
Modal extensions of Łukasiewicz logic for modelling coalitional power

TOMÁŠ KROUPA, *Institute of Information Theory and Automation of the Czech Academy of Sciences, Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic.*
E-mail: kroupa@utia.cas.cz

BRUNO TEHEUX, *University of Luxembourg, Faculté des Sciences, de la Technologie et de la Communication, 6, Rue Richard Coudenhove-Kalergi L-1359, Luxembourg.*
E-mail: bruno.teheux@uni.lu

Abstract

Modal logics for reasoning about the power of coalitions capture the notion of effectivity functions associated with game forms. The main goal of coalition logics is to provide formal tools for modelling the dynamics of a game frame whose states may correspond to different game forms. The two classes of effectivity functions studied are the families of playable and truly playable effectivity functions, respectively. In this article, we generalize the concept of effectivity function beyond the yes/no truth scale. This enables us to describe the situations in which the coalitions assess their effectivity in degrees, based on functions over the outcomes taking values in a finite Łukasiewicz chain. Then we introduce two modal extensions of Łukasiewicz finite-valued logic together with many-valued neighbourhood semantics in order to encode the properties of many-valued effectivity functions associated with game forms. As our main results we prove completeness theorems for the two newly introduced modal logics.

Keywords: Coalition Logic, Łukasiewicz modal logic, neighbourhood semantics, effectivity function, game form.

1 Introduction

Modelling collective actions of agents and capturing their effectivity is among the important research topics on the frontiers of game theory, computer science and mathematical logic. The main efforts are concentrated on answering the following question: what is the set of outcome states that can effectively be implemented by a coalition of agents? A game-theoretic framework for studying collective actions and their enforceability is based on the notion of game forms. Loosely speaking, a game form is a pure description of a game and its rules, without regard to the agents' preferences. Game frames enable us to capture a more general action model in which a game form is associated with every state of the frame and the outcome states of the game forms are the states of the frame. From the game-theoretic viewpoint, the game frames are extensive form games with simultaneous moves of the players; see [18, 23].

The concept of α -effectivity ([1, 21]) is one of the key approaches to characterize the coalitional effectivity within game form models. A coalition C is α -effective for a set of outcome states X if the players in C can choose a joint strategy that enforces an outcome in X no matter what strategies are adopted by the other players. The previous definition gives rise to the concept of a (truly) playable effectivity function. In his seminal paper [20], Pauly introduces Coalition Logic \mathbf{CL}_N to reason about α -effectivity in game forms with player set N . The axiomatization of \mathbf{CL}_N is an attempt to characterize the class of α -effectivity functions in a multi-modal language. Pauly also defined

a neighbourhood semantics with respect to which \mathbf{CL}_N is complete. The logic \mathbf{CL}_N was subsequently analysed and extended by many authors; see [2, 6]. In particular, Goranko *et al.* [10] found a gap in Pauly's characterization of playable effectivity functions (see [20, Theorem 3.2]), which led to the introduction of truly playable effectivity functions.¹ The reader is invited to consult Section 2 together with Appendix A, where we recall all the necessary notions regarding Boolean effectivity functions and the distinction between playable and truly playable functions, respectively.

In this article, we extend the results of Pauly [20] and Goranko *et al.* [10] to the situations in which the effectivity of coalitions is evaluated on a finer finite scale than $\{0, 1\}$. This generalization is based on several assumptions. First, we assume that coalitions evaluate their effectivity with respect to a certain family of $[0, 1]$ -valued functions over the state space and not only with respect to the sets of states. Second, the effectivity of coalitions comes in degrees rather than in Boolean values. Third, the underlying logical framework is that of finitely-valued Łukasiewicz logic, which offers a great expressive power, while preserving many desirable properties of logics at the same time.

Our main goal is to investigate the properties of many-valued modal logics \mathbf{P}_n and \mathbf{TP}_n devised for capturing the refined notion of effectivity. To this end, we proceed as follows. In Section 3 we generalize the notion of α -effectivity. Since we use finitely-valued Łukasiewicz logic, our scale is always the set

$$\mathbb{L}_n = \left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right\}, \text{ where } n \text{ is a positive integer.}^2 \quad (1.1)$$

The choice of Łukasiewicz logic is not only a design choice that is suitable for applying the usual operations of MV-algebras to the \mathbb{L}_n -valued functions over the state space, but also a matter of practical necessity since the homogeneity property (Definition 4.2) is among the key features of (playable) many-valued effectivity functions. On the logical side it corresponds to the axioms (1)–(2) of the logic \mathbf{P}_n in Definition 5.9. It is worth mentioning that homogeneity appears as an axiom in other modal languages as well; see [7, 19, 22]. The finiteness of our scale of truth degrees is essential for the completeness results, which usually fail badly for modal extensions of infinitely-valued Łukasiewicz logic [13].

We introduce the concept of \mathbb{L}_n -valued effectivity function whose purpose is to capture the effectivity of coalitions on the scale \mathbb{L}_n . In order to understand the relation between effectivity functions and game forms, we have to consider the class of playable and truly playable \mathbb{L}_n -valued effectivity functions, respectively, which are studied in Section 4. In particular, we establish a characterization of truly playable \mathbb{L}_n -valued effectivity functions (Theorem 4.9). In Section 5, we develop tools to capture the properties of \mathbb{L}_n -valued effectivity functions in a many-valued modal language. These developments rely not only on recent advances in modal extensions of Łukasiewicz logic (see [7, 13]), but they also require the introduction of neighbourhood semantics, which has never been considered in the modal many-valued setting before, to the best of our knowledge. The newly introduced logics \mathbf{P}_n and \mathbf{TP}_n axiomatize in the many-valued modal language the properties of playable and truly playable \mathbb{L}_n -valued effectivity functions, respectively. Our main results are Theorem 5.18 and Theorem 5.29, which show that the logics \mathbf{P}_n and \mathbf{TP}_n are complete with respect to the corresponding classes of \mathbb{L}_n -valued coalitional frames. The key ingredient in the proof of completeness of \mathbf{TP}_n is the many-valued generalization of the filtration technique for neighbourhood models [8, Chapter 7.5]. These mathematical constructions have their own merit and are among the main contributions of the article, since they provide a ‘bag of tricks’ that could be reused to develop neighbourhood semantics for other modal extensions of finite-valued Łukasiewicz logics.

¹We are grateful to Paolo Turrini who brought the paper [10] to our attention.

²Our notation for \mathbb{L}_n as a set of cardinality $n + 1$ follows [17].

2 Game forms and effectivity functions

We recall basic facts about game forms and effectivity functions; see [1]. In what follows, S denotes a nonempty set, $N = \{1, \dots, k\}$ is a finite set and, for any $i \in N$, Σ_i denotes a non-empty set. For any set Ω we denote by $\mathcal{P}\Omega$ the powerset of Ω .

DEFINITION 2.1 ([1, Definition 3.1])

A (*strategic*) *game form* is a tuple $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$, where N is a set of *players*, Σ_i is a set of *strategies* for each $i \in N$, S is a set of *outcome states*, and $o: \prod_{i \in N} \Sigma_i \rightarrow S$ is an *outcome function*.

The game forms are not to be confused with strategic games. While a preference relation over S must be defined for each player $i \in N$ in a strategic game [18], no such requirement exists for a game form. Below we provide some basic examples of game forms.

EXAMPLE 2.2

(i) Let $N = \{1, 2\}$ and Σ_1, Σ_2 be some strategy sets. Assume that the players choose their strategies simultaneously. Then we may set $S = \Sigma_1 \times \Sigma_2$ and define o as the identity function, which turns $(\{1, 2\}, \{\Sigma_1, \Sigma_2\}, \Sigma_1 \times \Sigma_2, o)$ into a game form. This game form just records an outcome as the pair of chosen strategies. (ii) Suppose, on the other hand, that Player 2 makes his choice only after observing the strategic choice of Player 1. This sequential procedure is modeled by a game form such that Σ_2 is the set of all functions $r: \Sigma_1 \rightarrow \Sigma'_2$, where Σ'_2 can be viewed as the set of all possible moves that can be played by Player 2. Hence, Σ_2 models the replies of Player 2 to the selection of a strategy by Player 1. The outcome function is given by $o(\sigma_1, r) = (\sigma_1, r(\sigma_1))$, where $(\sigma_1, r) \in \Sigma_1 \times \Sigma_2$ and the set of outcome states is $S = \Sigma_1 \times \Sigma'_2$.

An important example arises when the outcome function coincides with some social choice correspondence in the sense of [1, Chapter 1].

EXAMPLE 2.3

Let S' be any non-empty set of outcome states and $\Pi(S')$ be a set of admissible preference relations on S' . In most applications, $\Pi(S')$ will be either the set of *total preorders* (reflexive, transitive and complete binary relations) or the set of *linear orders*. A map $\pi: \Pi(S')^N \rightarrow \mathcal{P}S'$ is called a *social choice correspondence*. Social choice correspondences implement collective decision procedures mapping a preference profile $\sigma \in \Pi(S')^N$ of the agents into a set of outcome states that are considered equivalent with respect to σ . If an agent (or a group of agents) wants to enforce a specific outcome, his/her only possible strategy is to declare a preference relation that is likely to bring the collective decision into an outcome $\pi(\sigma)$ that contains the desired state. We can describe this scheme as a game form $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ in which $\Sigma_1 = \dots = \Sigma_k = \Pi(S')$, $S = \mathcal{P}S'$ and $o = \pi$.

The subsets $C \subseteq N$ are called *coalitions*. For every coalition C we denote by \bar{C} its set-complement in N . If $\sigma_C \in \prod_{i \in C} \Sigma_i$ and $\sigma_{\bar{C}} \in \prod_{i \in \bar{C}} \Sigma_i$, then $\sigma_C \sigma_{\bar{C}}$ is the strategy tuple in $\prod_{i \in N} \Sigma_i$ defined by $(\sigma_C \sigma_{\bar{C}})_i = (\sigma_C)_i$ if $i \in C$ and $(\sigma_C \sigma_{\bar{C}})_i = (\sigma_{\bar{C}})_i$ if $i \in \bar{C}$.

DEFINITION 2.4 ([1, Definition 4.1])

Let G be a game form. The *effectivity function* of G is the mapping $H_G: \mathcal{P}N \rightarrow \mathcal{P}\mathcal{P}S$ defined as follows: $X \in H_G(C)$ if there exists σ_C such that for every $\sigma_{\bar{C}}$, we have $o(\sigma_C \sigma_{\bar{C}}) \in X$.

In other words, the condition $X \in H_G(C)$ is true whenever the coalition C has the power to force the outcome to lie in X . We refer to [1, 20, 21] for a discussion and examples of effectivity functions in game theory and social choice.

The notion of effectivity function of a game form can be generalized as follows.

DEFINITION 2.5

A mapping $E: \mathcal{PN} \rightarrow \mathcal{PPS}$ is called an *effectivity function*.

The problem of characterizing effectivity functions that are effectivity functions of game forms is solved by introducing the notion of true playability. It was used by Goranko et al. [10] in order to fix the error in Pauly's characterization result [20, Theorem 3.2], which was based on the weaker notion of playability. We recall the two playability concepts in the next definition.

DEFINITION 2.6

Let $E: \mathcal{PN} \rightarrow \mathcal{PPS}$ be an effectivity function. We say that E

- (1) is *superadditive* if $C_1 \cap C_2 = \emptyset$ and $X \in E(C_1)$, $Y \in E(C_2)$ imply that $X \cap Y \in E(C_1 \cup C_2)$, for every $C_1, C_2 \in \mathcal{PN}$;
- (2) is *outcome monotonic* if $X \in E(C)$ and $X \subseteq Y$ imply $Y \in E(C)$, for every $C \in \mathcal{PN}$;
- (3) is *N -maximal* if $\bar{X} \notin E(\emptyset)$ implies $X \in E(N)$;
- (4) has the *liveness property* if $\emptyset \notin E(C)$, for every $C \in \mathcal{PN}$;
- (5) has the *safety property* if $S \in E(C)$, for every $C \in \mathcal{PN}$.

We call E *playable* whenever (1)–(5) are satisfied. We say that E is *truly playable* if it is playable and $E(\emptyset)$ is a principal filter in \mathcal{PS} .³

The following result, which was originally proved in [10, Theorem 1], amends the gap in the proof of Pauly's correspondence result in case of an infinite outcome space S . We provide an alternative proof of Theorem 2.7 in Appendix A, which shows that this result can be considered as a corollary of Peleg's Theorem [21, Theorem 3.5*].

THEOREM 2.7 ([10, Theorem 1])

Let $E: \mathcal{PN} \rightarrow \mathcal{PPS}$ be an effectivity function. There exists a game form $G = (N, \{\Sigma_i \mid i \in N\}, S, \rho)$ satisfying $E = H_G$ if and only if E is truly playable.

3 Many-valued effectivity functions

We are going to generalize the concept of effectivity function for an arbitrary game form $G = (N, \{\Sigma_i \mid i \in N\}, S, \rho)$ and Łukasiewicz chain \mathbb{L}_n as in (2). Our goal is to capture the degree or extent to which a coalition C can 'enforce' a function $f: S \rightarrow \mathbb{L}_n$. Before stating a formal definition, we will motivate this idea by two situations where such many-valued assessments f may arise.

- (1) A strategic game form G is made into a strategic game when a utility function (or a preference relation) over the outcome set S is introduced for every player $i \in N$. Thus an arbitrary function $f \in \mathbb{L}_n^S$ can be viewed as a utility function. However, this utility function is not necessarily attached to any player's preference relation.
- (2) When the state space S is too large or complex to deal with, the distinction between the subsets of S (equivalently, the functions $S \rightarrow \{0, 1\}$) and the functions $f \in \mathbb{L}_n^S$ may become immaterial. It is not against the spirit of neighbourhood semantics to draw a direct parallel with an analogous situation in topology: by Urysohn's lemma any two closed disjoint subsets in a normal topological space can be arbitrarily closely approximated by a $[0, 1]$ -valued continuous

³Our formulation of true playability is different from the original definition yet equivalent to it by [10, Proposition 5].

function. Thus, for a sufficiently large natural number n , we may think of functions $f \in \mathbb{L}_n^S$ as members of some limit sequence, which eventually encodes a subset of S . This interpretation of an originally finite object is not uncommon in game theory. Indeed, it was one of the motivations for the development of Aumann's theory of ideal coalitions in coalition games with continuum of players; see [5].

DEFINITION 3.1

Let $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ be a game form. The \mathbb{L}_n -valued effectivity function of G is the map $E_G: \mathcal{P}N \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$ defined by

$$E_G(C, f) = \max_{\sigma_C} \min_{\sigma_{\bar{C}}} f(o(\sigma_C \sigma_{\bar{C}})), \quad C \in \mathcal{P}N, f \in \mathbb{L}_n^S, \quad (3.2)$$

where σ_C and $\sigma_{\bar{C}}$ range through the set of all joint strategies of coalitions C and \bar{C} , respectively.

The meaning of definition (3.2) is the following: coalition C is effective for $f \in \mathbb{L}_n^S$ to the degree $E_G(C, f) \in \mathbb{L}_n$, disregarding the strategic options of players in the opposite coalition \bar{C} . Note that the usual Boolean α -effectivity function associated with G coincides with the \mathbb{L}_1 -valued effectivity function of G .

REMARK 3.2

In this article, we do not advocate any particular interpretation of the many-valued effectivity model (3.2) as suggested by (1)–(2) above, nor do we insist on a special meaning of truth degrees. From the purely mathematical point of view, any such interpretation is irrelevant since it yields the same underlying game form under the assumption of several playability conditions introduced in Section 4. This point will be explained in detail in Remark 4.10.

4 Playability of \mathbb{L}_n -valued effectivity functions

Analogously to the classical literature [16, 21] on effectivity functions, we can study the notion of effectivity in a setting independent of game forms. Let S be a set of outcomes and N be a finite player set. We always assume that $|S| \geq 2$ and $|N| \geq 2$.

DEFINITION 4.1

An \mathbb{L}_n -valued effectivity function is a mapping $E: \mathcal{P}N \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$.

Note that the \mathbb{L}_1 -valued effectivity functions are exactly the effectivity functions $\mathcal{P}N \times \{0, 1\}^S \rightarrow \{0, 1\}$ arising in the Boolean framework [1, 20]. Therefore we call any \mathbb{L}_1 -valued effectivity function a *Boolean effectivity function*.

Our goal is to characterize the class of \mathbb{L}_n -valued effectivity functions that are associated with game forms. This characterization is related to the properties of effectivity functions listed in Definition 4.2. We use the standard connectives of Łukasiewicz logic and respective operations of MV-algebras; see Appendix B. In particular, we always apply the operations of the MV-algebra \mathbb{L}_n to functions $f \in \mathbb{L}_n^S$ pointwise. For any \mathbb{L}_n -valued effectivity function E , we define $E(\emptyset, -)^{-1}(1) = \{f \in \mathbb{L}_n^S \mid E(\emptyset, f) = 1\}$.

DEFINITION 4.2

Let E be an \mathbb{L}_n -valued effectivity function. We say that E

- (1) is *outcome monotonic* whenever $f \geq g$ implies $E(C, f) \geq E(C, g)$, for every $C \in \mathcal{P}N$ and every $f, g \in \mathbb{L}_n^S$;

- (2) is *N-maximal* if $\neg E(\emptyset, \neg f) \leq E(N, f)$ for every $f \in \mathbb{L}_n^S$;
- (3) is *regular* if it satisfies $E(C, f) \leq \neg E(\overline{C}, \neg f)$ for every $C \in \mathcal{PN}$ and $f \in \mathbb{L}_n^S$;
- (4) is *superadditive* if $E(C_1, f) \wedge E(C_2, g) \leq E(C_1 \cup C_2, f \wedge g)$ for every $C_1, C_2 \in \mathcal{PN}$ such that $C_1 \cap C_2 = \emptyset$ and every $f, g \in \mathbb{L}_n^S$;
- (5) is *coalition monotonic* if $E(C, f) \leq E(C', f)$ for every $C \subseteq C' \in \mathcal{PN}$ and every $f \in \mathbb{L}_n^S$;
- (6) is *homogeneous* if $E(C, f \oplus f) = E(C, f) \oplus E(C, f)$ and $E(C, f \odot f) = E(C, f) \odot E(C, f)$ for every $C \in \mathcal{PN}$ and every $f \in \mathbb{L}_n^S$;
- (7) has the *liveness property* if $E(C, 1) = 1$ for every $C \in \mathcal{PN}$;
- (8) has the *safety property* if $E(C, 0) = 0$ for every $C \in \mathcal{PN}$;
- (9) is *principal* if there is a $g \in \mathbb{L}_n^S$ such that $\{f \in \mathbb{L}_n^S \mid E(\emptyset, f) = 1\} = \{f \in \mathbb{L}_n^S \mid f \geq \bigodot_{i=1}^n g\}$.

We say that E is *playable* whenever it is outcome monotonic, N -maximal, superadditive, homogeneous, and has liveness and safety properties. We say that E is *truly playable* if it is playable and principal.

If $n=1$, then the definitions of (truly) playable Boolean effectivity function coincide with the corresponding definitions used in the Boolean setting [10, 20]; cf. Definition 2.6.

Note that if E is an outcome monotonic and homogeneous \mathbb{L}_n -valued effectivity function, then $E(C, -)^{-1}(1)$ is an MV-filter of the MV-algebra \mathbb{L}_n^S [9]. Moreover, the \mathbb{L}_n -valued effectivity function E is principal whenever the MV-filter $E(\emptyset, -)^{-1}(1)$ is principal; see Appendix B.

It may be difficult to get some intuition about the definition of a homogeneous \mathbb{L}_n -valued effectivity function in the game form framework. We refer to Remark 5.19 for an equivalent formulation of this definition. The following result illustrates that homogeneity arises naturally in the context of game forms. For every set $Y \subseteq S$, we denote by χ_Y the characteristic function of Y .

PROPOSITION 4.3

If $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ is a game form, then E_G is a truly playable \mathbb{L}_n -valued effectivity function.

PROOF. It follows directly from Definition 3.1 that E_G is outcome monotonic, N -maximal and has liveness and safety. Homogeneity of E_G follows from the fact that the maps $\tau_{\oplus} : x \mapsto x \oplus x$ and $\tau_{\odot} : x \mapsto x \odot x$ are lattice homomorphisms of \mathbb{L}_n . Moreover, $E_G(\emptyset, -)^{-1}(1) = \{g \mid g \geq \chi_{\text{ran}(o)}\}$, which shows that E_G is principal since $\bigodot_{i=1}^n \chi_{\text{ran}(o)} = \chi_{\text{ran}(o)}$.

It remains to prove that E_G is superadditive. Let $C_1, C_2 \in \mathcal{PN}$ be such that $C_1 \cap C_2 = \emptyset$ and $f_1, f_2 \in \mathbb{L}_n^S$. Denote by $\sigma_{C_1}^*$ and $\sigma_{C_2}^*$ two strategy tuples satisfying $E(C_1, f_1) = \min_{\sigma_{C_1}^*} f_1(o(\sigma_{C_1}^* \sigma_{C_1}^-))$ and $E(C_2, f_2) = \min_{\sigma_{C_2}^*} f_2(o(\sigma_{C_2}^* \sigma_{C_2}^-))$. It follows that $E(C_1, f_1) \wedge E(C_2, f_2) = \min_{\sigma_{C_1}^*} \min_{\sigma_{C_2}^*} f_1(o(\sigma_{C_1}^* \sigma_{C_1}^-)) \wedge f_2(o(\sigma_{C_2}^* \sigma_{C_2}^-))$, which is not greater than $\min_{\sigma_{C_1}^- \cap \sigma_{C_2}^-} (f_1 \wedge f_2)(o(\sigma_{C_1}^* \sigma_{C_2}^* \sigma_{C_1}^- \cap \sigma_{C_2}^-))$. The conclusion follows from the fact that the latter is bounded above by $E_G(C_1 \cup C_2, f_1 \wedge f_2)$. ■

As the next result shows, a Boolean effectivity function can be associated with any homogeneous \mathbb{L}_n -valued effectivity function. The Boolean algebra $\mathbb{L}_1^S = \{0, 1\}^S$ is called the *Boolean skeleton* of \mathbb{L}_n^S . In other words, the Boolean skeleton of \mathbb{L}_n^S is the powerset of S if we identify the subsets of S with their characteristic functions on S . An element $f \in \mathbb{L}_n^S$ belongs to the Boolean skeleton of \mathbb{L}_n^S if and only if $f \oplus f = f$, and such an element is said to be *idempotent*. For every \mathbb{L}_n -valued effectivity function $E : \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$, we denote by E^\sharp the restriction of E to $\mathcal{PN} \times \mathbb{L}_1^S$.

LEMMA 4.4

If E is a homogeneous \mathbb{L}_n -valued effectivity function, then E^\sharp is a Boolean effectivity function. If in addition E is playable (respectively, truly playable), then so is E^\sharp .

PROOF. For any idempotent element $f \in \mathbb{L}_n^S$ and for every $C \in \mathcal{PN}$, we obtain

$$E(C, f) \oplus E(C, f) = E(C, f \oplus f) = E(C, f).$$

Therefore E^\sharp is a Boolean effectivity function.

If in addition E is playable (respectively, truly playable), then it satisfies conditions (1), (2), (4), (7) and (8) (respectively, conditions (1), (2), (4), (7), (8) and (9)) of Definition 4.2. It follows that E^\sharp also satisfies the analogous Boolean conditions (see Appendix A) since they do not involve any existential quantifier over the elements of \mathbb{L}_n^S . ■

In order to study the playability property, we need more technical preliminaries. To this end, put

$$\tau_{\oplus}(x) = x \oplus x \quad \text{and} \quad \tau_{\odot}(x) = x \odot x, \quad \text{for every } x \in \mathbb{L}_n.$$

DEFINITION 4.5

Let $i \in \{1, \dots, n\}$. We define the function $\tau_{\frac{i}{n}} : \mathbb{L}_n \rightarrow \mathbb{L}_n$ by

$$\tau_{\frac{i}{n}}(x) = \begin{cases} 0 & x < \frac{i}{n}, \\ 1 & x \geq \frac{i}{n}, \end{cases}$$

and we always assume that $\tau_{\frac{i}{n}}$ is the interpretation on \mathbb{L}_n of an algebraic term which is a composition of finitely many copies of the maps τ_{\oplus} and τ_{\odot} alone.⁴

Any mapping $\tau : \mathbb{L}_n \rightarrow \mathbb{L}_n$ can be composed with a function $f \in \mathbb{L}_n^S$. Thus we define $\tau(f)(s) = \tau(f(s))$ for every $s \in S$ and $f \in \mathbb{L}_n^S$.

LEMMA 4.6

Let $E, E' : \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$ be homogeneous \mathbb{L}_n -valued effectivity functions. Then $E = E'$ if and only if $E^\sharp = E'^\sharp$.

PROOF. Necessity is trivial. To prove sufficiency assume that $E^\sharp = E'^\sharp$. Let $C \in \mathcal{PN}$, $f \in \mathbb{L}_n^S$ and $i \in \{1, \dots, n\}$. Since $\tau_{\frac{i}{n}}(f)$ is idempotent and E and E' are homogeneous, we have $E(C, f) \geq \frac{i}{n}$ if and only if $1 = E(C, \tau_{\frac{i}{n}}(f)) = E'(C, \tau_{\frac{i}{n}}(f))$, which is equivalent to $E'(C, f) \geq \frac{i}{n}$. ■

The following lemma is straightforward. Its statement uses the notion of the Boolean effectivity function H_G associated with a game form G ; see Definition 2.4.

LEMMA 4.7

Let G be a game form. If H_G and E_G are the Boolean and the \mathbb{L}_n -valued effectivity function associated with G , respectively, then $H_G = E_G^\sharp$.

For any $r \in [0, 1]$, we denote by $\lceil r \rceil$ the element $\min\{a \in \mathbb{L}_n \mid a \geq r\}$. The following lemma will turn out to be crucial for understanding the limits of expressive power of the language associated with (truly) playable \mathbb{L}_n -valued effectivity functions; see Proposition 5.8.

⁴The proof of existence of such a term appears in [19].

LEMMA 4.8

If $H: \mathcal{PN} \times \mathbb{L}_1^S \rightarrow \mathbb{L}_1$ is a playable Boolean effectivity function, then the function $E: \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$ defined by

$$E(C, f) = \max \left\{ \frac{i}{n} \in \mathbb{L}_n \mid H(C, \tau_{\frac{i}{n}}(f)) = 1 \right\}, \quad (4.1)$$

for every $C \in \mathcal{PN}$ and $f \in \mathbb{L}_n^S$, is a playable \mathbb{L}_n -valued effectivity function that satisfies $E^\sharp = H$. If in addition H is truly playable, then so is E .

PROOF. Clearly $E^\sharp = H$. It follows from outcome monotonicity of H that E is outcome monotonic. Since H has liveness and safety, so does the function E . To prove that E is superadditive, assume on the contrary that there exist $f, g \in \mathbb{L}_n^S$ and $C, D \in \mathcal{PN}$ such that $C \cap D = \emptyset$ and $E(C \cup D, f \wedge g) < \frac{i}{n} \leq E(C, f) \wedge E(D, g)$ for some $i \in \{1, \dots, n\}$. On the one hand, it follows that $H(C \cup D, \tau_{\frac{i}{n}}(f) \wedge \tau_{\frac{i}{n}}(g)) = H(C \cup D, \tau_{\frac{i}{n}}(f \wedge g)) = 0$. On the other hand, we obtain $H(C, \tau_{\frac{i}{n}}(f)) = H(D, \tau_{\frac{i}{n}}(g)) = 1$ and by superadditivity of H we get $H(C \cup D, \tau_{\frac{i}{n}}(f) \wedge \tau_{\frac{i}{n}}(g)) = 1$, a contradiction.

Now, let $f \in \mathbb{L}_n^S$. Since $\tau_{\frac{i}{n}}(f \oplus f) = \tau_{\lceil \frac{i}{2n} \rceil}(f) \in \mathbb{L}_1^S$, it follows from the definition of E that $E(C, f \oplus f) \geq \frac{i}{n}$ if and only if $H(C, \tau_{\lceil \frac{i}{2n} \rceil}(f)) = 1$. Using again the definition of E , the latter is equivalent to $E(C, f) \geq \lceil \frac{i}{2n} \rceil$, which is the same as $E(C, f) \oplus E(C, f) \geq \frac{i}{n}$. We can proceed in a similar way to prove that $E(C, f \odot f) = E(C, f) \odot E(C, f)$ for every $f \in \mathbb{L}_n^S$.

To prove N -maximality of E , consider $f \in \mathbb{L}_n^S$ and $i \in \{0, \dots, n-1\}$. It follows from the definition of E and N -maximality of H that $E(N, f) \leq \frac{i}{n}$ if and only if $H(\emptyset, \neg \tau_{\frac{i+1}{n}}(f)) = 1$. Since $\neg \tau_{\frac{i+1}{n}}(f) = \tau_{\frac{n-i}{n}}(\neg f)$, the last identity is equivalent to $H(\emptyset, \tau_{\frac{n-i}{n}}(\neg f)) = 1$ and finally to $\neg E(\emptyset, \neg f) \leq \frac{i}{n}$.

Assume that H is truly playable. Definition 2.6 yields existence of $g \in \mathbb{L}_1^S$ such that $H(\emptyset, -)^{-1}(1) = \{h \in \mathbb{L}_1^S \mid h \geq g\}$. Since E is homogeneous, $E(\emptyset, -)^{-1}(1) = \{f \in \mathbb{L}_n^S \mid H(\emptyset, \tau_1(f)) = 1\} = \{f \in \mathbb{L}_n^S \mid \tau_1(f) \geq g\}$. The latter is equal to $\{f \in \mathbb{L}_n^S \mid f \geq g\}$ since $E(\emptyset, -)^{-1}(1)$ is an MV-filter of \mathbb{L}_n^S and g is idempotent. ■

The following result, which is the \mathbb{L}_n -valued generalization of [20, Theorem 3.2] and [10, Theorem 1], completes the characterization of truly playable \mathbb{L}_n -valued effectivity functions.

THEOREM 4.9

An \mathbb{L}_n -valued effectivity function $E: \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$ is truly playable if and only if there is a game form G such that $E = E_G$.

PROOF. By Proposition 4.3, E_G is truly playable. Conversely, assume that E is truly playable. By Lemma 4.4 and Theorem 2.7, there exists a game form G such that $H_G = E^\sharp$, where H_G is the Boolean effectivity function of G . We obtain by Lemma 4.7 that $E^\sharp = H_G = E_G^\sharp$, where E_G is the \mathbb{L}_n -valued effectivity function associated with G . The conclusion $E = E_G$ follows from Lemma 4.6. ■

REMARK 4.10

The previous theorem implies that the notions of playability for Boolean effectivity functions and \mathbb{L}_n -valued functions are equivalent on the game-theoretic level since any of those concepts leads to a uniquely determined game form. In our more general setting, we were able to maintain the correspondence with game forms by imposing homogeneity and the \mathbb{L}_n -version of the playability axioms in the case of the \mathbb{L}_n -valued effectivity functions. Admittedly, we do not arrive at a new concept of game form or extend the validity of Boolean effectivity functions to a larger class of objects. Nevertheless, the importance of our approach presented herein lies in an alternative

representation of the classical setting: allowing for a richer language and more truth degrees leaves the underlying class of game forms invariant. This situation is similar to the development of many-valued probability theory starting from Łukasiewicz logic: while every probability of an MV-algebra induces a unique ‘classical’ probability [14], the more general concept of MV-probability forms a solid basis for studying a number of stochastic phenomena, betting games among them; cf. [17, Chapter 1].

It is useful to introduce a weaker notion of playability. To this end, we need this preparatory result.

LEMMA 4.11

If E is a playable \mathbb{L}_n -valued effectivity function, then E is coalition monotonic and regular.

PROOF. We proceed by contradiction to prove that E is regular. Assume that there are $C \in \mathcal{PN}$, $f \in \mathbb{L}_n^S$ and $i \in \{1, \dots, n\}$ such that $\neg E(\overline{C}, \neg f) < \frac{i}{n} \leq E(C, f)$. Put $j = 1 - \frac{i-1}{n}$. Since E is homogeneous, it follows that $E(C, \tau_{\frac{i}{n}}(f)) = 1$ and $E(\overline{C}, \tau_{\frac{j}{n}}(\neg f)) = 1$. Thus we obtain by superadditivity $E(N, \tau_{\frac{i}{n}}(f) \wedge \tau_{\frac{j}{n}}(\neg f)) = 1$, which contradicts safety since $\tau_{\frac{i}{n}}(f) \wedge \tau_{\frac{j}{n}}(\neg f) = 0$.

We prove that E is coalition monotonic. Let $C \subseteq C' \in \mathcal{PN}$ and $f \in \mathbb{L}_n^S$. By applying superadditivity to $C_1 = C$ and $C_2 = C' \setminus C$ we obtain $E(C, f) \leq E(C', f)$, which is the desired result. ■

DEFINITION 4.12

An \mathbb{L}_n -valued effectivity function $E: \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$ is *semi-playable* if $E(C, f) \leq E(C, g)$ for every $f \leq g \in \mathbb{L}_n^S$ and every coalition $C \neq N$, if E has liveness and safety for coalitions $C \neq N$, and if it satisfies superadditivity for coalitions C_1 and C_2 such that $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 \neq N$.

PROPOSITION 4.13

An \mathbb{L}_n -valued effectivity function $E: \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$ is playable if and only if it is semi-playable, homogeneous, regular and N -maximal.

PROOF. The first implication follows from Lemma 4.11. Conversely, assume that E is semi-playable, homogeneous, regular and N -maximal. First we prove superadditivity. Let $C \in \mathcal{PN}$ with $C \neq N$. We have to verify that $E(C, f) \wedge E(\overline{C}, g) \leq E(N, f \wedge g)$. By way of contradiction, assume that there is $i \in \{1, \dots, n\}$ such that $E(N, f \wedge g) < \frac{i}{n} \leq E(C, f) \wedge E(\overline{C}, g)$. Since E is homogeneous, we obtain $E(C, \tau_{i/n}(f)) = 1 = E(\overline{C}, \tau_{i/n}(g))$, while, by N -maximality, $E(\emptyset, \neg \tau_{i/n}(f \wedge g)) = 1$. It follows from superadditivity that $E(C, \tau_{i/n}(f) \wedge \neg \tau_{i/n}(f \wedge g)) = 1$, which is equivalent to $E(C, \tau_{i/n}(f) \wedge \neg(\tau_{i/n}(f) \wedge \tau_{i/n}(g))) = 1$, since $\tau_{i/n}(f \wedge g) = \tau_{i/n}(f) \wedge \tau_{i/n}(g)$. From the fact that $\tau_{i/n}(f)$ and $\tau_{i/n}(g)$ belong to the Boolean skeleton of \mathbb{L}_n^S we deduce $E(C, \tau_{i/n}(f) \wedge \neg \tau_{i/n}(g)) = 1$. We conclude that $E(C, \neg \tau_{i/n}(g)) = 1$ by outcome monotonicity and finally that $E(\overline{C}, \tau_{i/n}(g)) = 0$ by regularity, which is the desired contradiction.

It is easy to check that the liveness and safety conditions are satisfied for $C = N$. Moreover, E is N -maximal and homogeneous by assumption. It remains to prove that E is outcome monotonic. If $f \leq g \in \mathbb{L}_n^S$, we obtain successively

$$E(N, f) \leq \neg E(\emptyset, \neg f) \leq \neg E(\emptyset, \neg g) \leq E(N, g),$$

where the first inequality is obtained by regularity, the second by monotonicity and the third by N -maximality. ■

5 \mathbb{L}_n -valued modal language and semantics for effectivity functions

In this section, we build a many-valued modal logic in the spirit of [10, 20] that captures the properties of (truly) playable \mathbb{L}_n -valued effectivity functions.

5.1 Neighbourhood semantics for playable \mathbb{L}_n -valued effectivity functions

Let \mathcal{L} be the language $\{\rightarrow, \neg, 1\} \cup \{[C] \mid C \in \mathcal{PN}\}$ where $\rightarrow, \neg, 1$ are binary, unary and constant, respectively, and $[C]$ is a unary modality for every $C \in \mathcal{PN}$. The set $\text{Form}_{\mathcal{L}}$ of formulas is defined inductively from the countably infinite set Prop of propositional variables by the following rules:

$$\phi ::= 1 \mid p \mid \phi \rightarrow \psi \mid \neg \phi \mid [C]\phi,$$

where $p \in \text{Prop}$ and $C \in \mathcal{PN}$. We use 0 as an abbreviation of $\neg 1$. The intended reading of the formula $[C]\phi$ is ‘coalition C can enforce ϕ ’. In the language \mathcal{L} , we also use the standard abbreviations for defined connectives in Łukasiewicz logic; see Appendix B.

We introduce a semantics for \mathcal{L} which is based on a class of action models called \mathbb{L}_n -frames. Such frames are \mathbb{L}_n -valued extensions of coalition frames introduced in [20]. The coalition frames are a very general model of interaction in which an effectivity function is associated with each outcome state in S . Under the assumption of true playability, this is equivalent to specifying a game form for every outcome state in S .

DEFINITION 5.1

An \mathbb{L}_n -frame is a tuple $\mathfrak{F} = (S, E)$, where S is a non-empty set of outcome states and E is a mapping sending each outcome state $u \in S$ to an \mathbb{L}_n -valued effectivity function $E(u): \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$. A tuple $\mathcal{M} = (\mathfrak{F}, \text{Val})$ is an \mathbb{L}_n -model (based on \mathfrak{F}) if $\mathfrak{F} = (S, E)$ is an \mathbb{L}_n -frame and $\text{Val}: S \times \text{Form}_{\mathcal{L}} \rightarrow \mathbb{L}_n$.

We use the Łukasiewicz interpretations of the connectives $\neg, \rightarrow, 1$ in \mathbb{L}_n ; see Appendix B. For every \mathbb{L}_n -model \mathcal{M} , the valuation map Val is extended inductively to $S \times \text{Form}_{\mathcal{L}}$ by setting

$$\text{Val}(u, \phi \rightarrow \psi) = \text{Val}(u, \phi) \rightarrow \text{Val}(u, \psi), \quad (5.1)$$

$$\text{Val}(u, \neg \psi) = \neg \text{Val}(u, \psi), \quad (5.2)$$

$$\text{Val}(u, 1) = 1, \quad (5.3)$$

$$\text{Val}(u, [C]\phi) = E(u)(C, \text{Val}(-, \phi)), \quad (5.4)$$

for every $C \in \mathcal{PN}$ and every $\phi, \psi \in \text{Form}_{\mathcal{L}}$. We use the standard notation and terminology. We say that a formula ϕ is *true* in $\mathcal{M} = (\mathfrak{F}, \text{Val})$ and write $\mathcal{M} \models \phi$ if $\text{Val}(u, \phi) = 1$ for every $u \in S$. A formula ϕ is *valid* in an \mathbb{L}_n -frame \mathfrak{F} if it is true in every \mathbb{L}_n -valued model based on \mathfrak{F} .

In order to proceed further, we need to generalize the technique of filtration [8, Chapter 7.5] for neighbourhood models.

DEFINITION 5.2

Let $\mathcal{M} = (S, E, \text{Val})$ be an \mathbb{L}_n -valued model and Γ be a set of formulas closed under subformulas and the unary connectives $\tau_{\oplus}: \phi \mapsto \phi \oplus \phi$ and $\tau_{\odot}: \phi \mapsto \phi \odot \phi$. Consider the equivalence relation \equiv_{Γ} defined on S by

$$u \equiv_{\Gamma} v \quad \text{if} \quad \forall \phi \in \Gamma \quad \text{Val}(u, \phi) = \text{Val}(v, \phi).$$

We denote by $|S|$ the set of equivalence classes $|u|$ for \equiv_Γ . An \mathbb{L}_n -model $\mathcal{M}^* = (|S|, E^*, \text{Val}^*)$ is a Γ -filtration of \mathcal{M} if the following conditions are satisfied for every $u \in S$:

- (1) $\text{Val}^*(|u|, p) = \text{Val}(u, p)$ for every $p \in \text{Prop} \cap \Gamma$,
- (2) $E(u)(C, \text{Val}(-, \phi)) = E^*(|u|)(C, |\text{Val}(-, \phi)|)$ for every $C \in \mathcal{PN}$ and $\phi \in \Gamma$,

where the map $|\text{Val}(-, \phi)| : |S| \rightarrow \mathbb{L}_n$ is defined by $|\text{Val}(|u|, \phi)| = \text{Val}(u, \phi)$.

LEMMA 5.3

Let $\mathcal{M} = (S, E, \text{Val})$ be an \mathbb{L}_n -valued model and Γ be a set of formulas closed under subformulas and the connectives τ_\oplus and τ_\odot . If $\mathcal{M}^* = (|S|, E^*, \text{Val}^*)$ is a Γ -filtration of \mathcal{M} , then

$$\text{Val}(u, \phi) = \text{Val}^*(|u|, \phi) \quad (5.5)$$

for every $\phi \in \Gamma$ and every $u \in S$.

PROOF. Note that identity (5.5) is equivalent to $\text{Val}^*(-, \phi) = |\text{Val}(-, \phi)|$. The proof is a standard induction argument on the length of $\phi \in \Gamma$. We consider only the case where $\phi = [C]\psi \in \Gamma$ for $C \in \mathcal{PN}$. By the definition of Val^* and the induction hypothesis, we obtain

$$\text{Val}^*(|u|, [C]\psi) = E^*(|u|)(C, \text{Val}^*(-, \psi)) = E^*(|u|)(C, |\text{Val}(-, \psi)|).$$

It follows from Definition 5.2(2) that

$$E^*(|u|)(C, |\text{Val}(-, \psi)|) = E(u)(C, \text{Val}(-, \psi)) = \text{Val}(u, [C]\psi). \quad \blacksquare$$

In what follows we focus on the relations between the language \mathcal{L} and the \mathbb{L}_n -frames in which the effectivity functions are (truly) playable.

DEFINITION 5.4

An \mathbb{L}_n -frame $\mathfrak{F} = (S, E)$ is said to be (*truly*) *playable* if $E(u)$ is (*truly*) *playable* for every $u \in S$. An \mathbb{L}_n -model \mathcal{M} is (*truly*) *playable* if it is based on a (*truly*) *playable* \mathbb{L}_n -frame.

Our first aim is to prove that, similarly as in the Boolean case [10], there is no set of \mathcal{L} -formulas that can define truly playable \mathbb{L}_n -frames inside the class of playable \mathbb{L}_n -frames. To this end, we show that for any playable \mathbb{L}_n -model \mathcal{M} and any formula ϕ , there is a finite playable \mathbb{L}_n -model \mathcal{M}_ϕ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M}_\phi \models \phi$. We use this property and the fact that finite playable \mathbb{L}_n -models are truly playable to prove Proposition 5.8. The construction of \mathcal{M}_ϕ is based on a refinement of filtration for playable \mathbb{L}_n -valued models. We proceed in two steps. The next definition constitutes the first step in this direction.

DEFINITION 5.5

Using the notation of Definition 5.2, an \mathbb{L}_n -valued model $\mathcal{M}^* = (|S|, E^*, \text{Val}^*)$ is an *intermediate* Γ -filtration of a playable \mathbb{L}_n -model $\mathcal{M} = (S, E, \text{Val})$ if \mathcal{M}^* is a Γ -filtration of \mathcal{M} that satisfies

$$E^*(|u|)(C, f) = \max\{E(u)(C, \text{Val}(-, \phi)) \mid \phi \in \Gamma \text{ and } |\text{Val}(-, \phi)| \leq f\}, \quad (5.6)$$

$$E^*(|u|)(N, f) = \neg E^*(|u|)(\emptyset, \neg f), \quad (5.7)$$

for every proper coalition $C \in \mathcal{PN}$ and every $f \in \mathbb{L}_n^{|S|}$.

Observe that since we have assumed playability (N -maximality, in particular) of \mathcal{M} in Definition 5.5, an intermediate Γ -filtration of \mathcal{M} is indeed a Γ -filtration in the sense of Definition 5.2.

For every formula μ we denote by $\text{Cl}(\mu)$ the closure of the set of subformulas of μ for the connectives \neg and \rightarrow . The next lemma shows that an intermediate $\text{Cl}(\mu)$ -filtration is an intermediate step in the construction of a playable $\text{Cl}(\mu)$ -filtration of a playable \mathbb{L}_n -model. Recall that by $E(u)^\sharp$ we denote the function that is the restriction of the \mathbb{L}_n -valued effectivity function $E(u): \mathcal{PN} \times \mathbb{L}_n^S \rightarrow \mathbb{L}_n$ to the domain $\mathcal{PN} \times \mathbb{L}_1^S$.

LEMMA 5.6

Let $\mu \in \text{Form}_{\mathcal{L}}$ and $\mathcal{M} = (S, E, \text{Val})$ be a playable \mathbb{L}_n -model. If $\mathcal{M}^* = (|S|, E^*, \text{Val}^*)$ is an intermediate $\text{Cl}(\mu)$ -filtration of \mathcal{M} , then the Boolean effectivity function $E^*(|u|)^\sharp: \mathcal{PN} \times \mathbb{L}_1^S \rightarrow \mathbb{L}_1$ is playable for every $u \in S$.

PROOF. By $n.\phi$ we denote the formula $\bigoplus_{i=1}^n \phi$. First, observe that if $u \in S$, $f \in \mathbb{L}_1^{|S|}$ and $C \neq N$ is a coalition, then

$$E^*(|u|)(C, f) = \max\{E(u)^\sharp(C, \text{Val}(-, n.\phi)) \mid \phi \in \Gamma \text{ and } |\text{Val}(-, n.\phi)| \leq f\},$$

which shows that $E^*(|u|)^\sharp$ is a Boolean effectivity function. We prove that $E^*(|u|)^\sharp$ is regular and semi-playable (see Definition 4.12). It is straightforward to show that $E^*(|u|)^\sharp(C, -)$ is monotonic and has liveness and safety for every coalition $C \neq N$. Moreover, N -maximality holds for $E^*(|u|)^\sharp$ according to (5.7).

Let us prove superadditivity for coalitions C, D such that $C \cap D = \emptyset$ and $C \cup D \neq N$. If $f, g \in \mathbb{L}_1^{|S|}$, then $E^*(|u|)(C, f) \wedge E^*(|u|)(D, g)$ is by definition equal to the maximum of the values $E(u)(C, \text{Val}(-, \psi)) \wedge E(u)(D, \text{Val}(-, \rho))$, where ψ and ρ run through the elements of $\text{Cl}(\mu)$ satisfying $|\text{Val}(-, \psi)| \leq f$ and $|\text{Val}(-, \rho)| \leq g$. By superadditivity of E we have

$$E(u)(C, \text{Val}(-, \psi)) \wedge E(u)(D, \text{Val}(-, \rho)) \leq E(u)(C \cup D, \text{Val}(-, \psi \wedge \rho)),$$

for every formula ψ and ρ . Thus it follows from the definition of E^* that

$$E^*(|u|)(C, f) \wedge E^*(|u|)(D, g) \leq E^*(|u|)(C \cup D, f \wedge g),$$

which is the desired result.

It remains to prove that for every $C \in \mathcal{PN}$ and every $f \in \mathbb{L}_1^{|S|}$ such that $E^*(|u|)(C, f) = 1$, we have $E^*(|u|)(\overline{C}, \neg f) = 0$. By condition (5.7), we may assume that $C \neq N$. Suppose for the sake of contradiction that $E^*(|u|)(\overline{C}, \neg f) = 1$. By (5.6) this means that there are some $\psi, \rho \in \text{Cl}(\mu)$ such that $|\text{Val}(-, \psi)| \leq f$ and $|\text{Val}(-, \rho)| \leq \neg f$, and $E(u)(C, \text{Val}(-, \psi)) = E(u)(\overline{C}, \text{Val}(-, \rho)) = 1$. By superadditivity of E , we obtain $E(N, \text{Val}(-, \psi \wedge \rho)) = 1$ with $\psi \wedge \rho \in \text{Cl}(\mu)$ satisfying $|\text{Val}(-, \psi \wedge \rho)| \leq f \wedge \neg f$. By (5.6), (5.7), Definition 5.2 (2), N -maximality of E and the fact that $f \in \mathbb{L}_1^{|S|}$, we obtain $E^*(|u|)(N, 0) = 1$. This is a contradiction since $E^*(|u|)(N, 0) = \neg E^*(|u|)(\emptyset, 1) = 0$ by (5.7) and liveness of E^* for the empty coalition. \blacksquare

We combine Lemma 5.6 together with Lemma 4.8 to construct $\text{Cl}(\mu)$ -filtrations that preserve playability.

PROPOSITION 5.7

If $\mu \in \mathbf{Form}_{\mathcal{L}}$ and $\mathcal{M} = (S, E, \text{Val})$ is a playable \mathbb{L}_n -model, then there is a playable $\text{Cl}(\mu)$ -filtration $\mathcal{M}^+ = (|S|, E^+, \text{Val}^+)$ of \mathcal{M} .

PROOF. Consider the intermediate $\text{Cl}(\mu)$ -filtration $\mathcal{M}^* = (|S|, E^*, \text{Val}^*)$ of \mathcal{M} . By Lemma 5.6, the Boolean effectivity function $E^*(|u|)^\sharp: \mathcal{PN} \times \mathbb{L}_1^{|S|} \rightarrow \mathbb{L}_1$ is playable for every $u \in S$. By Lemma 4.8, the map $E^+(|u|): \mathcal{PN} \times \mathbb{L}_n^{|S|} \rightarrow \mathbb{L}_n$ defined by

$$E^+(|u|)(C, f) = \max \left\{ \frac{i}{n} \in \mathbb{L}_n \mid E^*(|u|)^\sharp(C, \tau_{\frac{i}{n}}(f)) = 1 \right\}$$

is also playable. We prove that $\mathcal{M}^* := (|S|, E^*, \text{Val}^*)$ is a $\text{Cl}(\mu)$ -filtration of \mathcal{M} . It suffices to check that \mathcal{M}^+ satisfies condition (2) of Definition 5.2. Let $u \in S$, $C \in \mathcal{PN}$, $\phi \in \text{Cl}(\mu)$ and $i \in \{1, \dots, n\}$. We obtain by definition of E^+ that $E^+(|u|)(C, |\text{Val}(-, \phi)|) \geq \frac{i}{n}$ if and only if $E^*(|u|)^\sharp(C, |\text{Val}(-, \tau_{\frac{i}{n}}(\phi))|) = 1$. Since \mathcal{M}^* is an intermediate $\text{Cl}(\mu)$ -filtration of \mathcal{M} , the last identity is in turn equivalent to $E(u)(C, \text{Val}(-, \tau_{\frac{i}{n}}(\phi))) = 1$. Finally, this gives $E(u)(C, \text{Val}(-, \phi)) \geq \frac{i}{n}$. ■

The next result shows that the gain of expressive power induced by the many-valued nature of \mathcal{L} and of its associated semantics is not enough to single out those playable models that are truly playable.

PROPOSITION 5.8

There is no set Λ of \mathcal{L} -formulas such that a playable \mathbb{L}_n -frame \mathfrak{F} is truly playable if and only if every formula of Λ is valid in \mathfrak{F} .

PROOF. Assume that there exists a set Λ of \mathcal{L} -formulas such that a playable \mathbb{L}_n -frame \mathfrak{F} is truly playable if and only if every formula of Λ is valid in \mathfrak{F} . Let \mathfrak{F} be a playable \mathbb{L}_n -frame which is not truly playable. The existence of such \mathfrak{F} is a consequence of Lemma 4.8 applied to the effectivity function E defined in [10, Proposition 4]. For every $\phi \in \Lambda$ and every model \mathcal{M} based on \mathfrak{F} , Proposition 5.7 provides a playable $\text{Cl}(\phi)$ -filtration \mathcal{M}^+ . Since \mathcal{M}^+ has a finite set of outcome states, it is truly playable. It follows from the definition of Λ that $\mathcal{M}^+ \models \phi$ and from Lemma 5.3 that $\mathcal{M} \models \phi$. We have proved that every formula of Λ is true in every model based on \mathfrak{F} and we conclude that \mathfrak{F} is truly playable, which is the desired contradiction. ■

5.2 \mathbb{L}_n -valued playable logic for finite playable \mathbb{L}_n -frames

Proposition 5.8 says that \mathcal{L} is not adequate for capturing the properties of \mathbb{L}_n -valued effectivity functions associated with game forms. Indeed, this language is not even expressive enough to distinguish between the playable and the truly playable \mathbb{L}_n -frames. Nevertheless, when the set of outcome states S is finite, every playable \mathbb{L}_n -valued effectivity function is truly playable and it turns out that playability can be encoded by \mathcal{L} -formulas; see our completeness result, Theorem 5.18. We start with axiomatizing the properties of playable \mathbb{L}_n -valued effectivity functions.

DEFINITION 5.9

An \mathbb{L}_n -valued playable logic is a subset \mathbf{L} of $\mathbf{Form}_{\mathcal{L}}$ which is closed under Modus Ponens, Uniform Substitution and Monotonicity (if $\phi \rightarrow \psi \in \mathbf{L}$, then $[C]\phi \rightarrow [C]\psi \in \mathbf{L}$ for every $C \in \mathcal{PN}$) and that contains an axiomatic base of Łukasiewicz $(n+1)$ -valued logic (see [11] or [9, Section 8.5]) together with the following axioms:

The axioms of \mathbf{L}_n -valued playable logic

- | | |
|---|--|
| (1) $[C](p \odot p) \leftrightarrow [C]p \odot [C]p,$ | (4) $([C]p \wedge [C']q) \rightarrow [C \cup C'](p \wedge q),$ |
| (2) $[C](p \oplus p) \leftrightarrow [C]p \oplus [C]p,$ | (5) $[\emptyset]p \rightarrow \neg[N]\neg p,$ |
| (3) $\neg[C]0,$ | |

for every $C, C' \in \mathcal{PN}$ such that $C \cap C' = \emptyset$.

We denote by \mathbf{P}_n the *smallest \mathbf{L}_n -valued playable logic*, i.e. the intersection of all the \mathbf{L}_n -valued playable logics. We conform with common usage and we often write $\vdash_{\mathbf{P}_n} \phi$ instead of $\phi \in \mathbf{P}_n$.

REMARK 5.10

The use of the notation $\vdash_{\mathbf{P}_n}$ is justified by the observation that \mathbf{P}_n can be equivalently introduced through a Hilbert style proof system. Indeed, it suffices to consider the Hilbert system whose axioms are the axioms (1)–(5) above together with an axiomatic base of Łukasiewicz $(n+1)$ -valued logic, and whose inference rules are Modus Ponens, Uniform Substitution and Monotonicity. Clearly, a formula ϕ is a theorem in this system if and only if it belongs to \mathbf{P}_n .

The axioms (1)–(5) together with the Monotonicity rule reflect the properties defining playability. In Remark 5.19 at the end of this section, we give equivalent and more intuitive axioms that can replace (1)–(2) in the axiomatization of \mathbf{P}_n .

The following lemma can be proved by a standard induction argument.

LEMMA 5.11

Let \mathcal{M} be a playable \mathbf{L}_n -model. If $\vdash_{\mathbf{P}_n} \phi$, then $\mathcal{M} \models \phi$.

We will prove completeness of \mathbf{P}_n with respect to the class of playable \mathbf{L}_n -models. Our proof is based on the construction of the canonical model.

5.2.1 Construction of the canonical model

Let us denote by $\mathfrak{F}_{\mathbf{P}_n}$ the Lindenbaum-Tarski algebra of \mathbf{P}_n , i.e. the quotient of $\text{Form}_{\mathcal{L}}$ under the syntactic equivalence relation \equiv defined by

$$\phi \equiv \psi \quad \text{if} \quad \vdash_{\mathbf{P}_n} \phi \rightarrow \psi \text{ and } \vdash_{\mathbf{P}_n} \psi \rightarrow \phi,$$

equipped with the operations $1, \neg, \rightarrow$ and $[C]$ defined as $1 := 1/\equiv, \neg(\phi/\equiv) := \neg\phi/\equiv, \phi/\equiv \rightarrow \psi/\equiv := (\phi \rightarrow \psi)/\equiv$ and $[C](\phi/\equiv) := [C]\phi/\equiv$, for every $C \in \mathcal{PN}$ and every $\phi, \psi \in \text{Form}_{\mathcal{L}}$. By abuse of notation, we denote the class ϕ/\equiv by ϕ .

Since \mathbf{P}_n contains every tautology of Łukasiewicz $(n+1)$ -valued logic, the $\{\rightarrow, \neg, 1\}$ -reduct of $\mathfrak{F}_{\mathbf{P}_n}$ is an MV-algebra that belongs to the variety \mathcal{MV}_n generated by \mathbf{L}_n .

In the Boolean setting, one of the key ingredients of the construction of the canonical model is the ultrafilter theorem that allows us to separate by an ultrafilter any two different non-top elements of a Boolean algebra \mathbf{B} . We can rephrase this separation result using the bijective correspondence between the ultrafilters of \mathbf{B} and the homomorphisms of \mathbf{B} into the two-element Boolean algebra $\mathbf{2}$: for every $a \neq b \in \mathbf{B} \setminus \{1\}$, there is a homomorphism $u: \mathbf{B} \rightarrow \mathbf{2}$ such that $u(a) = 1$ and $u(b) = 0$. The variety \mathcal{MV}_n has an analogous property [9].

LEMMA 5.12

Let $\mathbf{A} \in \mathcal{MV}_n$. For every $a \neq b$ in $\mathbf{A} \setminus \{1\}$, there is a $\{\neg, \rightarrow, 1\}$ -homomorphism $u: \mathbf{A} \rightarrow \mathbf{L}_n$ such that $u(a) = 1$ and $u(b) \neq 1$.

This separation property explains our choice of the set W^c of $\{\neg, \rightarrow, 1\}$ -homomorphisms from $\mathfrak{F}\mathbf{P}_n$ to \mathbf{L}_n as the universe of the canonical model of \mathbf{P}_n .

We will use the following technique to associate an \mathbf{L}_n -valued effectivity function $E^c(u)$ with every $u \in W^c$. For each $i \in \{1, \dots, n\}$ we will define a subset P_i of $\mathcal{PN} \times W^c \times \mathbf{L}_n^{W^c}$, such that $P_1 \supseteq \dots \supseteq P_n$. Then we will safely set

$$E^c(u)(C, f) := \max \left\{ \frac{i}{n} \in \mathbf{L}_n \mid (C, u, f) \in P_i \right\}.$$

DEFINITION 5.13

For every $i \in \{1, \dots, n\}$ let P_i be the subset of $\mathcal{PN} \times W^c \times \mathbf{L}_n^{W^c}$ defined by

$$P_i = \left\{ (C, u, f) \mid \exists \phi (u([C]\phi) \geq \frac{i}{n} \text{ and } \forall v \in W^c (v(\phi) \geq \frac{i}{n} \implies f(v) \geq \frac{i}{n})) \right\}. \quad (5.8)$$

We use the convention $P_0 = \mathcal{PN} \times W^c \times \mathbf{L}_n^{W^c}$.

LEMMA 5.14

The inclusion $P_i \subseteq P_{i-1}$ holds for each $i \in \{1, \dots, n\}$.

PROOF. Assume that ϕ satisfies the condition defining P_i in (5.8) for C, f, u and $i > 0$ and put $\rho = \tau_i(\phi)$. Then $u([C]\rho) = \tau_i(u([C]\phi)) = 1 \geq \frac{i-1}{n}$. Moreover, if $v(\rho) \geq \frac{i-1}{n}$, then $v(\rho) = 1$ since ρ belongs to the Boolean skeleton of $\mathfrak{F}\mathbf{P}_n$. Therefore $v(\phi) \geq \frac{i}{n}$, which gives $f(v) \geq \frac{i}{n} \geq \frac{i-1}{n}$. ■

DEFINITION 5.15

The *canonical model* of \mathbf{P}_n is the \mathbf{L}_n -model $\mathcal{M} = (\mathfrak{F}, \text{Val}^c)$ with $\mathfrak{F} = (W^c, E^c)$ where $E^c(u)(C, f)$ is defined for every $u \in W^c$, every $C \in \mathcal{PN}$ and every $f \in \mathbf{L}_n^{W^c}$ by

$$E^c(u)(C, f) = \begin{cases} \max \left\{ \frac{i}{n} \in \mathbf{L}_n \mid (C, u, f) \in P_i \right\} & \text{if } C \neq N, \\ \neg E^c(u)(\emptyset, \neg f) & \text{if } C = N, \end{cases} \quad (5.9)$$

and where Val^c is defined by

$$\text{Val}^c(u, p) = u(p), \quad (5.10)$$

for every $p \in \text{Prop}$ and $u \in W^c$.

In particular, for every $(C, u, f) \in \mathcal{PN} \times W^c \times \mathbf{L}_n^{W^c}$ with $C \neq N$, we have

$$E^c(u)(C, f) \geq \frac{i}{n} \quad \text{if and only if} \quad (C, u, f) \in P_i. \quad (5.11)$$

The next proposition shows that the identity (5.10) remains true in the canonical model after replacing p by any formula $\mu \in \text{Form}_{\mathcal{L}}$.

PROPOSITION 5.16 (Truth Lemma)

The canonical model (W^c, E^c, Val^c) of \mathbf{P}_n satisfies $\text{Val}^c(u, \mu) = u(\mu)$ for every $\mu \in \text{Form}_{\mathcal{L}}$ and every $u \in W^c$.

PROOF. We proceed by induction on the number of connectives in μ . If $\mu \in \mathbf{Prop}$ or $\mu \in \{1, \neg\psi, \psi \rightarrow \rho\}$, then the result follows immediately from (5.1) to (5.3).

Let $\mu = [C]\psi$ for some $\psi \in \mathbf{Form}_{\mathcal{L}}$ and $C \in \mathcal{PN} \setminus \{N\}$. We will prove that for any $u \in W^c$ and $i \leq n$,

$$E^c(u)(C, \text{Val}^c(-, \psi)) \geq \frac{i}{n} \quad \text{if and only if} \quad u([C]\psi) \geq \frac{i}{n}. \quad (5.12)$$

First, assume $E^c(u)(C, \text{Val}^c(-, \psi)) \geq \frac{i}{n}$. Then, by (5.11) and (5.8), there is $\rho \in \mathbf{Form}_{\mathcal{L}}$ such that $u([C]\rho) \geq \frac{i}{n}$ and $\text{Val}^c(v, \psi) \geq \frac{i}{n}$ for any $v \in W^c$ satisfying $v(\rho) \geq \frac{i}{n}$. By the induction hypothesis, this means that for every $v \in W^c$ with $v(\rho) \geq \frac{i}{n}$, we have $v(\psi) \geq \frac{i}{n}$. It follows that $v(\tau_{\frac{i}{n}}(\rho) \rightarrow \tau_{\frac{i}{n}}(\psi)) = 1$ for every $v \in W^c$. This yields $\vdash_{\mathbf{P}_n} \tau_{\frac{i}{n}}(\rho) \rightarrow \tau_{\frac{i}{n}}(\psi)$ since the $\{\rightarrow, \neg, 1\}$ -reduct of $\mathfrak{F}_{\mathbf{P}_n}$ has the separation property (Lemma 5.12). As \mathbf{P}_n is closed under Monotonicity, we obtain

$$\vdash_{\mathbf{P}_n} [C]\tau_{i/n}(\rho) \rightarrow [C]\tau_{i/n}(\psi).$$

By axioms (1) and (2) of \mathbf{P}_n (Definition 5.9) and Uniform Substitution, this is equivalent to

$$\vdash_{\mathbf{P}_n} \tau_{\frac{i}{n}}([C]\rho) \rightarrow \tau_{\frac{i}{n}}([C]\psi).$$

Hence, if $v \in W^c$ and $v([C]\rho) \geq \frac{i}{n}$, then $v([C]\psi) \geq \frac{i}{n}$. We can thus conclude that $u([C]\psi) \geq \frac{i}{n}$.

Conversely, let $u([C]\psi) \geq \frac{i}{n}$. We obtain $(C, u, \text{Val}(-, \psi)) \in P_i$ by the induction hypothesis and by considering $\phi = \psi$ in the definition (5.8) of P_i . This proves (5.12).

Finally, assume $\mu = [N]\psi$ for some $\psi \in \mathbf{Form}_{\mathcal{L}}$. We will prove

$$E^c(u)(N, \text{Val}(-, \psi)) = u([N]\psi).$$

Indeed, on the one hand we obtain

$$E^c(u)(N, \text{Val}(-, \psi)) = \neg E^c(u)(\emptyset, \text{Val}(-, \neg\psi))$$

by (5.9), which is in turn equal to $u(\neg[\emptyset]\neg\psi)$ by the induction hypothesis and the first part of this proof. Axiom (5) of \mathbf{P}_n yields $E^c(u)(N, \text{Val}(-, \psi)) \leq u([N]\psi)$.

To prove the converse inequality, let us assume for the sake of contradiction that there is $i \in \{1, \dots, n\}$ such that $E^c(u)(N, \text{Val}(-, \psi)) < \frac{i}{n} = u([N]\psi)$. Therefore $u([N]\tau_{i/n}(\psi)) = 1$, while $E^c(u)(\emptyset, \text{Val}(-, \neg\tau_{i/n}(\psi))) = 1$ by (5.9). It follows from this identity that $u([\emptyset]\neg\tau_{\frac{i}{n}}(\psi)) = 1$ by induction hypothesis and the first part of the proof. From axiom (7) applied with $C = N$ and $C' = \emptyset$ we get $u([N](\tau_{\frac{i}{n}}(\psi) \wedge \neg\tau_{\frac{i}{n}}(\psi))) = 1$; however, this is in contradiction with axiom (4) since $\tau_{\frac{i}{n}}(\psi) \wedge \neg\tau_{\frac{i}{n}}(\psi) = 0$. ■

5.2.2 Completeness result for \mathbf{P}_n

In order to use the canonical model for the proof of completeness of \mathbf{P}_n with respect to the class of the playable \mathbf{L}_n -models, we need the following result.

LEMMA 5.17

The canonical model of \mathbf{P}_n is a playable \mathbf{L}_n -frame.

PROOF. Let $u \in W^c$. It suffices to prove that $E^c(u)$ is semi-playable, homogeneous, N -maximal and regular. It is easily checked that $E^c(u)(C, -)$ is monotonic for every coalition $C \neq N$. The property of N -maximality is obtained by (5.9).

For homogeneity, let $C \neq N \in \mathcal{P}N$ and $f \in \mathbb{L}_n^{W^c}$. We will prove that for any $i \in \{1, \dots, n\}$,

$$E^c(u)(C, f \oplus f) \geq \frac{i}{n} \quad \text{if and only if} \quad E^c(u)(C, f) \oplus E^c(u)(C, f) \geq \frac{i}{n}. \quad (5.13)$$

First, assume $E^c(u)(C, f) \oplus E^c(u)(C, f) \geq \frac{i}{n}$ or, equivalently, $E^c(u)(C, f) \geq \frac{i}{2n}$. By the definition of E^c , there is a formula ρ such that $u([C]\rho) \geq \frac{i}{2n}$ and $f(v) \geq \frac{i}{2n}$ for every v satisfying $v(\rho) \geq \frac{i}{2n}$. On the one hand, by axioms (1) and (2) of \mathbf{P}_n and considering $\phi = \tau_{\lceil \frac{i}{2n} \rceil}(\rho)$, we get $u([C]\phi) = \tau_{\lceil \frac{i}{2n} \rceil}(u([C]\rho)) = 1 \geq \frac{i}{n}$. On the other hand, if $v \in W^c$ is such that $v(\phi) \geq \frac{i}{n}$, then $v(\rho) = 1$ since ϕ is an idempotent element of $\mathfrak{F}_{\mathbf{P}_n}$. Therefore $v(\rho) \geq \frac{i}{2n}$. This implies $f(v) \geq \frac{i}{2n}$, or, equivalently, $(f \oplus f)(v) \geq \frac{i}{n}$. We conclude that $E^c(u)(C, f \oplus f) \geq \frac{i}{n}$.

Conversely, assume $E^c(u)(C, f \oplus f) \geq \frac{i}{n}$ for some $i > 0$. The definition of E^c yields a formula ρ such that $u([C]\rho) \geq \frac{i}{n}$ and $f(v) \geq \frac{i}{2n}$ for any $v \in W^c$ with $v(\rho) \geq \frac{i}{n}$. By considering $\phi = \tau_{i/n}(\rho)$, we obtain on the one hand that $u([C]\phi) = \tau_{\frac{i}{n}}(u([C]\rho)) = 1 \geq \frac{i}{2n}$. On the other hand, if $v \in W^c$ is such that $v(\phi) \geq \frac{i}{2n}$, then $v(\rho) = 1$, which means $v(\rho) \geq \frac{i}{n}$ so that $f(v) \geq \frac{i}{2n}$. We have proved that $E^c(u)(C, f) \geq \frac{i}{2n}$ or, equivalently, $E^c(u)(C, f) \oplus E^c(u)(C, f) \geq \frac{i}{n}$. This finishes the proof of (5.13).

Analogously, we can show that

$$E^c(u)(C, f \odot f) \geq \frac{i}{n} \quad \text{if and only if} \quad E^c(u)(C, f) \odot E^c(u)(C, f) \geq \frac{i}{n}.$$

Employing N -maximality and the first part of the proof, it is easy to prove that for every $f \in \mathbb{L}_n^{W^c}$, we have $E^c(u)(N, f \oplus f) = E^c(u)(N, f) \oplus E^c(u)(N, f)$ and $E^c(u)(N, f \odot f) = E^c(u)(N, f) \odot E^c(u)(N, f)$. The function $E^c(u)$ is hence homogeneous.

Let us prove that $E^c(u)$ has safety for $C \neq N$. By way of contradiction, assume that $E^c(u)(C, 0) \geq \frac{1}{n}$. There is a formula ϕ such that $u([C]\phi) \geq \frac{1}{n}$ and $v(\phi) = 0$ for every $v \in W^c$. We deduce that $\vdash_{\mathbf{P}_n} \phi \rightarrow 0$ and hence $\vdash_{\mathbf{P}_n} [C]\phi \rightarrow [C]0$ by Monotonicity. It follows that $\frac{1}{n} \leq u([C]\phi) \leq u([C]0) = 0$, a contradiction.

To prove that $E^c(u)$ has liveness for every $C \neq N$, it suffices to consider $\phi = 1$ in (5.8) in order to show $(C, u, 1) \in P_1$.

We have to prove coalition monotonicity for C_1 and C_2 such that $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 \neq N$. It is enough to prove the following: if $E^c(u)(C_1, f_1) \wedge E^c(u)(C_2, f_2) \geq \frac{i}{n}$ for some $i \in \{1, \dots, n\}$, then $E^c(u)(C_1 \cup C_2, f_1 \wedge f_2) \geq \frac{i}{n}$. Let $\ell \in \{1, 2\}$ and denote by ϕ_ℓ a formula such that $u([C_\ell]\phi_\ell) \geq \frac{i}{n}$ and $f_\ell(v) \geq \frac{i}{n}$, for every v satisfying $v(\phi_\ell) \geq \frac{i}{n}$. Thus we can consider $\phi = \phi_1 \wedge \phi_2$ in (5.8) to get $(C_1 \cup C_2, u, f_1 \wedge f_2) \in P_i$.

It remains to check that $E^c(u)$ is regular. By (5.9), it suffices to prove that it is C -regular for every $C \neq N$. For the sake of contradiction, assume that there exists $i \in \{1, \dots, n\}$ such that $-E^c(u)(\overline{C}, \neg f) < \frac{i}{n} \leq E^c(u)(C, f)$. It follows that $E^c(u)(C, \tau_{\frac{i}{n}}(f)) = 1$, while $E^c(u)(\overline{C}, \tau_{\frac{i}{n}}(\neg f)) = 1$ for $\frac{i}{n} = 1 - \frac{i-1}{n}$. By superadditivity, we obtain $E^c(u)(N, \tau_{\frac{i}{n}}(f) \wedge \tau_{\frac{i}{n}}(\neg f)) = 1$, which is a contradiction since $\tau_{\frac{i}{n}}(f) \wedge \tau_{\frac{i}{n}}(\neg f)$ is the constant map 0. ■

THEOREM 5.18 (Completeness of \mathbf{P}_n)

For any $\phi \in \text{Form}_{\mathcal{L}}$, the following assertions are equivalent:

- (1) $\vdash_{\mathbf{P}_n} \phi$.
- (2) ϕ is true in every playable \mathbb{L}_n -model.
- (3) ϕ is true in every finite playable \mathbb{L}_n -model.
- (4) ϕ is true in every truly playable \mathbb{L}_n -model.

PROOF. The implication (1) \implies (2) is the content of Lemma 5.11. The equivalences (2) \iff (3) and (3) \iff (4) follow from Proposition 5.7 and Lemma 5.3. Finally, it remains to argue for the implication (2) \implies (1). We know by Lemma 5.17 that \mathcal{M}^c is a playable \mathbb{L}_n -model. According to Proposition 5.16, $\mathcal{M}^c \models \phi$ means that the class of ϕ is equal to 1 in $\mathfrak{F}_{\mathbf{P}_n}$, or, equivalently, $\vdash_{\mathbf{P}_n} \phi$. ■

REMARK 5.19

We can use the formulas $\tau_{i/n}(p)$ to replace axioms (1) and (2) of \mathbf{P}_n (Definition 5.9) by a family of axioms, which are easier to understand. Indeed, put

$$\begin{aligned} A &= \{[C](p \star p) \leftrightarrow ([C]p \star [C]p) \mid \star \in \{\odot, \oplus\}, C \in \mathcal{PN}\}, \\ B &= \{[C]\tau_{i/n}(p) \leftrightarrow \tau_{i/n}([C]p) \mid i \in \{1, \dots, n\}, C \in \mathcal{PN}\}. \end{aligned}$$

It follows from the definition of an \mathbb{L}_n -valued playable logic that $B \subseteq \mathbf{P}_n$. A careful analysis of the proofs of Lemma 5.14, Proposition 5.16, Lemma 5.17 and Theorem 5.18 shows that we have only used the axioms in A in the form of substitutions in formulas of B . Denote by \mathbf{P}'_n the smallest set of formulas that contains an axiomatic base of Łukasiewicz $(n+1)$ -valued logic, the set B , the axioms (3)–(5) of Definition 5.9, and that is closed under Modus Ponens, Uniform Substitution and Monotonicity. It follows from the previous observation that for any $\phi \in \text{Form}_{\mathcal{L}}$ we have $\vdash_{\mathbf{P}'_n} \phi$ if and only if $\mathcal{M} \models \phi$ for every playable \mathbb{L}_n -model. Thus $\mathbf{P}'_n = \mathbf{P}_n$.

Thus the set of axioms A can be equivalently replaced by B . Hence, the content of axioms (1)–(2) of \mathbf{P}_n can be rephrased as follow.

For any $i \leq n$, the following two assertions are equivalent:

- The truth value of the statement ‘coalition C can enforce ϕ ’ is at least $\frac{i}{n}$.
- Coalition C can enforce an outcome state in which the truth value of ϕ is at least $\frac{i}{n}$.

5.3 \mathbb{L}_n -valued truly playable logics for truly playable enriched \mathbb{L}_n -frames

Theorem 5.18 says that \mathbf{P}_n is the logic of playable rather than truly playable effectivity functions. Moreover, by Proposition 5.8 there is no axiomatization of truly playable effectivity functions in the language \mathcal{L} . Thus the presented many-valued approach is a faithful generalization of the Boolean framework; see [10]. In fact, the authors of [10] go beyond this limitation in the Boolean setting by adding a new connective to \mathcal{L} and by enriching the neighbourhood semantics with a Kripke relation. We follow this idea by designing the modal equivalent of truly playable \mathbb{L}_n -valued effectivity functions.

Let \mathcal{L}^+ be the language $\mathcal{L} \cup \{[\mathcal{O}]\}$ where $[\mathcal{O}]$ is unary. The set $\text{Form}_{\mathcal{L}^+}$ of formulas is defined inductively from the countably infinite set Prop of propositional variables by the following rules:

$$\phi ::= 1 \mid p \mid \phi \rightarrow \phi \mid \neg\phi \mid [C]\phi \mid [\mathcal{O}]\phi,$$

where $p \in \text{Prop}$ and $C \in \mathcal{PN}$.

In order to interpret \mathcal{L}^+ -formulas, we enrich the \mathbb{L}_n -frames with a binary relation.

DEFINITION 5.20

A tuple $\mathfrak{F} = (S, E, R)$ is an *enriched \mathbb{L}_n -frame* if (S, E) is an \mathbb{L}_n -frame and $R \subseteq S \times S$. We say that $\mathfrak{F} = (S, E, R)$ is *standard* if $R = \{(u, v) \mid E(u)(\emptyset, \neg\chi_{\{v\}}) = 0\}$.

A tuple $\mathcal{M} = (S, E, R, \text{Val})$ is an *enriched \mathbb{L}_n -model (based on (S, E, R))* if (S, E, R) is an enriched \mathbb{L}_n -frame and $\text{Val} : S \times \text{Prop} \rightarrow \mathbb{L}_n$. An enriched \mathbb{L}_n -frame $\mathfrak{F} = (S, E, R)$ or an enriched \mathbb{L}_n -model $\mathcal{M} = (S, E, R, \text{Val})$ is called *playable (truly playable, respectively)* if (S, E) is a playable (truly playable, respectively) \mathbb{L}_n -frame.

In an enriched \mathbb{L}_n -model, the valuation map Val is extended inductively to $S \times \text{Form}_{\mathcal{L}^+}$ by using rules (5.1) to (5.3) for the connectives 1 , \neg and \rightarrow , by using rule (5.4) for the connectives $[C]$, where $C \in \mathcal{PN}$, and by putting

$$\text{Val}(u, [C]\phi) = \min\{\text{Val}(v, \phi) \mid (u, v) \in R\} \quad (5.14)$$

for any $\phi \in \text{Form}_{\mathcal{L}^+}$ and $u \in S$.

It turns out that the class of standard truly playable enriched \mathbb{L}_n -frames can be defined inside the class of standard playable enriched \mathbb{L}_n -frames by an \mathcal{L}^+ -formula. The next assertion is the \mathbb{L}_n -valued generalization of [10, Proposition 14].

PROPOSITION 5.21

A standard playable enriched \mathbb{L}_n -frame \mathfrak{F} is truly playable if and only if $[\emptyset]\phi \leftrightarrow [C]\phi$ is valid in \mathfrak{F} .

PROOF. First assume that \mathfrak{F} is truly playable and standard. It follows from Lemma 4.4 that $\mathfrak{F}^\sharp := (S, E^\sharp, R)$ is a standard truly playable enriched \mathbb{L}_1 -frame. By [10, Proposition 14], for every $\phi \in \text{Form}_{\mathcal{L}^+}$ the formula $[\emptyset]\phi \leftrightarrow [C]\phi$ is valid in \mathfrak{F}^\sharp and we must prove that $[\emptyset]\phi \leftrightarrow [C]\phi$ is also valid in \mathfrak{F} .

Assume that there is $\phi \in \text{Form}_{\mathcal{L}^+}$ and a model $\mathcal{M} = (S, E, R, \text{Val})$ based on \mathfrak{F} such that $\text{Val}(u, [\emptyset]\phi) < \frac{i}{n} \leq \text{Val}(u, [C]\phi)$ for some $i \leq n$. It follows from homogeneity of $E(u)$ that $\text{Val}(u, [\emptyset]\tau_{\frac{i}{n}}(\phi)) = 0$, while $\text{Val}(u, [C]\tau_{\frac{i}{n}}(\phi)) = 1$. Moreover, the map $\text{Val}(-, \tau_{\frac{i}{n}}(\phi))$ has range in \mathbb{L}_1 . Hence, any map $\text{Val}' : S \times \text{Prop} \rightarrow \mathbb{L}_n$ with $\text{Val}'(w, p) = \text{Val}(w, \tau_{\frac{i}{n}}(\phi))$ for every $w \in S$ defines an \mathbb{L}_1 -model based on \mathfrak{F}^\sharp that falsifies $[\emptyset]p \leftrightarrow [C]p$, which is the desired contradiction. We can derive a similar contradiction in case there is some $\phi \in \text{Form}_{\mathcal{L}^+}$ and a model $\mathcal{M} = (S, E, R, \text{Val})$ based on \mathfrak{F} such that $\text{Val}(u, [\emptyset]\phi) \geq \frac{i}{n} > \text{Val}(u, [C]\phi)$ for some $i \in \{1, \dots, n\}$.

Conversely, let $\mathfrak{F} = (S, E, R)$ be a standard playable enriched \mathbb{L}_n -frame in which $[\emptyset]\phi \leftrightarrow [C]\phi$ is valid for every $\phi \in \text{Form}_{\mathcal{L}^+}$. Then $\mathfrak{F}^\sharp = (S, E^\sharp, R)$ is a standard playable enriched \mathbb{L}_1 -frame such that $[\emptyset]\phi \leftrightarrow [C]\phi$ is valid for every $\phi \in \text{Form}_{\mathcal{L}^+}$. Since \mathfrak{F}^\sharp is truly playable by [10, Proposition 14], \mathfrak{F} is truly playable as well. ■

We have to adapt the filtration technique to fit in with the newly introduced language \mathcal{L}^+ . The next definition merges Definition 5.2 with [22, Definition 5.3].

DEFINITION 5.22

Let $\mathcal{M} = (S, E, R, \text{Val})$ be an enriched \mathbb{L}_n -model and Γ be a set of formulas closed under subformulas and the unary connectives τ_\oplus and τ_\odot . With the notation introduced in Definition 5.2, an enriched \mathbb{L}_n -model $\mathcal{M}^* = (|S|, E^*, R^*, \text{Val}^*)$ is a Γ -filtration of \mathcal{M} if it satisfies (1) and (2) of Definition 5.2 and the following conditions:

- (3) if $(u, v) \in R$, then $(|u|, |v|) \in R^*$,
- (4) if $(|u|, |v|) \in R^*$ and $u([\emptyset]\phi) = 1$ for every $\phi \in \Gamma$, then $v(\phi) = 1$.

LEMMA 5.23

Let $\mathcal{M} = (S, E, R, \text{Val})$ be an enriched \mathbf{L}_n -model and Γ be a set of formulas closed under subformulas and the connectives τ_{\oplus} and τ_{\odot} . If

$$\mathcal{M} \models \{[\mathcal{O}](\phi \oplus \phi) \leftrightarrow [\mathcal{O}]\phi \oplus [\mathcal{O}]\phi, [\mathcal{O}](\phi \odot \phi) \leftrightarrow [\mathcal{O}]\phi \odot [\mathcal{O}]\phi\}$$

and $\mathcal{M}^* = (|S|, E^*, R^*, \text{Val}^*)$ is a Γ -filtration of \mathcal{M} , then

$$\text{Val}(u, \phi) = \text{Val}^*(|u|, \phi) \quad (5.15)$$

for every $\phi \in \Gamma$.

PROOF. The proof is a routine induction argument on the length of $\phi \in \Gamma$. Considering the proof of Lemma 5.3, the only case we have to discuss is $\phi = [\mathcal{O}]\psi \in \Gamma$. First, we note that by our assumption on \mathcal{M} and condition (4) of Definition 5.22, we have

$$\text{Val}(u, [\mathcal{O}]\psi) \leq \text{Val}(v, \psi) \quad (5.16)$$

for every $(|u|, |v|) \in R^*$.

Let $u \in S$. Then the definition of Val^* and the induction hypothesis yield

$$\text{Val}^*(|u|, [\mathcal{O}]\psi) = \min\{\text{Val}(v, \psi) \mid (|u|, |v|) \in R^*\}. \quad (5.17)$$

The inequality $\text{Val}^*(|u|, [\mathcal{O}]\psi) \leq \text{Val}(u, [\mathcal{O}]\psi)$ holds true since $(|u|, |v|) \in R^*$ for every $(u, v) \in R$. The other inequality is obtained by (5.16) and (5.17). ■

DEFINITION 5.24

An \mathbf{L}_n -valued *truly playable logic* is a subset \mathbf{L} of $\text{Form}_{\mathcal{L}^+}$ that is closed under Modus Ponens, Uniform Substitution and Monotonicity for every $[C]$, where $C \in \mathcal{PN}$, and such that \mathbf{L} contains an axiomatic base of Łukasiewicz $(n+1)$ -valued logic together with axioms (1)–(5) of logic \mathbf{P}_n and the following axioms:

The additional axioms of \mathbf{L}_n -valued truly playable logic

(6) $[\mathcal{O}]1$

(7) $[\mathcal{O}]p \leftrightarrow [\mathcal{O}]p,$

(8) $[\mathcal{O}](p \rightarrow q) \rightarrow ([\mathcal{O}]p \rightarrow [\mathcal{O}]q).$

We denote by \mathbf{TP}_n the *smallest \mathbf{L}_n -valued truly playable logic*.

Contrary to the Boolean case, it is not known if axiom (8) can be removed from the axiomatization of \mathbf{TP}_n without changing \mathbf{TP}_n . Nevertheless, the following result holds true.

LEMMA 5.25

\mathbf{TP}_n is closed under the necessitation rule for $[\mathcal{O}]$.

PROOF. Let $\vdash_{\mathbf{TP}_n} \phi$. Then $\vdash_{\mathbf{TP}_n} 1 \rightarrow \phi$ and $\vdash_{\mathbf{TP}_n} [\mathcal{O}]1 \rightarrow [\mathcal{O}]\phi$ by Monotonicity, which is equivalent to $\vdash_{\mathbf{TP}_n} [\mathcal{O}]1 \rightarrow [\mathcal{O}]\phi$ by axiom (7). The conclusion follows from axiom (6) and Modus Ponens. ■

We prove completeness of \mathbf{TP}_n with respect to the standard truly playable enriched \mathbf{L}_n -models by the technique of the canonical model. We denote by $\mathfrak{F}_{\mathbf{TP}_n}$ the Lindenbaum-Tarski algebra of \mathbf{TP}_n .

DEFINITION 5.26

The *canonical model* of \mathbf{TP}_n is the enriched \mathbf{L}_n -model $\mathcal{M} = (\mathfrak{F}, \text{Val}^c)$ with $\mathfrak{F} = (W^c, E^c, R^c)$, where $W^c = \mathcal{M}\mathcal{V}(\mathfrak{F}_{\mathbf{TP}_n}, \mathbf{L}_n)$, E^c and Val^c are as in Definition 5.15, and R^c is defined by

$$R^c = \{(u, v) \mid \forall \phi \ u([\mathcal{O}]\phi) = 1 \implies v(\phi) = 1\}.$$

PROPOSITION 5.27 (Truth Lemma)

The canonical model $\mathcal{M} = (\mathfrak{F}, \text{Val}^c)$ of \mathbf{TP}_n satisfies $\text{Val}^c(u, \mu) = u(\mu)$ for every $\mu \in \text{Form}_{\mathcal{L}^+}$ and every $u \in W^c$.

PROOF. The proof is carried out by induction on the number of connectives in μ . The only case not considered in the proof of Proposition 5.16 is $\mu = [\mathcal{O}]\phi$. However, this case was considered in the proof of [13, Proposition 5.6] or in the proof of [12, Proposition 5.5]. ■

It is worth noticing that Proposition 5.27 relies on the fact that the modality $[\mathcal{O}]$ is normal.

PROPOSITION 5.28

Let $\mu \in \text{Form}_{\mathcal{L}^+}$ and let $\mathcal{M}^* = (|W^c|, E^*, \text{Val}^*)$ be an intermediate $\text{Cl}(\mu)$ -filtration of (W^c, E^c, Val^c) . If

$$R^* = \{(|u|, |v|) \mid \forall \phi \in \text{Cl}(\mu) \ u([\mathcal{O}]\phi) = 1 \implies v(\phi) = 1\},$$

then the model $\mathcal{M}^+ = (|W^c|, E^+, R^*, \text{Val}^*)$ is a standard truly playable $\text{Cl}(\mu)$ -filtration of the canonical model $\mathcal{M} = (W^c, E^c, R^c, \text{Val}^c)$ of \mathbf{TP}_n , where E^+ is obtained from E^* as in Proposition 5.7.

PROOF. By Proposition 5.7 we know that \mathcal{M} is playable. As \mathcal{M} is finite, it is truly playable. By Proposition 5.7 and by definition of R^* , the model \mathcal{M}^+ is a $\text{Cl}(\mu)$ -filtration of \mathcal{M} in the sense of Definition 5.22.

It remains to prove that \mathcal{M}^+ is standard. First, assume that $u, v \in W^c$ and $E^+(|u|)(\emptyset, \neg\chi_{\{|v|\}}) = 0$. It follows from the definition of E^+ and E^* that

$$E(u)(\emptyset, \text{Val}(-, \phi)) = 0$$

for every $\phi \in \text{Cl}(\mu)$ such that $\text{Val}(v, \phi) = 0$. We conclude that $(|u|, |v|) \in R^*$ by the definition of R^* .

Let $E(|u|, |v|) \in R^*$. We will prove that $E^+(|u|)(\emptyset, \neg\chi_{\{|v|\}}) = 0$. By way of contradiction, assume that there is $\phi \in \text{Cl}(\mu)$ such that $\text{Val}(v, \phi) = 0$ and

$$E(u)(\emptyset, \text{Val}(-, \phi)) = \frac{i}{n} > 0.$$

Since $\chi_{\{|v|\}}$ is idempotent and $E(u)$ is homogeneous, we may assume $i = n$. It follows from Definition 5.20 and Proposition 5.27 that $u([\emptyset]\phi) = 1$. By axiom (7) of \mathbf{TP}_n , we deduce $u([\mathcal{O}]\phi) = 1$. The last identity is a contradiction since $(|u|, |v|) \in R^*$ and $v(\phi) = 0$. ■

THEOREM 5.29 (Completeness of \mathbf{TP}_n)

For any $\phi \in \text{Form}_{\mathcal{L}^+}$, the following assertions are equivalent:

- (1) $\vdash_{\mathbf{TP}_n} \phi$.
- (2) ϕ is true in every standard truly playable enriched \mathbf{L}_n -model.
- (3) ϕ is true in every finite standard truly playable enriched \mathbf{L}_n -model.

PROOF. It is clear that (2) \implies (3). Moreover, (1) \implies (2) can be proved by a straightforward induction argument. To prove (2) \implies (1), we obtain by Proposition 5.28 and Proposition 5.27 that ϕ is true in the canonical model of \mathbf{TP}_n , which means $\vdash_{\mathbf{TP}_n} \phi$. ■

6 Conclusions and future research

In this article, we have studied some generalizations of Pauly's Coalition Logic in modal extensions of Łukasiewicz logic. Below we list some ideas for possible applications and topics for further investigations.

- (1) The gain of expressive power owing to the many-valued modal language that is used could be exploited to encode some properties of strategic or voting games, such as, for instance, the distribution of power among coalitions in weighted voting games.
- (2) In modal extensions of $(n + 1)$ -valued Łukasiewicz logics, two types of relational structures can naturally be considered, giving rise to two types of completeness results [13]. On the one hand, there is the class of frames (structures with binary accessibility relations), while, on the other hand, there is the class of \mathbb{L}_n -frames. The latter are frames in which the set of allowed truth values in a world is a prescribed subalgebra of \mathbb{L}_n for every world of the frame. Such a prescription could also be considered in the context of \mathbb{L}_n -valued (truly) playable logics, where the neighbourhood semantics replace the relational ones. The possible aim is to obtain new completeness results with respect to this enriched semantics.
- (3) We have based our generalizations of Coalition Logic on modal extensions of Łukasiewicz logic. Other families of many-valued logics could be considered as a basis for many-valued versions of Coalition Logic. For example, it would be interesting to compare expressive power between the language developed in this article and a many-valued coalitional language based on modal extensions of Gödel logics [15].
- (4) Coalition Logic is among many formal calculi developed to model the deductive aspects of games. Other systems have been considered, such as ATL [3, 4] and its epistemic extensions [24]. A natural task could be to design the many-valued versions of those calculi in order to capture wider classes of games or protocols in which errors are allowed; see [22], for instance.
- (5) We did not consider the complexity issue of the satisfiability problem for the many-valued modal languages and models introduced in this article. This topic becomes a subject of further investigation although we conjecture that the problem is PSPACE-hard as in the Boolean case [20], since the number of possible truth values remains finite.

Funding

T. K. was supported by grant GA ČR P402/12/1309 and B. T. was supported by the internal research project FIR-MTH-PUL-15MRO3 of the University of Luxembourg.

References

- [1] J. Abdou and H. Keiding. *Effectivity Functions in Social Choice*. Vol. 8 of *Theory and Decision Library. Series C: Game Theory, Mathematical Programming and Operations Research*. Kluwer Academic Publishers Group, 1991.

- [2] T. Ågotnes, W. van der Hoek and M. Wooldridge. Reasoning about coalitional games. *Artificial Intelligence*, **173**, 45–79, 2009.
- [3] R. Alur, T. Henzinger and O. Kupferman. Alternating-time temporal logic. In *Proceedings of FCS 1997*, pp. 100–109, 1997.
- [4] R. Alur, T. Henzinger and O. Kupferman. Alternating-time temporal logic. *Journal of the ACM*, **49**, 672–713, 2002.
- [5] R. J. Aumann and L. S. Shapley. *Values of Non-Atomic Games*. Princeton University Press, 1974.
- [6] G. Boella, D. M. Gabbay, V. Genovese and L. van der Torre. Higher-order coalition logic. In *ECAI*, H. Coelho, R. Studer and M. Wooldridge, eds, Vol. 215 of *Frontiers in Artificial Intelligence and Applications*, pp. 555–560. IOS Press, 2010.
- [7] F. Bou, F. Esteva, L. Godo and R.O. Rodríguez. On the minimum many-valued modal logic over a finite residuated lattice. *Journal of Logic and Computation*, **21**, 739–790, 2011.
- [8] B. F. Chellas. *Modal Logic. An Introduction*. Cambridge University Press, 1980.
- [9] R. L. O. Cignoli, I. M. L. D’Ottaviano and D. Mundici. *Algebraic Foundations of Many-Valued Reasoning*, Trends in Logic—Studia Logica Library. Kluwer Academic Publishers, 2000.
- [10] V. Goranko, W. Jamroga and P. Turrini. Strategic games and truly playable effectivity functions. *Autonomous Agents and Multi-Agent Systems*, **26**, 288–314, 2013.
- [11] R. S. Grigolia. Algebraic analysis of Łukasiewicz-Tarski’s n -valued logical systems. In *Selected Papers on Łukasiewicz Sentential Calculi*, R. Wójcicki and G. Malinowski, eds, pp. 81–92. Osolineum, 1977.
- [12] G. Hansoul and B. Teheux. Completeness results for many-valued Łukasiewicz modal systems and relational semantics. Available at <http://arxiv.org/abs/math/0612542>, 2006.
- [13] G. Hansoul and B. Teheux. Extending Łukasiewicz logics with a modality: algebraic approach to relational semantics. *Studia Logica*, **101**, 505–545, 2013.
- [14] T. Kroupa. Representation and extension of states on MV-algebras. *Archive for Mathematical Logic*, **45**, 381–392, 2006.
- [15] G. Metcalfe and N. Olivetti. Proof systems for a Gödel modal logic. In *Automated Reasoning with Analytic Tableaux and Related Methods*, M. Giese and A. Waaler, eds, Vol. 5607 of *Lecture Notes in Computer Science*, pp. 265–279, Springer, 2009.
- [16] H. Moulin. *The Strategy of Social Choice*. Vol. 18 of *Advanced Textbooks in Economics*. North Holland Publishing Company, 1983.
- [17] D. Mundici. *Advanced Łukasiewicz calculus and MV-Algebras*. Vol. 35 of *Trends in Logic*. Springer, 2011.
- [18] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. MIT Press, 1994.
- [19] P. Ostermann. Many-valued modal propositional calculi. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, **34**, 343–354, 1988.
- [20] M. Pauly. A modal logic for coalitional power in games. *Journal of Logic and Computation*, **12**, 149–166, 2002.
- [21] B. Peleg. Effectivity functions, game forms, games, and rights. *Social Choice and Welfare*, **15**, 67–80, 1998.
- [22] B. Teheux. Propositional dynamic logic for searching games with errors. *Journal of Applied Logic*, **12**, 377–394, 2014.
- [23] W. van der Hoek and M. Pauly. Modal logic for games and information. In *Handbook of Modal Logic*, P. Blackburn, J. van Benthem and F. Wolter, eds, Vol. 3 of *Studies in Logic and Practical Reasoning*, pp. 1077–1148. Elsevier, 2007.

[24] W. van Der Hoek and M. Wooldridge. Cooperation, knowledge, and time: alternating-time temporal epistemic logic and its applications. *Studia Logica*, **75**, 125–157, 2003.

Received 24 November 2014

Appendix

A Representation of Boolean effectivity functions

Let $N = \{1, \dots, k\}$ be a finite set of players with $k \geq 2$ and S be a (possibly infinite) set of outcomes such that $|S| \geq 2$. A family $\mathbf{B} \subseteq \mathcal{P}S$ that contains S is called a *structure* on S . We say that \mathbf{B} is *closed under finite intersections* if $X_1, \dots, X_j \in \mathbf{B}$ implies $\bigcap_{i=1}^j X_i \in \mathbf{B}$ for every $j \in \mathbb{N}$. An effectivity function $E: \mathcal{P}N \rightarrow \mathcal{P}\mathcal{P}S$ is said to be *compatible* with a structure \mathbf{B} on S if $E(C) \subseteq \mathbf{B}$ for every $C \in \mathcal{P}N$, E has liveness and safety, $E(\emptyset) = \{S\}$, and $E(N) = \mathbf{B} \setminus \{\emptyset\}$. An effectivity function is *outcome monotonic with respect to \mathbf{B}* when the following implication holds true for every $C \in \mathcal{P}N$: if $X \in E(C)$, $X \subseteq Y$ and $Y \in \mathbf{B}$, then $Y \in E(C)$.

THEOREM A.1 ([21, Theorem 3.5*])

Let \mathbf{B} be a structure on S closed under finite intersections and $E: \mathcal{P}N \rightarrow \mathcal{P}\mathcal{P}S$ be an effectivity function compatible with \mathbf{B} . Then the following conditions are equivalent:

- (1) E is superadditive and outcome monotonic w.r.t. \mathbf{B} .
- (2) There exists a game form $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ satisfying $E(C) = H_G(C) \cap \mathbf{B}$ for every $C \in \mathcal{P}N$.

As announced in Section 2, we will prove that the characterization of effectivity functions generated by game forms from [10, Theorem 1] can be obtained as a consequence of Peleg's Theorem 3.5* in [21]. We restate the theorem for reader's convenience. We use the notion of true playability introduced in Definition 2.6.

THEOREM A.2 ([10, Theorem 1])

Let $E: \mathcal{P}N \rightarrow \mathcal{P}\mathcal{P}S$ be an effectivity function. There exists a game form $G = (N, \{\Sigma_i \mid i \in N\}, S, o)$ satisfying $E = H_G$ if and only if E is truly playable.

PROOF. As for the first implication, it is easy to see that H_G is playable. Set $Z = \{z \in S \mid z = o(\sigma_N) \text{ for some strategy profile } \sigma_N\}$. Then $Z \in H_G(\emptyset)$. Clearly, for any set of outcomes $X \subseteq S$ we have $X \in H_G(\emptyset)$ if and only if X contains the 'range' Z . This means that $H_G(\emptyset)$ is the principal filter generated by Z and H_G is truly playable.

In order to show the converse implication, let E be truly playable and put $Z = \bigcap E(\emptyset)$. Since E has safety, $Z \neq \emptyset$, and since $E(\emptyset)$ is a principal filter, $E(\emptyset) = \{X \in \mathcal{P}S \mid Z \subseteq X\}$. Consider the mapping $E': \mathcal{P}N \rightarrow \mathcal{P}\mathcal{P}Z$ defined as follows:

$$X \in E'(C) \text{ if } X \in E(C), \text{ for every } X \in \mathcal{P}Z \text{ and every } C \in \mathcal{P}N.$$

We claim that E' is compatible with the structure $\mathcal{P}Z$ on Z . Indeed, it follows that $E'(\emptyset) = \{Z\}$ and $E'(C) \subseteq \mathcal{P}Z \setminus \{\emptyset\}$ for every $C \in \mathcal{P}N$. By monotonicity, $Z \in E'(C)$ for every $C \in \mathcal{P}N$. It remains to prove that $\mathcal{P}Z \setminus \{\emptyset\} \subseteq E(N)$. Let $\emptyset \neq X \subseteq Z$. If $X = Z$, then we already know that $X \in E(N)$. Otherwise, X is a non-empty proper subset of Z and thus $X \notin E(\emptyset)$. We obtain by N -maximality of E that $\bar{X} \in E(N)$ and by superadditivity that $\bar{X} \cap Z \in E(N)$. We have proved that the complement in Z of any nonempty proper subset of Z is in $E(N)$, which yields $\mathcal{P}Z \setminus \{\emptyset\} \subseteq E(N)$. We can conclude that $E'(N) = \mathcal{P}Z \setminus \{\emptyset\}$.

By Theorem A.1, there is a game form $G'=(N, \{\Sigma_i | i \in N\}, Z, o)$ such that $E'=H_{G'}$. Put $G=(N, \{\Sigma_i | i \in N\}, S, o)$. We will show that $E=H_G$. To this end, let $C \in \mathcal{P}N$ and $X \in E(C)$. By superadditivity, $X \cap Z \in E(C)$, therefore $X \cap Z \in E'(C)=H_{G'}(C)$. By the definition of $H_{G'}$ and H_G , we get $X \in H_G(C)$. For the converse inclusion $H_G \subseteq E$, assume that $X \in \mathcal{P}S$ belongs to $H_G(C)$. By the definition of H_G , there exists σ_C such that $o(\sigma_C \sigma_{\bar{C}}) \in X \cap Z$ for every $\sigma_{\bar{C}}$. This means $X \cap Z \in H_{G'}(C)=E'(C)$, which gives $X \cap Z \in E(C)$. Finally, $X \in E(C)$ follows from outcome monotonicity. ■

Since every filter on a finite set is principal, the class of truly playable functions and the class of playable functions coincide whenever the set of outcome states S is finite.

B Finite MV-algebras

For a general background on Łukasiewicz logic and MV-algebras see [9, 17]. In this Appendix, we recall the basic notions and facts about MV-algebras that are needed in this article.

An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$, where \oplus is a binary operation, \neg is a unary operation and 0 is a constant, such that the following equations are satisfied:

- (1) $(A, \oplus, 0)$ is an Abelian monoid,
- (2) $\neg(\neg x) = x$,
- (3) $\neg 0 \oplus x = \neg 0$,
- (4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

We introduce the new constant 1 and two additional operations \odot and \rightarrow as follows:

$$\begin{aligned} 1 &= \neg 0, \\ x \odot y &= \neg(\neg x \oplus \neg y), \\ x \rightarrow y &= \neg x \oplus y. \end{aligned}$$

We say that an MV-algebra $(A, \oplus, \neg, 0)$ is *finite* whenever A is finite. As usual we will say that ‘ A is an MV-algebra’ when no danger of confusion arises. For every MV-algebra A , the binary relation \leq on A given by

$$x \leq y \quad \text{whenever} \quad x \rightarrow y = 1$$

is a partial order. As a matter of fact, \leq is a lattice order induced by the join \vee and the meet \wedge operations defined by

$$\begin{aligned} x \vee y &= \neg(\neg x \oplus y) \oplus y, \\ x \wedge y &= \neg(\neg x \vee \neg y), \end{aligned}$$

respectively. Thus defined, the lattice reduct of A is a distributive lattice with top element 1 and bottom element 0 . If the order \leq of A is total, then A is said to be an *MV-chain*.

The algebraic semantics of finite-valued Łukasiewicz logics is given by finite MV-chains. The standard example of a finite MV-chain is a *finite Łukasiewicz chain* given by

$$\mathbb{L}_n = \left\{ 0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1 \right\}, \quad \text{where } n \text{ is a positive integer.}$$

For every $x, y \in \mathbb{L}_n$, put

$$\begin{aligned}\neg x &= 1 - x, \\ x \oplus y &= \min(x + y, 1).\end{aligned}$$

Then $(\mathbb{L}_n, \oplus, \neg, 0)$ becomes an MV-chain, where the lattice operations \wedge and \vee are the minimum and the maximum of $x, y \in \mathbb{L}_n$, respectively. Further derived operations \odot , \rightarrow and \leftrightarrow on \mathbb{L}_n are given by

$$\begin{aligned}x \odot y &= \max(x + y - 1, 0), \\ x \rightarrow y &= \min(1, 1 - x + y), \\ x \leftrightarrow y &= 1 - |x - y|.\end{aligned}$$

Observe that the choice $n = 1$ gives a two-element Łukasiewicz chain $\mathbb{L}_1 = \{0, 1\}$, in which \oplus coincides with \vee and \odot coincides with \wedge . The semantics of classical propositional logic is thus determined by \mathbb{L}_1 . On the other hand, the algebraic semantics of finite $(n + 1)$ -valued Łukasiewicz logic with $n \geq 2$ is given by the variety of MV-algebras \mathcal{MV}_n that is axiomatized by Grigolia's axioms [11]:

- (1) $\odot_{i=1}^n x = \odot_{i=1}^{n+1} x$,
- (2) $\oplus_{i=1}^{n+1} \odot_{j=1}^m x = \odot_{i=1}^{n+1} \oplus_{j=1}^m \odot_{k=1}^{m-1} x$,

for every integer $m \in \{2, \dots, n - 1\}$ that does not divide n . Moreover, it is known that \mathcal{MV}_n is generated (as a variety) by the Łukasiewicz chain \mathbb{L}_n .

An *MV-filter* (or a *filter*) in an MV-algebra A is a subset $F \subseteq A$ such that

- (1) $1 \in F$,
- (2) if $x, y \in F$, then $x \odot y \in F$,
- (3) if $x \in F$ and $x \leq y \in A$, then $y \in F$.

A *principal filter* in A is a filter F for which there exists $x \in A$ such that F coincides with the smallest filter containing x . If A is finite, this means simply $F = \{y \in A \mid y \geq \odot_{i=1}^n x\}$ for some $x \in A$.