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A New Hierarchy of Infinitary Logics in Abstract Algebraic Logic

Abstract. In this article we investigate infinitary propositional logics from the perspective of their completeness properties in abstract algebraic logic. It is well-known that every finitary logic is complete with respect to its relatively (finitely) subdirectly irreducible models. We identify two syntactical notions formulated in terms of (completely) intersection-prime theories that follow from finitarity and are sufficient conditions for the aforementioned completeness properties. We construct all the necessary counterexamples to show that all these properties define pairwise different classes of logics. Consequently, we obtain a new hierarchy of logics going beyond the scope of finitarity.

Keywords: Abstract algebraic logic, Consequence relations, Infinitary logics, Completeness properties.

1. Introduction

A big part of the literature on non-classical propositional logics is devoted to systems that, just like classical logic, are *finitary*, in the sense that whenever a proposition follows from a set of premises, it must also follow from a *finite* subset of these premises. Such restriction is due to the fact that finitarity is a technically convenient assumption that substantially simplifies the necessary mathematical framework. Moreover, it may also be argued, from a more philosophical point of view, that if mathematical logic is supposed to model *correct reasoning*, then it should provide systems that, like a finite rational being, can only perform finitely-many inference steps to justify a proposition. However, beyond that motivation, one can as well find many natural examples of infinitary logics in the literature, i.e. systems where a proposition may follow from an infinite set of premises, but not from any of its finite subsets, or equivalently, systems that need infinitary

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inference when presented in terms of a proof calculus.[1] A prominent one is the infinitely-valued Łukasiewicz logic $Ł_\infty$ [16]. Therefore, abstract algebraic logic, a discipline that intends to provide a very general and encompassing approach to the study of non-classical logics, cannot be restricted only to finitary logics.

The contribution to the study of infinitary systems in abstract algebraic logic contained in this paper is mostly concerned with their completeness properties. Mainly we focus on relatively (finitely) subdirectly irreducible models of a given logic and call the corresponding completeness properties RSI-*completeness* and RFSI-*completeness*. We study them via a syntactical property called *intersection-prime extension property* (IPEP), which says that, for a given logic, the family of *intersection-prime* theories (theories that cannot be decomposed as the intersection of two strictly larger theories) form a basis of the closure system of all theories. This property was first introduced in [6] where it turned out to be very useful for the study of general disjunctions, since intersection-prime theories coincide with the usual notion of prime theory (a theory such that, if it contains a disjunction of two formulas, then it also contains one of the disjuncts). Moreover, the IPEP also proved to be an essential property for the characterization of semilinear logics (logics complete with respect to a semantics of linearly ordered matrices) obtained in [7].

The present paper stems from the master thesis [15], devoted to the study of a hierarchy of infinitary logics given by the mentioned completeness properties, the IPEP and a variation of this property. Namely, we propose a natural strengthening of the IPEP, called *completely intersection-prime extension* property, CIPEP, which says that completely intersection-prime theories (those that cannot be decomposed as the intersection of an *arbitrary* family of strictly larger theories) form a basis.

These properties define corresponding classes of logics that extend that of finitary logics as depicted in Figure 1. IPEP (resp. CIPEP) are sufficient syntactical conditions for logic to be RFSI-complete (resp. RSI-complete). A natural question is whether this picture does indeed give a new meaningful hierarchy of logics in abstract algebraic logic, i.e. whether the classes are pairwise different or they collapse.

[1] In this paper we always use 'infinitary' to denote this property. We do not consider logics with infinitely long sentences, typically obtained by means of infinite conjunctions or disjunctions, also called *infinitary logics* in the corresponding literature.

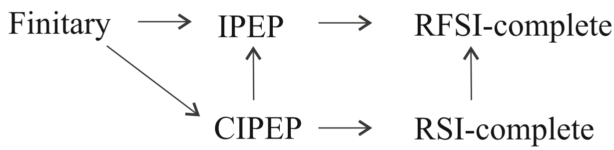

Figure 1. Inclusions between the defined classes of logics

The main contribution of this paper is that it gives a complete answer to that question. Namely, we will prove that all the mentioned classes are pairwise different by producing three examples of *infinitary* logics:

1. a logic with the CIPEP,
2. a logic with the IPEP which is not RSI-complete, and
3. an RSI-complete logic without the IPEP.

The paper is organized as follows. Section 2 gives some necessary preliminaries from abstract algebraic logic. In Section 3.1 we consider a notion of completeness with respect to surjective evaluations that allows us to show that the infinitely-valued Łukasiewicz logic $Ł_\infty$ and infinitely-valued product logic Π_∞ are infinitary logics with the CIPEP. In Section 3.2 we present an example of the second kind, a logic with the IPEP which is not RSI-complete, as a variant of the implicational fragment of the Gödel–Dummet fuzzy logic G [10] enriched with ω-many truth constants. Finally, Section 3.3 describes in details an example of an RSI-complete logic without the IPEP.

2. Preliminaries

2.1. Basic Notions

In this subsection we briefly recall the definitions and fix the notations of some basic notions of abstract algebraic logic that will be needed in the paper (for comprehensive monographs and a survey see [9,11–14,19]); we assume some familiarity with basic notions of universal algebra (see e.g. [2]).

A *propositional language* $\mathcal{L}$ is any type (with no restriction on the cardinality), by $\boldsymbol{Fm}_{\mathcal{L}}(\boldsymbol{Var})$ we denote the absolutely free term algebra in the language $\mathcal{L}$ over an arbitrary (but fixed) set of variables Var, by $Fm_{\mathcal{L}}(Var)$ we denote its universe. We usually assume Var is infinite countable and write simply $\boldsymbol{Fm}_{\mathcal{L}}$ and $Fm_{\mathcal{L}}$ respectively. For any sets of formulas Γ, Δ and a formula φ we often write 'Γ, Δ', and 'Γ, φ' for, respectively, '$\Gamma \cup \Delta$', and '$\Gamma \cup \{\varphi\}$'.

An $\mathcal{L}$-*consecution* is a pair $\Gamma \rhd \varphi$. Given a set of $\mathcal{L}$-consecutions L, we write $\Gamma \vdash_{\mathrm{L}} \varphi$ rather than $\Gamma \rhd \varphi \in \mathrm{L}$. A *consequence relation* L in the language $\mathcal{L}$ is a set of $\mathcal{L}$-consecutions satisfying:

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- $\Delta \vdash_{\mathrm{L}} \varphi$ and $\Delta \subseteq \Gamma$ then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Monotonicity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)

A *logic* is a structural consequence relation; i.e. a consequence relation with the following additional condition:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\mathcal{L}$-substitution σ. (Structurality)

Finally, a logic is *finitary* if it satisfies the following condition:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$. (Finitarity)

We write $\Gamma \vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \varphi$ for every $\varphi \in \Delta$. A *theory* of a logic L is a set of formulas closed under the consequence relation. The set of all theories of L is a closure system. By $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ we denote the theory generated by Γ.

An $\mathcal{L}$-matrix is a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra (the *algebraic reduct* of the matrix) and $F \subseteq A$ is a subset called the *filter* of the matrix. Given a class $\mathbb{K}$ of $\mathcal{L}$-matrices, the corresponding semantical consequence relation is defined as: $\Gamma \models_{\mathbb{K}} \varphi$ iff for each $\langle \boldsymbol{A}, F \rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e (i.e. a homomorphism $e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$) such that $e[\Gamma] \subseteq F$, we have $e(\varphi) \in F$. Clearly, $\models_{\mathbb{K}}$ is a logic. Moreover, if $\mathbb{K}$ is a finite set of finite $\mathcal{L}$-matrices, the logic $\models_{\mathbb{K}}$ is known to be finitary.

Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$, we say that a congruence θ of $\boldsymbol{A}$ is *compatible* with F iff for each $a, b \in A$, if $\langle a, b \rangle \in \theta$ and $a \in F$, then $b \in F$. Compatible congruences with F form a complete sublattice of the lattice of all congruences of $\boldsymbol{A}$, and thus there is a maximum congruence compatible with F, which is called the *Leibniz congruence* of $\mathbf{A}$ and denoted as $\Omega_{\boldsymbol{A}}(F)$. We say that $\mathbf{A}$ is a *reduced matrix* if $\Omega_{\boldsymbol{A}}(F) = Id_{\boldsymbol{A}}$.

A matrix $\mathbf{A}$ is a *model* of L if $\vdash_{\mathrm{L}} \subseteq \models_{\{\mathbf{A}\}}$. The class of (reduced) matrix models of a logic L is denoted as $\mathbf{MOD}(\mathrm{L})$ (or $\mathbf{MOD}^*(\mathrm{L})$ respectively). It is well-known that both of these classes are complete semantics for any logic L (in symbols: $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}(\mathrm{L})} = \models_{\mathbf{MOD}^*(\mathrm{L})}$); however it is common to consider meaningful subclasses of reduced models which may provide stronger completeness theorems. A matrix $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ is *relatively (finitely) subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$, in symbols $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ ($\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$), if it cannot be decomposed as a non-trivial subdirect product of an arbitrary (finite non-empty) family of matrices from

$\mathbf{MOD}^*(\mathrm{L})$. The class of algebraic reducts of $\mathbf{MOD}^*(\mathrm{L})$ is denoted as $\mathbf{ALG}^*(\mathrm{L})$.

Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$, we say that F is an L-*filter* provided that $\mathbf{A}$ is a model of L. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$; $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is also a closure system (and, consequently, a complete lattice) and hence it also induces a closure operator.

In this paper we will consider some logics belonging to the following implication-based class introduced in [3] (which generalizes implicative logics in sense of Rasiowa [18]):

Definition 2.1. (*Weakly implicative logic*) Let L be a logic in a language $\mathcal{L}$. We say that L is a *weakly implicative logic* if there is a binary connective $\to$ (primitive or definable by a formula in two variables) such that:

(R) $\vdash_{\mathrm{L}} \varphi \to \varphi$

(MP) $\varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi$

(T) $\varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi$

(sCng) $\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \to c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$
for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

We say that L is a *Rasiowa-implicative logic* if, moreover, it satisfies:

(W) $\varphi \vdash_{\mathrm{L}} \psi \to \varphi$

A very useful property of weakly implicative logics is that they enjoy an easy characterization of the Leibniz congruence via the connective $\to$:

Proposition 2.2. *Given a weakly implicative logic* L *and a matrix* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$, *we have:* $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ *if and only if* $\{a \to b, b \to a\} \subseteq F$. *In particular,* $\langle \boldsymbol{A}, F\rangle$ *is reduced if* $\{a \to b, b \to a\} \subseteq F$ *implies* $a = b$ *for every* $a, b \in A$.

Weakly implicative logics are a subclass of *protoalgebraic logics*, that will also be used in this paper. Protoalgebraic logics (as explained in [5]) can be introduced by requiring the same conditions as Definition 2.1, but for a generalized implication given by an arbitrary set of formulas with two variables (possibly infinite, possibly with parameters). The monograph [9] is the most extensive reference for the rich theory of protoalgebraic logics.

Further, we recall the notation for the usual algebraic class operators of isomorphisms, subalgebras, and direct products, denoted respectively by bold letters $\mathbf{I}, \mathbf{S}, \mathbf{P}$. *A generalized quasivariety* is a class of algebras axiomatized by a set of generalized quasiequations which are written using κ-many variables, for some infinite cardinal κ. Given a class of algebras $\mathbb{K}$ we denote $\mathbf{GQ}_\kappa(\mathbb{K})$ the class of algebras that validate all generalized quasiequations written in κ-many variables true in $\mathbb{K}$. It is known (see [1]) that, by means of operators $\mathbf{I}, \mathbf{S}, \mathbf{P}$, and $\mathbf{U}_\kappa$, where

$$\mathbf{U}_\kappa(\mathbb{K}) = \{\boldsymbol{A} \mid \text{every } \kappa\text{-generated subalgebra of } \boldsymbol{A} \text{ belongs to } \mathbb{K}\},$$

generalized quasivarieties can be characterized as follows:

$$\mathbf{GQ}_\kappa(\mathbb{K}) = \mathbf{U}_\kappa\mathbf{ISP}(\mathbb{K}).$$

In our case, considering the language restrictions (infinite countable set of variables), we simple write $\mathbf{GQ}$ and $\mathbf{U}$.

2.2. Intersection-Prime Filters and Classes of Infinitary Logics

In this subsection we first recall (from [6,9]) the definitions of the two kinds of filters that we will use in the rest of the paper and their corresponding extension properties; secondly we recall how they entail completeness with respect to the aforementioned classes of reduced matrix models.

Given a logic L, an algebra $\boldsymbol{A}$, and a filter $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, we say that F is *intersection-prime* if it is finitely meet-irreducible,[2] i.e. there is no pair of filters $F_1, F_2 \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ such that $F = F_1 \cap F_2$ and $F \subsetneq F_1, F_2$. Similarly, we say that F is *completely intersection-prime* if it is meet-irreducible, i.e. whenever $F = \bigcap_{i \in I} F_i$ for a family $\{F_i \mid i \in I\} \subseteq \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, there is $i_0 \in I$ such that $F = F_{i_0}$.

It is well-known [9, Proposition 1.3.4.] that

- $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ iff F is *intersection-prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$,
- $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ iff F is *completely intersection-prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

Recall that a family $\mathcal{B} \subseteq \mathcal{C}$ is a *basis* of a closure system $\mathcal{C}$ if for every $X \in \mathcal{C}$ there is a $\mathcal{D} \subseteq \mathcal{B}$ such that $X = \bigcap \mathcal{D}$ (which can be equivalently formulated as an extension property: for every $X \in \mathcal{C}$ and every $a \in A \setminus X$ there is $Y \in \mathcal{B}$ such that $X \subseteq Y$ and $a \notin Y$).

Using these notions one can define the following properties:[3]

Definition 2.3. We say that L has the (*completely*) *intersection-prime extension property*, (C)IPEP for short, if the (completely) intersection-prime theories form a basis of the closure system of theories of L. We say that a logic L is R(F)SI-complete if $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}}$.

Let us formulate two straightforward observations:

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[2] The tradition to call such a filter *prime* originated in [9, p. 147].

[3] The IPEP and the RFSI-completeness were already introduced explicitly in [6, Definition 2.5].

Proposition 2.4. *For every logic* L *we have:*

1. *If* L *has the* CIPEP*, then it has the* IPEP*.*
2. *If* L *is* RSI*-complete, then it is* RFSI*-complete.*

The properties in Definition 2.3 determine corresponding classes of logics that include that of finitary logics, as described by the next proposition.

Proposition 2.5. *For every logic* L *we have:*

1. *If* L *is finitary, then it has the* CIPEP*.*
2. *If* L *has the* IPEP*, then it is* RFSI*-complete.*
3. *If* L *has the* CIPEP*, then it is* RSI*-complete.*

Proof. The first claim is proved in [9, Corollary 1.3.3.]). The second claim is proved in [6, Lemma 2.6], and the last one is shown analogously. ■

Figure 1 depicts the inclusions between classes of logics stated by the two previous propositions. It is important to stress that this hierarchy does not exhaust completely the class of all propositional logics. Indeed, one can show that there exist non-RFSI-complete logics. An example of such a logic can be found in [6, Example 3.12]. This logic is shown to have a protodisjunction satisfying the PCP (*proof by cases property*) while not satisfying the sPCP (*strong proof by cases property*). From Section 4.4 of [6] it can be extracted that for any protoalgebraic RFSI-complete logic, the two proof by cases properties must coincide. Therefore, since this logic is weakly implicative (and hence protoalgebraic), it cannot be RFSI-complete.

3. Hierarchy of Infinitary Logics

Our aim is to show that the notions introduced in the previous section yield a hierarchy of finitary and infinitary propositional logics by showing that all the classes are pairwise different. This will be achieved by means of the results proved and the examples built in the following subsections.

3.1. Surjective Completeness

In this subsection we provide a semantical characterization of CIPEP and IPEP for protoalgebraic logics via a notion of *surjective completeness*. This notion already appeared in [8] in the context of equational consequence. We use this characterization to show that there exist infinitary logics with the CIPEP (and thus also with the IPEP).

Definition 3.1. (*Surjective semantical consequence*) A formula φ is a *surjective semantical consequence* of a set Γ of formulas w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each surjective $\boldsymbol{A}$-evaluation e (surjective as a function $e : \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$), we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models^s_{\mathbb{K}} \varphi$.

First notice that $\models_{\mathbb{K}} \subseteq \models^s_{\mathbb{K}}$. Moreover, it is easy to show that $\models^s_{\mathbb{K}}$ is a consequence relation. However, it is not necessarily structural, and hence not necessarily a logic, as shown by the following example.

Example 3.2. Let $\mathbf{A}$ be the matrix $\langle \mathbf{Ł}_3^{\to}, \{1\}\rangle$, where $\mathbf{Ł}_3^{\to}$ stands for the implication fragment of the standard 3-element Łukasiewicz algebra. Let $\Gamma = \{p \to q, q \to p\} \cup \{r \in \mathit{Var} \mid r \neq q \text{ and } r \neq p\}$. It can easily be seen that $\Gamma \models^s_{\{\mathbf{A}\}} p$, since there is no surjective evaluation satisfying Γ. On the other hand $\{p \to p, q\} \not\models^s_{\{\mathbf{A}\}} p$. Thus $\models^s_{\{\mathbf{A}\}}$ is not structural.

We use the notion of cardinality of L to characterize sufficient conditions under which $\models^s_{\mathbb{K}} = \models_{\mathbb{K}}$, hence conditions under which $\models^s_{\mathbb{K}}$ is indeed a logic.

Definition 3.3. The cardinality of a consequence relation L, card(L), is the smallest cardinal κ such that for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}(\mathit{Var})$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| < \kappa$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

Observe that a logic L is *finitary* if $\mathrm{card(L)} \leq \omega$.

Proposition 3.4. *Let κ be an infinite cardinal and $\mathbb{K}$ a class of $\mathcal{L}$-matrices. Assume that $|\mathit{Var}| = \kappa$, $\mathrm{card}(\models^s_{\mathbb{K}}) \leq \kappa$, and $|A| \leq \kappa$ for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$. Then, $\models^s_{\mathbb{K}} = \models_{\mathbb{K}}$ and, in particular, $\models^s_{\mathbb{K}}$ is structural.*

Proof. The inclusion $\supseteq$ trivially holds always. Suppose that $\Gamma \models^s_{\mathbb{K}} \varphi$. Then we obtain a set $\Gamma' \subseteq \Gamma$ of cardinality less than κ such that $\Gamma' \models^s_{\mathbb{K}} \varphi$. We claim that $\Gamma' \models_{\mathbb{K}} \varphi$ and consequently also $\Gamma \models_{\mathbb{K}} \varphi$. Consider any $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and any evaluation e on $\boldsymbol{A}$ such that $e[\Gamma'] \subseteq F$. Since $\Gamma' \cup \{\varphi\}$ contains less than κ variables, we can easily find a surjective evaluation e' which coincides with e on all variables occurring in $\Gamma' \cup \{\varphi\}$. Obviously, we have $e'[\Gamma'] \subseteq F$ and thus also $e(\varphi) = e'(\varphi) \in F$. ■

Note that in this proposition we have been more general than our original setting, since we have allowed sets of variables of uncountable cardinalities.

Further, in the next proposition, we refine the usual completeness results using surjective consequence relations.

Proposition 3.5. *Let* L *be a logic. Then:*

$$\mathrm{L} = \models^s_{\mathbf{MOD(L)}} = \models^s_{\mathbf{MOD^*(L)}}.$$

Moreover,

1. *if* L *has the* IPEP*, then* $\mathrm{L} = \models^s_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$,
2. *if* L *has the* CIPEP*, then* $\mathrm{L} = \models^s_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$.

Proof. It is enough to observe that the evaluations used in the proof of completeness w.r.t. reduced models (Lindenbaum–Tarski construction) are in fact surjective. In more details: If $\Gamma \nvdash_{\mathrm{L}} \varphi$, then by the IPEP there is an intersection-prime theory $T \supseteq \Gamma$ such that $T \nvdash_{\mathrm{L}} \varphi$. Then $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^*$, the Lindenbaum-Tarski reduction of $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle$, is the desired countermodel in $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ and the natural *surjective* projection e from $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle$ to $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^*$ is the desired evaluation. ■

Given a matrix $\langle A, F\rangle \in \mathbf{MOD}(\mathrm{L})$, note that the set of filters $[F, A] = \{G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid F \subseteq G\}$ can be seen as an interval in the lattice of L-filters over the algebra $\boldsymbol{A}$. For the next part we recall the well-known correspondence theorem for protoalgebraic logics (see e.g. [11, Theorem 6.19]).

Proposition 3.6. *Let* L *be a protoalgebraic logic. Take* $\langle A, F\rangle, \langle B, G\rangle \in \mathbf{MOD}(\mathrm{L})$ *and let* $h\colon \langle A, F\rangle \to \langle B, G\rangle$ *be a strict surjective homomorphism. Then the mapping* $\boldsymbol{h}$ *defined as* $\boldsymbol{h}(H) = h[H]$ *is an isomorphism between* $[F, A]$ *and* $[G, B]$.

Next we prove the characterization theorem for the CIPEP and the IPEP via surjective evaluations.

Proposition 3.7. *Let* L *be a protoalgebraic logic. Then:*

1. L *has the* IPEP *if and only if* $\mathrm{L} = \models^s_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$.
2. L *has the* CIPEP *if and only if* $\mathrm{L} = \models^s_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$.

Proof. We prove only the second part of the theorem (the first one is identical). $\Rightarrow$: This implication is given by Proposition 3.5.
$\Leftarrow$: Suppose $\Gamma \nvdash_{\mathrm{L}} \varphi$. There is $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ and a surjective evaluation $e\colon Fm_{\mathcal{L}} \to A$. We know that $T = e^{-1}[F]$ is an L-theory, in symbols $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle \in \mathbf{MOD}(\mathrm{L})$. By Proposition 3.6 we obtain an isomorphism $\boldsymbol{e}$ between $[T, Fm_{\mathcal{L}}]$ and $[F, A]$. It easily follows that T is intersection-prime (because F is intersection-prime). Moreover, since $\Gamma \subseteq T$ and $\varphi \notin T$, we are done. ■

In the next proposition we show that every usual consequence relation given by a class of matrices is actually a surjective consequence relation. We express this by means of the submatrix operator, which we denote in the same way as the algebraic one, $\mathbf{S}$. Recall that $\langle \boldsymbol{B}, G\rangle$ is a submatrix of $\langle \boldsymbol{A}, F\rangle$ if $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and $G = F \cap B$.

PROPOSITION 3.8. *For every class of $\mathcal{L}$-matrices $\mathbb{K}$ we have: $\models_{\mathbb{K}} = \models^{s}_{\mathbf{S}(\mathbb{K})}$.*

PROOF. Suppose that $\Gamma \not\models_{\mathbb{K}} \varphi$. Then there is $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and an evaluation e such that $e[\Gamma] \subseteq F$ and $e(\varphi) \notin F$. Take the theory $T = e^{-1}[F]$. We only need to prove that $\langle e[\boldsymbol{Fm}_{\mathcal{L}}], e[T]\rangle$ is a submatrix of $\langle \boldsymbol{A}, F\rangle$. Clearly $e[\boldsymbol{Fm}_{\mathcal{L}}]$ is a subalgebra of $\boldsymbol{A}$ and, since e is strict, it follows that $e[T] = e[Fm_{\mathcal{L}}] \cap F$. The converse direction follows easily from the fact that any evaluation on a submatrix is also an evaluation on the original matrix. ■

As an easy consequence of the previous proposition and Proposition 3.7, we obtain useful sufficient condition for a logic to have the CIPEP (resp. IPEP).

COROLLARY 3.9. *Let* L *be a protoalgebraic logic and suppose that $\mathbb{K}$ is a class of $\mathcal{L}$-matrices such that* $\mathrm{L} = \models_{\mathbb{K}}$. *Then:*

- *if* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$, *then* L *has the* IPEP,
- *if* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$, *then* L *has the* CIPEP.

Using this result, we can show that two prominent fuzzy logics (see e.g. [4]), namely the infinitely-valued Łukasiewicz logic $Ł_\infty$ and the infinitely-valued product fuzzy logic Π_∞, have the CIPEP (and consequently also the IPEP).[4]

EXAMPLE 3.10. Let us first recall a definition of $Ł_\infty$. The logic $Ł_\infty$ has three connectives $\to$, & and $\overline{0}$ and is given as the semantical consequence of the matrix $\mathbf{A} = \langle \boldsymbol{A}, \{1\}\rangle$, where $\boldsymbol{A} = \langle [0,1], \to, \&, \overline{0}\rangle$ such that $a \to b = \min\{1 - a + b, 1\}$, $a \,\&\, b = \max\{a + b - 1, 0\}$, and $\overline{0} = 0$.

It is well-known that every subalgebra $\boldsymbol{B}$ of $\boldsymbol{A}$ has only two filters, namely B and $\{1\}$. It then simply follows that $\mathbf{S}(\mathbf{A}) \subseteq \mathbf{MOD}^*(Ł_\infty)_{\mathrm{RSI}}$. The rest then follows by Corollary 3.9. Analogously we can prove the same for Π_∞.

THEOREM 3.11. *There is a logic which is not finitary and has the* CIPEP.

PROOF. Both Π_∞ and $Ł_\infty$ are well-known to be infinitary and, by the previous example, they have the CIPEP. ■

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[4]We thank our colleague Petr Cintula for pointing at Łukasiewicz logic as an example of an infinitary logic with the CIPEP.

3.2. A Non-RSI-Complete Logic with the IPEP

The aim of this subsection is to separate the top layer of Figure 1 from the lower one. Consider a language $\mathcal{L} = \{\to\} \cup \{\bar{q} \mid q \in (0,1] \cap \mathbb{Q}\}$. For every $0 < q \leq 1$ define an $\mathcal{L}$-algebra $\boldsymbol{A_q}$ with domain $[0,q]$, $a \to b = q$ if $a \leq b$ and $a \to b = b$ otherwise (i.e. it is a Gödel implication); and for constants $\bar{r}^{\boldsymbol{A}_q} = \min\{r,q\}$. Define $\mathbb{K} = \{\mathbf{A}_q = \langle \boldsymbol{A}_q, \{q\}\rangle \mid q \in (0,1] \cap \mathbb{Q}\}$ and let L be the logic of this class of matrices, i.e. $\mathrm{L} = \models_{\mathbb{K}}$. It is easy to check it is Rasiowa-implicative. The idea behind this definition is to have a logic with as many filters as possible. Moreover the presence of constants will enable us to prove that there are no subdirectly irreducible matrices in $\mathbf{MOD}^*(\mathrm{L})$.

Let $\mathbb{K}^a$ denote the class of algebraic reducts of matrices in $\mathbb{K}$. Knowing that L is Rasiowa-implicative, we obtain two important consequences. First that $\mathbf{ALG}^*(\mathrm{L}) = \mathbf{GQ}(\mathbb{K}^a)$, i.e. $\mathbf{ALG}^*(\mathrm{L})$ is the generalized quasivariety generated by $\mathbb{K}^a$. Second that for every $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ there is a unique filter F making $\langle \boldsymbol{A}, F\rangle$ reduced, namely $F = \{\bar{1}^{\boldsymbol{A}}\}$.

Therefore we obtain the following simple characterization of all reduced matrix models of L:

$$\mathbf{MOD}^*(\mathrm{L}) = \{\langle \boldsymbol{A}, \{\bar{1}^{\boldsymbol{A}}\}\rangle \mid \boldsymbol{A} \in \mathbf{UISP}(\mathbb{K}^a)\}$$

Thus on every algebra $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ we have a canonical ordering given by the unique filter on $\boldsymbol{A}$: $a \leq^{\boldsymbol{A}} b$ iff $a \to^{\boldsymbol{A}} b = \bar{1}^{\boldsymbol{A}}$, for each $a, b \in A$ (we will often omit the superscript, when there can be no confusion.

Till the end of this section we will use letters q, r, s as variables for rational numbers in the interval $[0,1]$. Given any two matrices $\mathbf{A}_q$ and $\mathbf{A}_r$ with $q < r$ and given any $\mathbf{A}_r$-evaluation e, we define an $\mathbf{A}_q$-evaluation e^q as follows: $e^q(p) = \min\{e(p), q\}$ for each variable p.

Using induction on the complexity of formulas we can easily prove the following observation.

Consider two matrices $\mathbf{A}_q$ and $\mathbf{A}_r$ such that $q < r$. Then for every $\mathbf{A}_r$-evaluation e and every formula φ we have:

1. $e(\varphi) \in [q,r]$ iff $e^q(\varphi) = q$,
2. if $e(\varphi) \leq q$ then $e(\varphi) = e^q(\varphi)$.

For every algebra $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ we will denote as $\uparrow^{\boldsymbol{A}} q$ the set of all elements in A bigger than $\bar{q}^{\boldsymbol{A}}$ (w.r.t. the canonical order induced by $\to$), in symbols: $\uparrow^{\boldsymbol{A}} q = \{a \in A \mid \bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} a\}$.

Corollary 3.12. *For any $\boldsymbol{A}_r$ and $q < r$, $\uparrow^{\boldsymbol{A}_r} q = [q,r]$ is an* L*-filter on $\boldsymbol{A}_r$. In particular, $\mathbf{A}_r \notin \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$.*

PROOF. Suppose that $\Gamma \vdash_{\mathrm{L}} \varphi$ and $e[\Gamma] \subseteq [q, r]$. From the previous observation we know that $e^q[\Gamma] \subseteq \{q\}$ and, since $\mathbf{A}_q$ is a model, we obtain $e^q(\varphi) = q$ and again, from the observation, $e(\varphi) \in [q, r]$. ■

PROPOSITION 3.13. *The unique reduced matrix based on each algebra from* $\mathbf{SP}(\mathbb{K}^a)$ *is not relatively subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$.

PROOF. Let $\boldsymbol{B} \in \mathbf{SP}(\mathbb{K}^a)$. $\boldsymbol{B}$ is a subalgebra of some direct product of algebras $\boldsymbol{C} = \prod_{i \in I} \boldsymbol{C}_i$ for $\boldsymbol{C}_i \in \mathbb{K}^a$. The only filter that makes $\boldsymbol{B}$ reduced is $\{\bar{1}^{\boldsymbol{C}}\}$; we show it is completely intersection-prime: it is easy to observe that for any system of filters $F_i \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{C_i})$ we have $\prod_{i \in I} F_i \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{C})$. In particular, if we choose $F_i = {\uparrow}^{\boldsymbol{C_i}} q$, then $\prod_{i \in I} F_i = {\uparrow}^{\boldsymbol{C}} q$ is a filter on $\boldsymbol{C}$.

Define $Z = \{q \in (0, 1) \mid \text{there is some } \boldsymbol{C_i} \text{ with domain } [0, r] \text{ and } q < r\}$. Observe that for every $q \in Z$ we have $\{\bar{1}^{\boldsymbol{C}}\} \subsetneq {\uparrow}^{\boldsymbol{C}} q$ and moreover $\{\bar{1}^{\boldsymbol{C}}\} = \bigcap_{q \in Z} {\uparrow}^{\boldsymbol{C}} q$.

Further ${\uparrow}^{\boldsymbol{B}} q = {\uparrow}^{\boldsymbol{C}} q \cap B$ is an L-filter on $\boldsymbol{B}$ and, since, for every $q \in Z$: $\bar{1}^{\boldsymbol{B}} = \bar{1}^{\boldsymbol{C}} \neq \bar{q}^{\boldsymbol{C}} = \bar{q}^{\boldsymbol{B}} \in {\uparrow}^{\boldsymbol{B}} q$ we conclude $\{\bar{1}^{\boldsymbol{B}}\} \subsetneq {\uparrow}^{\boldsymbol{B}} q$ and finally $\{\bar{1}^{\boldsymbol{B}}\} = \bigcap_{q \in Z} {\uparrow}^{\boldsymbol{B}} q$. Thus $\langle \boldsymbol{B}, \{\bar{1}^{\boldsymbol{B}}\} \rangle \notin \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$. ■

Now we are heading towards the same claim for $\mathbf{UISP}(\mathbb{K}^a)$. We first show some properties of chains in $\mathbf{SP}(\mathbb{K}^a)$:

LEMMA 3.14. *For any chain* $\boldsymbol{A} \in \mathbf{SP}(\mathbb{K}^a)$ *and any* $a \in A$ *such that* $a < \bar{1}^{\boldsymbol{A}}$ *there is some* q *such that* $a < \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$.

PROOF. Start with a subalgebra $\boldsymbol{A}$ of a direct product of algebras $\boldsymbol{B} = \prod_{i \in I} \boldsymbol{B_i}$. Let us have $a \in A$ such that $a < \bar{1}^{\boldsymbol{A}}$. Clearly there is i such that $a(i) < \bar{1}^{\boldsymbol{A_i}}$ and consequently some $q \in (0, 1)$ such that $a(i) < \bar{q}^{\boldsymbol{A_i}} < \bar{1}^{\boldsymbol{A_i}}$. From linearity we know that either $a < \bar{q}^{\boldsymbol{A}}$ or $\bar{q}^{\boldsymbol{A}} \leq a$ is true. Clearly, the second possibility would lead to contradiction. Thus, since obviously $\bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$, we are done. ■

PROPOSITION 3.15. *The unique reduced matrix based on each algebra from* $\mathbf{UISP}(\mathbb{K}^a)$ *is not relatively subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$.

PROOF. In pursuit of a contradiction suppose that there is $\boldsymbol{A}$ in $\mathbf{UISP}(\mathbb{K}^a)$ such that the unique L-filter $\{\bar{1}^{\boldsymbol{A}}\}$ is completely intersection-prime. First note that this implies that $\langle \boldsymbol{A}, \leq \rangle$ is linear with maximum element $\bar{1}^{\boldsymbol{A}}$ (see [7, Theorem 3]).

CLAIM 1. For every $a \in A$ such that $a < \bar{1}^{\boldsymbol{A}}$ there is $q \in (0, 1)$ such that $a < \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}$.

Proof (of Claim 1): Let $\langle a \rangle$ be the subalgebra generated by the element a. Since it is countable generated, we have

$$i : \langle a \rangle \simeq \boldsymbol{B} \hookrightarrow \prod_{i \in I} \boldsymbol{B_i}.$$

Further, since $\boldsymbol{B}$ is a chain (due to the isomorphism i) and $\boldsymbol{B} \in \mathbf{SP}(\mathbb{K}^a)$, we can find the desired q by applying i and Lemma 3.14.

Claim 2. $\uparrow^{\boldsymbol{A}} q$ is a filter on $\boldsymbol{A}$ for every $q \in (0, 1)$.

Proof (of Claim 2): Suppose $\Gamma \vdash_{\mathrm{L}} \varphi$ and $e[\Gamma] \subseteq \uparrow^{\boldsymbol{A}} q$. It is clear that $e[\boldsymbol{Fm_{\mathcal{L}}}]$ is a countably generated subalgebra of $\boldsymbol{A}$ thus we have

$$i : e[\boldsymbol{Fm_{\mathcal{L}}}] \simeq \boldsymbol{B} \hookrightarrow \prod_{i \in I} \boldsymbol{B_i}.$$

For any $\psi \in \Gamma$ we have $\bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} e(\psi)$. Since i is an isomorphism, also $i(\bar{q}^{\boldsymbol{A}}) = \bar{q}^{\boldsymbol{B}} \leq^{\boldsymbol{B}} i(e(\psi))$. We know that $\uparrow^{\boldsymbol{B}} q$ is a filter on $\boldsymbol{B}$ (see the proof of Proposition 3.13), which implies $\bar{q}^{\boldsymbol{B}} \leq^{\boldsymbol{B}} i(e(\varphi))$. Thus it follows that $\bar{q}^{\boldsymbol{A}} \leq^{\boldsymbol{A}} e(\varphi)$, as we wanted.

To finish the proof observe that if $\boldsymbol{A}$ is not trivial then it is, by Claim 1, infinite. Define $Z = \{q \in (0, 1) \mid \bar{q}^{\boldsymbol{A}} < \bar{1}^{\boldsymbol{A}}\}$. Now, using both claims, we can easily decompose $\{\bar{1}^{\boldsymbol{A}}\}$ by means of the collection of $\uparrow^{\boldsymbol{A}} q$ ranging over Z. ■

Theorem 3.16. *There is a logic with the* IPEP *which is not* RSI-*complete.*

Proof. Proposition 3.15 tells us that the logic L defined in this section has in fact no subdirectly irreducible reduced models. Thus, in particular, it is not RSI-complete.

On the other hand we can easily argue, using Corollary 3.9, that L has the IPEP. ■

3.3. An RSI-Complete Logic Without the IPEP

As we have seen in the previous section, when we want to determine whether a given logic is RSI-complete or RFSI-complete, the notions of CIPEP or IPEP are useful sufficient conditions. It is enough to check whether the logic satisfies one of these extension properties (or finitarity). The rest of this paper is devoted to the problem of separating the classes of logics with the IPEP from RFSI-complete logics, and the classes of logics with the CIPEP from RSI-complete logics. This will be achieved by producing a single example, rather difficult to construct, of an RSI-complete logic which does not enjoy the IPEP. This way we prove that CIPEP and IPEP are not trivial notions, which, as conclusion, allows us to obtain a hierarchy of infinitary logics.

3.3.1. Introducing the Example. We are going to describe an RSI-complete *weakly implicative* logic L which does not belong to the IPEP class. Our logic will be given semantically by a suitable matrix $\langle \boldsymbol{A}, F\rangle$. This approach will turn out be very useful in proving RSI-completeness: we only need to check that the matrix is reduced and F is completely intersection-prime filter in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, i.e. that $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$.

In order to falsify the IPEP in L, we will implement a full binary tree of height ω into $\boldsymbol{A}$. The motivation is that every node in the tree is $\wedge$-reducible (i.e. can be expressed as a meet of its two immediate successors). To benefit from this idea we make sure that every node s of the tree will correspond to some theory T_s of L. Moreover, we make sure that for every node s and its two immediate successors s_1 and s_2 their corresponding theories T_s, T_{s_1}, T_{s_2} will satisfy $T_s \subsetneq T_{s_1}$, $T_s \subsetneq T_{s_2}$ and $T_s = T_{s_1} \cap T_{s_2}$ (thus ensuring T_s is not intersection-prime for every s). Then we define a set of formulas Γ and a formula φ such that $\Gamma \nvdash_{\mathrm{L}} \varphi$, in such a way that any theory extending Γ and not containing φ will correspond to some node in the tree. This way we conclude that Γ cannot be extended to any intersection-prime theory.

To this end, we add a unary connective for each node s, which will allow us to capture (within the logic) which nodes are above s. Interestingly enough, in order to follow through we need to let the logic 'know' something about itself, that is, we include in the algebra $\boldsymbol{A}$ also some substantial subset of $Fm_{\mathcal{L}}(\{p\})$ (algebra of formulas in one variable p). Therefore, an interesting feature of this logic is that its semantics is partially based on its own syntax.

Fix $\mathbf{T} = \langle \mathrm{T}, \leq_{\mathrm{T}}\rangle$ to be the full binary tree of height ω. Note that we can view $\mathbf{T}$ as e.g. the collection of all functions which have a natural number n as a domain and a subset of 2 as range, where $\leq_{\mathrm{T}}$ is the inclusion order ($\emptyset$ is the root of this tree). We will use variables s, r, t, u (possibly with indexes, superscripts) for the nodes, moreover we write $\mathbf{r}$ for the root of $\mathbf{T}$.

Let us next focus on the language of L. $\mathcal{L} = \{\overline{0}, \to, B\} \cup \{B_s \mid s \in \mathrm{T}\}$, where $\overline{0}$ is a nullary connective, $\to$ is binary and the rest are unary connectives; read B_s as 'bigger than the node s'. Moreover we define a nullary connective $\overline{1}$ as $\overline{0} \to \overline{0}$.

Next we describe the subset of $Fm_{\mathcal{L}}(\{p\})$ that we will include in $\boldsymbol{A}$. We will call this set $\mathbf{Fm^p}$ and define it recursively as the least set satisfying

1. (a) $\{\boldsymbol{B_s(p)} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,
   (b) $\{\boldsymbol{B_s(\overline{0})} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,
   (c) $\{\boldsymbol{B_s(\overline{1})} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,

Table 1. Interpretation of the connective $\to$

	Operation $\to$
T, T	$s \to t = s$ if $s = s$ and $s \to t = 0$ otherwise
T, $\mathbf{Fm^p}$	$s \to \boldsymbol{\varphi} = \boldsymbol{p}{\Rightarrow}\boldsymbol{\varphi}$ and $\boldsymbol{\varphi} \to s = \boldsymbol{\varphi}{\Rightarrow}\boldsymbol{p}$
T, 0, 1	$s \to 0 = 0 \to s = 0$ and $s \to 1 = 1 \to s = 0$
$\mathbf{Fm^p}$, $\mathbf{Fm^p}$	$\boldsymbol{\varphi} \to \boldsymbol{\psi} = \boldsymbol{\varphi}{\Rightarrow}\boldsymbol{\psi}$
$\mathbf{Fm^p}$, 0	$0 \to \boldsymbol{\varphi} = \overline{\mathbf{0}}{\Rightarrow}\boldsymbol{\varphi}$ and $\boldsymbol{\varphi} \to 0 = \boldsymbol{\varphi}{\Rightarrow}\overline{\mathbf{0}}$
$\mathbf{Fm^p}$, 1	$1 \to \boldsymbol{\varphi} = \overline{\mathbf{1}}{\Rightarrow}\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}{\Rightarrow}1 = \boldsymbol{\varphi}{\Rightarrow}\overline{\mathbf{1}}$
0, 1	$0 \to 0 = 1 \to 1 = 1$ and $1 \to 0 = 0 \to 1 = 0$
$\star$	As if $\star \in$ T (e.g. $\star \to \boldsymbol{\varphi} = \boldsymbol{p}{\Rightarrow}\boldsymbol{\varphi}$)

Table 2. Interpretation of B_i's

	Operations B_s
T	$B_s(t) = t$ if $t \geq_{\mathrm{T}} s$ and $B_s(t) = \boldsymbol{B_s(p)}$ otherwise
$\mathbf{Fm^p}$	$B_s(\boldsymbol{\varphi}) = \boldsymbol{B_s(\varphi)}$
0, 1	$B_s(0) = \boldsymbol{B_s(\overline{0})}$ and $B_s(1) = \boldsymbol{B_s(\overline{1})}$
$\star$	$B_s(\star) = \boldsymbol{B_s(p)}$

2. moreover for every $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbf{Fm^p}$ also
   (a) $\{\boldsymbol{B_s(\varphi)} \mid s \in \mathrm{T}\} \subseteq \mathbf{Fm^p}$,
   (b) $\overline{\mathbf{0}}{\Rightarrow}\boldsymbol{\varphi} \in \mathbf{Fm^p}$ and $\boldsymbol{\varphi}{\Rightarrow}\overline{\mathbf{0}} \in \mathbf{Fm^p}$,
   (c) $\overline{\mathbf{1}}{\Rightarrow}\boldsymbol{\varphi} \in \mathbf{Fm^p}$ and $\boldsymbol{\varphi}{\Rightarrow}\overline{\mathbf{1}} \in \mathbf{Fm^p}$,
   (d) $\boldsymbol{p}{\Rightarrow}\boldsymbol{\varphi} \in \mathbf{Fm^p}$ and $\boldsymbol{\varphi}{\Rightarrow}\boldsymbol{p} \in \mathbf{Fm^p}$,
   (e) $\boldsymbol{\varphi}{\Rightarrow}\boldsymbol{\psi} \in \mathbf{Fm^p}$.

We write the formulas in $\mathbf{Fm^p}$ in boldface (and we use $\Rightarrow$ instead of $\to$) to have a better distinction between formulas that are part of the syntax and those that we use to build the algebra $\boldsymbol{A}$. Note that $\mathbf{Fm^p}$ is a subalgebra of the $\{B, \overline{0}, \overline{1}\}$-free reduct of $Fm_{\mathcal{L}}(\{p\})$. Let us mention that we have presented $\mathbf{Fm^p}$ in such a way that it has a well-arranged structure for inductive proofs on $\mathbf{Fm^p}$, as we will see later. Having described the two main components of the algebra $\boldsymbol{A}$ we are finally ready to define the logic L.

Definition 3.17. *(The logic* L) Suppose that all the elements in $\mathbf{Fm^p}$, T and $\{0, 1, \star\}$ are mutually distinct objects and denote $\mathrm{D} = \{\boldsymbol{\varphi} \in \mathbf{Fm^p} \mid$ there is $\boldsymbol{\psi} \in \mathbf{Fm^p}$ such that $\boldsymbol{\varphi} = \boldsymbol{\psi}{\Rightarrow}\boldsymbol{\psi}\}$. We define:

$$A = \mathbf{Fm^p} \cup \mathrm{T} \cup \{0, 1, \star\} \text{ and } F = \mathrm{D} \cup \mathrm{T} \cup \{1, \star\}.$$

For every $s, t \in \mathrm{T}$ and every $\boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathbf{Fm^p}$ the operations of the algebra $\boldsymbol{A}$ are given by the Tables 1 and 2.

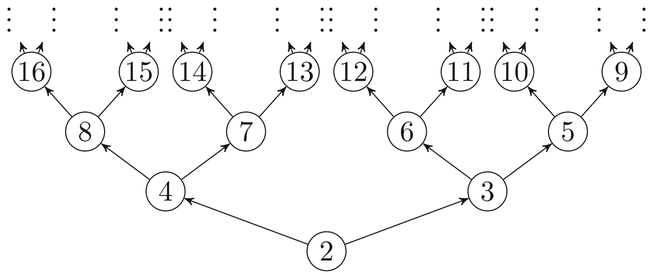

Figure 2. The tree T

Moreover $B(\star) = \star$ and $B(a) = B_{\mathbf{r}}(a)$, where $\mathbf{r}$ is the root of $\mathbf{T}$ and $a \neq \star$. The constant $\overline{0}$ is interpreted as 0. We define L as the logic of this matrix, i.e. $\mathrm{L} = \models_{\mathbf{A}}$.

To get better acquainted with this definition let us first compute a few values of formulas. For this example we will use labels for the nodes as depicted on Figure 2.

Example 3.18. Consider an evaluation e and a formula

$$\varphi = B_6(q \to q) \to q$$

(a) if $e(q) = 11$, then $e(\varphi) = 11$;

(b) if $e(q) = 3$ (or $5, 10, 14, \dots$ or $\star$), then $e(\varphi) = \boldsymbol{B_6(p) \Rightarrow p}$,

(c) if $e(q) = 0$, then $e(\varphi) = \boldsymbol{B_6(\overline{1}) \Rightarrow \overline{0}}$;

(d) if $e(q) = \boldsymbol{\psi} \in \mathbf{Fm^p}$, then $e(\varphi) = \boldsymbol{B_6(\psi \Rightarrow \psi) \Rightarrow \psi}$.

Observe that for any evaluation e and any formula φ it is true that

$$e(\varphi) \in \mathbf{Fm^p} \text{ iff there is a subformula } B_s(\psi) \text{ of } \varphi \text{ and } e(\psi) \not\geq_{\mathrm{T}} s.$$

Thus, whenever $e(\varphi) \in \mathbf{Fm^p}$ we can say that the formula φ is in some sense falsified by the evaluation e.

It is very easy to see that L is a weakly implicative logic using the following observation:

Observation 3.19. For every $a, b \in A$ it holds: $a \to b \in F$ iff $a = b$.

Note that it relies on the fact that we included some part of $\mathbf{Fm^p}$, namely the set D, in the filter.

Corollary 3.20. L *is a weakly implicative logic and* $\langle \boldsymbol{A}, F \rangle$ *is reduced.*

3.3.2. Failure of the IPEP. In this section we prove that L does not satisfy the IPEP. The proof of this claim is divided into two main parts. The goal of each part is to prove a key property of the logic: *upward persistency* (Proposition 3.25) and *infimum property* (Proposition 3.32). To prove each of these two properties we will need several lemmata first. Also during this section we will use the following notational shortcuts:

CONVENTION 3.21. Every formula mentioned in this section is assumed to contain only the variable p. For every such a formula φ, we denote by φ^s the value of φ under an evaluation e such that $e(p) = s \in \mathrm{T}$. We also write $\varphi =_s \psi$ meaning that $\varphi^s = \psi^s$.

Recall the convention from the beginning of this section that we use variables u, s, r, t for the nodes of $\mathbf{T}$.

LEMMA 3.22. *For every formula φ and every $s \leq_{\mathrm{T}} t$, we have the following: if $\varphi^s \in \mathrm{T} \cup \{\star\}$, then $\varphi^s = s$ and $\varphi^t = t$.*

PROOF. The proof proceeds by induction on the complexity of φ. The base step where $\varphi = p$ (or $\varphi = \overline{0}$) is obvious. Induction step:

- If $\varphi = B_u(\psi)$ and $\varphi^s \in \mathrm{T} \cup \{\star\}$, then obviously $\psi^s \in \mathrm{T}$ and, therefore, by induction assumption, $\psi^s = s$ and $\psi^t = t$. It is also clear that $u \leq_{\mathrm{T}} \psi^s \leq_{\mathrm{T}} \psi^t$. We can thus conclude that $\varphi^s = \psi^s = s$ and $\varphi^t = \psi^t = t$.
- The case of $\varphi = B(\psi)$ follows easily because for any $u \in \mathrm{T}$ we have $B(u) = u$.
- Assume that $\varphi = \varphi_1 \to \varphi_2$ and $\varphi^s \in \mathrm{T} \cup \{\star\}$. Since ${\varphi_1}^s = {\varphi_2}^s \in \mathrm{T} \cup \{\star\}$, we obtain the result simply from the induction assumption. ■

Notice that by this lemma for any formula φ and node s, the value of φ^s can only be $s, 0, 1$, or $\boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$.

LEMMA 3.23. *For every formula φ and every $s \leq_{\mathrm{T}} t$ it holds:*

1. *if $\varphi^s = 0$, then $\varphi^t = 0$,*
2. *if $\varphi^s = 1$, then $\varphi^t = 1$.*

PROOF. We prove both cases simultaneously using induction over the complexity of the formula φ. The base step is again obvious.

- If $\varphi = B_u(\psi)$, it is trivial (φ^s can be neither 0 nor 1).
- If $\varphi = B(\psi)$, it is trivial for the same reasons.
- If $\varphi = \varphi_1 \to \varphi_2$ and $\varphi^s = 0$, then we have the following possibilities:

1. ${\varphi_1}^s = 0$ and ${\varphi_2}^s = s$ (or the other way around),
2. ${\varphi_1}^s = 1$ and ${\varphi_2}^s = s$ (or the other way around),
3. ${\varphi_1}^s = 0$ and ${\varphi_2}^s = 1$ (or the other way around),

All these cases are easy to prove using the induction assumption and Lemma 3.22.

- If $\varphi = \varphi_1 \to \varphi_2$ and $\varphi^s = 1$, then both ${\varphi_1}^s$ and ${\varphi_2}^s$ are either 0 or 1. Thus the result is a simple consequence of induction assumption. ■

LEMMA 3.24. *For every formulas φ and ψ, for every $s \leq_{\mathrm{T}} t$ and for every $\boldsymbol{\chi} \in \mathbf{Fm^P}$, it holds: if $\varphi =_s \psi = \boldsymbol{\chi}$, then $\varphi =_t \psi$.*

PROOF. We prove it by induction over the complexity of $\mathbf{Fm^P}$. In the upcoming proof we will not deal with formulas of the form $B(\varphi')$ because the proof for these instances proceeds exactly in the same way as the proof for $B_s(\varphi')$.

1. (a) If $\varphi =_s \psi = \boldsymbol{B_u(p)}$, then, obviously, $\varphi = B_u(\varphi_1)$ and $\psi = B_u(\psi_1)$ such that $s = {\varphi_1}^s \not\geq_{\mathrm{T}} u$ and $s = {\psi_1}^s \not\geq_{\mathrm{T}} u$. We use Lemma 3.22 to derive: ${\varphi_1}^t = t = {\psi_1}^t$. The rest is straightforward.
   (b) If $\varphi =_s \psi = \boldsymbol{B_u(\overline{0})}$, then $\varphi = B_u(\varphi_1)$, $\psi = B_u(\psi_1)$ and ${\varphi_1}^s = {\psi_1}^s = 0$. The rest is an easy consequence of Lemma 3.23.
   (c) If $\varphi =_s \psi = \boldsymbol{B_u(\overline{1})}$, we do it similarly.
2. (a) If $\varphi =_s \psi = \boldsymbol{B_u(\chi)}$, then $\varphi = B_u(\varphi_1)$, $\psi = B_u(\psi_1)$ and ${\varphi_1}^s = {\psi_1}^s = \boldsymbol{\chi}$. The rest follows from the induction assumption.
   (b) If $\varphi =_s \psi = \boldsymbol{\overline{0}{\Rightarrow}\chi}$, then $\varphi = \varphi_1 \to \varphi_2$, $\psi = \psi_1 \to \psi_2$ and ${\varphi_1}^s = {\psi_1}^s = 0$ and ${\varphi_2}^s = {\psi_2}^s = \boldsymbol{\chi}$. The rest follows from the induction assumption and Lemma 3.23. (The same proof applies to $\boldsymbol{\chi{\Rightarrow}\overline{0}}$).
   (c) If $\varphi =_s \psi = \boldsymbol{\overline{1}{\Rightarrow}\chi}$, we do it similarly.
   (d) If $\varphi =_s \psi = \boldsymbol{p{\Rightarrow}\chi}$, then $\varphi = \varphi_1 \to \varphi_2$, $\psi = \psi_1 \to \psi_2$ and ${\varphi_1}^s = {\psi_1}^s = s$ and ${\varphi_2}^s = {\psi_2}^s = \boldsymbol{\chi}$. We apply Lemma 3.22 and the induction assumption.
   (e) If $\varphi =_s \psi = \boldsymbol{\chi_1{\Rightarrow}\chi_2}$, we do it similarly. ■

These lemmata allow us to obtain the first of the two ingredients we need for disproving the IPEP:

PROPOSITION 3.25. (Upward persistency). *For every formula φ and any evaluations $s \leq_{\mathrm{T}} t$, it holds: if $\varphi^s \in F$, then $\varphi^t \in F$.*

PROOF. Let us consider the three possible cases. First, if $\varphi^s = s$, it follows from Lemma 3.22. Second, if $\varphi^s = 1$, it follows by Lemma 3.23. Finally, if $\varphi^s = \boldsymbol{\chi{\Rightarrow}\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$, it follows that $\varphi = \varphi_1 \to \varphi_2$ and ${\varphi_1}^s = {\varphi_2}^s = \boldsymbol{\chi}$; then we just apply Lemma 3.24. ■

Next we focus on the *infimum property*. We start introducing several technical lemmata:

LEMMA 3.26. *For every formula φ and any $s \leq_{\mathrm{T}} t$, it holds: if $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$, then there is a subformula $B_u(\psi)$ of φ such that $s <_{\mathrm{T}} u \leq_{\mathrm{T}} t$.*

PROOF. Suppose we are given s and t satisfying the conditions of this lemma. We prove the conclusion by induction over the complexity of φ. The base step holds trivially. For the induction step we consider the following cases:

- $\varphi = B_u(\psi)$ and suppose $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$. Then either also ψ satisfies the conditions of this lemma and we are done by the induction assumption or $\psi^s = s$. In the second case, from Lemma 3.22, we get $\psi^t = t$. Finally, since $\varphi^s \neq \varphi^t = \boldsymbol{B_u(p)}$, we conclude that $s <_{\mathrm{T}} u \leq_{\mathrm{T}} t$.
- $\varphi = B(\psi)$ and suppose $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$. This case is very similar to the previous one; the only difference is that in this case the second possibility cannot happen.
- $\varphi = \varphi_1 \to \varphi_2$ and suppose $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ and $\varphi^s \neq \varphi^t$. There are many cases to discuss (based on the form of $\boldsymbol{\chi}$), however all of them are easy to check (using Lemmas 3.22 and 3.23 and the induction assumption). For example $\boldsymbol{\chi} = \overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{\chi'}$ for some $\boldsymbol{\chi'} \in \mathbf{Fm^P}$, meaning that ${\varphi_1}^s = 0$ and ${\varphi_2}^s = \boldsymbol{\chi'}$. By Lemma 3.23 ${\varphi_1}^s = {\varphi_1}^t$, therefore, since, $\varphi^s \neq \varphi^s$, we obtain ${\varphi_2}^s \neq {\varphi_2}^s$; the rest follows by the induction applied to φ_2. ■

LEMMA 3.27. *If $B_s(\psi)$ is a subformula of φ and $t \not\geq_{\mathrm{T}} s$, then $\varphi^t = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$.*

PROOF. Because of Lemma 3.22, we know that ψ^t has one of these values: $t, 0, 1$, or $\boldsymbol{\varphi}$ for some $\boldsymbol{\varphi} \in \mathbf{Fm^P}$. In all these cases we get $B_s(\psi)^t = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$. The rest is easy (cf. comments right below the definition of the logic L). ■

The next auxiliary lemma shows the relation between the presence of $\overline{0}$ in φ and certain values of φ^s.

LEMMA 3.28. *For every formula φ and any node $s \in \mathrm{T}$ it holds that $\overline{0}$ is a subformula of φ iff $\varphi^s \in \{0, 1\}$ or $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ such that $\overline{\mathbf{0}}$ is a subformula of $\boldsymbol{\chi}$.*

Proof. We prove it by induction over the complexity of φ. The base step is easy. Now let us write Prop0(φ) if $\varphi^s \in \{0,1\}$ or $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ such that $\overline{\mathbf{0}}$ is a subformula of $\boldsymbol{\chi}$.

- If $\varphi = B_u(\psi)$, it is easy: $\overline{0}$ is a subformula of φ iff it is a subformula of ψ iff (by the induction assumption) Prop0(ψ) iff Prop0(φ).
- If $\varphi = B(\psi)$, it works similarly.
- Assume that $\varphi = \varphi_1 \to \varphi_2$. Then: $\overline{0}$ is a subformula of φ iff $\overline{0}$ is a subformula of φ_1 or φ_2 iff Prop0(φ_1) or Prop0(φ_2) iff Prop0(φ). ∎

Lemma 3.29. *Let φ be a formula and take any $s_1, s_2 \in \mathrm{T}$. Suppose that $s = \inf\{s_1, s_2\}$.*[5] *Then:*

1. *if $\varphi^{s_1} = s_1$ and $\varphi^{s_2} = s_2$, then $\varphi^s = s$,*
2. *if $\varphi^{s_1} = \varphi^{s_2} = 0$ (resp. $\varphi^{s_1} = \varphi^{s_2} = 1$), then $\varphi^s = 0$ (resp. $\varphi^s = 1$),*
3. *any other combination of these values is not possible, i.e. the following cannot happen:*
   - *$\varphi^{s_1} = s_1$ and $\varphi^{s_2} \in \{0,1\}$,*
   - *$\varphi^{s_1} = 0$ and $\varphi^{s_2} = 1$.*

Proof. 1. By the way of contradiction suppose that $\varphi^{s_1} = s_1$, $\varphi^{s_2} = s_2$, and $\varphi^s \neq s$. Therefore, $\varphi^s \in \{0,1\}$ or $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$, but the first possibility is not true because of Lemma 3.23, thus $\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$. Now we use twice Lemma 3.26 to obtain two subformulas of φ: $B_{t_1}(\psi_1)$ and $B_{t_2}(\psi_2)$ such that $s <_{\mathrm{T}} t_1 \leq_{\mathrm{T}} s_1$ and $s <_{\mathrm{T}} t_2 \leq_{\mathrm{T}} s_2$. Since s is the infimum of s_1, s_2, we obtain that t_1 and t_2 are $\leq_{\mathrm{T}}$-incomparable. Thus we obtain a contradiction from Lemma 3.27.

2. It follows by an analogous argument.
3. First point: we argue using Lemma 3.28. If $\varphi^{s_2} \in \{0,1\}$ we obtain that $\overline{0}$ is a subformula of φ and thus $\varphi^{s_1} \neq s_1$.

   Second point: we prove it by induction over the complexity of φ. The base step is immediate.

   - If $\varphi = B_u(\psi)$ (or $= B(\psi)$) then we are done (φ^s can be neither 0 nor 1 for any s).
   - $\varphi = \varphi_1 \to \varphi_2$ and suppose $\varphi^{s_1} = 0$. Now there are two possibilities. First: ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = s_1$. From the induction assumption and from the previous point we get: ${\varphi_1}^{s_2} = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$ (or

[5] $\inf\{s_1, s_2\}$ is the infimum of s_1 and s_2 w.r.t. $\leq_{\mathrm{T}}$. Note that it always exists.

${\varphi_1}^{s_2} = 0$). The case of $\boldsymbol{\chi}$ is obvious. In the other one in order to obtain $\varphi^{s_2} = 1$ we would need ${\varphi_2}^{s_2} = 0$, but it is not possible, by the previous point (because ${\varphi_2}^{s_1} = s_1$). Second: ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = 1$. From the induction assumption (and from the previous point) we obtain: ${\varphi_1}^{s_2} = \boldsymbol{\chi}$ (or ${\varphi_1}^{s_2} = 0$) and ${\varphi_2}^{s_2} = \boldsymbol{\chi'}$ (or ${\varphi_2}^{s_2} = 1$). However, in none of these cases we get $\varphi^{s_2} = 1$. ■

Lemma 3.30. *For every nodes s, t and every formulas φ, ψ: if $\varphi^s = 0$ and $\varphi^t = \psi^t = 0$ (or $\varphi^t = \psi^t = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P}$), then $\psi^s \neq 1$.*

Proof. We prove it by induction over the complexity of formulas φ and ψ.

- $\varphi = p$: trivial.
- $\varphi = \overline{0}$: if we have $\varphi^t = \psi^t = 0$, then we use Lemma 3.29 to conclude that $\psi^s \neq 1$.
- $\varphi = B_s(\varphi')$ (or $\varphi = B(\varphi')$): trivial.
- $\varphi = \varphi_1 \to \varphi_2$: we discuss two cases. First: if $\varphi^t = 0 = \psi^t$ then, we can again easily use Lemma 3.29 to obtain the conclusion. Second, if $\varphi^t = \boldsymbol{\chi} = \psi^t$, we have that $\psi = \psi_1 \to \psi_2$. We must again deal with two possibilities:
    1. ${\varphi_1}^s = 0$ and ${\varphi_2}^s = s$. We argue that ${\psi_2}^s = s$ or ${\psi_2}^s = \boldsymbol{\chi'}$ for some $\boldsymbol{\chi'} \in \mathbf{Fm^P}$. Suppose not, i.e. ${\psi_2}^s = 0$ (or $= 1$), by Lemma 3.28 we can conclude that $\overline{0}$ is a subformula also of φ_2 and thus again, by the same lemma, ${\varphi_2}^s \neq s$, a contradiction. From this we infer that it cannot be the case that $\psi^s = 1$.
    2. ${\varphi_1}^s = 0$ and ${\varphi_2}^s = 1$. Suppose for contradiction that $\psi^s = 1$, i.e. ${\psi_1}^s = {\psi_2}^s = 0$ (or both are equal to 1). Since the preconditions of this lemma are satisfied for φ_2 and ψ_2 (because, by Lemma 3.29, neither $\psi_2^t = t$ nor $\psi_2^t = 1$), we obtain by the induction assumption that ${\varphi_2}^s \neq 1$, contradiction (the other case is similar). ■

Now we have everything set up to prove the most important lemma of this section:

Lemma 3.31. *For every formula φ, ψ and every s_1, s_2, it holds: if $\varphi =_{s_1} \psi$ and $\varphi =_{s_2} \psi$, then $\varphi =_s \psi$, where $s = \inf\{s_1, s_2\}$.*

Proof. We prove it by induction over the complexity of φ and ψ.

- $\varphi = p$. It must hold that $\psi^{s_1} = s_1$ and $\psi^{s_2} = s_2$ and, by Lemmas 3.29 and 3.22, we get $\psi^s = s$. Thus we have verified $\varphi =_s \psi$.
- $\varphi = \overline{0}$. It must hold that $\psi^{s_1} = 0$ and $\psi^{s_2} = 0$. Again, by Lemma 3.29, we get $\psi^s = 0$, and conclude $\varphi =_s \psi$.

- $\varphi = B_t(\varphi')$. Let us inspect what values φ can take. Note that for any s, φ^s can be neither 0 nor 1. Therefore, we have the following possibilities:[6]

Cases	Value of φ^{s_1}	Value of φ^{s_2}
(i)	s_1	s_2
(ii)	$\boldsymbol{\chi}$	s_2
(iii)	s_1	$\boldsymbol{\chi}$
(iv)	$\boldsymbol{\chi_1}$	$\boldsymbol{\chi_2}$

  (i) From Lemma 3.29, we get $\varphi^s = s$ and, since also $\psi^{s_1} = s_1$ and $\psi^{s_2} = s_2$, we can use again Lemma 3.29 and get $\psi^s = s$, i.e. $\varphi =_s \psi$.

  (ii) Since $\psi^{s_1} = \boldsymbol{\chi}$, we can infer that $\psi = B_t(\psi')$ for some ψ'. Now it is not difficult to show that $\varphi' =_{s_1} \psi'$ and $\varphi' =_{s_2} \psi'$. Therefore, by the induction assumption, we get $\varphi' =_s \psi'$, thus clearly also $\varphi =_s \psi$. (Note that in case of $t = \mathbf{r}$ it can also happen $\psi = B(\psi')$, which however behaves in a similar way).

  (iii) Identical.

  (iv) Similar to (ii).

- $\varphi = B(\varphi')$: almost the same.
- $\varphi = \varphi_1 \to \varphi_2$. Here we need to discuss even more cases (realize that we have already rejected a few possibilities in Lemma 3.29):[7]

Cases	Value of φ^{s_1}	Value of φ^{s_2}
(i)	s_1	s_2
(ii)	s_1	$\boldsymbol{\chi}$
(iii)	0	0
(iv)	0	$\boldsymbol{\chi}$
(v)	1	1
(vi)	1	$\boldsymbol{\chi}$
(vii)	$\boldsymbol{\chi_1}$	$\boldsymbol{\chi_2}$

  (i) Again, easily using Lemma 3.29, we obtain $\varphi^s = s$ and $\psi^s = s$; in other words, $\varphi =_s \psi$.

  (ii) Since $\varphi^{s_2} = \boldsymbol{\chi}$ and $\varphi = \varphi_1 \to \varphi_2$, we infer that also $\psi = \psi_1 \to \psi_2$. Obviously, $\varphi_1 =_{s_1} \psi_1 = s_1$ and $\varphi_2 =_{s_1} \psi_2 = s_1$. We can also derive $\varphi_1 =_{s_2} \psi_1$ and $\varphi_2 =_{s_2} \psi_2$: we need to distinguish cases based on the

[6]Note that, thanks to Lemma 3.22, we know that s is the only element in T that can be the value of φ^s.

[7]This time we do not mention symmetric cases.

formula $\boldsymbol{\chi}$, whether it is: $\boldsymbol{\chi_1 \Rightarrow \chi_2}$, $\boldsymbol{\overline{0} \Rightarrow \chi_1}$, $\boldsymbol{\overline{1} \Rightarrow \chi_1}$, or $\boldsymbol{p \Rightarrow \chi_1}$ (or some of its symmetric variants). All these cases are easy to check. Now we can apply the induction assumption and obtain $\varphi_1 =_s \psi_1$ and $\varphi_2 =_s \psi_2$ and, hence, conclude $\varphi =_s \psi$.

(iii) As in (i), it follows easily from Lemma 3.29.

(iv) First, since $\varphi^{s_2} = \boldsymbol{\chi}$, we infer $\psi = \psi_1 \to \psi_2$. Now we need to cover two cases based on φ^{s_1}.

1. ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = s_1$: we argue that also ${\psi_1}^{s_1} = 0$ and ${\psi_2}^{s_1} = s_1$. First, if ${\psi_2}^{s_1} = 0$ (or $= 1$), then, by Lemma 3.28, we would get that $\overline{0}$ is a subformula of ψ_2 and, since ${\psi_2}^{s_2} = {\varphi_2}^{s_2}$, we would obtain by the same lemma that $\overline{0}$ is also a subformula of φ_2. Therefore again, by Lemma 3.28, we know ${\psi_2}^{s_1} = s_1$. By Lemma 3.30, we get that ${\psi_1}^{s_1} \neq 1$ thus it must be the case that ${\psi_1}^{s_1} = 0$. Thus we can conclude $\varphi_1 =_{s_1} \psi_1$ and $\varphi_2 =_{s_1} \psi_2$. It is easy to derive that $\varphi_1 =_{s_2} \psi_1$ and $\varphi_2 =_{s_2} \psi_2$. The rest is an easy consequence of the induction assumption.
2. ${\varphi_1}^{s_1} = 0$ and ${\varphi_2}^{s_1} = 1$. If we show that also ${\psi_1}^{s_1} = 0$ and ${\psi_2}^{s_1} = 1$, we are done simply by using the induction assumption. For contradiction suppose it is not the case, i.e. ${\psi_1}^{s_1} = 1$ and ${\psi_2}^{s_1} = 0$ (note that using Lemma 3.28, as in the previous point, we have ${\psi_1}^{s_1} \neq s_1$ and ${\psi_2}^{s_1} \neq s_1$). We get a contradiction from Lemma 3.30 applied on ψ_1 and φ_1.

(v) It is similar to (i) and (iii).

(vi) Again we first argue that also $\psi = \psi_1 \to \psi_2$ (because $\psi^{s_2} = \boldsymbol{\chi}$). Suppose ${\varphi_1}^{s_1} = {\varphi_2}^{s_1} = 0$. Again it is enough to argue that also ${\psi_1}^{s_1} = {\psi_2}^{s_1} = 0$. This is easy to prove, just realize that the only other possibility would be ${\psi_1}^{s_1} = {\psi_2}^{s_1} = 1$, which is by Lemma 3.30 not possible.

(vii) This case is a straightforward application of the induction assumption. ■

Finally we are ready to obtain the second key component, the *infimum property*:

PROPOSITION 3.32. (Infimum property) *For every formula φ and every s_1, s_2, it holds: if $\varphi^{s_1} \in F$ and $\varphi^{s_2} \in F$, then also $\varphi^s \in F$, where $s = \inf\{s_1, s_2\}$.*

PROOF. For a contradiction suppose that $\varphi^s \notin F$ and both $\varphi^{s_1} \in F$ and $\varphi^{s_2} \in F$. First we use Lemma 3.23 to argue that $\varphi^s \neq 0$ which implies

$\varphi^s = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^P} \setminus \mathrm{D}$. Since $\varphi^s \neq \varphi^{s_1}$ and $\varphi^s \neq \varphi^{s_2}$, we can use Lemma 3.26 to infer that there are nodes t_1 and t_2 such that $s <_{\mathrm{T}} t_1 \leq_{\mathrm{T}} s_1$ and $s <_{\mathrm{T}} t_2 \leq_{\mathrm{T}} s_2$ and subformulas $B_{t_1}(\psi_1)$ and $B_{t_2}(\psi_2)$ of the formula φ. Then, since $s = \inf\{s_1, s_2\}$, we obtain that $t_1 \not\leq_{\mathrm{T}} s_2$ and $t_2 \not\leq_{\mathrm{T}} s_1$, therefore by Lemma 3.27 it follows $\varphi^{s_1} = \boldsymbol{\chi_1} \in \mathrm{D}$ and $\varphi^{s_2} = \boldsymbol{\chi_2} \in \mathrm{D}$. Thus it follows that $\boldsymbol{\chi_1} = \boldsymbol{\chi_1}'\Rightarrow\boldsymbol{\chi_1}'$ and $\boldsymbol{\chi_2} = \boldsymbol{\chi_2}'\Rightarrow\boldsymbol{\chi_2}'$ for some $\boldsymbol{\chi_1}', \boldsymbol{\chi_2}' \in \mathbf{Fm^P}$. We can now easily conclude that $\varphi = \varphi_1 \to \varphi_2$ and $\varphi_1 =_{s_1} \varphi_2 = \boldsymbol{\chi_1}'$ and $\varphi_1 =_{s_2} \varphi_2 = \boldsymbol{\chi_2}'$ thus by Lemma 3.31 also $\varphi_1 =_s \varphi_2$, contradiction (Observation 3.19). ■

Now to disprove the IPEP we need to describe a suitable set of formulas Γ_0 and a formula φ such that $\Gamma_0 \nvdash_{\mathrm{L}} \varphi$ and for every theory $T \supseteq \Gamma_0$: if $T \nvdash_{\mathrm{L}} \varphi$, then T is not intersection-prime, i.e. there are two theories T_1 and T_2 strictly containing T such that $T = T_1 \cap T_2$.

Definition 3.33. (Γ_0) Let us enumerate all propositional variables. We then define $\Gamma_0 = \{p_i \to p_j \mid i, j \in \mathbb{N}\} \cup \{B_{\mathbf{r}}(p_1)\}$.[8]

We extend the notation from Convention 3.21 to sets of formulas: we write T^s for the set $\{\varphi^s \mid \varphi \in T\}$.

Theorem 3.34. *The logic* L *does not satisfy the* IPEP*.*

Proof. We will denote as e_a the evaluation that sends every variable to a fixed element $a \in A$. First observe that $\Gamma_0 \nvdash_{\mathrm{L}} \overline{0}$, which can be stated as follows: there is an evaluation e which satisfies Γ_0, i.e. $e[\Gamma_0] \subseteq F$. Moreover all the evaluations that satisfy Γ_0 are exactly of the form e_s for some $s \in \mathrm{T}$ (evaluations satisfying Γ_0 can be identified with nodes of T). This allows us to treat formulas as being written in one variable, call it p, and use the notation φ^s instead of $e_s(\varphi)$: because we have for any $T \supseteq \Gamma_0$

$$e_s[T] \subseteq F \text{ if and only if } \sigma[T]^s \subseteq F,$$

where σ is the substitution sending every variable to p.

Next let us have a theory T containing Γ_0 such that $T \nvdash_{\mathrm{L}} \overline{0}$. Denote the set of all nodes satisfying T as $\mathrm{Sat}(T) = \{s \in \mathrm{T} \mid T^s \subseteq F\}$. Note that $\mathrm{Sat}(T)$ is nonempty, $\mathrm{Sat}(T) \subseteq \mathrm{T}$, and it contains all the evaluations satisfying T.

We first show that there is a $\leq_{\mathrm{T}}$-least element in $\mathrm{Sat}(T)$: pick any $s \in \mathrm{Sat}(T)$ and consider a set $\downarrow s = \{t \leq_{\mathrm{T}} s \mid t \in \mathrm{Sat}(T)\}$ and let s_0 be the $\leq_{\mathrm{T}}$-least element in $\downarrow s$ (such an element always exists, because T is a tree). We show that s_0 is the $\leq_{\mathrm{T}}$-least element in $\mathrm{Sat}(T)$. Suppose it is not, then there is $s_1 \in \mathrm{Sat}(T)$ such that $s_0 \not\leq_{\mathrm{T}} s_1$. Let t be the infimum of $\{s_0, s_1\}$

[8] Recall that $\mathbf{r} \in \mathrm{T}$ is the root of T.

(obviously, $t <_{\mathrm{T}} s_0$). To arrive at contradiction it remains to show that $t \in \mathrm{Sat}(T)$, which is, however, an easy consequence of the infimum property (Proposition 3.32): since for every $\varphi \in T$ we have $\varphi^{s_0} \in F$ and $\varphi^{s_1} \in F$ thus also $\varphi^t \in F$.

Since s_0 is the least element in $\mathrm{Sat}(T)$ we obtain:

$$T \vdash_{\mathrm{L}} \varphi \text{ if and only if } \varphi^{s_0} \in F \tag{1}$$

The direction from left to right is obvious. For the other let $\varphi^{s_0} \in F$. Then, since for every $s \in \mathrm{Sat}(T)$ we have $s_0 \leq_{\mathrm{T}} s$, but $\varphi^s \in F$ (by the upward persistency, Proposition 3.25). Therefore every evaluation which satisfies T also satisfies φ.

Now let s_1, s_2 be the two distinct immediate successors of s_0. Obviously, $T \nvdash_{\mathrm{L}} B_{s_1}(p)$ and $T \nvdash_{\mathrm{L}} B_{s_2}(p)$ (this fact is witnessed by the evaluation e_{s_0}). Therefore, both $T_1 = \mathrm{Th}_{\mathrm{L}}(T \cup \{B_{s_1}(p_1)\})$ and $T_2 = \mathrm{Th}_{\mathrm{L}}(T \cup \{B_{s_2}(p_1)\})$ strictly contain T. Finally we prove that for every formula φ:

$$\text{if } T_1 \vdash_{\mathrm{L}} \varphi \text{ and } T_2 \vdash_{\mathrm{L}} \varphi, \text{ then } T \vdash_{\mathrm{L}} \varphi \tag{2}$$

Suppose $T_1 \vdash_{\mathrm{L}} \varphi$ and $T_2 \vdash_{\mathrm{L}} \varphi$. By (1), we need to show $\varphi^{s_0} \in F$. It is easy to see that $T_1^{s_1} \subseteq F$, and consequently $\varphi^{s_1} \in F$, similarly $\varphi^{s_2} \in F$. Thus the desired result is a consequence of the infimum property.

In particular, the fact (2) tells us that the theory T is not intersection-prime ($T = T_1 \cap T_2$), which is exactly what we wanted. ■

3.3.3. Proof of RSI-Completeness. Finally we prove that L is RSI-complete. From Corollary 3.20 we know that $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is reduced. Moreover, by definition, $\mathbf{A}$ is a complete semantics for L. Therefore to prove RSI-completeness, it is enough to show that $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$. Let us now prove that F is completely intersection-prime in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. To obtain this result we recursively define for every $\boldsymbol{\chi} \in \mathbf{Fm}^{\mathbf{P}}$ a corresponding formula $\varphi_{\boldsymbol{\chi}}$ written in a fixed variable p:

Definition 3.35. *(Formulas $\varphi_{\boldsymbol{\chi}}$)* We define formulas $\varphi_{\boldsymbol{\chi}}$ recursively as follows:

1. (a) $\varphi_{\boldsymbol{B_s(p)}} = B_s(B(p))$,
   (b) $\varphi_{\boldsymbol{B_s(\overline{0})}} = B_s(0 \to B(p))$,
   (c) $\varphi_{\boldsymbol{B_s(\overline{1})}} = B_s(\overline{0} \to (\overline{0} \to B(p)))$,
2. (a) $\varphi_{\boldsymbol{B_s(\psi)}} = B_s(\varphi_{\boldsymbol{\psi}})$,
   (b) $\varphi_{\boldsymbol{\overline{0} \Rightarrow \psi}} = \overline{0} \to \varphi_{\boldsymbol{\psi}}$ and $\varphi_{\boldsymbol{\psi \Rightarrow \overline{0}}} = \varphi_{\boldsymbol{\psi}} \to \overline{0}$,
   (c) $\varphi_{\boldsymbol{\overline{1} \Rightarrow \psi}} = \overline{1} \to \varphi_{\boldsymbol{\psi}}$ and $\varphi_{\boldsymbol{\psi \Rightarrow \overline{1}}} = \varphi_{\boldsymbol{\psi}} \to \overline{1}$,

(d) $\varphi_{\boldsymbol{p}\Rightarrow\boldsymbol{\psi}} = p \to \varphi_{\boldsymbol{\psi}}$ and $\varphi_{\boldsymbol{\psi}\Rightarrow\boldsymbol{p}} = \varphi_{\boldsymbol{\psi}} \to p$,
(e) $\varphi_{\boldsymbol{\psi}\Rightarrow\boldsymbol{\psi'}} = \varphi_{\boldsymbol{\psi}} \to \varphi_{\boldsymbol{\psi'}}$.

Using an easy induction over the complexity of the set $\mathbf{Fm}^{\mathbf{p}}$ we obtain:

Observation 3.36. For every $\boldsymbol{\chi} \in \mathbf{Fm}^{\mathbf{p}}$ and every evaluation e such that $e(p) = \star$, it holds that $e(\varphi_{\boldsymbol{\chi}}) = \boldsymbol{\chi}$.

Note that in the previous observation we benefited from the connective B (namely from the fact that $B(\star) = \star$). In order to obtain the RSI-completeness, we need the following proposition:

Proposition 3.37. *For every* $\boldsymbol{\chi} \in \mathbf{Fm}^{\mathbf{p}} \setminus \mathrm{D}$*:* $\varphi_{\boldsymbol{\chi}} \vdash_{\mathrm{L}} B_{\mathbf{r}}(p)$*.*

We postpone its proof till later and proceed immediately to the main theorem of this section.

Theorem 3.38. *The logic* L *is* RSI-*complete.*

Proof. We show that F is completely intersection-prime in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$. Let us consider a non-trivial $F' \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ which strictly contains F. First observe that $0 \notin F'$ (because $\overline{0} \vdash_{\mathrm{L}} p$, which would imply that F' is trivial). It follows that there is some $\boldsymbol{\chi} \in \mathbf{Fm}^{\mathbf{p}} \setminus \mathrm{D}$ which is also in F'. However, from the previous proposition, we know that $\varphi_{\boldsymbol{\chi}} \vdash_{\mathrm{L}} B_{\mathbf{r}}(p)$; thus, if we consider an evaluation e such that $e(p) = \star$, we obtain, by Observation 3.36, that $e(\varphi_{\boldsymbol{\chi}}) = \boldsymbol{\chi} \in F'$, which implies that also $e(B_{\mathbf{r}}(p)) = \boldsymbol{B_{\mathbf{r}}(p)} \in F'$. Therefore any non-trivial filter strictly above F contains $\boldsymbol{B_{\mathbf{r}}(p)}$. In other words: F is completely intersection-prime, which completes the proof. ■

Now it remains to prove Proposition 3.37, but first we need some technical lemmata. In upcoming proofs we will tacitly use the next statement (easily provable by induction):

Observation 3.39. Let us have an evaluation e such that $e(p) \notin \mathrm{T}$. Then, for every $\boldsymbol{\chi} \in \mathbf{Fm}^{\mathbf{p}}$, it holds that $e(\varphi_{\boldsymbol{\chi}}) \in \mathbf{Fm}^{\mathbf{p}}$.

Further we define a subformula order on $\mathbf{Fm}^{\mathbf{p}}$: we write $\boldsymbol{\chi} \leq_{\mathbf{Fm}^{\mathbf{p}}} \boldsymbol{\chi'}$ iff $\boldsymbol{\chi}$ is a subformula of $\boldsymbol{\chi'}$. It takes again an easy induction argument to prove that for every $\boldsymbol{\chi}, \boldsymbol{\chi'} \in \mathbf{Fm}^{\mathbf{p}}$ and any evaluation e sending p to $\boldsymbol{\chi}$ that

$$\boldsymbol{\chi} <_{\mathbf{Fm}^{\mathbf{p}}} e(\varphi_{\boldsymbol{\chi'}}) \text{ and thus } e(\varphi_{\boldsymbol{\chi'}}) \neq \boldsymbol{\chi} \tag{3}$$

Lemma 3.40. *For every* $\boldsymbol{\chi} \in \mathbf{Fm}^{\mathbf{p}}$ *and for every evaluation* e *we have:*

1. If $e(p) = 0$, then $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B_{\mathbf{r}}(\overline{0})}$, $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{\overline{0}\Rightarrow B_{\mathbf{r}}(\overline{0})}$, and $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{\overline{0}\Rightarrow(\overline{0}\Rightarrow B_{\mathbf{r}}(\overline{0}))}$.

2. If $e(p) = 1$, then $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{1}})$, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{1}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{1}})$, and $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{1}}\boldsymbol{\Rightarrow}(\overline{\mathbf{1}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{1}}))$.
3. If $e(p) = \boldsymbol{\chi}'$ for some $\boldsymbol{\chi}' \in \mathbf{Fm^p}$, then $e(\varphi_{\boldsymbol{\chi}'}) \neq \boldsymbol{B}_{\mathbf{r}}(\boldsymbol{\chi}')$, $e(\varphi_{\boldsymbol{\chi}'}) \neq \overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\boldsymbol{\chi}')$ and $e(\varphi_{\boldsymbol{\chi}'}) \neq \overline{\mathbf{0}}\boldsymbol{\Rightarrow}(\overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\boldsymbol{\chi}'))$.

PROOF. We prove only the first point; the others are even simpler. Case of $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}})$: obvious. Second, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}})$: this possibly happens only if (1) $\boldsymbol{\chi} = \boldsymbol{p}\boldsymbol{\Rightarrow}\boldsymbol{\chi}'$ (in this case $\varphi_{\boldsymbol{\chi}} = p \to \varphi_{\boldsymbol{\chi}'}$), then $e(\varphi_{\boldsymbol{\chi}}) = \overline{\mathbf{0}}\boldsymbol{\Rightarrow}e(\varphi_{\boldsymbol{\chi}'})$, the rest easily follows from the first inequation; the second possibility $\boldsymbol{\chi} = \overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{\chi}'$ is proven analogously. Finally, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}}\boldsymbol{\Rightarrow}(\overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}}))$: similar but using the second inequation. ■

LEMMA 3.41. *For every $\boldsymbol{\chi_1}, \boldsymbol{\chi_2} \in \mathbf{Fm^p}$ and for any evaluation e such that $e(p) = 0$ or $e(p) = 1$ or $e(p) = \boldsymbol{\chi}$ for some $\boldsymbol{\chi} \in \mathbf{Fm^p}$, we have: $e(\varphi_{\boldsymbol{\chi_1}}) = e(\varphi_{\boldsymbol{\chi_2}})$ iff $\boldsymbol{\chi_1} = \boldsymbol{\chi_2}$.*

PROOF. We prove this lemma only for evaluations e such that $e(p) = 0$, the other cases follow almost in the same way (they are only using different points from the previous lemma). This lemma is then proved by induction over the complexity of $\boldsymbol{\chi_1}, \boldsymbol{\chi_2}$ according to the definition of $\mathbf{Fm^p}$:

1. (a) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\boldsymbol{p})$: we get $\varphi_{\boldsymbol{\chi_1}} = B_s(B(p))$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}}))$. Now it is easy to see that the conclusion follows for base step for $\boldsymbol{\chi_2}$ (i.e. points 1.(a),(b),(c)). Moreover, for the induction step there is only one more complicated variant, namely 2.(a) (for the rest obviously $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$): suppose $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi})$ for some $\boldsymbol{\chi} \in \mathbf{Fm^p}$, but, by Lemma 3.40, $e(\varphi_{\boldsymbol{\chi}}) \neq \boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}})$, i.e. $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.
   (b) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\overline{\mathbf{0}})$: $\varphi_{\boldsymbol{\chi_1}} = B_s(\overline{0} \to B(p))$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}}))$; we again deal only with the case 2.(a). So let $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi})$ for some $\boldsymbol{\chi} \in \mathbf{Fm^p}$. However, again by Lemma 3.40, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}})$ and therefore $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.
   (b) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\overline{\mathbf{1}})$: then $\varphi_{\boldsymbol{\chi_1}} = B_s(\overline{0} \to (\overline{0} \to B(p)))$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\overline{\mathbf{0}}\boldsymbol{\Rightarrow}(\overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}})))$. Again 2.(a): $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi})$ for some $\boldsymbol{\chi} \in \mathbf{Fm^p}$. However, by Lemma 3.40, $e(\varphi_{\boldsymbol{\chi}}) \neq \overline{\mathbf{0}}\boldsymbol{\Rightarrow}(\overline{\mathbf{0}}\boldsymbol{\Rightarrow}\boldsymbol{B}_{\mathbf{r}}(\overline{\mathbf{0}}))$.
2. (a) $\boldsymbol{\chi_1} = \boldsymbol{B_s}(\boldsymbol{\chi_1}')$: $\varphi_{\boldsymbol{\chi_1}} = B_s(\varphi_{\boldsymbol{\chi_1}'})$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \boldsymbol{B_s}(\boldsymbol{e}(\boldsymbol{\varphi}_{\boldsymbol{\chi_1}'}))$ the base step for $\boldsymbol{\chi_2}$ follows by the first part of this proof. Case 2.(a): $\boldsymbol{\chi_2} = \boldsymbol{B_s}(\boldsymbol{\chi_2}')$ and $e(\varphi_{\boldsymbol{\chi_2}}) = \boldsymbol{B_s}(\boldsymbol{e}(\boldsymbol{\varphi}_{\boldsymbol{\chi_2}'}))$. We obtain the result easily from the induction assumption. For the other cases we trivially get $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.

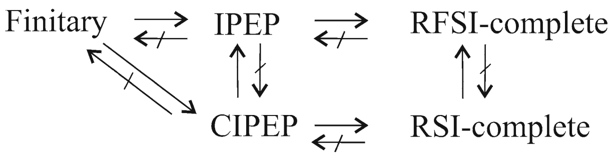

Figure 3. Separation of the classes in the hierarchy

(b) $\boldsymbol{\chi_1} = \overline{0}\Rightarrow\boldsymbol{\chi_1}'$: $\varphi_{\boldsymbol{\chi_1}} = \overline{0} \to \varphi_{\boldsymbol{\chi_1}'}$ and $e(\varphi_{\boldsymbol{\chi_1}}) = \overline{\mathbf{0}}\Rightarrow e(\varphi_{\boldsymbol{\chi_1}'})$. The base step for $\boldsymbol{\chi_2}$ is trivial. Moreover the only interesting induction cases for $\boldsymbol{\chi_2}$ are 2.(b) and 2.(d)-which are treated in the same way: we obtain $\boldsymbol{\chi_2} = \overline{\mathbf{0}}\Rightarrow\boldsymbol{\chi_2}'$ and $e(\varphi_{\boldsymbol{\chi_2}}) = \overline{\mathbf{0}}\Rightarrow e(\varphi_{\boldsymbol{\chi_2}'})$. The rest easily follows by induction assumption. In the remaining cases we can simply observe $e(\varphi_{\boldsymbol{\chi_1}}) \neq e(\varphi_{\boldsymbol{\chi_2}})$.

(c),(d),(e) Similar. ■

Now we have all the ingredients to prove the key proposition:

Proof of Proposition 3.37. It is enough to show that for any evaluation e such that $e(p) \in \mathbf{Fm^P} \cup \{0, 1, \star\}$ we have $e(\varphi_{\boldsymbol{\chi}}) \notin F$. If $e(p) = \star$ we argue using Observation 3.36. For other evaluations we distinguish two possible scenarios in which we could get $e(\varphi_{\boldsymbol{\chi}}) \in F$ (note that by Observation 3.39 we would get $e(\varphi_{\boldsymbol{\chi}}) \in \mathrm{D}$, therefore the only possible cases are 2.(d) and (e) of Definition 3.35). First, if $\boldsymbol{\chi} = \boldsymbol{p}\Rightarrow\boldsymbol{\chi}'$, we get $e(\varphi_{\boldsymbol{\chi}}) = e(p) \to^{\mathbf{A}} e(\varphi_{\boldsymbol{\chi}'})$. If $e(p) = 0$ or $e(p) = 1$, then obviously $e(\varphi_{\boldsymbol{\chi}}) \notin F$ and, if $e(p) = \boldsymbol{\chi}''$, we conclude $e(\varphi_{\boldsymbol{\chi}}) \notin F$ by (3). Second, assume that $\boldsymbol{\chi} = \boldsymbol{\chi_1}\Rightarrow\boldsymbol{\chi_2}$. Since $\boldsymbol{\chi_1} \neq \boldsymbol{\chi_2}$, we can use Lemma 3.41 to obtain $e(\varphi_{\boldsymbol{\chi}}) \notin F$. ■

Hence, we have successfully proven that L is an RSI-complete logic (Theorem 3.38) without the IPEP (Theorem 3.34), thus finally showing the separation of all classes of logics studied in the paper. Therefore, we have obtained a new hierarchy of infinitary logics depicted in Figure 3.

4. Conclusion and Remarks

We have discussed four properties (IPEP, CIPEP, RFSI-completeness, and RSI-completeness) that naturally arise as generalizations of finitarity in the study of propositional logical systems, especially those with disjunction or semilinear implication (fuzzy logics). As the main result of the paper, we have proved that the corresponding four classes of logics are pairwise different.

However some questions about these notions still remain open. The most important one concerns the existence of natural separating examples. In particular, is there a natural RFSI-complete logic without the IPEP, besides the rather complicated system that we have constructed *ad hoc* in this paper? Actually, we lack natural examples of logics that do not satisfy the IPEP.

On the other hand, a measure of the good behavior of a logical system, from the point of view of abstract algebraic logic, is given by its position in the Leibniz hierarchy. From this perspective, the presented examples feature quite well. The first two are at the very top of the hierarchy (they are Rasiowa-implicative) and the last one is a weakly implicative logic (hence finitely equivalential), though we know it is not even order-algebraizable (in the sense of [17]). Therefore, another interesting open problem is whether the four classes can be separated using only algebraizable logics. Also, notice that the two *ad hoc* examples that we have built are formulated in languages with infinitely-many connectives. This observation yields the question whether there are actually logics with similar properties expressed in finite signatures.

Another question, typically studied in abstract algebraic logic works, regards the transferability of syntactical properties to the general matrix semantics, that is, whether a property defined for theories can be also proved to hold for filters over any algebra of the language of the logic. In our context, the corresponding open problems would be to determine whether the IPEP and the CIPEP can be transferred in this sense.

Furthermore, it would be very useful to have syntactical characterizations for the studied notions. For example, finitary logics are precisely those that have some axiomatic system which has only rules with finitely-many premises; can we similarly describe IPEP (resp. CIPEP) placing some restriction on the infinitary rules of their corresponding axiomatic systems? To answer these questions it might be helpful to obtain first axiomatic presentations of the constructed counterexamples.

Finally, a related issue is to determine whether the studied properties are preserved in expansions.

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