop assuming (7) (i.e. let Y, I be stochastic again). Then, however, (25) and (26), and consequently (20), hold by Lemma 2 (notice that $(\Xi_\tau)_{\tau\geq 1} \perp\!\!\!\perp (Y_\tau, I_\tau)_{\tau\geq 1}$ and that there are no other random elements than

$$(\Xi_\tau, Y_\tau, I_\tau)_{\tau\geq 1}$$

in our setting). The differentiability follows from (iii) Proposition 1 (iii) and Proposition 2 (iii). The one-to-one property follows from the fact that the image of Φ_t is a subset of $(0,1)^{2\times t}$ and from the proof of (ii), during which it was shown that a unique $\Phi_t^{-1}(\nu)$ exists for any $\nu \in (0,1)^{2\times t}$.
Ad (iii). The assertion follows from from Proposition 1 (vi) and Proposition 2 (vi). ☐

Given the mutual independence of the factors, the situation simplifies:

Corollary 2. *If $U \perp\!\!\!\perp V$ then*

$$g_\tau(\iota_\tau, y_\tau, \dots, y_1, \theta) = \gamma_\tau(\iota_\tau, \theta), \qquad \tilde{\iota}_t(x_\tau, z_\tau, \dots, x_1; \theta) = \gamma_\tau^{-1}(x_\tau, \theta), \qquad \tau \geq 1$$

(see Corollary 1 for the definition of γ_τ).

Proof. The assertion is a direct consequence of Corollary 1 (i). ☐

Next we show that the one-to-one property holds also between $Y's$ and $Q's$ alone:

Corollary 3. *For any $t \geq 1$,*

$$(Q_1, \dots, Q_t) = \Phi_t^1(Y_1, \dots, Y_t; \theta) \tag{27}$$

where

$$\Phi_t^1(y_1, \dots, y_t; \theta) = (q_\tau(y_\tau, \dots, y_1; \theta))_{1\leq\tau\leq t} \tag{28}$$

is a bijection between $\mathbb{R}^t$ and $(0,1)^t$, is continuously differentiable in $(y_1, \dots, y_t, \theta)$ and has inversion

$$\Psi_1^1(z_1, \dots, z_t; \theta) = (\tilde{y}_\tau(z_\tau, \dots, z_1; \theta))_{1\leq\tau\leq t}. \tag{29}$$

which is continuously differentiable in $(z_1, \dots, z_\tau, \theta)$. Given (10), (11) and (12), Φ_t^1 and Ψ_t^1 are twice continuously differentiable in $(y_1, \dots, y_t, \theta)$, $(z_1, \dots, z_t, \theta)$, respectively.

Proof. Formula (27) and the differentiability of both Φ^1 and Ψ_t^1 follow directly from Theorem 1. The fact that Ψ_t^1 is the inversion of Φ_t^1 may be proved analogously to the proof of Theorem 1. ☐

Before leaving the topic of Φ_t, let us present several formulas suitable for working with individual loans.

Theorem 2. *For any $t \geq 1$, any $i \in \mathbb{N}$, and any $y_1, \iota_1, \dots, y_t, \iota_t \in \mathbb{R}$,*

(i)

$$\mathbb{P}[A_t^i \le a, P_t^i \le e | S_{t-1}^i = 1, Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t] \\ = F_t(\log a - y_t, \log e - \iota_t; y_{t-1}, \dots, y_1, \theta),$$

(see Lemma 1 for the definition of F_t)

(ii) $\mathbb{P}[Q_t^i = 1 | S_{t-1}^i = 1, Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t] = q_t(y_t, \dots, y_1, \theta)$,

(iii)

$$\mathbb{E}(L_t^i | S_{t-1}^i = 1, Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t) \\ = \begin{cases} q_t(y_t, \dots, y_1, \theta)\gamma_t(\iota_t, \theta) & \textit{if } U \perp\!\!\!\perp V \\ \lambda_t(\iota_t, y_t, \dots, y_1, \theta) & \textit{otherwise,} \end{cases}$$

(iv)

$$\mathbb{E}(G_t^i | Q_t^i = 1, S_{t-1}^i = 1, Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t) \\ = \begin{cases} \gamma_t(\iota_t, \theta) & \textit{if } U \perp\!\!\!\perp V \\ \frac{\lambda_t(\iota_t, y_t, \dots, y_1, \theta)}{q_t(y_t, \dots, y_1, \theta)} & \textit{otherwise.} \end{cases}$$

Proof. Ad (i). By basic probability calculus, applied to the conditional distribution given (Y,I),

$$\mathbb{P}[A_t^i \le a, P_t^i \le e | S_{t-1}^i = 1, Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t] \\ = \frac{\mathbb{P}[A_t^i \le a, P_t^i \le e, S_{t-1}^i = 1 | Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t]}{\mathbb{P}[S_{t-1}^i = 1 | Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t]}. \quad (30)$$

Given (7), we would have

$$\mathbb{P}[S_{t-1}^i = 1, Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t] \\ = \mathbb{P}[S_{t-1}^i = 1] = \prod_{1 \le \tau \le t-1} \mathbb{P}[S_\tau^i = 1 | S_{\tau-1}^i = 1] = \prod_{1 \le \tau \le t-1} (1 - q_\tau)$$

and

$$\mathbb{P}[A_t^i \le a, P_t^i \le e, S_{t-1}^i = 1, Y_1 = y_1, I_1 = \iota_1, \dots, Y_t = y_t, I_t = \iota_t] \\ = \mathbb{P}[A_t^i \le a, P_t^i \le e, S_{t-1}^i = 1] \\ = \mathbb{P}[A_t^i \le a, P_t^i \le e | S_{t-1}^i = 1]\mathbb{P}[S_{t-1}^i = 1] \\ F_t(\log a - y_t, \log e - \iota_t; y_{t-1}, \dots, y_1, \theta) \prod_{1 \le \tau \le t-1} (1 - q_\tau).$$

As, by Lemma 2, the last two formulas hold with (7) released, (i) follows by plugging these formulas into (30).
Ad (iv). We have

$$\mathbb{E}(L_t^i S_{t-1}^i | Y_1 = y_1, \dots) = \mathbb{E}(G_t^i Q_t^i S_{t-1}^i | Y_1 = y_1, \dots) = \\ \mathbb{E}(G_t^i | Q_t^i S_{t-1}^i = 1, Y_1 = y_1, \dots)\mathbb{P}[Q_t^i S_{t-1}^i = 1 | Y_1 = y_1, \dots] \\ + \underbrace{\mathbb{E}(G_t^i | Q_t^i S_{t-1}^i = 0, Y_1 = y_1, \dots)}_{=0} \mathbb{P}[Q_t^i S_{t-1}^i = 0 | Y_1 = y_1, \dots]$$

giving

$$\mathbb{E}(G_t^i|Q_t^iS_{t-1}^i = 1, Y_1 = y_1, \dots) = \frac{\mathbb{E}(L_t^iS_{t-1}^i|Y_1 = y_1, \dots)}{\mathbb{P}[Q_t^iS_{t-1}^i = 1|Y_1 = y_1, \dots]}.$$

As, by Lemma 2,

$$\begin{aligned}\mathbb{E}(L_t^iS_{t-1}^i|Y_1 = y_1, \dots) &= \mathbb{E}(L_t^iS_{t-1}^i) \\ &= \mathbb{E}(L_t^i|S_{t-1}^i = 1)\mathbb{P}[S_{t-1}^i = 1] + \underbrace{\mathbb{E}(L_t^i|S_{t-1}^i = 0)}_{=0}\mathbb{P}[S_{t-1}^i = 0] = \lambda_t \prod_{1\le\tau\le t-1}(1 - q_\tau)\end{aligned}$$

and

$$\mathbb{P}[Q_t^i = 1, S_{t-1}^i = 1|Y_1 = y_1, \dots] = q_t \prod_{1\le\tau\le t-1}(1 - q_\tau);$$

(iv) is proved (see Corollary 1 for the case $U \perp\!\!\!\perp V$).
The proofs of (ii) and (iii) are analogous.

□

Finally let us construct mapping Λ, promised by (6).

Theorem 3. *It holds that*

$$L = \Lambda(Y_1, I_1, \dots, Y_m, I_m; \theta) \tag{31}$$

where

$$\Lambda(y_1, \iota_1, \dots, y_m, \iota_m; \theta) = \sum_{t=1}^{m} \beta^{t-1}\lambda_t(\iota_t, y_t, \dots y_1, \theta) \prod_{\tau=1}^{t-1}[1 - q_\tau(y_\tau, \dots, y_1, \theta)]. \tag{32}$$

Moreover, Λ is continuously differentiable in all $y_1, \iota_1, \dots, y_m, \iota_m, \theta$. If, in addition, (10), (11) and (12) hold true then Λ is twice continuously differentiable in all $y_1, \iota_1, \dots, y_m, \iota_m, \theta$ (see Proposition 1, Proposition 2 and Corollary 1 for definitions of q_t and λ_t, respectively).

Proof. The proof is similar to that of Theorem 1: If Y and I were deterministic (assumption (7)) then we would get, using Lemma 3, the Law of Large Numbers and Proposition 2 (i), that

$$\begin{aligned}L = \text{plim}_{N\to\infty} L_N &= \sum_{i=1}^{m} \beta^{t-1}\text{plim}_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} L_t^i, = \sum_{t=1}^{m} \beta^{t-1}\mathbb{E}(L_t^1) \\ &= \sum_{t=1}^{m} \beta^{t-1}\left(\mathbb{E}(L_t^1|S_{t-1}^1 = 1)\mathbb{P}[S_{t-1}^1 = 1] + \underbrace{\mathbb{E}(L_t^1|S_{t-1}^1 = 0)}_{=0}\mathbb{P}[S_{t-1}^1 = 0]\right) \\ &= \sum_{t=1}^{m} \beta^{t-1}\lambda_t\mathbb{P}[S_{t-1}^1 = 1].\end{aligned} \tag{33}$$

As, for any $\tau \ge 1$,

$$\begin{aligned}\mathbb{P}[S_\tau^1 = 1] &= \mathbb{P}[S_\tau^1 = 1|S_{\tau-1}^1 = 1]\mathbb{P}[S_{\tau-1}^1 = 1] + \underbrace{\mathbb{P}[S_\tau^1 = 1|S_{\tau-1}^1 = 0]}_{=0}[S_{\tau-1}^1 = 0] \\ &= (1 - q_\tau)\mathbb{P}[S_{\tau-1}^1 = 1]\end{aligned}$$

(by Proposition 1 (i)), it is

$$\mathbb{P}[S_{t-1}^1 = 1] = \prod_{\tau=1}^{t-1}(1 - q_\tau),$$

which, together with (33) gives (31) for deterministic Y, I. Similarly to the proof of Theorem 1, the validity of (31) given stochastic Y, I follows by Lemma 2. The differentiability if Λ is clearly inherited from that of the q's and theλ's (see Propositions 1 and 2). $\square$

Remark 4. *If $U \perp\!\!\!\perp V$ then we may use the fact that*

$$\lambda_t(\iota_t, y_t, \ldots y_1, \theta) = q_t(y_t, \ldots y_1, \theta)\gamma_t(\iota_t, \theta), \qquad 1 \leq t \leq m,$$

in computation of (32) (see Corollary 1).

5 Numerical Computation of the Transformation

In the present Section, we discuss practical computation of mapping Φ_t^1 transforming Y's to Q's, and its inverse. The transformation between I's and G's is left aside because its computation is relatively easy in the case of independent individual factors, requiring no broader discussion, while its treatment in the general case of dependent factors, despite it could be done similarly as in the case of Φ^1 (see also Remark 5 at the end of the Section), would add undue complexity to the present text.

Let $y_1, \ldots, y_t \in \mathbb{R}$ be constants. We start by describing the dynamics of the conditional distribution of the individual wealth factor.

Proposition 3. *Denote $\mathcal{U}_1$ and $\mathcal{U}$ the distributions of $\sigma_1 \cdot U_1$, $\sigma \cdot U_2$, respectively. Denote*

$$\mathcal{Z}_t = \mathcal{Z}_t(y_1, \ldots, y_t) = \mathcal{L}(Z_t | S_{t-1} = 1, Y_1 = y_1, \ldots, Y_t = y_t), \qquad t \geq 1.$$

It holds that

$$\mathcal{Z}_1 = \mathcal{U}_1, \tag{34}$$

and, for any $t > 1$,

(i)

$$\mathcal{Z}_t = \phi \cdot \mathcal{T}(\mathcal{Z}_{t-1}, -y^\star_{t-1}) \circ \mathcal{U}, \qquad t > 1, \tag{35}$$

where $\circ$ denotes convolution, $\mathcal{T}(\bullet; y)$ is an operator of truncation[8] *at y and $a \cdot \bullet$ stands for scaling, i.e. multiplication of a corresponding random variable by constant a ($y^\star_{t-1}$ is defined in Lemma 1).*

(ii) *Alternatively*

$$\mathcal{Z}_t = \phi \cdot \mathcal{M}(\mathcal{Z}_{t-1}, q_t(y_1, \ldots, y_t)) \circ \mathcal{U}, \qquad t > 1, \tag{36}$$

where

$$\mathcal{M}(\mathcal{Z}, p) = \mathcal{T}(\mathcal{Z}, \chi(\mathcal{Z}, p))$$

and, for any distribution $\mathcal{Z}$,

$$\chi(\mathcal{Z}, \alpha) = \inf(x; \mathcal{Z}(-\infty, x] \geq \alpha)$$

denotes its α-quantile (q_t is defined by in Proposition 1).

[8] By *truncation* of a distribution with c.d.f. F at a we understand a distribution with c.d.f. $\hat{F}(x) = \max\left(0, \frac{F(x)-F(a)}{1-F(a)}.\right)$

Proof. Formula (34) is straightforward.
Ad (i). For any τ, denote $G_\tau(\bullet) = F_\tau(\bullet, \infty; y_1, \dots y_{t-1})$ the c.d.f. of $\mathcal{Z}_\tau$. First, note that, by (i) of Lemma 1, G_t may be rewritten as

$$G_t(z) = \frac{\int_{\{x \geq -y^\star_{t-1}, s \in \mathbb{R}\}} W(\frac{z-\phi x}{\sigma}, \infty) dF_{t-1}(x,s)}{1 - F_{t-1}(-y^\star_{t-1}, \infty)}$$
$$= \frac{\int_{\{x \geq -y^\star_{t-1}\}} W(\frac{z-\phi x}{\sigma}, \infty) dG_{t-1}(x)}{1 - G_{t-1}(-y^\star_{t-1})}$$
$$= \int_{\{x \geq -y^\star_{t-1}\}} W(\frac{z-\phi x}{\sigma}, \infty) d\frac{G_{t-1}(x)}{1 - G_{t-1}(-y^\star_{t-1})} = \int W(\frac{z-\phi x}{\sigma}, \infty) d\tilde{G}_{t-1}(x) \quad (37)$$

where

$$\tilde{G}_{t-1}(x) = \max\left(0, \frac{G_{t-1}(x) - G_{t-1}(-y^\star_{t-1})}{1 - G_{t-1}(-y^\star_{t-1})}\right)$$

(we have used the facts that adding constants to c.d.f.'s does not change integrals). As (37) is exactly the formula for convolution of $\phi \cdot X$ and $\sigma \cdot U$ where U is a variable with c.d.f. $W(\bullet, \infty)$ and X is a variable with c.d.f. $\hat{G}_{t-1}$, and as $\tilde{G}_{t-1}$ is nothing else but the c.d.f. of $\mathcal{Z}_{t-1}$ truncated at $-y^\star_{t-1}$, (i) is proved.
Ad (ii). The assertion holds thanks to (i) and because $\chi(\mathcal{Z}_t, q_t(y_t, \dots y_1)) = -y^\star_t$ by the definitions of q_t and $\mathcal{Z}_t$. $\square$

Because formula for $\mathcal{Z}_t$ is intractable already starting from t=2, we use an approximation - namely discretization - to evaluate the dynamics of $\mathcal{Z}_t$. In particular, we replace $\mathcal{U}_1$ and $\mathcal{U}$ by atomic distributions $\mathcal{W}_1$,$\mathcal{W}$, respectively, defined by

$$\mathcal{W}_1 = \mathcal{D}_{h,-n,n}(\mathcal{U}_1), \qquad \mathcal{W} = \mathcal{D}_{h,-n,n}(\mathcal{U}) \tag{38}$$

where h is small enough and n is large enough and where

$$\mathcal{D}_{\eta,n_1,n_2}(\mathcal{P}) = \sum_{i=n_1}^{n_2} \delta_{i\eta} \mathcal{P}\left[I_i\right], \qquad I_i = \begin{cases} (-\infty, n_1\eta + \frac{\eta}{2}] & i = n_1 \\ (i\eta - \frac{\eta}{2}, i\eta + \frac{\eta}{2}] & n_1 < i < n_2 \\ (n_2\eta - \frac{\eta}{2}, \infty) & i = n_2 \end{cases}$$

is an operator of discretization of a distribution $\mathcal{P}$, with bandwidth η and range (n_1, n_2) (here, δ_x stands for the Dirac measure concentrated in x).

First, let us discuss the computation of Φ^1 in the case when Y's are known and Q's are computed; here, we approximate $\mathcal{Z}_t$, $t \geq 1$, by distributions $\mathcal{A}_t$, $t \geq 1$, such that

$$\mathcal{A}_1 = \mathcal{W}_1$$

and, for each $t > 1$, $\mathcal{A}_t$ is computed by the following steps:

1. $\mathcal{A}^\star_t \leftarrow \mathcal{T}(\mathcal{A}_{t-1}, -Y^\star_{t-1})$ (truncation at $-Y^\star_{t-1}$)
2. $\mathcal{A}^S_t \leftarrow \mathcal{S}(\mathcal{A}^\star_t, \phi)$ where $\mathcal{S}(\bullet, \phi)$ is an operator of scaling by ϕ and consequent re-discretization so that the distance between atoms is h
3. $\mathcal{A}^\circ_t \leftarrow \mathcal{A}^S_t \circ \mathcal{W}$ (convolution with $\mathcal{W}$)

4. $\mathcal{A}_t \leftarrow \mathcal{C}(\mathcal{A}_t^\circ, \epsilon_t)$ where $\mathcal{C}(\bullet, \epsilon_t)$ is an operator of two-sided censoring[9] done so that the censored mass is no greater than a prescribed ϵ_t and simultaneously the range of the resulting distribution is minimal (this step is done in order to save the computational time by accumulating tails of negligible probability).

Consequently, Q_t's are approximated by

$$\tilde{Q}_t = \mathcal{A}_t(-\infty, -Y_t^\star), \qquad t \geq 1.$$

The following Proposition estimates the accuracy of this approximation; before formulating it, recall that the Kolmogorov distance of two distributions $\mathcal{F}, \mathcal{G}$ with c.d.f.'s F, G, respectively, is defined as

$$\varrho(\mathcal{F}, \mathcal{G}) = \sup_x |F(x) - G(x)|.$$

Proposition 4. *For any $t \geq 1$,*

$$|\tilde{Q}_t - Q_t| \leq \delta_t,$$

where $\delta_1 = \varrho(\mathcal{W}_1, \mathcal{U}_1)$ and, for each $t > 1$,

$$\delta_t = \frac{2\delta_{t-1}}{\mathcal{A}_{t-1}(-Y_{t-1}^\star, \infty)} + \varrho(\mathcal{A}_t^S, \phi \cdot \mathcal{A}_t^\star) + \varrho(\mathcal{W}, \mathcal{U}) + \epsilon_t.$$

.

Proof. See Appendix C.1. □

When, on the other hand, rates Q_t are known and Y's unknown, then Y's may be approximated by

$$\tilde{Y}_t = -\chi(\mathcal{B}_t, Q_t) + \log t + \log b, \qquad t \geq 1, \tag{39}$$

where $\mathcal{B}_t$ follows the same dynamics as $\mathcal{A}_t$ with the difference that the truncation (step 1.) is done at $-\tilde{Y}_{t-1}^\star$ instead of at $-Y_{t-1}^\star$.

Proposition 5. *For any $t \geq 1$,*

$$\left|\tilde{Y}_t - Y_t\right| \leq \chi(\mathcal{B}_t, Q_t + \eta_t) - \chi(\mathcal{B}_t, Q_t - \eta_t)$$

where $\eta_1 = \varrho(\mathcal{W}_1, \mathcal{U}_1)$ and, for any $t > 0$,

$$\eta_t = \frac{|Q_{t-1} - \hat{Q}_t| + \eta_{t-1}}{1 - \max(Q_{t-1}, \hat{Q}_t)} + \varrho(\mathcal{B}_t^S, \phi \cdot \mathcal{B}_t^\star) + \varrho(\mathcal{W}, \mathcal{U}) + \epsilon_t, \qquad \hat{Q}_t = \mathcal{B}_{t-1}(-\infty, -\tilde{Y}_{t-1}^\star].$$

Proof. See Appendix C.2. □

Finally, let us conclude that our numerical method is convergent:

Theorem 4. *If $h \to 0$, $nh \to \infty$ and if $\epsilon_t \to 0$ for each t then $\tilde{Y} \to Y$ and $q \to Q$ in distribution.*

Proof. The assertion is easy to prove by showing, by induction, that, for each t, $\delta_t \to 0$ almost sure. □

[9] By *two-sided censoring* of a distribution with c.d.f. F at $a < b$ we mean a distribution with c.d.f.

$$\hat{F}(x) = \begin{cases} 0 & x < a \\ F(x) & a \leq x < b \\ 1 & x \geq b \end{cases}.$$

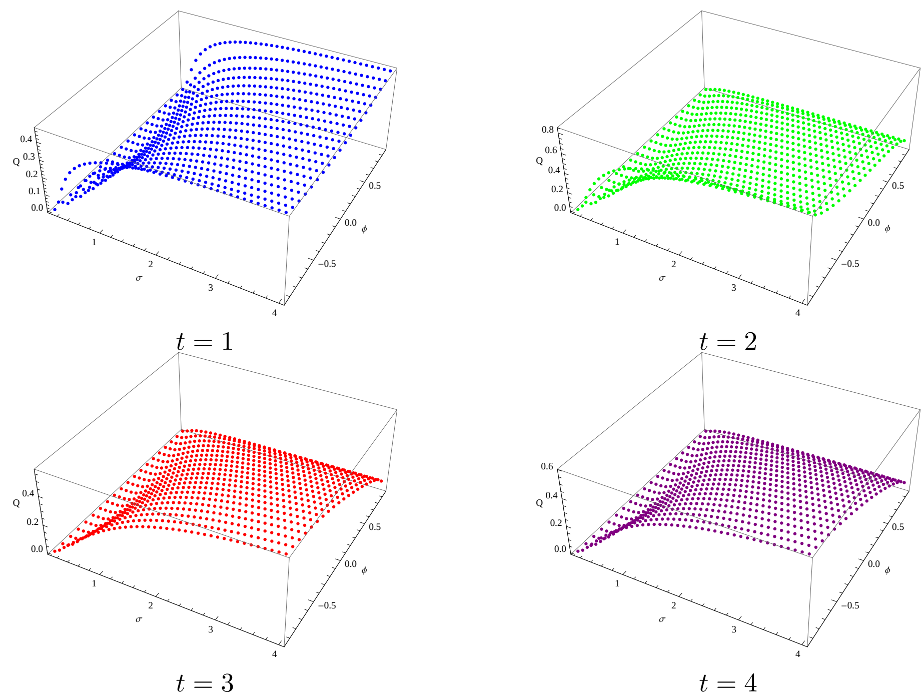

Figure 1: Q_1, Q_2, Q_3, Q_4 given that $Y_1^\star = Y_2^\star = Y_3^\star = Y_4^\star = 1$ and normal stationary Z_t for various ϕ and σ

Remark 5. *The general case of dependent U and V might be treated as well, requiring iterative evaluation of $\mathcal{L}(Z_t, E_t | S_{t-1} = 1)$ instead of sole $\mathcal{Z}_t$.*

Remark 6. *The source code of our C++ program computing Φ and Ψ for normal independent factors can be found at* `https://github.com/cyberklezmer/phi` *under branch* `v16`.

6 Numerical Illustration

In the present Section, we demonstrate the non-linearity and the non-triviality of transformation Φ^1. In addition, we show that the actual errors of our numerical technique is far less than their bounds given by Proposition 4.

First, assume that Y's are such that

$$Y_t^\star = 1, \qquad t \geq 1.$$

Figure 1 shows values of $\Phi^1(Y_1, Y_2, Y_3, Y_4)$ with U_1, U_2 normal and for various values of ϕ and σ. The initial variance is set to $\sigma_1^2 = \frac{\sigma^2}{1-\phi^2}$, which corresponding to stationarity of Z. The values n and h are set so that the Kolmogorov error of the discretizations of U_1 and U_2 is less than 0.0001. From the graphs it is clear that, even in this simple setting, the mapping is far from trivial and it is neither convex nor concave.

Figures 2 and 3 show the evolution of $\mathcal{Z}_t$ (the distribution of individual part of a debtor's wealth). Here, we consider two different parametric settings:

(a) ϕ=-0.5, σ=1, σ_1=1.15, (b) ϕ=0.5, σ=1, σ_1=1.15,

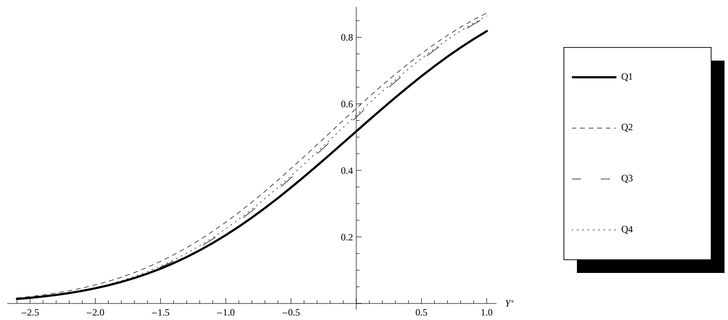

Figure 2: Evolution of the c.d.f.'s of Z_t given $Y_1^\star = Y_2^\star = Y_3^\star = Y_4^\star = 1$ and normal stationary Z_t with $\sigma = 1, \phi = -0.5$.

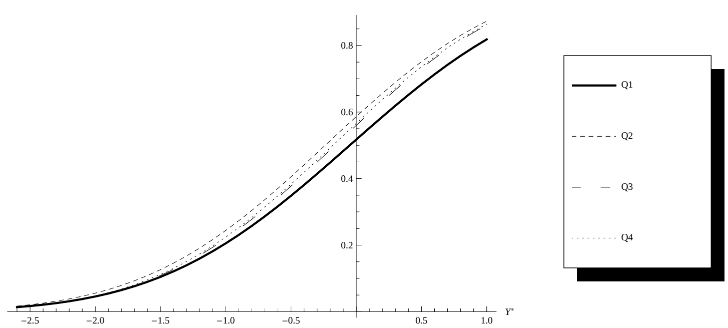

Figure 3: Evolution of the c.d.f.'s of Z_t given $Y_1^\star = Y_2^\star = Y_3^\star = Y_4^\star = 1$ and normal stationary Z_t with $\sigma = 1, \phi = 0.5$.

t	$m = 8, \phi = -0.5$	$m = 8, \phi = 0.5$	$m = 1$ (repeated)
1	0.193	0.193	0.193
2	0.230	0.136	0.193
3	0.206	0.129	0.193
4	0.211	0.127	0.193
5	0.210	0.126	0.193
6	0.210	0.126	0.193
7	0.210	0.126	0.193
8	0.210	0.126	0.193

Table 1: Default rates given $Y_1^\star = Y_2^\star = \cdots = 1$ and stationary Z_t with $\sigma = 1$ for multi-period models with $\phi = -0.5$ and $\phi = 0.5$ in comparison with a corresponding single period model.

respectively (Z_t is stationary in both the cases). In Table 1, Q_t, $t = 1, 2, \dots, 8$, are shown for both (a) and (b) as well as for the case of repeated application of a single period version of the model (i.e. the one with $m = 1$, $\sigma_1 = 1.15$, coinciding with [24] up to scaling).

We can observe that the results from both the multi-period versions stabilize soon, each, however, at different value, different from the result of the single period model.

A different course of Q_t's in each of the cases - the decrease of the PDs for $\phi > 0$ and their oscillation for $\phi < 0$ - is also worth of noting, documenting non-triviality of the model. Namely. An intuitive explanation for this is following: When $\phi > 0$, the worst debts are cut off at each period so the default rate continuously declines. In the case of $\phi < 0$, on the other hand, some of the debtors, who were rich at $t = 1$, become poor at $t = 2$ and vice versa. These, however, who were poor initially and could be rich subsequently, are cut off by defaults at $t = 1$ so they are missing from the portfolio at $t = 2$ when, in addition, a new bunch of poor debtors appears, formed by those who were rich at $t = 1$. Consequently, the second portfolio is “poorer” than the first one and its default rate is greater; the further oscillation could be explained similarly.

Interestingly, the case $\phi > 0$ replicates the empirically documented decrease of the default rates short after the beginning of a mortgage (see e.g. [1]).

Finally, Figure 4 shows that our estimates of the approximation error, given by Proposition 4, are largely overvalued in comparison with the approximation errors in a concrete cases with $\phi \in [-1, 1]$ and $Y_t^\star = [0, 1]$, $t \leq 4$.[10] In particular, with increasing t, the difference between the upper bound and the actual error grows rapidly while the actual error grows moderately in comparison with the bound.

7 Example - Optimal Scoring

In this Section, we present a simple example application of our model: the optimal selection of the loan portfolio from a (large) set of loan candidates, the i-th of which disposes with wealth

$$\hat{A}_t = \exp\left\{\hat{Y}_t + \hat{Z}_t\right\}, \qquad t \geq 1,$$

where $\hat{Y} \perp\!\!\!\perp \hat{Z}$ are stochastic processes such that $\hat{Y}$ is general and

[10] Our graphs were obtained as follows: For each of value of $\phi \in [-1, 1]$, 100 evaluations of Φ^1 by means of our numerical technique and its Monte Carlo estimate was made, each with $\sigma = 0.9$, with σ_1 corresponding to the stationarity and with different arguments Y corresponding to $Y_t^\star$ drawn from a uniform distribution (each MC estimate was computed by 10 times running a simulation of the portfolio with 500.000 debts).

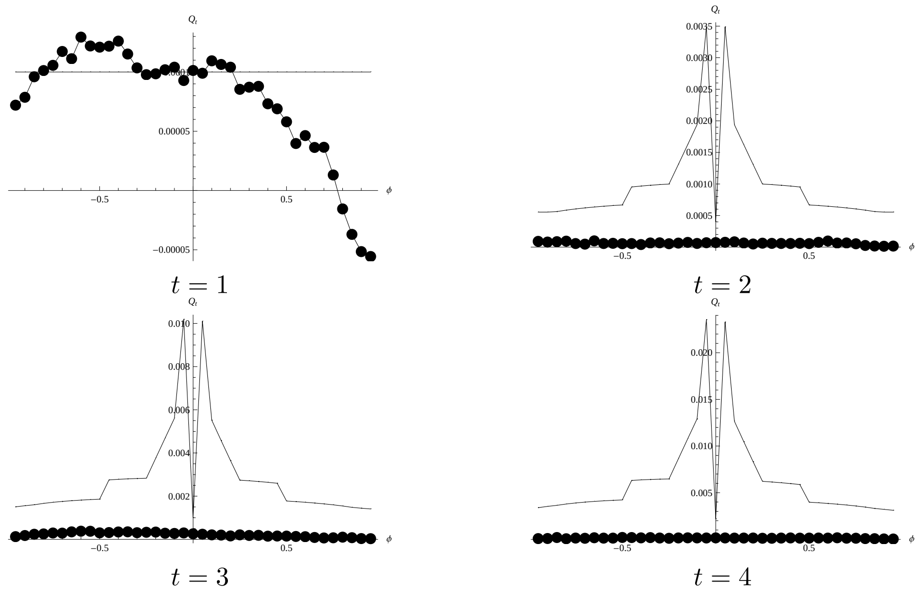

Figure 4: Estimated vs. actual approximation error: solid line - bound given by Proposition 4, circles - Monte Carlo estimate of the error.

- $\hat{Z}_0^i = \sigma_0 \hat{U}_0$ where the distribution of $\hat{U}_0$ meets the requirements for U_1,[11]
- $\hat{Z}_t = \hat{\phi}\hat{Z}_{t-1} + \hat{\sigma}\hat{U}_t$, $t \geq 1$, where $\hat{\phi} \in \mathbb{R}$, $\hat{\sigma} > 0$ and where $\hat{U}_1, \hat{U}_2, \dots$ are i.i.d. with their distribution fulfilling our requirements for U_2.

Assume that the creditor decides to refuse the loan to the candidates with their initial wealth less than a certain threshold ϑ, i.e., such candidates will be accepted for which

$$\hat{S}_0 = 1, \qquad \hat{S}_0 = \mathbf{1}[\hat{Y}_0 + \hat{Z}_0 < \log \vartheta].$$

In this case, the individual part of the wealth of the accepted candidates will have distribution

$$\mathcal{Z}^\theta = \mathcal{T}(\hat{\mathcal{Z}}_0, \log \vartheta - \hat{Y}_0)$$

where $\hat{\mathcal{Z}}_0$ is the distribution of $\hat{Z}_0$. It is easy to verify that, once we put

$$\sigma_1^2 = \hat{\phi}^2 \mathrm{var}(Z^\theta) + \hat{\sigma}^2, \qquad U_1 = \frac{1}{\hat{\sigma}_1}(\hat{\phi}(Z^\theta - \mathbb{E}Z^\theta) + \hat{\sigma}\hat{U}_1), \qquad Z^\theta \sim \mathcal{Z}^\theta, \tag{40}$$

$$Y_t = \hat{Y}_t + \frac{\hat{\phi}\hat{\sigma}}{\sigma_1}\mathbb{E}Z^\theta, \qquad U_t = \hat{U}_t, \qquad t \geq 1, \tag{41}$$

$$\phi = \hat{\phi}, \quad \sigma = \hat{\sigma}, \tag{42}$$

then the assumptions of our model are met and $A_t^i = \hat{A}_t$, $t \geq 1$ on set $[\hat{S}_0 = 1]$.

Now, if we assume, that the (discounted) profit B of the creditor stemming from a single debt given that it will not default until m is deterministic and identical for all the debts, the the maximal expected overall profit from the portfolio will be

$$\max_{\vartheta} P(\vartheta), \qquad P(\vartheta) = \Big(1 - F_0(\log \vartheta - \hat{Y}_0)\Big)\left(B - \mathbb{E}\Lambda^\vartheta(Y_1, I_1, \dots, Y_m, I_m)\right) \tag{43}$$

[11] I.e., it is standardized with its c.d.f. strictly increasing, having continuous uniformly bounded first- and second-order derivatives and fulfilling an analog of (1).

where F_0 is the c.d.f. of $\hat{Z}_0$ and where the index ϑ at Λ stresses the fact that $\Lambda(\bullet)$ is dependent on ϑ through (40)-(42).

As the truncation of $\hat{\mathcal{Z}}_0$ (at $\log\vartheta - \hat{Y}_0$), caused by refusing the debts to inappropriate candidates, is completely analogous to the truncations of $\mathcal{Z}_t$, $t \geq 1$, caused by defaults and as all $\sigma_1, Y_1, \dots Y_m$ are differentiable in ϑ, mapping Λ^ϑ, and consequently $P(\vartheta)$, is easy to be shown to be differentiable in ϑ. Consequently, its maximum may be found by usual gradient methods. However, in light of the results for (a) of Section 6, we cannot generally expect $\mathbb{E}\Lambda^\vartheta$ to be decreasing in ϑ so the possibility of local maxima in (43) cannot be easily excluded.

8 Conclusions

In this paper, we formulated a general multi-period factor model of a large portfolio of debts secured by collaterals. In addition, a simple numerical technique of the model's computation was proposed and an example application was presented.

As a whole, our paper provides a ready-to-use technology for modeling multi-period loans portfolio, suitable both for theorists and the practitioners. In particular, our results may serve e.g. for an analysis of time series of aggregate losses or for the particular portfolio management.

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A Auxiliary Results

Let us start by the citation of two useful probabilistic results.

Lemma 2. *([11] 6.8.14.) Let R and S be independent random vectors and let f be a measurable function such that $\mathbb{E}f(R,S)$ exists and $\mathbb{E}f(R,s)$ exists for any s. Then*

$$\mathbb{E}(f(R,S)|S=s)=\phi(s), \qquad \phi(s)=\mathbb{E}(f(R,s)).$$

Lemma 3. *([15], Corollary 4.5.) Let $R, R_1, R_2, \dots$ and $S, S_1, S_2, \dots$ be random variables such that $R_n \to R$ and $S_n \to S$ in probability. Then $R_nS_n \to RS$ in probability. If, in addition, all S, S_1, S_2, $\dots$ are non-zero, then also $R_n/S_n \to R/S$ in probability.*

The following result, on the other hand, is a generalization of the well known theorem, guaranteeing the interchangeability of integration and differentiation given that an integrable upper bound of the integrand's derivative exists.

Lemma 4.

(i) *Let T be an open subset of $\mathbb{R}$. Let $f : \mathbb{R}^k \times T \to \mathbb{R}$ be a measurable function continuous in its first k parameters. Let f be continuously differentiable in its last parameter outside $M = N \times T$ where $N \subset \mathbb{R}^k$ is a set of zero measure. Let there exists integrable h such that $\frac{\partial}{\partial \beta} f(r, \beta) \leq h(r)$ for any $r \in \mathbb{R}^k$, $\beta \in T$. Then a continuous $\frac{\partial}{\partial \beta}(\int f(r, \beta) dr)$ exists and equals to$\int \frac{\partial}{\partial \beta} f(r, \beta) dr$.*

(ii) *Point (i) keeps holding if $M = (N \times T) \cup \{(r, \beta) \in \mathbb{R}^k \times T : r_1 = g(\beta)\}$ where g is differentiable.*

Proof. (i) follows from the well known rules for interchanging integrals and derivatives ([18], Theorem 9.2.) and [18], Theorem 9.1.

As for (ii), we have, from absolute continuity of the Lebesgue measure, that $\int f(r, \beta) dr = H(\beta, \beta, \beta)$ where $H(\beta_1, \beta_2, \beta) = \int_{\{r_1 \leq g(\beta_1)\}} f(r, \beta) dr + \int_{\{r_1 \geq g(\beta_2)\}} f(r, \beta) dr$. By the Leibnitz Rule, we have

$$\frac{\partial}{\partial \beta_1} H = g'(\beta_1) \int f(g(\beta_1), r_2, \dots, r_k, \beta) dr_{2,} \dots dr_k,$$

$$\frac{\partial}{\partial \beta_2} H = -g'(\beta_2) \int f(g(\beta_2), r_2, \dots, r_k \beta) dr_{2,} \dots dr_k,$$

while, by ([18], Theorem 9.2.), we get

$$\frac{\partial}{\partial \beta} H = \int_{\{r_1 \leq g(\beta_1)\}} \frac{\partial}{\partial \beta} f(r, \beta) dr + \int_{\{r_1 \geq g(\beta_1)\}}^{\infty} \frac{\partial}{\partial \beta} f(r, \beta) dr$$

Finally, by the Chain Rule for Multivariate Functions,

$$\int f(r, \beta) dr = \left(\frac{\partial}{\partial \beta_1} + \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta} \right) H(\beta, \beta, \beta) = \frac{\partial}{\partial \beta} H = \int \frac{\partial}{\partial \beta} f(r, \beta) dr$$

because the first two terms of the sum cancel out. The continuity of the derivative follows from [18], Theorem 9.1. □

Finally, we recall two useful results from analysis.

Lemma 5. *([13], Theorem 80) Let f be continuous and finite in $[a - \Delta, a + \Delta]$. Then*

$$f'(a) = \lim_{x \to a} f'(x)$$

provided that the limit exists.

Lemma 6. *([13], Theorem 192) Let $f(x, y)$ be a function such that $\frac{\partial}{\partial x} f$ and $\frac{\partial}{\partial y} f$ exists on some neighborhood of (x_0, y_0) and $\frac{\partial}{\partial x \partial y} f$ is continuous in (x_0, y_0). Then*

$$\frac{\partial}{\partial x \partial y} f(x_0, y_0) = \frac{\partial}{\partial x \partial y} f(y_0, x_0).$$

B Proof of Lemma 1

Ad (i). If $t = 1$ then (i) follows from the definition of Ξ_1. Let $t > 1$ and assume (i) to hold for $t - 1$. Using the facts that

$$[S_{t-1} = 1] \Rightarrow [S_{t-2} = 1]$$

and that

$$\mathbb{P}(A|BC) = \frac{\mathbb{P}(ABC)}{\mathbb{P}(BC)} = \frac{\mathbb{P}(ABC)}{\mathbb{P}(BC)} \cdot \frac{\mathbb{P}(C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(AB|C)}{\mathbb{P}(B|C)}$$

for any random events A, B, C fulfilling $\mathbb{P}(BC) > 0$, we get

$$\mathbb{P}[Z_t \leq z, E_t \leq e | S_{t-1} = 1] = \mathbb{P}[Z_t \leq z, E_t \leq e | S_{t-1} = 1, S_{t-2} = 1] \\ = \frac{G_t(z, e)}{\mathbb{P}[S_{t-1} = 1 | S_{t-2} = 1]} \quad (44)$$

where

$$G_t(z, e) = \mathbb{P}[Z_t \leq z, E_t \leq e, S_{t-1} = 1 | S_{t-2} = 1] \\ \mathbb{P}[\phi Z_{t-1} + \sigma U_t \leq z, \psi E_{t-1} + \rho V_t \leq e, Z_{t-1} \geq -y^\star_{t-1} | S_{t-2} = 1]. \quad (45)$$

From the independence of (U_t, V_t) and $(Z_{t-1}, E_{t-1}, S_{t-2})$ we further get, by Lemma 2, that

$$G_t(z, e) = \int \mathbf{1}[\phi r + \sigma u \leq z, \psi s + \rho v \leq e] \mathbf{1}[r \geq -y^\star_{t-1}] dW(u, v) dF_{t-1}(r, s) \\ = \int \left(\int \mathbf{1}[u \leq \frac{z - \phi r}{\sigma}, v \leq \frac{e - \phi t}{\rho}] dW(u, v) \right) \mathbf{1}[r \geq -y^\star_{t-1}] dF_{t-1}(r, s) = A_t.$$

Finally, as, by our induction hypothesis and (3),

$$\mathbb{P}[S_{t-1} = 1 | S_{t-2} = 1] = C_t,$$

(i) is proved.
Ad (iii)-(iv). We shall proceed by induction to prove (iii)-(iv). Meanwhile, we prove that

$\mathbf{D}_t$ density $f_t(z, e; \xi_{t-1}) = \frac{\partial}{\partial z \partial e} F_t(z, e; \xi_{t-1})$ exist and is bounded by a finite continuous function $\varsigma_t(\xi_{t-1})$,

$\mathbf{D}^1_t$ density $f^1_t(z; \xi_{t-1}) = \frac{\partial}{\partial z} F_t(z, \infty; \xi_{t-1})$ exist and is bounded by a finite continuous function $\varsigma^1_t(\xi_{t-1})$,

$\mathbf{EZ}_t$ $\mathbb{E}(Z^2_t | S_{t-1} = 1)$ is bounded by a finite continuous function $\epsilon_t(\xi_{t-1})$,

$\mathbf{EE}_t$ $\mathbb{E}(E^2_t | S_{t-1} = 1)$ is bounded by a finite continuous function $\varepsilon_t(\xi_{t-1})$.

Until the end of the proof, we shall abbreviate $\frac{\partial}{\partial x} g$ as $g^{(x)}$ for any function g and variable x. During the proof, we shall frequently use the fact that

$$0 \leq W^{(u)}_1(u, v) = \int^v w_1(u, x) dx \leq \int w_1(u, x) dx = w^1_1(u) \quad (46)$$

(recall that w_1 and w^1_1 are the densities of (U_1, V_1), U_1, respectively) as and its analogs involving $W^{(v)}_1$,$W^{(u)}$ and $W^{(v)}$.

Coming to the proof itself, let $t = 1$ first. By differentiating (8), we get

$$F^{(z)}_1 = \frac{1}{\sigma_1} W^{(u)}_1 \left(\frac{z}{\sigma_1}, \frac{e}{\rho} \right), \qquad |F^{(z)}_1| \leq \frac{\alpha_{1,1}}{\sigma_1} \quad (47)$$

where $\alpha_{1,1}$ is the bound of the first derivatives of W_1. Further,

$$F^{(\sigma_1)}_1 = -\frac{z}{\sigma^2_1} W^{(u)}_1 \left(\frac{z}{\sigma_1}, \frac{e}{\rho} \right), \quad (48)$$

and, by (46),

$$\left|F_1^{(\sigma_1)}\right| \leq \frac{1}{\sigma_1^2}|z|w_1^1(\frac{z}{\sigma_1}) \leq \frac{\sup_x |x|w_1^1(x)}{\sigma_1}$$

where the r.h.s. is finite by (1). As the cases of e and ρ are analogous, and as derivatives in the remaining parameters are zero, we have proved (iii) and (iv) for $t = 1$.

Further, as $\mathbb{E}Z_1^2 = \sigma_1^2$ and $\mathbb{E}E_1^2 = \rho$ EZ_1, EE_1, respectively, hold true.

As for D_1, note that $f_1 = \frac{\partial}{\partial e}F_1^{(z)}(z,e) = \frac{w_1(z/\sigma_1, e/\rho)}{\sigma_1\rho}$ is bounded by $\frac{\alpha_{1,2}}{\sigma_1\rho}$, where $\alpha_{1,2}$ is the bound of the second derivatives of W_1. Finally, as $f_1^1(\bullet) = F_1^{(z)}(\bullet,\infty) = \int f_t(\bullet,y)dy = \lim_z \int^z f(\bullet,y)dy = \lim_z F^{(z)}(\bullet,z) \leq \frac{\alpha_{1,1}}{\sigma_1}$ by (47), D_1^1 is proved, too.

Let $t > 1$, let D_{t-1}, D_{t-1}^1 EZ_{t-1} and EE_{t-1} hold true and let (iii) and (iv) hold for $t-1$. First, let us prove the continuous differentiability (later abbreviated as c.d.) and the local boundedness (in the sense of (iv), later abbreviated as l.b.) of G_t. To this end, let us rewrite (45) as

$$G_t(z,e;\xi_{t-1}) = D(z,e,\phi,\psi,\sigma,\rho,-y_{t-1}^\star(y_t,\epsilon),\xi_{t-2}) \tag{49}$$

where

$$\begin{aligned} &D(e,z,\phi,\psi,\sigma,\rho,\eta,\xi) \\ &\qquad = \mathbb{P}[\phi Z_{t-1} + \sigma U_t \leq z, \psi E_{t-1} + \rho V_t \leq e, Z_{t-1} \geq \eta | S_{t-2} = 1; \xi_{t-2} = \xi] \\ &\qquad = \int \mathbf{1}[r \geq \eta] W(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}) dF_{t-1}(r,s;\xi) \end{aligned}$$

and prove the c.d. and l.b. of D in all its arguments. Starting with η, we get, by the Leibnitz Rule,

$$D^{(\eta)} = -\int W(\frac{z-\phi\eta}{\sigma}, \frac{e-\psi s}{\rho}) f_{t-1}(\eta,s;\xi)ds \tag{50}$$

with

$$|D^{(\eta)}| \leq \int f_{t-1}(\eta,s;\xi)ds = f_{t-1}^1(\eta;\xi)ds \leq \varsigma_{t-1}^1(\xi).$$

In all the remaining cases, we shall use Lemma 4 to prove the c.d. The required integrable upper bound will be always denoted by m, symbol γ, if used, will always denote a bound of $|D^{(\bullet)}|$.

Starting with z, we are getting

$$D^{(z)} = \frac{1}{\sigma}\int_{\{r\geq\eta\}} W^{(u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$m = \alpha_1, \qquad \gamma = \frac{\alpha_1}{\sigma}, \tag{51}$$

where α_1 is the bound of the first derivatives of W.

Coming to ϕ, we have

$$D^{(\phi)} = -\frac{1}{\sigma}\int_{\{r\geq\eta\}} rW^{(u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi) \tag{52}$$

$$m = |r|\alpha_1, \qquad |D^{(\phi)}| = \frac{1}{\sigma}\int |r|dF_{t-1}(r,s;\xi) = \frac{1}{\sigma}\mathbb{E}(|Z_{t-1}||S_{t-2} = 1) \leq \frac{\sqrt{\epsilon_{t-1}}}{\sigma}$$

by the Schwarz Inequality.

Further,

$$D^{(\sigma)} = -\int_{\{r\geq\eta\}} \frac{z-\phi r}{\sigma^2} W^{(u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$\gamma = m = \frac{\sup_x |x| w^1(x)}{\sigma} \tag{53}$$

(we have used (46)).

Before proceeding to ξ^i, $1 \leq i \leq \dim(\xi)$, observe that we can alternatively write

$$D = E, \qquad E(e,z,\phi,\psi,\sigma,\rho,\eta,\xi) = \int \tilde{F}(z-\sigma u, e-\rho v, \eta, \xi) dW(u,v) \tag{54}$$

where

$$\tilde{F}(r,s,\phi,\psi,\eta,\xi) = \mathbb{P}[\phi Z_{t-1} \leq r, \psi E_{t-1} \leq s, Z_{t-1} \geq \eta | S_{t-2} = 1; \xi_{t-2} = \xi]$$

$$= \begin{cases} \mathbb{P}[\phi Z_{t-1} \in (\eta \frac{r}{\phi}], E_{t-1} \leq \frac{s}{\psi}, | S_{t-2} = 1] & \\ = F_{t-1}(\frac{r}{\phi} \vee \eta, \frac{s}{\psi}; \xi) - F_{t-1}(\eta, \frac{s}{\psi}; \xi) & \phi > 0, \psi > 0 \\ \mathbb{P}[Z_{t-1} \geq \frac{r}{\phi}, \psi E_{t-1} \leq s, Z_{t-1} \geq \eta | S_{t-2} = 1] = & \\ \quad = \mathbb{P}[Z_{t-1} \geq \frac{r}{\phi} \vee \eta, E_{t-1} \leq \frac{s}{\psi} | S_{t-2} = 1] & \\ \quad = F_{t-1}(\infty, \frac{s}{\psi}; \xi) - F_{t-1}(\frac{r}{\phi} \vee \eta, \frac{s}{\psi}; \xi)) & \phi < 0, \psi > 0 \\ \mathbf{1}[r \geq 0] \left(F_{t-1}(\infty, \frac{s}{\psi}; \xi) - F_{t-1}(\eta, \frac{s}{\psi}; \xi)\right) & \phi = 0, \psi > 0 \\ \mathbf{1}[s \geq 0] \left(F_{t-1}(\frac{r}{\phi} \vee \eta, \infty; \xi) - F_{t-1}(\eta, \infty; \xi)\right) & \phi > 0, \psi = 0 \\ \mathbf{1}[s \geq 0](1 - F_{t-1}(\frac{r}{\phi} \vee \eta, , \infty; \xi)) & \phi < 0, \psi = 0 \\ \mathbf{1}[r \geq 0]\mathbf{1}[s \geq 0]\left(1 - F_{t-1}(\eta, \infty; \xi)\right) & \phi = 0, \psi = 0 \\ \mathbb{P}[Z_{t-1} \in (\eta, \frac{r}{\phi}], E_{t-1} \geq \frac{s}{\psi} | S_{t-2} = 1] & \\ = (F_{t-1}(\frac{r}{\phi} \vee \eta, \infty; \xi) - F_{t-1}(\eta, \infty; \xi)) & \\ \qquad -(F_{t-1}(\frac{r}{\phi} \vee \eta, \frac{s}{\psi}; \xi) - F_{t-1}(\eta, \frac{s}{\psi}; \xi)) & \phi > 0, \psi < 0 \\ = \mathbb{P}[Z_{t-1} \geq \eta \vee \frac{r}{\phi}, E_{t-1} \geq \frac{s}{\psi} | S_{t-2} = 1] & \\ = (1 - F_{t-1}(\infty, \frac{s}{\psi}; \xi)) & \\ \qquad -(F_{t-1}(\eta \vee \frac{r}{\phi}, \infty; \xi) - F_{t-1}(\eta \vee \frac{r}{\phi}, \frac{s}{\psi}; \xi))s & \phi < 0, \psi < 0 \\ \mathbf{1}[r \geq 0]\mathbb{P}[Z_{t-1} \geq \eta, E_{t-1} \geq \frac{s}{\psi} | S_{t-2} = 1] & \\ = \mathbf{1}[r \geq 0][(1 - F_{t-1}(\infty, \frac{s}{\psi}; \xi)) & \\ \qquad -(F_{t-1}(\eta, \infty; \xi) - F_{t-1}(\eta, \frac{s}{\psi}; \xi))] & \phi = 0, \psi < 0 \end{cases}$$

$$= \sum_{k=1}^{4} a_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(r,s) F_{t-1}(b_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(r,\phi,\eta), c_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(s,\psi); \xi). \tag{55}$$

Here, for each k, s_1 and s_2,

$$a_{k,s_1,s_2}(\bullet) \in \{0, 1, -1\},$$

is constant on each open quadrant, and

$$b_{k,s_1,s_2}(r,\phi,\eta) = \begin{cases} \frac{r}{\phi} \vee \eta & \text{for certain } k, s_1, s_2 \\ \eta & \text{for certain } k, s_1, s_2 \\ \infty & \text{for the rest of } k, s_1, s_2 \end{cases}$$

$$c_{k,s_1,s_2}(s,\psi) = \begin{cases} \frac{s}{\psi} & \text{for certain } k, s_1, s_2 \\ \infty & \text{for the rest of } k, s_1, s_2. \end{cases}$$

By differentiating we get

$$\tilde{F}^{(i)} = \sum a_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(r,s) F^{(i)}_{t-1}(b_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(r,\phi,\eta), c_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(s,\psi);\xi),$$
$$\left|\tilde{F}^{(i)}\right| = 4\alpha^{(i)}_{t-1}, \quad (56)$$

(recall that $\alpha^{(i)}_{t-1}$ is the bound of $F^{(i)}_{t-1}$ guaranteed by (iv)), and, consequently,

$$E^{(i)} = \sum \int a_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(z-\sigma u, e-\rho v)$$
$$F^{(i)}_{t-1}(b_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(z-\sigma u,\phi,\eta), c_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(e-\rho v,\psi);\xi)dW(u,v),$$
$$\gamma = m = 4\alpha^{(i)}_{t-1}. \quad (57)$$

As the cases of e, ψ and ρ are symmetric to those of z, ϕ and σ, we have proved the c.d. and l.b. of D in all its arguments, which immediately yields the c.d. and l.b. of G_t in z, e as well as in y_{t-1} (note that $G_t^{(y_{t-1})} = -y^{\star(y_{t-1})}_{t-1}(y_{t-1})D^{(\eta)}(y^\star_{t-1}) = -D^{(\eta)}(y^\star_{t-1})$). Further, as

$$G_t^{(\epsilon)}(\dots,-y^{\star(\epsilon)},\dots) = -y^{\star(\epsilon)}D^{(\eta)}(\dots,-y^\star(\epsilon),\dots) = \frac{b^{(\epsilon)}}{b}D^{(\eta)}(\dots,-y^{\star(\epsilon)},\dots), \quad (58)$$

the c.d. and l.b. of G_t in ϵ follows from the strict positivity and c.d. of b.[12] The c.d. and l.b. of G_t in the rest of the parameters follows from the Chain Rule for Multivariate Functions (if i is the index of ϕ within ξ_{t-1}, for instance, then $G_t^{(\phi)} = D_t^{(\phi)} + D_t^{(i)}$).

Consequently, by the strict positivity and continuity of C_t, we get that

$$F_t^{(\bullet)} = \frac{G_t^{(\bullet)}C_t - C_t^{(\bullet)}G_t}{C_t^2}$$

proving (iii) and, as C_t is continuous by our induction hypothesis, also (iv).

To prove D_t, differentiate (8) twice to get

$$\begin{aligned} f_t(z,e;\xi_{t-1}) &= F^{(z,e)}(z,e;\xi_{t-1}) \\ &= \frac{1}{C_t}\left(\int_{\{r\geq -y^\star_{t-1}\}} \frac{1}{\sigma}W^{(u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi_{t-2})\right) \\ &= \frac{1}{C_t}\int_{\{r\geq -y^\star_{t-1}\}} \frac{1}{\rho\sigma} w\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi_{t-2}) \end{aligned}$$
$$m^z = \frac{1}{\sigma}\alpha^1, \qquad m^{z,e} = \frac{\alpha_2}{\rho\sigma}, \qquad \gamma^{z,e} = \frac{m^{z,e}}{C_t}, \quad (59)$$

where α_2 is the upper bound of the second derivatives of W (recall that w is the density of (U,V)). Validity of D^1_t follows from the fact that $f_t(\bullet) = F_t(\bullet,\infty)$ similarly to the case of $t=1$.

[12] See Footnote 4.

To prove EZ_t, employ (59) to get

$$\mathbb{E}(Z_t^2|S_{t-1}=1) = \int z^2 f_t(z,e;\xi_{t-1})dzde$$
$$\leq \frac{1}{C_t}\int z^2 \int \frac{1}{\rho\sigma} w\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi_{t-1})dzde$$
$$= \frac{\mathbb{E}\left((\phi Z_{t-1}+\sigma U_t)^2 \middle| S_{t-2}=1\right)}{C_t}$$
$$= \frac{\phi^2\mathbb{E}(Z_{t-1}^2|S_{t-2}=1) + 2\phi\sigma\mathbb{E}(U_t)\mathbb{E}(Z_{t-1}|S_{t-2}=1)+\sigma^2\mathbb{E}U_t^2}{C_t}$$
$$= \frac{\phi^2\epsilon_{t-1}+\sigma^2}{C_t}.$$

The proof of EE_t is analogous.
Ad (ii). If $t=1$ then (ii) follows from our assumptions concerning W_1. If $t>1$ then, according to (51), $D_t^{(z)}$ is strictly positive (which is because integrand is positive and the c.d.f. is strictly increasing). Consequently, by (49), $G_t^{(z)}$ is strictly positive, which proves the monotonicity of F_t in z because C_t is constant in z. The case of e is symmetric.
Ad. (v) and (vi). Similarly as above, we shall abbreviate $\frac{\partial}{\partial x\partial y}f$ as $f^{(x,y)}$ for any function f and variables x,y. Before starting the proof, let us note that

$$\left|W_1^{(u,u)}(u,v)\right| = \left|\left(W_1(u,v)^{(u)}\right)^{(u)}\right| = \left|\left(\int^v w_1(u.y)dy\right)^{(u)}\right|$$
$$= \left|\left(\int^v w_1^1(u)w^{2|1}(y|u)dy\right)^{(u)}\right| \leq \left|w_1^{1,(u)}(u)\right|\int w_1^{2|1}(y|u)dy = \left|w_1^{1,(u)}(u)\right| \quad (60)$$

and that it follows from (10) that

$$\left|xw_1^{1,(u)}(x)\right| \text{ and } \left|w_1^{1,(u)}(x)\right| \text{ are bounded integrable,} \quad (61)$$

which is because, for any continuous function f and any $0\leq m<n$,

$$\int |x^m f(x)|dx \leq \int_{\{x\notin(0,1)\}}|x^n f(x)|dx + \int_{-1}^1 |f(x)|dx \leq \int |x^n f(x)|dx + 2\max_{|x|\leq 1} f(x). \quad (62)$$

Moreover, (60) and (61) hold with W instead of W_1 and/or for u instead of v due to the symmetry of our assumptions.

During the proof, we shall use the fact, which we prove later, that, for any $t\geq 1$ and any i corresponding to some component of ξ_{i-1},

ZBI$_t$ there exist continuous functions $m_t^{i,z}(z,\xi_{t-1})$, $r_t^{i,z}(z,\xi_{t-1})$ and $n_t^{i,z}(z,e,\xi_{t-1})$ bounded by finite continuous functions $\overline{m}_t^{i,z}(\xi_{t-1})$, $\overline{r}_t^{i,z}(\xi_{t-1})$ and $\overline{n}_t^{i,z}(\xi_{t-1})$, respectively, such that

$$(1+|z|)—F_t^{(i,z)}(z,e;\xi_{t-1})| \leq m_t^{i,z}(z,\xi_{t-1}) + n_t^{i,z}(z,e,\xi_{t-1}),$$
$$(1+|z|)—F_t^{(i,z)}(z,\infty;\xi_{t-1})| \leq r_t^{i,z}(z;\xi_{t-1}).$$

and that

$$\int m_t^{i,z}(x,\xi_{t-1})dx \leq M_t^{i,z}(\xi_{t-1}), \qquad \int n_t^{i,z}(x,y,\xi_{t-1})dxdy \leq N_t^{i,z}(\xi_{t-1})$$

and

$$\int r_t^{i,z}(x,\xi_{t-1})dx \le R_t^{i,z}(\xi_{t-1})$$

for some finite continuous $M_t^{i,z}$, $N_t^{i,z}$ and $R_t^{i,z}$,

and

EBI$_t$ there exist continuous functions $m_t^{i,e}(e,\xi_{t-1})$, $r_t^{i,e}(e,\xi_{t-1})$ and $n_t^{i,e}(z,e,\xi_{t-1})$ bounded by finite continuous functions $\overline{m}_t^{i,e}(\xi_{t-1})$, $\overline{r}_t^{i,e}(\xi_{t-1})$ and $\overline{n}_t^{i,e}(\xi_{t-1})$, respectively, such that

$$(1+|e|)—F_t^{(i,e)}(z,e;\xi_{t-1})| \le m_t^{i,e}(e,\xi_{t-1}) + n_t^{i,e}(z,e,\xi_{t-1}),$$
$$(1+|e|)—F_t^{(i,e)}(e,\infty;\xi_{t-1})| \le r_t^{i,e}(e;\xi_{t-1}).$$

and that

$$\int m_t^{i,e}(x,\xi_{t-1})dx \le M_t^{i,e}(\xi_{t-1}), \qquad \int n_t^{i,e}(x,y,\xi_{t-1})dxdy \le N_t^{i,e}(\xi_{t-1})$$

and

$$\int r_t^{i,e}(x,\xi_{t-1})dx \le R_t^{i,e}(\xi_{t-1})$$

for some finite continuous $M_t^{i,e}$, $N_t^{i,e}$ and $R_t^{i,e}$.

The proof will be carried out by induction. To this end, let $t=1$. As F_1 is constant in all the variables except for σ_1,ρ,e,z, it suffices to prove the bounded continuous differentiability only for pairs containing σ_1,ρ,e,z.

By differentiating (48), we get:

(σ_1,σ_1)

$$F_1^{(\sigma_1,\sigma_1)} = \frac{2z}{\sigma_1^3}W_1^{(u)}\left(\frac{z}{\sigma_1},\frac{e}{\rho}\right) + \frac{z^2}{\sigma_1^4}W^{(u,u)}\left(\frac{z}{\sigma_1},\frac{e}{\rho}\right)$$
$$|F_1^{(\sigma_1,\sigma_2)}| \le \left|\frac{2z}{\sigma_1^3}w^1\left(\frac{z}{\sigma_1}\right)\right| + \frac{z^2}{\sigma_1^4}\left|w^{1,(u)}\left(\frac{z}{\sigma_1},\frac{e}{\rho}\right)\right|$$
$$\le \frac{2\sup_x|x|w_1^1(x) + \sup_x x^2|w_1^{1,(u)}(x)|}{\sigma_1^2},$$

where the r.h.s. is finite by (1) and (10),

(σ_1,ρ)

$$F_1^{(\sigma_1,\rho)} = \frac{ze}{\sigma_1^2\rho^2}w_1\left(\frac{z}{\sigma_1},\frac{e}{\rho}\right)$$
$$\left|F_1^{(\sigma_1,\rho)}\right| = \frac{|ze|}{\sigma_1^2\rho^2}w_1\left(\frac{z}{\sigma_1},\frac{e}{\rho}\right) \le \frac{\sup_{u,v}|uv|w_1(u,v)}{\sigma_1\rho}$$

where the r.h.s. is finite by (10),

(σ_1,z)

$$F_1^{(\sigma_1,z)} = -\frac{1}{\sigma_1^2}W_1^{(u)}\left(\frac{z}{\sigma_1},\frac{e}{\rho}\right) - \frac{z}{\sigma_1^3}W^{(u,u)}\left(\frac{z}{\sigma_1},\frac{e}{\rho}\right) \tag{63}$$
$$\left|F_1^{(\sigma_1,z)}\right| \le \frac{1}{\sigma_1^2}\left[w_1^1\left(\frac{z}{\sigma_1}\right) + \left|\frac{z}{\sigma_1}w^{1,(u)}\left(\frac{z}{\sigma_1}\right)\right|\right] \le \frac{\alpha_{1,2} + \sup_x|xw^{1,(u)}(x)|}{\sigma_1^2}$$

where $\alpha_{1,1}$ is the bound of the first derivatives of W_1 (the second term in the numerator is finite by (61)),

(σ_1, e)

$$F_1^{(\sigma_1,e)}(z,e) = -\frac{z}{\sigma_1^2\rho} w_1\left(\frac{z}{\sigma_1}, \frac{e}{\rho}\right),$$

$$\left|F_1^{(\sigma_1,e)}\right| \leq \frac{\sup_{x,y}|x|\, w_1(x,y)}{\sigma_1\rho} \tag{64}$$

where the finiteness of the r.h.s. follows from the boundedness of $|uv|w_1(u,v)$.

By differentiating (47) we get

(z, z)

$$F_1^{(z,z)}(z,e) = \frac{1}{\sigma_1^2} W_1^{(u,u)}\left(\frac{z}{\sigma_1}, \frac{e}{\rho}\right), \qquad \left|F_1^{(z,z)}\right| \leq \frac{a_{1,2}}{\sigma_1^2}$$

where $\alpha_{1,2}$ is the bound of the second derivatives of W_1

(z, e)

$$F_1^{(z,e)}(z,e) = \frac{1}{\sigma_1\rho} w_1\left(\frac{z}{\sigma_1}, \frac{e}{\rho}\right), \qquad \left|F_1^{(z,z)}\right| \leq \frac{a_{1,2}}{\sigma_1\rho}.$$

As the proofs for the rest of the combinations are symmetric, or the symmetry of second derivatives may be used (Lemma 6), (v) and (vi) are proved for $t = 1$.

Let $t > 1$ and let (v) and (vi) hold for $t - 1$. Similarly to the proof of (iii), we start with the bounded continuous differentiability of D_t or, alternatively, E_t, for all the pairs of their variables and/or parameters; we shall keep our convention that m denotes an integrable upper bound and γ stands for the bound of the derivative. Until the end of the proof, i and j will denote indices corresponding to components of ξ_{t-2}.

By differentiation of (51), we get

(z, z)

$$D^{(z,z)} = \frac{1}{\sigma^2}\int_{\{r\geq\eta\}} W^{(u,u)}(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho})dF_{t-1}(r,s;\xi), \qquad m = \alpha_2, \qquad \gamma = \frac{\alpha_2}{\sigma^2}$$

where α_2 is the bound of the second derivatives of W,

(z, e)

$$D^{(z,e)} = \frac{1}{\sigma\rho}\int_{\{r\geq\eta\}} w(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho})dF_{t-1}(r,s;\xi), \qquad m = \alpha_2, \qquad \gamma = \frac{\alpha_2}{\sigma\rho}.$$

From (52), we gradually get

(ϕ, z)

$$D^{(\phi,z)} = -\frac{1}{\sigma^2}\int_{\{r\geq\eta\}} rW^{(u,u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right)dF_{t-1}(r,s;\xi)$$

$$m = |r|, \qquad \gamma = \frac{\alpha_2\sqrt{\epsilon_{t-1}}}{\sigma^2},$$

(ϕ, ϕ)

$$D^{(\phi,\phi)} = \frac{1}{\sigma^2}\int_{\{r\geq\eta\}} r^2 W^{(u,u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right)dF_{t-1}(r,s;\xi)$$

$$m = r^2\alpha_2 \qquad \gamma = \frac{\alpha_2\epsilon_{t-1}}{\sigma^2},$$

(ϕ, ψ)

$$D^{(\phi,\psi)} = \frac{1}{\sigma\rho}\int_{\{r\geq\eta\}} rsw\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi),$$

$$m = |r||s|\alpha_2, \qquad \gamma = \frac{\alpha_2\sqrt{\epsilon_{t-1}\varepsilon_{t-1}}}{\sigma\rho}$$

by the Schwarz Inequality,

(ϕ, ρ)

$$D^{(\phi,\rho)} = \frac{1}{\sigma\rho}\int_{\{r\geq\eta\}} (\frac{e-\psi s}{\rho}) rw\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$m = |r|\omega^{1|2}\sup_v |v—w^2(v), \qquad \gamma = \frac{\sqrt{\epsilon_{t-1}}\omega^{1|2}\sup_v |v—w^2(v)}{\sigma\rho}.$$

Next several cases is obtained by differentiating (53):

(σ, z)

$$D^{(\sigma,z)} = -\frac{1}{\sigma^2}\int_{\{r\geq\eta\}} \left[\frac{z-\phi r}{\sigma}W^{(u,u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) + W^{(u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right)\right] dF_{t-1}(r,s;\xi)$$

$$m = \sup_x |x||w^{1,u}(x)| + \alpha_1, \qquad \gamma = \frac{m}{\sigma^3}$$

where α_1 is the bound of the first derivatives of W,

(σ, σ)

$$D^{(\sigma,\sigma)} = \int_{\{r\geq\eta\}} \left[\frac{1}{\sigma^2}(\frac{z-\phi r}{\sigma})^2 W^{(u,u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right)\right.$$
$$\left. +2\frac{z-\phi r}{\sigma^3}W^{(u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right)\right] dF_{t-1}(r,s;\xi)$$

$$m = \gamma = \frac{\sup_u u^2\left|w^{1,(u)}(u)\right|}{\sigma^2} + 2\frac{\sup_u |u|\, w^1(u)}{\sigma^2},$$

(σ, ρ)

$$D^{(\sigma,\rho)} = \frac{1}{\sigma\rho}\int_{\{r\geq\eta\}} (\frac{z-\phi r}{\sigma})(\frac{e-\psi s}{\rho}) w\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$m = \sup_{u,v} |uv|w(u,v), \qquad \gamma = \frac{m}{\sigma\rho},$$

(σ, ϕ)

$$D^{(\sigma,\phi)} = \frac{1}{\sigma^2}\int_{\{r\geq\eta\}} (\frac{z-\phi r}{\sigma}) rW^{(u,u)}\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$m = |r|\sup_u |uw^{1,(u)}(u)|, \qquad \gamma = \frac{\sqrt{\epsilon_{t-1}}\sup_u |uw^{1,(u)}(u)|}{\sigma^2}.$$

Next two cases are got by differentiating analogs of (52), (53), respectively:

(ψ, z)

$$D^{(\psi,z)} = -\frac{1}{\sigma\rho}\int_{\{r\geq\eta\}} sw\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$m = |s|\alpha_2, \qquad \gamma = \frac{\sqrt{\varepsilon_{t-1}}\alpha_2}{\sigma\rho},$$

(ρ, z)

$$D^{(\rho,z)} = -\int_{\{r\geq\eta\}} \frac{(e-\psi s)}{\sigma\rho^2} w\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$m = \gamma = \frac{\sup_{u,v}|v|w(u,v)}{\rho^2}.$$

Next several cases are obtained by differentiating (50):

(η, z)

$$D^{(\eta,z)} = -\frac{1}{\sigma}\int W^{(u)}(\frac{z-\phi\eta}{\sigma}, \frac{e-\psi s}{\rho})f_{t-1}(\eta,s;\xi)ds,$$

$$m = \alpha_1 f_{t-1}(\eta,s;\xi), \qquad \gamma = \frac{\alpha_1 f^1_{t-1}(\eta;\xi)}{\sigma},$$

(η, e)

$$D^{(\eta,e)} = -\frac{1}{\rho}\int W^{(v)}(\frac{z-\phi\eta}{\sigma}, \frac{e-\psi s}{\rho})f_{t-1}(\eta,s;\xi)ds$$

$$m = \alpha_1 f_{t-1}(\eta,s;\xi), \qquad \gamma = \frac{\alpha_1 f^1_{t-1}(\eta;\xi)}{\rho},$$

(η, ϕ)

$$D^{(\eta,\phi)} = \frac{\eta}{\sigma}\int W^{(u)}\left(\frac{z-\phi\eta}{\sigma}, \frac{e-\psi s}{\rho}\right) f_{t-1}(\eta,s;\xi)ds,$$

$$m = \alpha_1 f_{t-1}(\eta,s;\xi), \qquad \gamma = \frac{\eta\alpha_1 f^1_{t-1}(\eta;\xi)}{\sigma},$$

(η, σ)

$$D^{(\eta,\sigma)} = \frac{z-\phi\eta}{\sigma^2}\int W^{(u)}\left(\frac{z-\phi\eta}{\sigma}, \frac{e-\psi s}{\rho}\right) f_{t-1}(\eta,s;\xi)ds,$$

$$m = \alpha_1 f_{t-1}(\eta,s;\xi), \qquad \gamma = \frac{f^1_{t-1}(\eta;\xi)\sup_u|u|w^1(u)}{\sigma},$$

(we have used (46)),

(η, ρ)

$$D^{(\eta,\rho)} = \int \frac{e-\psi s}{\rho^2} W^{(v)}(\frac{z-\phi\eta}{\sigma}, \frac{e-\psi s}{\rho}) f_{t-1}(\eta, s; \xi) ds$$

$$m = \frac{\sup_v |v| w^2(v)}{\rho} f_{t-1}(\eta, s; \xi), \qquad \gamma = \frac{\sup_v |v| w^2(v)}{\rho} f^1(\eta; \xi),$$

(we have used an analog of (46)),

(η, ψ)

$$D^{(\eta,\psi)} = \frac{1}{\rho} \int s W^{(v)} \left(\frac{z-\phi\eta}{\sigma}, \frac{e-\psi s}{\rho} \right) f_{t-1}(\eta, s; \xi) ds, \qquad m = |s| \alpha_1 f_{t-1}(\eta, s; \xi);$$

we show that m is integrable and $\left| D^{(\eta,\psi)} \right|$ bounded: If $t > 2$ then

$$\begin{aligned} \int \alpha_1 |s| f_{t-1}(\eta, s; \xi) ds &= \alpha_1 \int |s| \int w(\eta - x, s - y) dF_{t-2}(x, y) ds \\ &\leq \alpha_1 \omega^{1|2} \int |s| \int w^2(s-y) dF_{t-2}(x,y) ds = \alpha_1 \omega^{1|2} \int |s| f^2_{t-1}(s) ds \leq \alpha_1 \omega^{1|2} \sqrt{\varepsilon_{t-1}}. \end{aligned} \tag{65}$$

so $\left| D^{(\eta,\psi)} \right| \leq \frac{\alpha_1 \omega^{1|2} \sqrt{\varepsilon_{t-1}}}{\rho}$. If $t = 2$, on the other hand, then

$$\int \alpha_1 |s| f_{t-1}(\eta, s; \xi) ds = \alpha_1 \int |s| w_1^{1|2}(\eta|s) w_1^2(s) ds = \alpha_1 \omega^{1|2} \mathbb{E}|V_1| \leq \alpha_1 \omega^{1|2} \tag{66}$$

so $\left| D^{(\eta,\psi)} \right| \leq \frac{\alpha_1}{\rho} \omega_1^{1|2}$.

The next case we get by differentiating (57):

(i, j)

$$E^{(i,j)} = \sum \int a_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(z - \sigma u, e - \rho v) F^{(i,j)}_{t-1}(b_k(z - \sigma u, \phi, \eta), c_k(e - \rho v \psi)) dW(u, v)$$

$$\gamma = m = 4\alpha^{(i,j)}_{t-1}. \tag{67}$$

The next several cases are more complicated:

(i, z) Let $\phi \neq 0$ first. In this case,

$$\begin{aligned} \tilde{F}^{(i,r)} &= \frac{1}{\phi} \times \begin{cases} \begin{cases} \mathbf{1}[r \geq \eta\phi] F^{(i,z)}(\frac{r}{\phi}, \frac{s}{\psi}, \xi) & \phi > 0 \\ -\mathbf{1}[r \leq \eta\phi] F^{(i,z)}(\frac{r}{\phi}, \frac{s}{\psi}, \xi) & \phi < 0 \end{cases} & \psi > 0 \\ \begin{cases} \mathbf{1}[s \geq 0] \mathbf{1}[r \geq \eta\phi] F^{(i,z)}(\frac{r}{\phi}, \infty, \xi) & \phi > 0 \\ -\mathbf{1}[s \geq 0] \mathbf{1}[r \leq \eta\phi] F^{(i,z)}(\frac{r}{\phi}, \infty, \xi) & \phi < 0 \end{cases} & \psi = 0 \\ \begin{cases} \mathbf{1}[r \geq \eta\phi] \left(F^{(i,z)}_{t-1}(\frac{r}{\phi}, \infty; \xi) + F^{(i,z)}_{t-1}(\frac{r}{\phi}, \frac{s}{\psi}; \xi) \right) & \phi > 0 \\ -\mathbf{1}[r \leq \eta\phi] \left(F^{(i,z)}_{t-1}(\frac{r}{\phi}, \infty; \xi) + F^{(i,z)}_{t-1}(\frac{r}{\phi}, \frac{s}{\psi}; \xi) \right) & \phi < 0 \end{cases} & \psi < 0 \end{cases} \\ &= \frac{1}{|\phi|} \mathbf{1}[\frac{r}{\phi} \geq \eta] \sum_{k=1}^{2} \alpha_{k,\mathrm{sgn}(\psi)}(s) F^{(i,z)}_{t-1}(\frac{r}{\phi}, d_k(s, \psi); \xi) \end{aligned} \tag{68}$$

on set $\{(r,\phi) : r/\phi \neq \eta\}$ for some $\alpha_{k,s_2}(s) \in \{-1,0,1\}$ and $d_k \in \mathbb{R}^\star$ equal to either $\frac{s}{\psi}$ or to ∞. Therefore and because $|\tilde{F}^{(i,r)}| \leq \frac{2\alpha_{t-1}^{(i,z)}}{|\phi|}$, we get, by Lemma 4 (ii),

$$E^{(i,z)} = \int \tilde{F}^{(i,r)}(z-\sigma u, e-\rho v)dW(u,v), \qquad m = \frac{2\alpha_{t-1}^{(i,z)}}{|\phi|}.$$

Therefore, by (68) and by ZBI_{t-1},

$$\begin{aligned}|E^{(i,z)}| &\leq \frac{1}{|\phi|}\int \sum_{k=1}^{2} |F_{t-1}^{(i,z)}(\frac{z-\sigma u}{\phi}, d_k(s,\psi);\xi)|dW(u,v)\\ &= \frac{1}{\sigma}\int \sum_{k=1}^{2} |F_{t-1}^{(i,z)}(x, d_k(s,\psi))|w(\frac{z-\phi x}{\sigma}, v)dxdv\\ &\leq \frac{\alpha_1}{\sigma}\sum_{k=1}^{2}\int m_k(x,v)dxdv \end{aligned} \tag{69}$$

where

$$m_k(x,v) = \begin{cases} r_{t-1}^{i,z}(x,\xi)w^{2|1}(v|\frac{z-\phi x}{\sigma}) & \text{if } d_k = \infty\\ m_{t-1}^{i,z}(x,\xi)w^{2|1}(v|\frac{z-\phi x}{\sigma}) + n_{t-1}^{i,z}(x,\frac{e-\rho v}{\psi},\xi)\omega^{2|1} & \text{otherwise,}\end{cases}$$

As

$$\int m_k(x,v)dxdv \leq \gamma_k \tag{70}$$

$$\gamma_k = \begin{cases} R_{t-1}^{i,z}(\xi) & \text{if } d_k = \infty\\ M_{t-1}^{i,z}(\xi) + \omega^{1|2}\frac{|\psi|}{\rho}N_{t-1}^{i,z}(\xi) & \text{otherwise,}\end{cases} \tag{71}$$

$k=1,2$, the boundedness of $|E^{(i,z)}|$ given $\phi \neq 0$ is proved (we made substitution $\int n_{t-1}^{i,z}(x,\frac{e-\rho v}{\psi},\xi)dxdv = \frac{|\psi|}{\rho}\int n_{t-1}^{i,z}(x,y,\xi)dxdy$ to prove (70)).
If, on the other hand, $\phi = 0$, then, for any k and s_2,

$$a_{k,\mathrm{sgn}(\phi),s_2}(r,s) = \mathbf{1}[r \geq 0]\tilde{a}_{k,s_2}(s), \qquad b_{k,\mathrm{sgn}(\phi),s_2}(\bullet,\phi,\eta) = \tilde{b}_{k,s_2}(\eta),$$

for some $\tilde{a}_{k,s_2} \in \{-1,0,1\}$ and $\tilde{b}_{k,s_2}$ which equals either η or ∞ so

$$E^{(i)} = \int \mathbf{1}[z-\sigma u \geq 0]g(v)w(u,v)dudv$$

where

$$g(v) = \sum_{k=1}^{4}\tilde{a}_{k,s_2}(e-\rho v)F_{t-1}^{(i)}(\tilde{b}_{k,s_2}(\eta), c_{k,\mathrm{sgn}(\phi),\mathrm{sgn}(\psi)}(e-\rho v,\psi);\xi).$$

Consequently, by the Leibnitz Rule,

$$E^{(i,z)} = \left(\int^{\frac{z}{\sigma}}\int g(v)w(u,v)dudv\right)^{(z)} = \frac{1}{\sigma}\int g(v)w\left(\frac{z}{\sigma},v\right)dv;$$

continuity of which is guaranteed by continuity of the integrand and by its boundedness by $\omega^{1|2}4\alpha_{t-1}^{(i)}w^2(v)$.
As

$$E^{(i,z)}| \leq \gamma_0 \qquad \gamma_0 = \frac{\omega^{1|2}\sup_x|g(x)|}{\sigma}\int w^2(v)dv = \frac{\omega^{1|2}4\alpha_{t-1}^{(i)}}{\sigma}$$

given $\phi = 0$, we get that $|E^{(i,z)}| \leq \max(\frac{\alpha_1}{\sigma}(\gamma_1+\gamma_2),\gamma_0)$ given any ϕ.

(i,ϕ) Let $\phi \neq 0$ first. It this case, similarly to (68), we get that

$$\tilde{F}^{(i,\phi)}(r,s;\xi) = -\frac{r}{|\phi|\phi}\mathbf{1}[\frac{r}{\phi} \geq \eta]\sum_{k=1}^{2}\alpha_{k,\mathrm{sgn}(\psi)}(s)F_{t-1}^{(i,z)}(\frac{r}{\phi}, d_k(s,\psi);\xi)$$

outside set $\{r = \eta\phi\}$ so, by Lemma 4 (ii),

$$E^{(i,\phi)} = -\frac{1}{|\phi|}\sum_{k=1}^{2}\int \frac{z-\sigma u}{\phi}\mathbf{1}[\frac{z-\sigma u}{\phi} \geq \eta]\alpha_{k,\mathrm{sgn}(\psi)}(e-\rho v)F_{t-1}^{(i,z)}(\frac{z-\sigma u}{\phi}, d_k(e-\rho v);\xi)dW(u,v)$$

$$m = \max(\overline{m}_{t-1}^{i,z}(\xi) + \overline{n}_{t-1}^{i,z}(\xi), \overline{r}_{t-1}^{i,z}(\xi))$$

To prove the boundedness, note that, by substitution

$$E^{(i,\phi)} = \frac{1}{\sigma}\int\sum_{k} x\mathbf{1}[x \geq \eta]\alpha_{k,\mathrm{sgn}(\psi)}(e-\rho v)F_{t-1}^{(i,z)}(x, d_k(e-\rho v);\xi)w(\frac{z-\phi x}{\sigma}, v)dxdv \quad (72)$$

implying

$$|E^{(i,\phi)}| \leq \frac{1}{\sigma}\sum_{k}\int |xF_{t-1}^{(i,z)}(x, d_k(e-\rho v);\xi)w(\frac{z-\phi x}{\sigma}, v)|dxdv \leq \frac{\alpha_1}{\sigma}(\gamma_1+\gamma_2)$$

(see (71) for the definition of γ_k).
Let $\phi = 0$. As $E^{(i)}$ is continuous in ϕ by (iii), we have, according to Lemma 5, that

$$\frac{\partial}{\partial\xi^i\partial\phi}E = \lim_{\phi\to 0+}E_k^{(i,\phi)}$$

given that the limit exist, which is however true in our case because, by a limit transition in (72),

$$\lim_{\phi\to 0}E^{(i,\phi)}(\phi) = -\frac{1}{\sigma}\sum\int x\mathbf{1}[x \geq \eta]F_{t-1}^{(i,z)}(x, d_k(e-\rho v,\psi);\xi)w(\frac{z}{\sigma}, v)dxdv$$

exists (the required integrable upper bound being $\frac{\alpha_1}{\sigma}\sum_{k=1}^{2} m_k(x,v)$), so $E^{(i,\phi)}$ exists, is continuous and bounded by $\frac{\alpha_1}{\sigma}(\gamma_1+\gamma_2)$ for $\phi = 0$.

(i,σ) The proof is similar to that of (i,z): If $\phi \neq 0$ then

$$E^{(i,\sigma)} = \frac{1}{|\phi|}\sum_{k}\int u\mathbf{1}[\frac{z-\sigma u}{\phi} \geq \eta]\alpha_{k,\mathrm{sgn}(\psi)}(e-\rho v)F_{t-1}^{(i,z)}(\frac{z-\sigma u}{\phi}, d_k(e-\rho v,\psi);\xi)dW(u,v),$$

$$m = |u|\max(\overline{m}_{t-1}^{i,z}(\xi) + \overline{n}_{t-1}^{i,z}(\xi), \overline{r}_{t-1}^{i,z}(\xi)).$$

As. by substitution,

$$E^{(i,\sigma)} = \frac{1}{\sigma}\int\sum_{k}\frac{z-\phi x}{\sigma}\mathbf{1}[x \geq \eta]\alpha_{k,\mathrm{sgn}(\psi)}(e-\rho v)F_{t-1}^{(i,z)}(x, d_k(e-\rho v,\psi);\xi)w(\frac{z-\phi x}{\sigma}, v)dxdv \quad (73)$$

we have

$$|E^{(i,\sigma)}| \leq \int |\frac{z-\phi x}{\sigma}|w^1(\frac{z-\phi x}{\sigma})\sum_{k} m_k(x,v)dxdv \leq \left(\sup_u |u|w^1(u)\right)\sum_{k}\gamma_k,$$

which proves the boundedness. If $\phi = 0$, then

$$E^{(i,\sigma)} = (\int^{\frac{z}{\sigma}} g(v)w(u,v)dudv)^{(\sigma)} = -\frac{z}{\sigma^2}\int g(v)w\left(\frac{z}{\sigma},v\right)dv,$$

$$|E^{(i,\sigma)}| \leq \frac{1}{\sigma}\left|\frac{z}{\sigma}\right| w^1(\frac{z}{\sigma})\int g(v)w^{2|1}\left(v|\frac{z}{\sigma}\right)dv \leq \frac{4\alpha_{t-1}^{(i)}\sup_u |u|w^1(u)}{\sigma},$$

which, combined with the results for $\phi \neq 0$, proves both the continuous differentiability and the boundedness.

(η,η),(η,i) First, note that, substituting $r = z - \sigma u$ and $s = e - \rho v$ in (54),

$$E = \sum_{k=1}^{4} E_k$$

$$E_k = \frac{1}{\sigma\rho}\int a_k(r,s)F_{t-1}(b_k(r,\phi,\eta),c_k(s,\psi))w\left(\frac{z-r}{\sigma},\frac{e-s}{\rho}\right)drds. \quad (74)$$

Assume $\phi \geq 0$ first. If $b_k(r) = \eta \vee \frac{r}{\phi}$ (which implies that $\phi \neq 0$ see (55)), then, by Lemma 4 (ii),

$$E_k^{(\eta)} = \frac{1}{\sigma\rho}\int a_k(r,s)\frac{\partial}{\partial\eta}F_{t-1}(\eta \vee \frac{r}{\phi},c_k(s,\psi))w\left(\frac{z-r}{\sigma},\frac{e-s}{\rho}\right)drds$$

$$= \frac{1}{\sigma\rho}\int^{\phi\eta}\int a_k(r,s)F_{t-1}^{(z)}(\eta,c_k(s,\psi))w(\frac{z-x}{\sigma},\frac{e-y}{\rho})dsdr$$

$$m = \alpha_{t-1}^{(z)}w\left(\frac{z-r}{\sigma},\frac{e-s}{\rho}\right), \qquad \gamma = \alpha_{t-1}^{(z)};$$

If $b_k = \eta$ then

$$E_k^{(\eta)} = \frac{1}{\sigma\rho}\int a_k(r,s)F_{t-1}^{(z)}(\eta,c_k(s,\psi))w(\frac{z-r}{\sigma},\frac{e-s}{\rho})drds$$

$$m = \alpha_{t-1}^{(z)}w\left(\frac{z-r}{\sigma},\frac{e-s}{\rho}\right), \qquad \gamma = \alpha_{t-1}^{(z)}.$$

Finally, if b_k is constant then $E_k^{(\eta)} = 0$. Thus, we may summarize

$$E_k^{(\eta)} = \frac{1}{\sigma\rho}\int_{q_k}\int a_k(r,s)F_{t-1}^{(z)}(\eta,c_k(s,\psi))w(\frac{z-r}{\sigma},\frac{e-s}{\rho})drds, \qquad \gamma = \alpha_{t-1}^{(z)} \quad (75)$$

where $q_k = q_k(\eta,\phi)$ is either $\phi\eta, -\infty$ or ∞, from which we get

$$E_k^{(\eta,\eta)} = \frac{1}{\sigma\rho}\int_{q_k}\int a_k(r,s)F_{t-1}^{(z,z)}(\eta,c_k(s,\psi))w(\frac{z-r}{\sigma},\frac{e-s}{\rho})drds$$

$$-\frac{q_k^{(\eta)}}{\sigma\rho}\int a_k(q_k,s)F_{t-1}^{(z)}(\eta,c_k(s,\psi))w(\frac{z-q_k}{\sigma},\frac{e-s}{\rho})ds$$

where the upper bound, required in the first integral, is $\alpha_{t-1}^{(z,z)}w(\frac{z-x}{\sigma},\frac{e-y}{\rho})$. As $q^{(\eta)} \in \{0,\phi\}$, we have

$$|E_k^{(\eta,\eta)}| \leq 1 + \frac{|\phi|\alpha_{t-1}^{(z)}\omega^{1|2}}{\sigma}.$$

Further, by differentiation of (75),

$$E_k^{(\eta,i)} = \frac{1}{\sigma\rho}\int_q\int a_k(r,s)F_{t-1}^{(z,i)}(\eta, c_k(s,\psi))w(\frac{z-r}{\sigma}, \frac{e-s}{\rho})drds, \qquad m = \gamma = \alpha_{t-1}^{(z,i)}.$$

The proofs for $\phi \leq 0$ are symmetric.

As the cases (e,e), (i,e), (ψ,e), (ϕ,e), (σ,e), (ρ,e), (i,ψ), (i,ρ), (ψ,ψ), (ρ,ρ), (ψ,σ), (ψ,ρ) are symmetric to (z,z), (i,z), (ϕ,z), (ψ,z), (ρ,z), (σ,z), (i,ϕ), (i,σ), (ϕ,ϕ), (σ,σ), (ϕ,ρ), (ϕ,σ), respectively, and as $D^{(x,y)} = D^{(y,x)}$ for any parameters x, y thanks to Lemma 6, we have proved continuous second differentiability and boundedness of D in all its parameters except for ϵ, implying the same properties for G_t by the Chain Rule for Multivariate Functions (see the similar discussion in the proof of (ii)-(iii). As for ϵ, note that, from (58),

$$G_t^{(\epsilon,\epsilon)} = -\left(\frac{b^{(\epsilon)}}{b}\right)^2 D^{(\eta,\eta)}(\dots, -y^{\star(\epsilon)}, \dots) + \frac{b^{(\epsilon,\epsilon)}b - (b^{(\epsilon)})^2}{b^2}D^{(\eta)}(\dots, -y^{\star(\epsilon)}, \dots)$$

and

$$G_t^{(\epsilon,x)} = \frac{b^{(\epsilon)}}{b}D^{(\eta,x)}(\dots, -y^{\star(\epsilon)}, \dots)$$

when $x \neq \epsilon$ where the second continuous differentiability of b follows by Lemma 5. Finally, as

$$\begin{aligned} F_t^{(x,y)} &= \left(\frac{G_t^{(x)}C_t - C_t^{(x)}G_t}{C_t^2}\right)^{(y)} \\ &= \frac{[G_t^{(x,y)}C_t + G_t^{(x)}C_t^{(y)} - C_t^{(x,y)}G_t - C_t^{(x)}G_t^{(y)}]C_t^2 - 2C_t^{(y)}[G_t^{(x)}C_t - C_t^{(x)}G_t]}{C_t^4} \end{aligned} \tag{76}$$

we are getting (v) and (vi) thanks to (iii) and (iv) and the strict positivity and the second continuous differentiability of C_t.

Now, we can proceed with ZBI_t. Unusually, we start with the case $t > 1$, assuming ZBI_{t-1}. Similarly as in the proof of (iii), we start with $D_t^{(\bullet,z)}$ or, alternatively, $E_t^{(\bullet,z)}$. Again we will go through all the variables and parameters:

(ϕ, z)

$$\begin{aligned} (1+|z|)|D^{(\phi,z)}| &\leq \frac{1}{\sigma^2}\int |r|w^{1,(u)}\left(\frac{z-\phi r}{\sigma}\right)dF_{t-1}(r,s;\xi) \leq m^\phi(z), \\ m^\phi(z) &= \frac{1}{\sigma}\int |r|\left(\frac{1}{\sigma}(1+|\phi r|) + \left|\frac{z-\phi r}{\sigma}\right|\right)\left|w^{1,(u)}\left(\frac{z-\phi r}{\sigma}\right)\right|dF_{t-1}(r,s;\xi) \\ &\leq \frac{\sqrt{\epsilon_{t-1}} + |\phi|\epsilon_{t-1}}{\sigma^2}\sup_u\left|w^{1,(u)}(u)\right| + \frac{\sqrt{\epsilon_{t-1}}}{\sigma}\sup_u\left|uw^{1,(u)}(u)\right|, \end{aligned}$$

$$\begin{aligned} \int m^\phi(z)dz &= \int |r|\left(\frac{1}{\sigma}(1+|\phi r|) + |x|\right)\left|w^{1,(u)}(x)\right|dF_{t-1}(r,s;\xi)dx \\ &\leq \frac{\sqrt{\epsilon_{t-1}} + |\phi|\epsilon_{t-1}}{\sigma}\int\left|w^{1,(u)}(x)\right|dx + \sqrt{\epsilon_{t-1}}\int\left|xw^{1,(u)}(x)\right|dx. \end{aligned}$$

(σ, z)

$$(1+|z|)|D^{(\sigma,z)}| \le m^\sigma(z),$$

$$m^\sigma(z) = \frac{1}{\sigma}\int \left(\frac{1}{\sigma} + \left|\frac{\phi r}{\sigma}\right| + \left|\frac{z-\phi r}{\sigma}\right|\right)\left(\left|\frac{z-\phi r}{\sigma} w^{1,(u)}\left(\frac{z-\phi r}{\sigma}\right)\right| + w^1\left(\frac{z-\phi r}{\sigma}\right)\right) dF_{t-1}(r,s;\xi)$$

$$\le \frac{1+|\phi|\sqrt{\epsilon_{t-1}}}{\sigma^2}\left(\sup_u |uw^{1,(u)}(u)| + \sup_u w^1(u)\right)$$

$$+ \frac{1}{\sigma}\left(\sup_u |u^2 w^{1,(u)}(u)| + \sup_u |u| w^1(u)\right),$$

$$\int m^\sigma(z)dz = \int \left(\frac{1}{\sigma} + \left|\frac{\phi r}{\sigma}\right| + |x|\right)\left||x|w^{1,(u)}(x) + w^1(x)\right| dx dF_{t-1}(r,s;\xi)$$

$$\le \frac{1+|\phi|\sqrt{\epsilon_{t-1}}}{\sigma}\left(\int \left|xw^{1,(u)}(x)\right| dx + 1\right) + \left(\int \left|x^2 w^{1,(u)}(x)\right| dx + \mathbb{E}|U_2|\right).$$

(ρ, z)

$$(1+|z|)|D^{(\rho,z)}| \le n^\rho(z,e),$$

$$n^\rho(z,e) = \frac{1}{\sigma\rho}\int \left|\frac{e-\psi s}{\rho}\right|\left(1+|\phi r| + \sigma\left|\frac{z-\phi r}{\sigma}\right|\right) w\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right) dF_{t-1}(r,s;\xi)$$

$$\le \frac{(1+|\phi|\sqrt{\epsilon_{t-1}})\omega^{1|2}}{\sigma\rho}\sup_v |v| w^2(v) + \frac{1}{\rho}\sup_{u,v}|uv| w(u,v),$$

$$\int n^\rho(e,z)dzde = \int |y|\left(1+|\phi r| + \sigma|x|\right) w(x,y)\, dF_{t-1}(r,s;\xi)dxdy$$

$$\le (1+|\phi|\sqrt{\epsilon_{t-1}})\mathbb{E}|V_2|\omega^{1|2} + \sigma\int |xy| w(x,y)dxdy.$$

Further,

$$(1+|z|)|D^{(\rho,z)}(z,\infty)| = \text{limsup}_{e\to\infty} n^\rho(e,z)$$

$$\le \frac{1}{\sigma\rho}\int \text{limsup}_{e\to\infty}\left(\left|\frac{e-\psi s}{\rho}\right|\left(1+|\phi r| + \sigma\left|\frac{z-\phi r}{\sigma}\right|\right) w\left(\frac{z-\phi r}{\sigma}, \frac{e-\psi s}{\rho}\right)\right) dF_{t-1}(r,s;\xi)$$

$$\le \frac{1}{\sigma}\int \left(1+|\phi r| + |z-\phi r|\right)\omega^{1|2}\left(\lim_{e\to\infty}\left|\frac{e-\psi s}{\rho}\right| w^2\left(\frac{e-\psi s}{\rho}\right)\right) dF_{t-1}(r,s;\xi) = 0.$$

(ψ, z)

$$(1+|z|)|D^{(\psi,z)}| \le m^\psi(z),$$

$$m^\psi(z) = \frac{\omega^{2|1}}{\sigma\rho}\int \left(1+|\phi r| + \sigma\left|\frac{z-\phi r}{\sigma}\right|\right)|s| w^1\left(\frac{z-\phi r}{\sigma}\right) dF_{t-1}(r,s;\xi)$$

$$\le \frac{\alpha_1\omega^{2|1}}{\rho\sigma}(\sqrt{\varepsilon_{t-1}} + |\phi|\sqrt{\epsilon_{t-1}}\sqrt{\varepsilon_{t-1}}) + \frac{\sqrt{\varepsilon_{t-1}}\omega^{2|1}}{\rho}\sup_u |u| w^1(u),$$

$$\int m^\psi(z)dz = \frac{\omega^{2|1}}{\rho}\int \left(1+|\phi r| + \sigma|x|\right)|s| w^1(x)\, dF_{t-1}(r,s;\xi)dx$$

$$\le \frac{\omega^{2|1}(\sqrt{\varepsilon_{t-1}} + |\phi|\sqrt{\epsilon_{t-1}}\sqrt{\varepsilon_{t-1}})}{\rho} + \frac{\sigma\omega^{2|1\sqrt{\varepsilon_{t-1}}}}{\rho}\mathbb{E}|U_2|.$$

(η, z)

$$(1+|z|)|D^{(\eta,z)}| \le m^\eta(z)$$

$$m^\eta(z) = \frac{1}{\sigma}\left(1 + |\phi\eta| + \sigma\left|\frac{z-\phi\eta}{\sigma}\right|\right) w^1\left(\frac{z-\phi\eta}{\sigma}\right) f^1_{t-1}(\eta;\xi)$$
$$\le \left[\frac{1}{\sigma}\left(1+|\phi\eta|\right)\alpha_1 + \sup_u |u|w^1(u)\right] f^1_{t-1}(\eta;\xi),$$

$$\int m^\eta(z)dz \le \left[(1+|\phi\eta|)\int w^1(x)dx + \sigma\int |x|w^1(x)dx\right] f^1_{t-1}(\eta;\xi)$$
$$[(1+|\phi\eta|) + \sigma\mathbb{E}|U_2|]\, f^1_{t-1}(\eta;\xi).$$

(i, z) Let $\phi \neq 0$ first. By substitution, we get, similarly as in (69),

$$(1+|z|)|E^{(i,z)}| = \frac{1}{\sigma}\int (1+|z|)|F^{(i,z)}_{t-1}(x, d_k(s,\psi))|w\left(\frac{z-\phi x}{\sigma}, v\right) dxdv. \tag{77}$$

Further, as

$$1+|z| \le 1+|\phi x| + |z-\phi x| \le (1+|\phi|)(1+|x|) + |z-\phi x|$$
$$\le [(1+|\phi|) + |z-\phi x|](1+|x|)$$

we may estimate

$$(1+|z|)|E^{(i,z)}|$$
$$\le \sum_{k=1}^{2}\int \frac{1}{\sigma}\left((1+|\phi|) + |z-\phi x|\right)(1+|x|)|F^{(i,z)}_{t-1}(x, d_k(s,\psi))|w\left(\frac{z-\phi x}{\sigma}, v\right) dxdv$$
$$\le \sum_{k=1}^{2}[m^i_k(z) + n^i_k(z,e)] \tag{78}$$

where, for $d_k(\bullet) = \frac{\bullet}{\psi}$,

$$m^i_k(z) = \int \frac{1}{\sigma}\left((1+|\phi|) + |z-\phi x|\right) w^1(\frac{z-\phi x}{\sigma}) w^{2|1}(v|\frac{z-\phi x}{\sigma}) m^{i,z}_{t-1}(x) dxdv$$
$$\le \left(\frac{1+|\phi|}{\sigma}\alpha_1 + \sup_u |u|w^1(u)\right) M^{i,z}_{t-1}, \tag{79}$$

$$n^i_k(z,e) = \int \frac{1}{\sigma}\left((1+|\phi|) + |z-\phi x|\right) w^1(\frac{z-\phi x}{\sigma})\omega^{2|1} n^{i,z}_{t-1}(x, \frac{e-\sigma v}{\psi}) dxdv$$
$$= \frac{|\psi|}{\rho}\int \frac{1}{\sigma}\left((1+|\phi|) + |z-\phi x|\right) w^1(\frac{z-\phi x}{\sigma})\omega^{2|1} n^{i,z}_{t-1}(x,y) dxdy$$
$$\le \frac{|\psi|}{\rho}\left(\frac{1+|\phi|}{\sigma}\alpha_1 + \sup_u |u|w^1(u)\right) N^{i,z}_{t-1}$$

with

$$\int m_k^i(z)dz = \int \frac{1}{\sigma}\left((1+|\phi|)+|z-\phi x|\right) w^1(\frac{z-\phi x}{\sigma})m_{t-1}^{i,z}(x)w^{2|1}(v|\frac{z-\phi x}{\sigma})dvdxdz$$
$$= \int \frac{1}{\sigma}\left((1+|\phi|)+|z-\phi x|\right) w^1(\frac{z-\phi x}{\sigma})m_{t-1}^{i,z}(x)dzdx$$
$$= \int \left((1+|\phi|)+\sigma|u|\right) w^1(u)m_{t-1}^{i,z}(x)dudx$$
$$\leq \left(1+|\phi|+\sigma\mathbb{E}|U_2|\right)\overline{m}_{t-1}^{i,z}$$

$$\int n_k^i(z,e)dz = \int \frac{1}{\sigma}\left((1+|\phi|)+|z-\phi x|\right) w^1(\frac{z-\phi x}{\sigma})n_{t-1}^{i,z}(x,y)\omega^{2|1}dxdydz$$
$$\leq \left(1+|\phi|+\sigma\mathbb{E}|U_2|\right)\omega^{2|1}\overline{n}_{t-1}^{i,z}.$$

If $d_k=\infty$, on the other hand, then we may put $n_k^z\equiv 0$ and define m_k^i by (79) with $r_{t-1}^{i,z}$ instead of $m_{t-1}^{i,z}$, bounding it using $R_{t-1}^{i,z}$ instead of $M_{t-1}^{i,z}$.
Similarly we get that

$$(1+|z|)|E^{(i,z)}(z,\infty)| \leq 2r^i(z),$$
$$r^i(z) = \int \frac{1}{\sigma}\left(1+|\phi|+|z-\phi x|\right)(1+|x|)|F_{t-1}^{(i,z)}(x,\infty)|w\left(\frac{z-\phi x}{\sigma},v\right)dxdv$$
$$\leq 2\left(\frac{1+|\phi|}{\sigma}\alpha_1+\sup_u |u|w^1(u)\right)R_{t-1}^{i,z} \quad (80)$$

$$\int r^i(z)dz \leq [(1+|\phi|)+\sigma\mathbb{E}|U_2|]\overline{r}_{t-1}^{i,z}.$$

If $\phi=0$, on the other hand, then relations (78) and (80) follow by a limit transition, which may be done thanks to the continuity of the integrand from (77) in ϕ and its dominatedness by integrable bound

$$\left((1+\phi_0)\alpha_1+\sigma\sup_u |u|w^1(u)\right)(1+|x|)|F_{t-1}^{(i,z)}(x,d_k(s,\psi))$$

whenever $|\phi|\leq\phi_0$ where ϕ_0 is small.

Now, similarly as in the proof of (iii)-(iv), we may use the Chain Rule to get that $(1+|z|)G_t^{(i,z)}(z,e)\leq \tilde{m}^i(z)+\tilde{n}^i(z,e)$ where $\tilde{m}^i$ and $\tilde{n}^i$ have properties analogous to $m_t^{i,z}$ and $n_t^{i,z}$ from ZBI_t; for instance, if i corresponds to ϕ , then $\tilde{m}^i(z)=m^\phi(z)+m_1^i(z)+m_2^i(z)$, $\tilde{n}^i(z,e)=n_1^i(z,e)+n_2^i(z,e)$ (similarly with $G_t^{(i,z)}(z,\infty)$ and r^i).

Finally, from (76),

$$(1+|z|)|F_t^{(i,z)}(z,e)| = (1+|z|)\left|\frac{G_t^{(i,z)}(z,e)C_t - G_t^{(z)}(z,e)C_t^{(i)}}{C_t^2}\right|$$
$$= \frac{1}{C_t}(1+|z|)(|G_t^{(i,z)}(z,e)|+|C_t^{(i)}F_t^{(z)}(z,e)|)$$
$$\leq \frac{1}{C_t}(\tilde{m}^i(z)+\tilde{n}^i(z,e)+|C_t^{(i)}|(1+|z|)f_t^1(z))$$

(the last inequality may be got similarly as (46)), so we may put

$$m_t^{i,z}(z) = \frac{\tilde{m}^i(z)}{C_t} + |C_t^{(i)}|(1+|z|)f_t^1(z), \qquad n_t^{i,z}(z,e) = \frac{1}{C_t}\tilde{n}^i(z,e)$$

(note that $\int (1+|z|)f_t^1(z)dz \le 1 + \sqrt{\epsilon_t}$).
Now, let us proceed to ZBI_1, This is, however, easy because the $F_1^{\phi,z} \equiv F_1^{\psi,z} \equiv F_1^{\sigma,z} \equiv F_1^{\epsilon,z} \equiv 0$ and the proofs for $F_1^{\sigma_1,z}$and $F_1^{\rho,z}$ are the same that those for $F_t^{\sigma,z}$ and $F_t^{\rho,z}$ with $\phi = \psi = 0$, with σ_1 instead of σ and W_1 instead of W.

The proof of EBI_t is symmetric except for the case of (η, e), which reads as

$$(1+|e|)|D^{(\eta,e)}| \le m^\eta(e),$$

$$\begin{aligned} m^\eta(e) &= \int \left(\frac{1}{\rho}(1+|\psi s|) + \left|\frac{e-\psi s}{\rho}\right|\right) w^2\left(\frac{e-\psi s}{\rho}\right) f_{t-1}(\eta, s; \xi) ds \\ &\le f_{t-1}^1(\eta;\xi)\left(\frac{1}{\rho} + \sup_v |v| w^2(v)\right) + \frac{|\psi|\alpha_1}{\rho}\int |s| f_{t-1}(\eta, s; \xi) ds, \end{aligned}$$

$$\int m^\eta(e) de \le f_{t-1}^1(\eta;\xi)(1+\mathbb{E}|V_2|) + |\psi|\int |s| f_{t-1}(\eta, s;\xi) ds,$$

where the finiteness and boundedness of $\int |s| f_{t-1}(\eta, s; \xi) ds$ is proved by (65) and (66).

C Proofs of Propositions from Section 5

We start with several auxiliary results.

Lemma 7. *Let $\mathcal{F}$ and $\mathcal{G}$ be distributions with c.d.f.'s F, G, respectively and let $a, b \in \mathbb{R}$ such that $F(a) < 1$,$G(b) < 1$. Let $\varrho(\mathcal{F}, \mathcal{G}) \le \delta$ for some $\delta > 0$. Then*
(i)

$$\varrho(\mathcal{T}(\mathcal{F}, a), \mathcal{T}(\mathcal{G}, b)) \le \frac{\delta + |F(a) - G(b)|}{1 - \max(F(a), G(b))};$$

(ii) if, in addition, $a = b$, then

$$\varrho(\mathcal{T}(\mathcal{F}, a), \mathcal{T}(\mathcal{G}, a)) \le \frac{\delta + |F(a) - G(a)|}{1 - \min(F(a), G(a))}.$$

Proof. Denote $\tilde{F}$ and $\tilde{G}$ the c.d.f.'s of $\mathcal{T}(\mathcal{F}, a)$, $\mathcal{T}(\mathcal{G}, b)$, respectively. For $x \le a, x \le b$, clearly

$$|\tilde{F}(x) - \tilde{G}(x)| = 0. \tag{81}$$

Ad (ii). If $x \geq a$, $x \geq b$, then

$$\begin{aligned}
|\tilde{F}(x) - \tilde{G}(x)| &= \left| \frac{F(x) - F(a)}{1 - F(a)} - \frac{G(x) - G(b)}{1 - G(b)} \right| \\
&= \left| \left(1 - \frac{G(x) - G(b)}{1 - G(b)}\right) - \left(1 - \frac{F(x) - F(a)}{1 - F(a)}\right) \right| = \left| \frac{1 - G(x)}{1 - G(b)} - \frac{1 - F(x)}{1 - F(a)} \right| \\
&\leq \left| \frac{1 - G(x)}{1 - G(b)} - \frac{1 - G(x)}{1 - F(a)} \right| + \left| \frac{1 - G(x)}{1 - F(a)} - \frac{1 - F(x)}{1 - F(a)} \right| \\
&= (1 - G(x)) \left| \frac{1}{1 - G(b)} - \frac{1}{1 - F(a)} \right| + \frac{|F(x) - G(x)|}{1 - F(a)} \\
&\leq (1 - G(b)) \left| \frac{1}{1 - G(b)} - \frac{1}{1 - F(a)} \right| + \frac{\delta}{1 - F(a)} \\
&= \left| 1 - \frac{1 - G(b)}{1 - F(a)} \right| + \frac{\delta}{1 - F(a)} = \frac{|F(a) - G(b)|}{1 - F(a)} + \frac{\delta}{1 - F(a)} = \frac{\delta + \gamma}{1 - F(a)}, \quad (82)
\end{aligned}$$

where $\gamma = |F(a) - G(b)|$, which, together with symmetric formula

$$|\tilde{F}(x) - \tilde{G}(x)| \leq \frac{\delta + \gamma}{1 - G(b)} \quad (83)$$

and (81), proves (ii).
Ad (i). If $a < x < b$ then $\tilde{G}(x) = 0$, $F(x) \geq F(a)$ and $G(x) \leq G(b)$ soi

$$\begin{aligned}
|\tilde{F}(x) - \tilde{G}(x)| &= \frac{F(x) - F(a)}{1 - F(a)} = \frac{(F(x) - G(x)) + (G(x) - F(a))}{1 - F(a)} \\
&\leq \frac{(F(x) - G(x)) + (G(b) - F(a))}{1 - F(a)} \leq \frac{|F(x) - G(x)| + |G(b) - F(a)|}{1 - F(a)} \leq \frac{\delta + \gamma}{1 - F(a)};
\end{aligned}$$

which, together with (82) and (83) proves (i) for $a < b$. The proof of the case $a > b$ is symmetrical. □

Lemma 8. *Let $\mathcal{F}$ and $\mathcal{G}$ be as in Lemma 7 with $\varrho(\mathcal{F}, \mathcal{G}) \leq \delta$ and let $\mathcal{N}$ and $\mathcal{P}$ be distributions such that $\varrho(\mathcal{N}, \mathcal{P}) \leq \gamma$. Then,*

$$\varrho(\mathcal{F} \circ \mathcal{N}, \mathcal{G} \circ \mathcal{P}) \leq \delta + \gamma$$

Proof. By the Triangular inequality,

$$\varrho(\mathcal{F} \circ \mathcal{N}, \mathcal{G} \circ \mathcal{P}) \leq \varrho(\mathcal{F} \circ \mathcal{N}, \mathcal{G} \circ \mathcal{N}) + \varrho(\mathcal{G} \circ \mathcal{N}, \mathcal{G} \circ \mathcal{P}).$$

Further, denoting N the c.d.f. of $\mathcal{N}$, we get

$$\begin{aligned}
\varrho(\mathcal{F} \circ \mathcal{N}, \mathcal{G} \circ \mathcal{N}) &= \sup_x |\int F(x - y) dN(y) - \int G(x - y) dN(y)| \\
&= \sup_x |\int [F(x - y) - G(x - y)] dN(y)| \\
&\leq \sup_x \int |F(x - y) - G(x - y)| dN(y) \leq \int \varrho(\mathcal{F}, \mathcal{G}) dN(y) = \varrho(\mathcal{F}, \mathcal{G}).
\end{aligned}$$

Applying the same procedure to $\varrho(\mathcal{G} \circ \mathcal{N}, \mathcal{G} \circ \mathcal{P})$, the Lemma is proved. □

Lemma 9. For any c.d.f. G and any $\pi \in (0, 1)$, $G(\gamma(\pi)) \geq \pi$ and $G(\gamma(\pi)-) \leq \pi$ where γ the quantile function of G.

Proof. The assertion may be proved directly from the definition of quantile □

Lemma 10. *Let $p \in (0.1)$ and let $\mathcal{F}$ and $\mathcal{G}$ be as in Lemma 7. Let F be continuous strictly increasing and let*

$$\varrho(\mathcal{F}, \mathcal{G}) \leq \delta. \tag{84}$$

Then

$$|\chi(\mathcal{F}, p) - \chi(\mathcal{G}, p)| \leq \overline{u} - \underline{u}, \tag{85}$$

where

$$\underline{u} = \chi(\mathcal{G}, p - \delta), \qquad \overline{u} = \chi(\mathcal{G}, p + \delta).$$

Proof. From the monotonicity of the quantile,

$$\underline{u} \leq \chi(\mathcal{G}, p) \leq \overline{u};$$

thus, to prove (85), it suffices to show that

$$\underline{u} \leq \chi(\mathcal{F}, p) \leq \overline{u}. \tag{86}$$

We do it by contradiction: First, assume that $\overline{u} < \chi(\mathcal{F}, p)$. Then, however, by the strict monotonicity of F and its continuity,

$$F(\overline{u}) < F(\chi(\mathcal{F}, p)) = p,$$

and, by Lemma 9,

$$G(\overline{u}) \geq p + \delta,$$

which, together, gives

$$G(\overline{u}) - F(\overline{u}) \geq p + \delta - F(\overline{u}) > (p + \delta) - p = \delta.$$

which is a contradiction.

Now, assume $\chi(\mathcal{F}, p) < \underline{u}$. Then, however, by the continuity and monotonicity of F, $F(\underline{u}-) = F(\underline{u}) > F(\chi(\mathcal{F}, p)) = p$, and, by Lemma 9, $G(\underline{u}-) \leq p - \delta$, which gives a contradiction $F(\underline{u}-) - G(\underline{u}-) > \delta$.

Summed up, (86) is proved implying (85). □

C.1 Proof of Proposition 4

Let

$$\varrho(\mathcal{A}_{t-1}, \mathcal{Z}_{t-1}) \leq \delta_{t-1}. \tag{87}$$

Denote T

$$\mathcal{Z}_t^\star = \mathcal{T}(\mathcal{Z}_{t-1}, -Y_{t-1}^\star), \qquad \mathcal{Z}_t^S = \phi \cdot \mathcal{Z}_t^\star, \qquad \mathcal{Z}_t^\circ = \mathcal{Z}_t^S \circ \mathcal{W}$$

By the Triangular Inequality and Lemma 8, and Triangular Inequality again

$$\begin{aligned}\varrho(\mathcal{A}_t, \mathcal{Z}_t^\circ) \leq \varrho(\mathcal{A}_t, \mathcal{A}_t^\circ) + \varrho(\mathcal{A}_t^\circ, \mathcal{Z}_t^\circ) \leq \varrho(\mathcal{A}_t, \mathcal{A}_t^\circ) + \varrho(\mathcal{A}_t^S, \mathcal{Z}_t^S) + \varrho(\mathcal{W}, \mathcal{U}) \\ \leq \varrho(\mathcal{A}_t, \mathcal{A}_t^\circ) + \varrho(\mathcal{A}_t^S, \phi \cdot \mathcal{A}_t^\star) + \varrho(\phi \cdot \mathcal{A}_t^\star, \mathcal{Z}_t^S) + \varrho(\mathcal{W}, \mathcal{U}).\end{aligned}$$

Further, by Lemma 7 (i),

$$\varrho(\phi \cdot \mathcal{A}_t^\star, \mathcal{Z}_t^S) = \varrho(\phi \cdot \mathcal{A}_t^\star, \phi \cdot \mathcal{Z}_t^\star) = \varrho(\mathcal{A}_t^\star, \mathcal{Z}_t^\star) \leq \frac{2\delta_{t-1}}{1 - \mathcal{A}_{t-1}(-\infty, -Y_{t-1}^\star]}$$

As, in addition, $\varrho(\mathcal{A}_t, \mathcal{A}_t^\circ) \leq \epsilon_t$ the Proposition is proved.

C.2 Proof of Proposition 5

The proof is analogous to that of Proposition 4 with the difference that Lemma 7 (ii) is used to estimate

$$\varrho(\mathcal{B}_t^\star, \mathcal{Z}_t^\star) \le \frac{\left|\mathcal{Z}_{t-1}(-\infty, -Y_t^\star] - \mathcal{B}_{t-1}(-\infty, -\tilde{Y}_t^\star]\right| + \eta_{t-1}}{1 - \max(\mathcal{Z}_{t-1}^\star(-\infty, -Y_t^\star]_{t-1}, \mathcal{B}_{t-1}(-\infty, -\tilde{Y}_t^\star])}.$$