Hidden Auto-Conflict in the Theory of Belief Functions

Milan DANIEL[1] and Václav KRATOCHVÍL[2]

[1] *Jan Becher - Karlovarská Becherovka, a.s., Pernod Ricard group*
Luxembourg Plaza, Přemyslovská 43, Prague 3, CZ - 130 00
Czech Republic
milan.daniel@pernod-ricard.com

[2] *Institute of Information Theory and Automation, Czech Academy of Sciences*
Pod vodárenskou věží 4, Prague 8, CZ - 182 08
Czech Republic
velorex@utia.cas.cz

Abstract

Hidden conflicts of belief functions in some cases where the sum of all multiples of conflicting belief masses being equal to zero were observed. Relationships of hidden conflicts and auto-conflicts of belief functions are pointed out. We are focused on hidden auto-conflicts here — on hidden conflicts appearing when three or more numerically same belief functions are combined. Hidden auto-conflict is a kind of internal conflict. Degrees of hidden auto-conflicts and full non-conflictness are defined and analysed. Finally, computational issues of hidden auto-conflicts and non-conflictness are presented.

Keywords: Belief functions; Dempster-Shafer theory; Uncertainty; Conflicting belief masses; Internal conflict; Conflict between belief functions; Auto-Conflict; Hidden conflict; Hidden auto-conflict; Full non-conflictness.

1 Introduction

When combining belief functions (BFs) by the conjunctive rules of combination, some conflicts often appear (they are assigned either to $\emptyset$ by non-normalised conjunctive rule $\copyright$ or distributed among other belief masses by normalisation in Dempster's rule of combination $\oplus$). Combination of conflicting BFs and interpretation of their conflicts are often questionable in real applications. Thus a series of papers related to conflicts of BFs was published, e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. A new interpretation of conflicts of belief functions was introduced in [11]: important distinction of internal conflicts of individual BFs (due to their inconsistency) from conflicts between BFs (due to conflict/contradiction of evidences represented by the BFs) was introduced there. Note that zero sum of all multiples of conflicting belief masses (denoted by $m_{\copyright}(\emptyset)$) is usually considered as non-conflictness of the belief functions in all these approaches.

When analyzing the conflict between belief functions based on their non-conflicting parts[1] defined by Daniel in [4] a positive value of conflict was observed even in a situation when sum of all multiples of conflicting belief masses equals to zero. This arose a series of new questions: how to interpret the sum of conflicting masses, is the conflict based on non-conflicting parts of belief functions correct? These questions are studied in [13]. The answers are positive in favour of the conflict based on non-conflicting parts. This led to a definition of a hidden conflict of BFs there.

Different levels / degrees of hidden conflicts are defined and investigated there. In correspondence to the degrees of hidden conflict, there are studied different degrees of non-conflictness, including full non-conflictness and conditions, under which belief functions are fully non-conflicting. In accordance with the original Daniel's approach from [11], there are observed and investigated not only hidden conflicts between two belief functions, but also internal hidden conflicts of individual BFs. The research covers also computational aspects of hidden conflict.

[1] Conflicting and non-conflicting parts of belief functions originally come from [12].

By investigating a hidden conflict, we have noticed a hidden auto-conflict. Auto-conflict is a term describing hidden conflict defined by Martin's et al. in [14, 15, 8]. It is a sum of multiples of conflicting belief masses when two or more numerically same BFs are conjunctively combined. An idea of auto-conflict of any positive order was defined and briefly presented in 2006 [14] and further studied in [15] two years later. Our current contribution is focused to investigation of hidden auto-conflict and its relation to original Martin's et al. results.

2 Preliminaries

We assume classic definitions of basic notions from theory of *belief functions* [16] on finite exhaustive frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is *power-set* of Ω. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. Because there is a unique correspondence among m and corresponding Bel and Pl thus we often speak about m as of belief function.

A *focal element* is a subset of the frame of discernment $X \subseteq \Omega$, such that $m(X) > 0$. In the case of $0 < |X| < n$ it is a proper focal element. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF); in fact, it is a probability distribution on Ω. If there are only focal elements such that $|X| = 1$ or $|X| = n$ we speak about *quasi-Bayesian BF* (qBBF). In the case of $m(\Omega) = 1$ we speak about *vacuous BF* (VBF) and otherwise about *a non-vacuous BF*; in the case of the only focal element $\emptyset \neq X \subset \Omega$, i.e., if $m(X) = 1$, we speak about *a categorical BF*. If all focal elements have a non-empty intersection, we speak about *a consistent BF*; and if all of them are nested, about *a consonant BF*.

Dempster's (normalized conjunctive) rule of combination $\oplus$ is given as

$$(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$$

for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [16]. Putting $K = 1$ and $(m_1 \circledcirc m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\circledcirc$, see e. g. [17].

Smets's *pignistic probability* is given by $BetP(\omega_i) = \sum_{\omega_i \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1-m(\emptyset)}$, see e.g. [17]. *Normalized plausibility of singletons*[2] of Bel is a probability distribution Pl_P such that $Pl\_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [18, 19].

A conflict of BFs Bel', Bel'' based on their non-conflicting parts is defined by the expression $Conf(Bel', Bel'') = (m_0' \circledcirc m_0'')(\emptyset)$, where non-conflicting part Bel_0 (of a BF Bel) is unique consonant BF such that $Pl\_P_0 = Pl\_P$ (normalized plausibility of singletons corresponding to Bel_0 is the same as that corresponding to Bel). For an algorithm to compute Bel_0 see [4].

The auto-conflict of order s of a belief function Bel given by bba m if defined by

$$a_s(m) = (\circledcirc_{i=1}^{s} m)(\emptyset),$$

where $s \geq 1$ and $\circledcirc$ is the non-normalized conjunctive combination; and simply $a_2(m) = (m \circledcirc m)(\emptyset)$, [15, 14].

The basic properties of auto-conflict are the following[15, 14]:

$$a_s(m) \leq a_{s+1}(m),$$

and

$$a(m) = a_2(m) > 0 \quad \text{implies} \quad lim_{s \to \infty} a_s(m) = 1.$$

[2] A plausibility of singletons is called a *contour function* by Shafer in [16], thus $Pl\_P(Bel)$ is a normalization of a contour function in fact.

3 Hidden Conflicts of Belief Functions

3.1 An Introductory Example

Let us suppose two simple consistent belief functions Bel' and Bel'' on a three-element frame of discernment $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ given by the bbas $m'(\{\omega_1, \omega_2\}) = 0.6$, $m'(\{\omega_1, \omega_3\}) = 0.4$, and $m''(\{\omega_2, \omega_3\}) = 1.0$. Then $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$ what seems — and it is usually considered — to be a non-conflictness of m' and m'', but there is positive conflict based on non-conflicting parts $Conf(Bel', Bel'') = (m'_0 \textcircled{\scriptsize$\cap$} m''_0)(\emptyset) = 0.4 > 0$. (This holds true despite of Theorem 4 from [4] which should be revised in a future).

We can easily verify this situation: the only focal element of m'' has a non-empty intersection with both focal elements of m', thus $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = \sum_{X \cap Y = \emptyset} m'(X)m''(Y) = \text{(empty sum)} = 0$; Bel'' is already consonant itself, thus $Bel''_0 = Bel''$, $m''_0 = m''$, $Pl'(\{\omega_1\}) = 1$, $Pl'(\{\omega_2\}) = 0.6$, $Pl'(\{\omega_3\}) = 0.4$, thus $m'_0(\{\omega_1\}) = 0.4$, $m'_0(\{\omega_1, \omega_2\}) = 0.2$, $m'_0(\{\omega_1, \omega_2, \omega_3\}) = 0.4$, hence $Conf(Bel', Bel'') = (m'_0 \textcircled{\scriptsize$\cap$} m''_0)(\emptyset) = m'_0(\{\omega_1\}) m''_0(\{\omega_2, \omega_3\}) = 0.4 \cdot 1 = 0.4$.

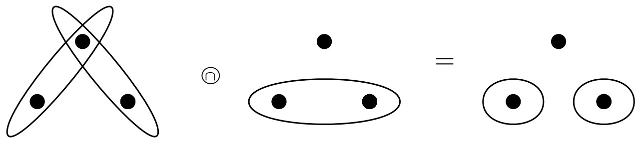


Figure 1: Introductory Example: focal elements of m', m'', and of $m' \textcircled{\scriptsize$\cap$} m''$.

3.2 Observation of Hidden Conflict

The following questions arise: Does $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$ really represent non-conflictness of respective BFs? Is the definition of conflict based on non-conflicting parts correct? Are m' and m'' conflicting or non-conflicting? What does $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$ mean?

We can formalize our assumptions from [13] as follows:

Assumption As1 Conjunctive combination of two mutually non-conflicting BFs Bel' and Bel'' is mutually non-conflicting with any of the individual BFs Bel' and Bel''.

By induction, Assumption **As1** can be extended as follows:

Assumption As1* Conjunctive combination of two mutually non-conflicting BFs Bel' and Bel'' is mutually non-conflicting with any number of combinations with individual BF Bel' and also with any number of combinations with the other individual BF Bel'', thus with $\textcircled{\scriptsize$\cap$}_1^k Bel'$ and with $\textcircled{\scriptsize$\cap$}_1^l Bel'$, for any $k, l \geq 1$.

Supposing symmetry of conflictness / non-conflictness we obtain **As1**** as it follows:

Assumption As1** Conjunctive combination of two mutually non-conflicting BFs Bel' and Bel'' is mutually non-conflicting with any number of combinations of any of both the individual BFs Bel' and/or Bel'', thus with $\textcircled{\scriptsize$\cap$}_1^k Bel' \textcircled{\scriptsize$\cap$} \textcircled{\scriptsize$\cap$}_1^l Bel'$, for any $k, l \geq 0$, $k + l \geq 1$.

Thus Assumption **As1*** is just a reformulation of **As1** and supposing symmetry of conflictness / non-conflictness also Assumption **As1**** is just a reformulation of **As1**.

Based on Assumption **As1**, we can easily show that the BF from Introductory Example are not non-conflicting: Let us suppose that Bel' and Bel'' are non-conflicting now. Thus their combination $Bel' \textcircled{\scriptsize$\cap$} Bel''$ should be also non-conflicting with both of them. Does this hold for BFs from our example? This holds true when we combine $m' \textcircled{\scriptsize$\cap$} m''$ with m'' one more time. It follows from the idempotency of categorical m'': $m' \textcircled{\scriptsize$\cap$} m'' \textcircled{\scriptsize$\cap$} m'' = m' \textcircled{\scriptsize$\cap$} m''$ and therefore $(m' \textcircled{\scriptsize$\cap$} m'' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0$ again. On the other hand, we obtain positive $(m' \textcircled{\scriptsize$\cap$} m'' \textcircled{\scriptsize$\cap$} m')(\emptyset) = (m' \textcircled{\scriptsize$\cap$} m' \textcircled{\scriptsize$\cap$} m'')(\emptyset) = 0.48$. See Table 1 and Figure 2. When m'' and m' are combined once, then we observe $m_{\textcircled{\scriptsize$\cap$}}(\emptyset) = 0$. When combining m'' with m' twice then $m_{\textcircled{\scriptsize$\cap$}}(\emptyset) = 0.48$. We observe some kind of a hidden conflict. And we have an argument for correctness of positive value of $Conf(Bel', Bel'')$.

Analogous result was obtained using a decisional interpretation of the BFs [13]. Hence $(m' \textcircled{\scriptsize$\cap$} m'')(\emptyset)$ really does not mean non-conflictness of the BFs. It means a simple or partial compatibility of respective focal elements only.

Table 1: Hidden conflict in the Introductory Example

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3	$\emptyset$
$m'(X)$:	0.0	0.0	0.0	0.60	0.40	0.00	0.00	—
$m''(X)$:	0.0	0.0	0.0	0.00	0.00	1.00	0.00	—
$(m' \circledcap m'')(X)$:	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m' \circledcap m'' \circledcap m'')(X)$:	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m' \circledcap m'' \circledcap m')(X)$:	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48
$(m' \circledcap m'' \circledcap m' \circledcap m'')(X)$:	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48

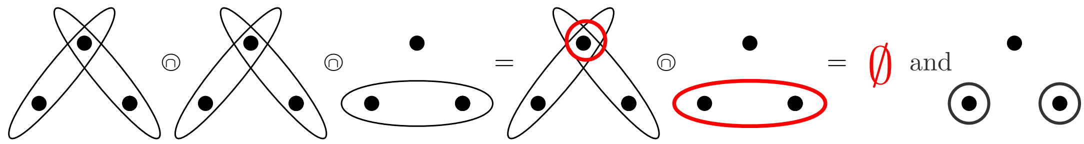


Figure 2: Arising of a hidden conflict between BFs in the Introductory Example: focal elements of m', m', m'' — $m' \circledcap m', m''$ and of $(m' \circledcap m') \circledcap m''$.

3.3 A Simple Definition of Hidden Conflict and its Relationship to Auto-Conflict

Definition 1 *Let us suppose two BFs Bel', Bel'' defined by bbas m', m'', such that $(m' \circledcap m'')(\emptyset) = 0$. If there further holds $(m' \circledcap m'' \circledcap m')(\emptyset) > 0$ or $(m' \circledcap m'' \circledcap m'')(\emptyset) > 0$ we say that there is* a hidden conflict *of the BFs.*

Observation 1 *A condition $(m' \circledcap m'' \circledcap m')(\emptyset) > 0$ or $(m' \circledcap m'' \circledcap m'')(\emptyset) > 0$ from Definition 1 is equivalent to the following condition $(m' \circledcap m'' \circledcap m' \circledcap m'')(\emptyset) > 0$.*

We have to note that a hidden conflict is quite a new phenomenon first time defined in [13], it is qualitatively different from the other referred approaches, even different form ideas of all Daniel's works on conflict of belief functions from previous years. Till now, it was supposed that $m_{\circledcap}(\emptyset)$ includes both all conflicts between BFs and also all internal conflicts of individual BFs. Thus conflict between BFs was supposed to be less or equal to $m_{\circledcap}(\emptyset)$. Here, we deal with a situation of a positive conflict between BFs while $m_{\circledcap}(\emptyset) = 0$. The presented approach is new and different from all previous ones, but it is definitely not against all of the previous approaches, especially not against the conflict between BFs based on their non-conflicting parts, which has enabled observation of hidden conflict and which is supported by the existence of a hidden conflict.

We have already observed that $m_{\circledcap}(\emptyset) = 0$ does not mean full non-conflictness of BFs and that the condition $(m' \circledcap m'' \circledcap m' \circledcap m'')(\emptyset) > 0$ together with $(m' \circledcap m'')(\emptyset) = 0$ defines a hidden conflict. What about the condition $(m' \circledcap m'' \circledcap m' \circledcap m'')(\emptyset) = 0$? Is this condition sufficient for full non-conflictness of BFs Bel' and Bel''? May some conflict be still hidden there?

There are repeated combinations of m' and m'' which resembles Martin's auto-conflict. There is $a(m') = (m' \oplus m')(\emptyset) = 0$ and also $a(m'') = (m'' \oplus m'')(\emptyset) = 0$, thus there is no auto-conflict of the individual input BFs. But $a(m' \oplus m'') = ((m' \oplus m'') \oplus (m' \oplus m'')(\emptyset) = (m' \circledcap m'' \circledcap m' \circledcap m'')(\emptyset) > 0$. The simple definition of the hidden conflict is: $(m' \circledcap m'')(\emptyset) = 0$ and $a(m' \oplus m'') > 0$. Thus the question from the previous example may be reformulated: Is the condition $a(m' \oplus m'') = 0$ sufficient for full non-conflictness of BFs Bel' and Bel''?

We have shown that validity of the condition $(m' \circledcap m'' \circledcap m' \circledcap m'')(\emptyset) = 0$ is sufficient for full non-conflictness of BFs Bel' and Bel'' only on Ω_3 - on the frame of discernment of the Introductory Example. It is not sufficient in general. To solve the question in general, we have to consider a larger frame of discernment.

3.4 Little Angel Example

For Ω_5 one can find the following Little Angel Example (see Table 2 and Figure 3). Similarly to Introductory Example, we have two consistent BFs Bel^i and Bel^{ii} with disjoint sets of max-

Table 2: Hidden Conflict in the Little Angel Example

X :	$A=\{\omega_1,\omega_2,\omega_5\}$	$B=\{\omega_1,\omega_2,\omega_3,\omega_4\}$	$C=\{\omega_1,\omega_3,\omega_4,\omega_5\}$	$X=\{\omega_2,\omega_3,\omega_4,\omega_5\}$	$\emptyset$
$m^i(X)$:	0.1	0.30	0.60	0.00	—
$m^{ii}(X)$:	0.0	0.00	0.00	1.00	—

X :	$A\cap X$	$B\cap X$	$C\cap X$	$A\cap B\cap X$	$A\cap C\cap X$	$B\cap C\cap X$	$\emptyset$
$(m^i \circledcirc m^{ii})(X)$:	0.1	0.3	0.6	0.0	0.0	0.0	0.000
$(m^i \circledcirc m^{ii} \circledcirc m^{ii})(X)$:	0.10	0.30	0.60	0.00	0.00	0.00	0.00
$(m^i \circledcirc m^i \circledcirc m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i \circledcirc m^i \circledcirc m^{ii} \circledcirc m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i \circledcirc m^{ii} \circledcirc m^{ii} \circledcirc m^{ii})(X)$:	0.010	0.090	0.360	0.060	0.120	0.360	0.000
$(m^i \circledcirc m^i \circledcirc m^i \circledcirc m^{ii})(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108
$m^*(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108

where $m^* = m^i \circledcirc m^i \circledcirc m^i \circledcirc m^{ii} \circledcirc m^{ii} \circledcirc m^{ii}$.

plausibility elements and where zero condition $(m^i \circledcirc m^{ii})(\emptyset) = 0$ holds true. Moreover here holds also $(m^i \circledcirc m^{ii} \circledcirc m^i \circledcirc m^{ii})(\emptyset) = 0$ (see Table 2) while $Conf(Bel^i, Bel^{ii}) = 0.1$ is positive again. Positiveness of the $Conf$ value can be easily seen from the fact that sets of max-plausibility elements are disjoint for Pl^i and Pl^{ii}. Numerically, we have again $Bel_0^{ii} = Bel^{ii}$, and $Pl\_P^i = (10/39, 4/39, 9/39, 9/39, 7/39)$. We obtain $m_0^i(\{\omega_1\}) = 0.1, m_0^i(\{\omega_1, \omega_3, \omega_4\}) = 0.2, m_0^i(\{\omega_1, \omega_3, \omega_4, \omega_5\}) = 0.3, m_0^i(\{\Omega_5\}) = 0.4$, and $Conf(Bel^i, Bel^{ii}) = m_0^i(\{\omega_1\})m^{ii}(X) = 0.1$.

Analogous arguments hold true for the positive $Conf$ and hidden conflict again (of the 2nd degree this time): (i) $BetP^i = (0.2583, 0.1083, 0.225, 0.225, 0.1833)$ which is not numerically the same as Pl_P^i, but both prefer ω_1, whereas $BetP^{ii} = Pl\_P^{ii} = (0.00, 0.25, 0.25, 0.25, 0.25)$.
(ii) Despite to the assumption As1* there is $((m^i \circledcirc m^{ii}) \circledcirc (m^i \circledcirc m^i))(\emptyset) = 0.108 > 0$, see Table 2.

In language of auto-conflicts there is $a_2(m^i \circledcirc m^{ii}) = 0$ and $a_3(m^i \circledcirc m^{ii}) > 0$.

For an existence of a hidden conflict, it is the structure of focal elements that is important — not their belief masses. Belief masses are important for the size of a conflict. In general, we can take $m^i(A) = a$, $m^i(B) = b$, $m^i(C) = c$, for A, B, C defined in Table 2 and for any $a, b, c > 0$, such that $a + b + c = 1$ and we obtain $m(\emptyset) = 6abc$ as a hidden conflict of the 2nd degree (in our numeric case there is $6abc = 6 \cdot 0.1 \cdot 0.3 \cdot 0.6 = 0.108$). For a graphical representation of the Little Angel Example, see Figure 3.

Degrees of hidden conflict, its maximal value, and the issue of full non-conflictness will be briefly introduced in the following subsection.

3.5 Degrees of Hidden Conflict and Full Non-conflictness

When analyzing examples from the previous section, we have observed different levels where conflicts were hidden. We can formalize degrees of hidden conflict:

Definition 2 *Assume two BFs Bel^i, Bel^{ii} defined by bbas m^i, m^{ii}, such that for some $k > 0$ $(\bigcirc_{j=1}^{k}(m^i \circledcirc m^{ii}))(\emptyset) = 0$. If there further holds $(\bigcirc_{j=1}^{k+1}(m^i \circledcirc m^{ii}))(\emptyset) > 0$ we say that there is* a hidden conflict of the k-th degree *of BFs Bel^i and Bel^{ii}.*

Analogously to a particular degree of hidden conflict, there are degrees of non-conflictness as well. Particular degree of non-conflictness is not important. However, there is an important question whether there is some hidden conflict or not, i.e. whether or not the BFs in question are fully non-conflicting.

Definition 3 *We say that BFs Bel^i and Bel^{ii} are* fully non-conflicting *if there is no hidden conflict of any degree. I.e. if $(\bigcirc_{j=1}^{k}(m^i \circledcirc m^{ii}))(\emptyset) = 0$ for any $k > 0$.*

Thus there is a question of how many times we have to combine $(m^i \circledcirc m^{ii})$, i.e., for which k value of $(\bigcirc_{j=1}^{k}(m^i \circledcirc m^{ii}))(\emptyset)$ shows whether there is some hidden conflict of the BFs Bel^i and Bel^{ii} or not.

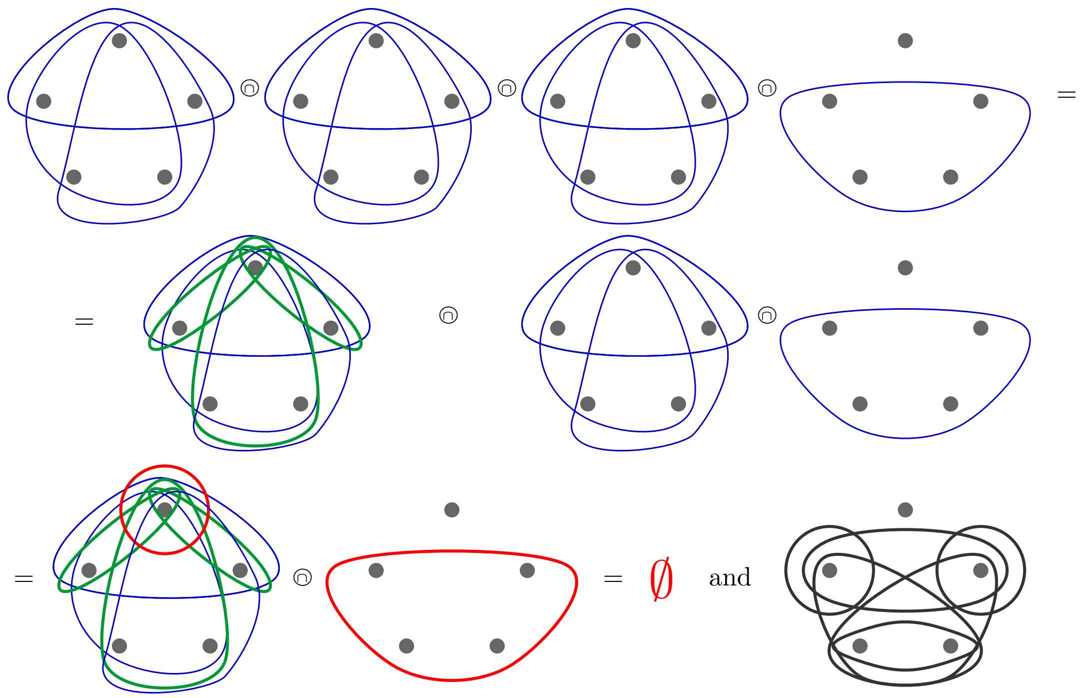


Figure 3: Arrising of a hidden conflict between BFs in the Little Angel Example. Focal elements of m^i, $m^i \odot m^i$, $m^i \odot m^i \odot m^i$ and of $(m^i \odot m^i \odot m^i) \odot m^{ii}$. Red-colored focal elements are those responsible for creation of the empty-set in the last step.

Theorem 1 (maximal degree of hidden conflict) *For any non-vacuous BFs Bel^i and Bel^{ii} on any frame Ω_n it holds that*

$$(\bigodot_{j=1}^{n-1}(m^i \odot m^{ii}))(\emptyset) = 0 \quad \textit{iff} \quad (\bigodot_{j=1}^{k}(m^i \odot m^{ii}))(\emptyset) = 0 \tag{1}$$

for any $k > n-2$.

Considering the notion of auto-conflict we obtain the following theorem:

Theorem 2 *For any two BFs Bel^i and Bel^{ii} defined by bbas m^i and m^{ii} the following holds:*
(i) There is a hidden conflict of the k-th degree *of BFs Bel^i and Bel^{ii} if and only if* $a_k(m^i \odot m^{ii}) = 0$ & $a_{k+1}(m^i \odot m^{ii}) > 0$.
(ii) Bel^i and Bel^{ii} are fully non-conflicting if and only if auto-conflict of any order of their conjunctive combination is zero, i.e., if and only if $a_k(m^i \odot m^{ii}) = 0$ for any $k > 0$.
(iii) If Bel^i and Bel^{ii} are non-vacuous BFs on any finite frame of discernment Ω_n it holds that

$$a_{n-1}(m^i \odot m^{ii}) = 0 \quad \textit{iff} \quad a_k(m^i \odot m^{ii}) = 0 \;\; \textit{for any } k > n-2.$$

4 Hidden Auto-Conflict of a Belief Function

4.1 Examples and Definition of Hidden Auto-Conflict

Let us return to properties of auto-conflict. In general, the following holds true:

$$a_s(m) \le a_{s+1}(m) \;\; \& \;\; a_2(m) > 0 \Rightarrow lim_{s\to\infty} a_s(m) = 1$$

by [14]. It is stated in [15] that $lim_{s\to\infty} a_s(m) = 1$ holds true (without any assumption). This is not correct generally, (any consistent BF is a counter-example) but under non-explicitly stated assumption of BFs considered in [15]. I.e. a special subclass of quasi-Bayesian BFs, BFs with all singletons plus Ω_n as their focal elements. I.e. qBBFs with exactly $n+1$ focal elements.

Such BFs have always $a_2(m) > 0$. On the other hand for all the consistent BFs it hods that $a(m) = a_2(m) = 0 = a_k(m)$, for any $k > 0$. Even $lim_{k\to\infty} a_k(m) = 0$.

For any other BFs it holds that $a(m) \geq 0$ and $a_s(m) \leq a_{s+1}(m)$. Moreover it holds that $a_{k-1}(m) < a_k(m)$ implies $a_k(m) < a_{k+1}(m)$. This follows the number and cardinalities of focal elements of $a_k(m)$; $a_k(m)$ has all focal elements of $a_{k-1}(m)$, plus possibly some additional focal elements (defined by intersection of focal elements of $a_{k-1}(m)$). Furthermore, if $a_{k-1}(m)$ and $a_k(m)$ have the same focal elements, this holds true also for any $a_s(m)$ where $s > k$. Specially, whenever $\emptyset$ is a focal element of $a_k(m)$ it is also a focal element of any $a_s(m)$ where $s > k$ and $a_s(m) < a_{s+1}(m)$.

If $a(m) = a_s(m) = 0$ for a non-consistent BF Bel and some $s > 1$, then there exists some $k > s$, such that $a_k(m) > 0$. We can call such an auto-conflict as hidden auto-conflict. There was not hidden auto-conflict in our Introductory example of hidden conflict, but there is a hidden auto-conflict in the Little Angel Example:

Hidden auto-conflict in the Little Angel Example: Let us compute conjunctive combination of the input BFs Bel^i and Bel^{ii}: $m = m^i \oplus m^{ii}$. We obtain $m(\{\omega_2, \omega_5\}) = 0.1$, $m(\{\omega_2, \omega_3, \omega_4\}) = 0.3$, $m(\{\omega_3, \omega_4, \omega_5\}) = 0.6$, $m(\emptyset) = 0$ (See line $(m^i \oplus m^{ii})(X)$ in Table 2.)

There is no auto-conflict: $a(m) = 0$ (See line $(m^i \circledcap m^i \circledcap m^{ii} \circledcap m^{ii})(X)$ in Table 2.) There is a (positive) auto-conflict of degree 3: $a_3(m) = 0.108$ (See line $(m^*(X)$ in Table 2.) Thus the positive auto-conflict of the order 3 $a_3(m) = 0.108$ is hidden by zero auto-conflict $a(m)$.

For focal elements and arising of the hidden auto-conflict of $m = m^i \oplus m^{ii}$ see Figure 4.

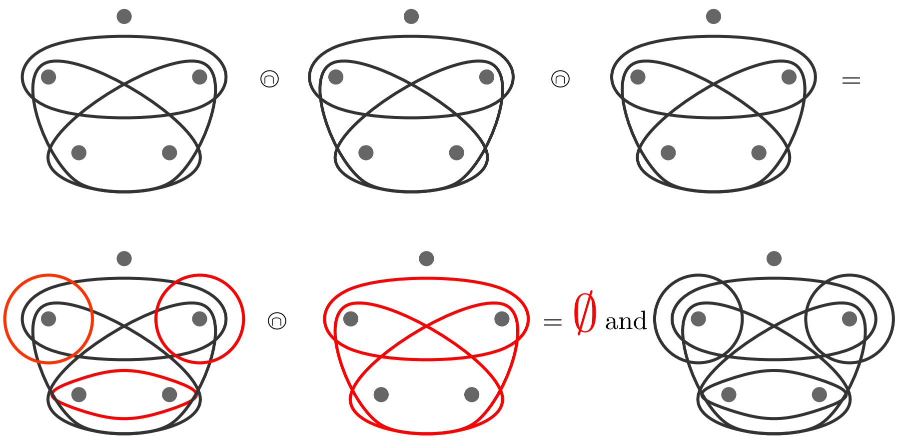


Figure 4: Arising of a hidden auto-conflict of BF $Bel^i \circledcap Bel^{ii}$, focal elements of $Bel^i \circledcap Bel^{ii}$, $\circledcap_1^2(Bel^i \circledcap Bel^{ii})$, and $\circledcap_1^3(Bel^i \circledcap Bel^{ii})$.

Definition 4 *We say that BF Bel defined by bba m has a* hidden auto-conflict, *if its auto-conflict* $a(m) = a_2(m) = 0$ *and if* $a_s(m) > 0$ *for some* $s > 0$.

We say that BF Bel defined by bba m have hidden auto-conflict of degree s *if* $a_{s+1}(m) = 0$ *and* $a_{s+2}(m) > 0$.

Example 1 *We can find an example of hidden auto-conflict even on Ω_3 frame of discernment. Let us suppose BF given by bba m^{iii} from Table 1; we obtain $a(m^{iii}) = 0$ and $a_3(m^{iii}) = 0.18$ there. Hence again, the positive auto-conflict of the order 3 $a_3(m^{iii}) = 0.18$ is hidden by zero auto-conflict $a(m^{iii})$. For focal elements and arising of the hidden auto-conflict of $m = m^i \oplus m^{ii}$ see Figure 6.*

Example 2 (A general example) *We can take conjunctive sum of any two BFs with hidden conflict of 2nd degree, thus $Bel = Bel^i \circledcap Bel^{ii}$: there is $(m^i \circledcap m^i \circledcap m^{ii} \circledcap m^{ii})(\emptyset) = 0 = a(m^i \circledcap m^{ii})$ and $a_3(m^i \circledcap m^{ii}) = (m^i \circledcap m^i \circledcap m^i \circledcap m^{ii} \circledcap m^{ii} \circledcap m^{ii})(\emptyset) > 0$. Thus the positive $a_3(m)$ is hidden by $a(m) = 0$.*

4.2 Maximal Degree of Hidden Auto-Conflict

The following theorem provides a solution to the question whether BF Bel on Ω_n has any auto-conflict or whether it is completely non-conflicting.

Table 3: Example of a hidden auto-conflict on Ω_3

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3	$\emptyset$
$m^{iii}(X)$:	0.0	0.0	0.0	0.50	0.30	0.20	0.00	—
$(m^{iii} \odot m^{iii})(X)$:	0.30	0.20	0.12	0.25	0.09	0.04	0.00	0.00
$(m^{iii} \odot m^{iii} \odot m^{iii})(X)$:	0.36	0.21	0.09	0.125	0.027	0.008	0.00	0.18

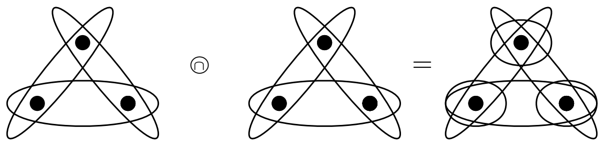


Figure 5: Focal elements of m^{iii} and $m^{iii} \odot m^{iii}$; no auto-conflict of non-consistent BF Bel^{iii}: $a_2(m^{iii}) = 0$.

Theorem 3 *For any belief function Bel on Ω_n the following holds:*

$$a_n(Bel) = 0 \quad iff \quad a_k(Bel) = 0 \;\; for\ all \;\; k \geq n. \tag{2}$$

Thus, maximal degree of hidden auto-conflict is equal to $n - 2$.

Idea of proof. Number of focal elements (f.e.) is decreased until it is fixed. Thus there is at most $n-1$ decreases (creations of less f.e.); n-times Bel, $(n-1)$-times $\odot$.

For a given BF Bel it may be sufficient to compute auto-conflict of even lesser degree:

Theorem 4 *For a given belief function Bel on Ω_n the following holds:*

$$a_{s+1}(Bel) = 0 \quad iff \quad a_k(Bel) = 0 \;\; for\ all \;\; k \geq s + 1, \tag{3}$$

where s is maximal cardinality of a proper focal element of Bel (focal element different from Ω_n). Thus Bel may have hidden auto-conflict up to degree $s - 1$.

Idea of proof. Analogously to the proof of Theorem 3, there is at most s decreases for $s < n$.

According to this theorem we can see that a special class of qBBFs with just $n+1$ focal element has maximal cardinality of a proper focal element 1 and maximal degree of hidden auto-conflict 0. Thus there is no hidden auto-conflict on the special class of BFs on which auto-conflicts were studied by Martin et al. [15]. Similarly, according to Theorem 3, we can see that there is no hidden auto-conflict of any BF on Ω_2.

For a general BF on Ω_n, there are proper focal elements up to cardinality $s = n - 1$. Thus $s + 1 = n$ and we can consider Theorem 3 to be a special case of Theorem 4.

We have an upper bound for a degree of hidden conflict. And we can ask the following questions: May be this upper bound reached? How the BFs with maximal degree of auto-conflict looks like?

Example 3 (A general example) *Similarly to hidden conflicts in general, structure of focal elements is important for existence and particular degree of hidden auto-conflicts (not values of belief masses - values are important for resulting value of hidden auto-conflicts). Thus we can present a BF $Bel^{(n)}$ on Ω_n, having all subsets of Ω_n of cardinality $n-1$ as its focal elements. For simplicity we may assume $m^{(n)}(X) = 1/n$ for all $|X| = n - 1$. We obtain $a_{n-1}(Bel^{(n)}) = 0$ and $a_n(Bel^{(n)}) > 0$. There is a hidden conflict of $n - 2$ degree.*

Example 4 *[Specific examples] (i) Simple example of maximal degree of hidden auto-conflict on Ω_3 (thus maximal degree is 1 there) was already presented in Example 1. There is $a(m^{iii}) = 0$ and $a_3(m^{iii}) = 0.18$.*
(ii) Having $Bel^{(3)}$ from Example 3 with $m^{(3)}(X) = 1/3$, we obtain $a(m^{(3)}) = 0$ and $a_3(m^{(3)}) = 2/9 = 0.22\overline{2}$. ($(\bigodot_1^3 m^{(3)})(\{\omega_i\}) = 2/9$ and $(\bigodot_1^3 m^{(3)})(X) = 1/27$ for $|X| = 2$).

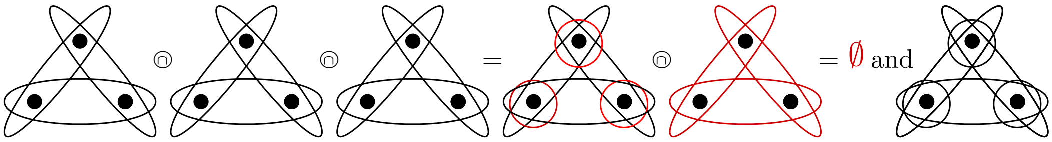


Figure 6: Arising of a hidden auto-conflict of BF Bel^{iii} on Ω_3; focal elements of m^{iii}, $m^{iii} \circledcirc m^{iii}$ and $m^{iii} \circledcirc m^{iii} \circledcirc m^{iii}$.

(iii) Let us suppose $Bel^{(16)}$ *on* Ω_{16} *now:* $m^{(16)}(X) = 1/16 = 0.0625$ *for all* $|X| = 15$ *and* $m^{(16)}(X) = 0$ *otherwise. We obtain* $a_{15}(Bel^{(16)}) = 0$ *and* $a_{16}(Bel^{(16)}) = 1.134227 \cdot 10^{-6}$. *Thus there is very small hidden auto-conflict of* 14^{th} *degree. Value of the auto-conflict rapidly grows up with order:* $a_{18}(Bel^{(16)}) = 4.428801 \cdot 10^{-5}$, $a_{24}(Bel^{(16)}) = 5.460487 \cdot 10^{-3}$, $a_{36}(Bel^{(16)}) = 0.1480689$.
(iv) Supposing m^{xvi} *with bbms* $m^{xvi}(X)$*:* 0.005, 0.010, 0.015, 0.020, 0.030, 0.040, 0.050, 0.060 0.065, 0.075, 0.085, 0.095, 0.105, 0.110, 0.115, 0.120 *for* $|X| = 15$ *and* $m^{xvi}(X) = 0$ *otherwise we obtain:* $a_{15}(Bel^{xvi}) = 0$ *and* $a_{16}(Bel^{xvi}) = 7.089108 \cdot 10^{-9}$. *Thus even less value of auto-conflict of* 14^{th} *degree than in (iii). Value of the auto-conflict rapidly grows up with degree again:* $a_{18}(Bel^{xvi}) = 2.789497 \cdot 10^{-7}$, $a_{24}(Bel^{xvi}) = 4.029782 \cdot 10^{-5}$, $a_{36}(Bel^{xvi}) = 0.001939817$.

Theorem 5 *Belief functions on* Ω_n *with maximal degree* $n-2$ *of hidden auto-conflict are just BF from one of the following categories:*
(i) $Bel^{(n)}$ *with structure as in Example 3*
(ii) $Bel^{(n)}$ *with structure as in Example 3 extended with focal element* Ω_n
There are no other BFs on Ω_n *with hidden auto-conflict of degree* $n-2$.

Idea of proof. Both the classes are obvious. Decreasing number of focal elements removes the hidden auto-conflict; decreasing cardinality of any of the focal elements decreases degree of hidden conflict/auto-conflict.

Remark We can compare very similar equations (1) and (2) in Theorems 1 and 3 differing just in one order of auto-conflict. Is this difference correct? YES, it is. From Theorem 5 we can see that for obtaining maximal degree of a hidden auto-conflict, we need combination of n pieces of BF $Bel^{(n)}$, thus we need multiples of n focal elements of cardinality $(n-1)$. The same we need in the situation from Theorem 1: for maximal degree of hidden conflict all focal elements X of both the BFs have to satisfy $|X| \geq n-1$; but in this case $(m^i \oplus m^{ii})$ is an argument of $\circledcirc_{j=1}^{n-1}$, thus, (at least) one of the focal elements s.t. $X = |n-1|$ is a focal element of let us say Bel^i and therefore it is enough to combine only from $n-1$ pieces of BF Bel^{ii} to obtain the intersection of all n focal elements s.t. $X = |n-1|$. Thus there is really less order of auto-conflict sufficient in Theorem 1.

4.3 Characterisation and Overview of Properties of Auto-Conflict and of Hidden Auto-Conflict

Simply, any consistent belief function has no auto-conflict of any order and therefore no hidden auto-conflict as well. (Any intersection of any number of focal elements of a consistent BF is non-empty, thus $(\circledcirc_1^k m)(\emptyset) = 0$ for any $k > 0$).

For non-consistent belief function Bel on Ω_n there are possible auto-conflicts of some degrees. From the examples shown earlier in this section, we have seen that auto-conflict $a(Bel)$ may be equal to zero, thus that there is a hidden auto-conflict. From Theorem 3 we know that degree of hidden auto-conflict is at most $n-2$, i.e., that any non-consistent BF has (positive) auto-conflict of order n.

Auto-conflict is utilized by Martin et al. as an alternative measure of internal conflict for belief functions. But, values of auto-conflicts of higher orders have no reasonable interpretation and a way how to compare them with values of auto-conflict (of 2nd order). Hidden auto-conflict may be considered as an extension of this measure of internal conflict. Unfortunately, analogously to values of higher order auto-conflicts, we have no procedure to compare the values of hidden auto-conflicts of different degrees.

Theorem 6 *For auto-conflicts of a belief function Bel on Ω_n, the following holds:*
(i) Auto-conflict of a consistent BF of any order is zero (there is no auto-conflict in fact). Also $lim_{s\to\infty} a_s(Bel) = 0$ for any consistent BF Bel.
(ii) Auto-conflict of order 1 is e equal to $m(\emptyset)$, hence it is zero for any normalised BF Bel.
(iii) Auto-conflicts of orders from 2 to $k+1$ are zero for any non-consistent BF, where $k \le n-2$ is a degree of hidden auto-conflict; specially,
for $k = 0$ $a_2(Bel) > 0$ and also $a_s(Bel) > 0$ for all $s > 2$,
for $k = 1$ $a_2(Bel) = 0$ and $a_s(Bel) > 0$ for all $s > 2$,
for $k = 2$ $a_2(Bel) = a_3(Bel) = 0$ and $a_s(Bel) > 0$ for all $s > 3$, etc...
(iv) Positive auto-conflict is increasing: if $a_s(Bel) > 0$ then $a_s(Bel) < a_{s+1}(Bel)$, thus $a_1(Bel) = a_2(Bel) = ... = a_{k+1}(Bel) = 0 < a_{k+2}(Bel) < a_{k+3}(Bel) < ...$, for $0 \le k \le n-2$. And $lim_{s\to\infty} a_s(Bel) = 1$ for any non-consistent BF.

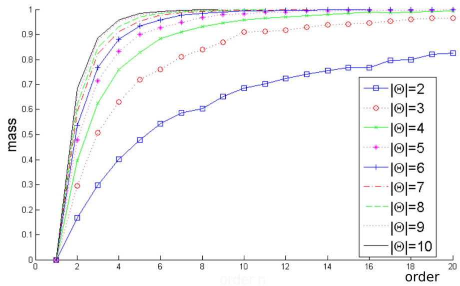
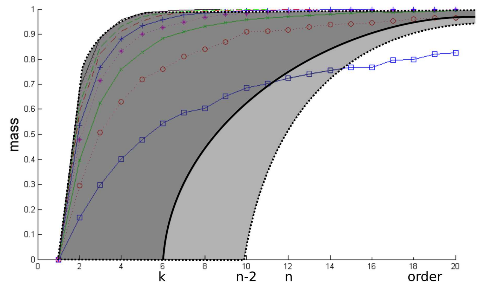


Figure 7: Average auto-conflicts of qBBFs from [15], $|\Theta| = |\Omega| = 2 - 10$. ($|X| = 1$ for all proper focal elements).



.Figure 8: Auto-conflicts of general non-consistent .BFs and of non-consistent BFs with proper focal elements X up to $|X| = k$ (darker) on $|\Omega_{12}| = 12$

Hidden auto-conflicts of qBBFs from [15] are just around the curves from $[1, 0]$ to $[\infty, 1]$ see Figure 7; for proper focal elements holds $|X| = 1$ there. Whereas hidden auto-conflicts of general non-consistent BFs are in the entire grey area on Figure 8, BF with proper focal elements X up to $|X| = k$ in darker grey area above the curve from $[k, 0]$ to $[\infty, 1]$, and zero auto-conflicts of consistent BFs are on straight line $[0, 1]$ to $[\infty, 0]$.

Yes, auto-conflict is an intrinsic property of belief functions as it is stated in [15] but, $a(Bel) \ge 0$ in general. Hence it may be equal to zero while there may be no or some hidden auto-conflict.

We have seen, that $a(Bel) > 0$ is an intrinsic property of a class of quasi-Bayesian BFs with just $n+1$ focal elements, which was studied in [15]. More generally, it is also an intrinsic property of a class of non-consistent BFs with two or more disjunctive focal element e.g. $X \cap Y = \emptyset$; there always holds that $a(Bel) > 0$.

5 Computational Issues

Based on Definition 4 and Theorem 3, the complexity of computation of the degree of hidden auto-conflict of BF Bel is — on a general Ω_n — $O(n)$ of $\textcircled{\cap}$ operations. In the case of checking existence of a hidden auto-conflict of a BF we obtain the complexity $O(log_2(n))$ of $\textcircled{\cap}$ operations utilizing a simplification of computation based on $\textcircled{\cap}_{j=1}^{2k} m = \textcircled{\cap}_{j=1}^{k} m \; \textcircled{\cap} \; \textcircled{\cap}_{j=1}^{k} m$. Note that the complexity of $\textcircled{\cap}$ operation depends on the number and structure of focal elements.

During our analysis of hidden conflicts a series of example computations was performed on frames of discernment of cardinality from 5 up to 16. A number of focal elements rapidly grows up to $|\mathcal{P}(\Omega)| = 2^{|\Omega|} - 1$ when conjunctive combination $\textcircled{\cap}$ is repeated; see e.g. 65534 focal elements of the presented BFs $Bel^{(16)}$ and Bel^{xvi} on Ω_{16} in Example 4. Because the degree of the hidden auto-conflict and existence of the hidden auto-conflict depends on the number and the structure of focal elements not on their bbms, we have usually used same bbms for all focal elements of a BF in our computations on frames of cardinality greater than 10.

All our experiments were performed in R Language and Environment [20] using R Studio [21]. We are currently developing an R package for dealing with belief functions on various frames of discernment. It is based on a relational database approach — nicely implemented in R in package called data.table [22].

6 Several Important Remarks

We have to underline that neither hidden conflict of belief functions nor hidden auto-conflict of a belief function are a new measure of conflict. These notions are intended for understanding conflictness / non-conflictness, they enable to point out the conflict also in situations where conflicts had not been expected, in situations where $m_{\bigcirc\!\!\!\!\cap}(\emptyset) = 0$; hence to point out and to help to understand the conflicts which are hidden by $m_{\bigcirc\!\!\!\!\cap}(\emptyset) = 0$.

The values either of hidden conflict or hidden auto-conflict have not yes any enough reasonable interpretation. We are only interesting whether they are zero (thus no hidden conflict is there) or whether they are positive (thus hidden conflicts appear there).

Both degrees of hidden conflict and degrees of hidden auto-conflict do not present any size or a strength of the conflict. They present the level / degree how the conflict is hidden. Thus they are rather degrees of hiddenness of the conflicts / auto-conflicts. The higher degree, the higher hiddenness, thus less conflict and less strength of the same value. We continue with notation from [13] here. It seems that conflict / auto-conflict hidden in degree k is better formulation than k-th degree of hidden (auto-)conflict or (auto-)conflict of degree k.

Repeating application of conjunctive combination $\oplus$ of a BF with itself is used here to simulate situation where different independent believers have numerically the same bbm. Thus this has nothing to do with idempotent belief combination (where no conflict between two BFs is possible).

There is brand new idea of hidden conflicts in [13] and in this contribution. The assumption of non-conflictness when $m_{\bigcirc\!\!\!\!\cap}(\emptyset) = 0$ was relaxed, due to observation of conflict even in the cases where $m_{\bigcirc\!\!\!\!\cap}(\emptyset) = 0$. Both these studies want to point out the existence of hidden (auto-)conflicts in situations where no conflict was expected till now. Thus the definitions of hidden conflict and hidden auto-conflict are not anything against the previous Daniel's research and results on conflict of belief functions e.g. [11, 2, 4]. Of course, same parts of the previous approaches should be updated to be fully consistent with the new presented results on hidden (auto-)conflicts.

7 Conclusion

Following recently observed and analysed hidden conflicts of belief functions [13], a relationship of hidden conflicts of BFs and auto-conflict of a BF has been pointed out and analysed in this study. Further hidden auto-conflict of a belief function has been defined and analysed here. New results has been compared with Martin's et al. results on auto-conflict.

These qualitatively new phenomena of conflicts of BFs moves us to better understanding of nature of conflicts of belief functions in general and bring a challenge to elaborate and update existing approaches to conflicts of BFs.

Acknowledgement

This work was supported by the Czech Science Foundation (project 16-12010S).

References

[1] Thomas Burger. Geometric views on conflicting mass functions: From distances to angles. *International Journal of Approximate Reasoning*, 70:36–50, 2016.

[2] Milan Daniel. Properties of plausibility conflict of belief functions. In *International Conference on Artificial Intelligence and Soft Computing*, pages 235–246. Springer, 2013.

[3] Milan Daniel and Jianbing Ma. Conflicts of belief functions: continuity and frame resizement. In *International Conference on Scalable Uncertainty Management*, pages 106–119. Springer, 2014.

[4] Milan Daniel. Conflict between belief functions: a new measure based on their non-conflicting parts. In *International Conference on Belief Functions*, pages 321–330. Springer, 2014.

[5] Sébastien Destercke and Thomas Burger. Toward an axiomatic definition of conflict between belief functions. *IEEE transactions on cybernetics*, 43(2):585–596, 2013.

[6] Eric Lefèvre and Zied Elouedi. How to preserve the conflict as an alarm in the combination of belief functions? *Decision Support Systems*, 56:326–333, 2013.

[7] Weiru Liu. Analyzing the degree of conflict among belief functions. *Artificial Intelligence*, 170(11):909–924, 2006.

[8] Arnaud Martin. About conflict in the theory of belief functions. In *Belief Functions: Theory and Applications*, pages 161–168. Springer, 2012.

[9] Johan Schubert. The internal conflict of a belief function. In *Belief Functions: Theory and Applications*, pages 169–177. Springer, 2012.

[10] Philippe Smets. Analyzing the combination of conflicting belief functions. *Information fusion*, 8(4):387–412, 2007.

[11] Milan Daniel. Conflicts within and between belief functions. In *International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pages 696–705. Springer, 2010.

[12] Milan Daniel. Non-conflicting and conflicting parts of belief functions. In *ISIPTA*, volume 11, pages 149–158. Citeseer, 2011.

[13] Milan Daniel and Václav Kratochvíl. Hidden conflict of belief functions. In *In preparation*, 2017.

[14] Christophe Osswald and Arnaud Martin. Understanding the large family of dempster-shafer theory's fusion operators — a decision-based measure. In *International Conference on Information Fusion*, 2006.

[15] Arnaud Martin, Anne-Laure Jousselme, and Christophe Osswald. Conflict measure for the discounting operation on belief functions. In *International Conference on Information Fusion*, pages 1003–1010, 2008.

[16] Glenn Shafer. *A mathematical theory of evidence*, volume 1. Princeton University Press Princeton, 1976.

[17] Philippe Smets. Decision making in the tbm: the necessity of the pignistic transformation. *International Journal of Approximate Reasoning*, 38(2):133–147, 2005.

[18] Barry R Cobb and Prakash P Shenoy. On the plausibility transformation method for translating belief function models to probability models. *International journal of approximate reasoning*, 41(3):314–330, 2006.

[19] Milan Daniel. Probabilistic transformations of belief functions. In *European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty*, pages 539–551. Springer, 2005.

[20] R Core Team. *R: A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Vienna, Austria, 2016.

[21] RStudio Team. *RStudio: Integrated Development Environment for R.* RStudio, Inc., Boston, MA, 2015.

[22] Matt Dowle and Arun Srinivasan. *data.table: Extension of 'data.frame'*, 2016. R package version 1.10.0.