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On the Aubin property of a class of parameterized variational systems

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Abstract The paper deals with a new sharp condition ensuring the Aubin property of solution maps to a class of parameterized variational systems. This class encompasses various types of parameterized variational inequalities/generalized equations with fairly general constraint sets. The new condition requires computation of directional limiting coderivatives of the normal-cone mapping for the so-called critical directions. The respective formulas have the form of a second-order chain rule and extend the available calculus of directional limiting objects. The suggested procedure is illustrated by means of examples.

Keywords Solution map · Aubin property · Graphical derivative · Directional limiting coderivative

Mathematics Subject Classification 49J53 · 90C31 · 90C46

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1 Introduction

In Gfrerer and Outrata (2016), the authors have developed a new sufficient condition ensuring the Aubin property of solution maps to general implicitly defined set-valued maps. This property itself has been introduced in Aubin (1984) and became gradually one of the most important stability notions for multifunctions. It is widely used in *post-optimal analysis*, as a useful *qualification condition* in generalized differentiation and it is closely connected with several important classical results like, e.g., the theorems of Lyusternik and Graves (Dontchev and Rockafellar 2014, pp. 275–276).

To verify the Aubin property of practically important mappings, various primal and dual methods have been developed, cf. e.g. Rockafellar and Wets (1998, Chapter 9), Mordukhovich (2006, Chapter 4), Dontchev and Rockafellar (2014, Chapters 3, 4), Dontchev et al. (2006) and the references therein. A particularly efficient tool is the so-called Mordukhovich criterion which characterizes the Aubin property of a set-valued mapping via the local boundedness of the respective limiting coderivative. These conditions work typically well, e.g., in case of parameterized constraint or variational systems whenever one has to do with *ample* parameterizations (Dontchev and Rockafellar 2001, Definition 1.1). This is notably the case of a canonically perturbed Karush–Kuhn–Tucker (KKT) system which has been thoroughly investigated in Bonnans and Sulem (1995) and Dontchev and Rockafellar (1996). The parameterizations arising in post-optimal analysis or in problems with equilibrium constraints are, however, typically non-ample and then the standard characterizations for the Aubin property of the respective solution maps became only sufficient conditions which may be very far from necessity. This drawback was the main motivation for the development in Gfrerer and Outrata (2016) where, among other things, substantially weaker yet sufficient conditions have been derived for the Aubin property of implicitly defined set-valued maps.

The aim of this paper is to work out a weak (non-restrictive) sufficient condition from Gfrerer and Outrata (2016) to obtain a workable tool for ensuring the Aubin property of solution maps to a broad class of parameterized variational systems. This class includes, in particular, multiplier-free optimality conditions for optimization problems with parameter-dependent objectives or stationarity conditions of a Nash game with parameters entering the objectives of the single players. Further, our condition is applicable to KKT systems related to nonlinear programs, where the parameters arise both in the objective as well as in the constraints. An efficient usage of the new condition requires our ability to compute graphical derivatives and directional limiting coderivatives of normal-cone mappings to the considered constraint sets. Unfortunately, the calculus of directional limiting objects is not yet sufficiently developed and also in computation of graphical derivatives of normal-cone mappings one often meets various too restrictive assumptions. In this paper we will compute graphical derivatives and directional limiting coderivatives of normal cone mappings associated with the sets Γ of the form

$$\Gamma = g^{-1}(D) \tag{1}$$

under reasonable assumptions imposed on the mapping g and the set D. To this aim we will significantly improve the results from Mordukhovich et al. (2015a, b) concerning the graphical derivative and from Mordukhovich et al. (2015a, Theorem 4.1) concerning the regular coderivative of the normal-cone mapping associated with (1). The resulting new second-order chain rules are valid under substantially relaxed reducibility and nondegeneracy assumptions compared with the preceding results of this type and are thus important for their own sake, not only in the context of this paper. Concretely, the new formula for the graphical derivative could be used, e.g., in testing the so-called isolated calmness of solution maps to variational systems (Henrion et al. 2013; Mordukhovich et al. 2015a, b).

The main result (Theorem 5) represents a variant of Gfrerer and Outrata (2016, Theorem 4.4) tailored to the mentioned broad class of parameterized variational systems. As documented by examples, it substantially improves the efficiency of the currently available sufficient conditions for the Aubin property in the case when the considered parametrization is not *ample*.

The plan of the paper is as follows. In Sect. 2 we summarize the needed notions from variational analysis, state the main problem and recall Gfrerer and Outrata (2016, Theorem 4.4) which is the basis for our development. Section 3 is devoted to the new results concerning the mentioned graphical derivatives and directional limiting coderivatives of the normal-cone mapping related to Γ. In Sect. 4 we formulate the resulting new sufficient condition for the Aubin property of the considered solution maps and illustrate its application by means of an example, where Γ is given by nonlinear programming (NLP) constraints. Section 5 contains some amendments which may be useful for genuine conic constraints. In particular, we consider the case when D amounts to the Carthesian product of Lorentz cones.

Our notation is standard. For a set A, $\operatorname{lin} A$ denotes the *lineality space* of A, i.e., the largest linear space contained in A, $\operatorname{sp} A$ is the linear hull of A and $P_A(\cdot)$ stands for the mapping of metric projection onto A. For a set-valued map F, $\operatorname{gph} F$ denotes its graph and $\operatorname{rge} F$ denotes its range, i.e., $\operatorname{rge} F := \{y | y \in F(x) \text{ for } x \in \operatorname{dom} F\}$. For a cone K, K° is the (negative) polar cone, $\mathbb{B}$, $\mathbb{S}$ are the unit ball and the unit sphere, respectively, and for a vector a, $[a]$ stands for the linear subspace generated by a. Given a vector-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, differentiable at $\bar{x}$, the Jacobian of f at $\bar{x}$, denoted by $\nabla f(\bar{x})$, amounts to the $m \times n$ matrix, whose rows are the gradients of the components $f_i, i = 1, 2, \ldots, m$. $f'(\bar{x}; h)$ stands for the directional derivative of f at $\bar{x}$ in direction h. Finally, $\xrightarrow{A}$ means the convergence within a set A.

2 Problem formulation and preliminaries

In the first part of this section we introduce some notions from variational analysis which will be extensively used throughout the whole paper. Consider first a general closed-graph set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^z$ and its inverse $F^{-1} : \mathbb{R}^z \rightrightarrows \mathbb{R}^n$ and assume that $(\bar{u}, \bar{v}) \in \operatorname{gph} F$.

Definition 1 We say that F has the *Aubin property* around $(\bar{u}, \bar{v})$, provided there are neighborhoods U of $\bar{u}$, V of $\bar{v}$ and a constant $\kappa > 0$ such that

$$F(u_1) \cap V \subset F(u_2) + \kappa \|u_1 - u_2\| \mathbb{B} \quad \text{for all } u_1, u_2 \in U.$$

F is said to be *calm* at $(\bar{u}, \bar{v})$, provided there is a neighborhood V of $\bar{v}$ and a constant $\kappa > 0$ such that

$$F(u) \cap V \subset F(\bar{u}) + \kappa \|u - \bar{u}\| \mathbb{B} \quad \text{for all } u \in \mathbb{R}^n.$$

It is clear that the calmness is substantially weaker (less restrictive) than the Aubin property. Furthermore, it is known that F is calm at $(\bar{u}, \bar{v})$ if and only if F^{-1} is *metrically subregular at* $(\bar{u}, \bar{v})$, i.e., there is a neighborhood V of $\bar{v}$ and a constant $\kappa > 0$ such that

$$d(v, F(\bar{u})) \leq \kappa d(\bar{u}, F^{-1}(v)) \quad \text{for all } v \in V,$$

cf. Dontchev and Rockafellar (2014, Exercise 3H.4).

To conduct a thorough analysis of the above stability notions one typically makes use of some basic notions of generalized differentiation, whose definitions are presented below.

Definition 2 Let A be a closed set in $\mathbb{R}^n$ and $\bar{x} \in A$.

(i)

$$T_A(\bar{x}) := \operatorname*{Lim\,sup}_{t \searrow 0} \frac{A - \bar{x}}{t}$$

is the *tangent (contingent, Bouligand) cone* to A at $\bar{x}$ and

$$\hat{N}_A(\bar{x}) := (T_A(\bar{x}))^\circ$$

is the *regular (Fréchet) normal cone* to A at $\bar{x}$.

(ii)

$$N_A(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \hat{N}_A(x)$$

is the *limiting (Mordukhovich) normal cone* to A at $\bar{x}$ and, given a direction $d \in \mathbb{R}^n$,

$$N_A(\bar{x}; d) := \operatorname*{Lim\,sup}_{\substack{t \searrow 0 \\ d' \to d}} \hat{N}_A(\bar{x} + td')$$

is the *directional limiting normal cone* to A at $\bar{x}$ *in direction* d .

In case when $\bar{x} \notin A$ we define $T_A(\bar{x}) := \hat{N}_A(\bar{x}) := N_A(\bar{x}) := N_A(\bar{x}; d) := \emptyset$.

The symbol "Limsup" stands for the outer (upper) set limit in the sense of Painlevé-Kuratowski, cf. Rockafellar and Wets (1998, Chapter 4B). If A is convex, then both the regular and the limiting normal cones coincide with the classical normal cone in the sense of convex analysis. Therefore we will use in this case the notation N_A.

By the definition, the limiting normal cone coincides with the directional limiting normal cone in direction 0, i.e., $N_A(\bar{x}) = N_A(\bar{x}; 0)$, and $N_A(\bar{x}; d) = \emptyset$ whenever $d \notin T_A(\bar{x})$.

The above listed cones enable us to describe the local behavior of set-valued maps via various generalized derivatives. Consider again the set-valued map F and the point $(\bar{u}, \bar{v}) \in \operatorname{gph} F$.

Definition 3 (i) The set-valued map $DF(\bar{u}, \bar{v}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^z$, defined by

$$DF(\bar{u}, \bar{v})(d) := \{h \in \mathbb{R}^z | (d, h) \in T_{\operatorname{gph} F}(\bar{u}, \bar{v})\}, \quad d \in \mathbb{R}^n$$

is called the *graphical derivative* of F at $(\bar{u}, \bar{v})$;

(ii) The set-valued map $\hat{D}^* F(\bar{u}, \bar{v}) : \mathbb{R}^z \rightrightarrows \mathbb{R}^n$, defined by

$$\hat{D}^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in \hat{N}_{\operatorname{gph} F}(\bar{u}, \bar{v})\}, \quad v^* \in \mathbb{R}^z$$

is called the *regular (Fréchet) coderivative* of F at $(\bar{u}, \bar{v})$.

(iii) The set-valued map $D^* F(\bar{u}, \bar{v}) : \mathbb{R}^z \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\operatorname{gph} F}(\bar{u}, \bar{v})\}, \quad v^* \in \mathbb{R}^z$$

is called the *limiting (Mordukhovich) coderivative* of F at $(\bar{u}, \bar{v})$.

(iv) Finally, given a pair of directions $(d, h) \in \mathbb{R}^n \times \mathbb{R}^z$, the set-valued map $D^* F((\bar{u}, \bar{v}); (d, h)) : \mathbb{R}^z \rightrightarrows \mathbb{R}^n$, defined by

$$D^* F((\bar{u}, \bar{v}); (d, h))(v^*) := \{u^* \in \mathbb{R}^n | (u^*, -v^*) \in N_{\operatorname{gph} F}((\bar{u}, \bar{v}); (d, h))\}, v^* \in \mathbb{R}^z \tag{2}$$

is called the *directional limiting coderivative* of F at $(\bar{u}, \bar{v})$ in direction (d, h).

For the properties of the cones $T_A(\bar{x})$, $\hat{N}_A(\bar{x})$ and $N_A(\bar{x})$ from Definition 2 and generalized derivatives (i), (ii) and (iii) from Definition 3 we refer the interested reader to the monographs Rockafellar and Wets (1998) and Mordukhovich (2006). The directional limiting normal cone and coderivative were introduced by the first author in Gfrerer (2013) and various properties of these objects can be found in Gfrerer and Outrata (2016) and the references therein. Note that $D^* F((\bar{u}, \bar{v})) = D^* F((\bar{u}, \bar{v}); (0, 0))$ and that $\operatorname{dom} D^* F((\bar{u}, \bar{v}); (d, h)) = \emptyset$ whenever $h \notin DF(\bar{u}, \bar{v})(d)$.

Let now $M : \mathbb{R}^l \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a given set-valued map with a closed graph and $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ be the associated *implicit set-valued map* given by

$$S(p) := \{x \in \mathbb{R}^m | 0 \in M(p, x)\}. \tag{3}$$

In what follows, p will be called the *parameter* and x will be the *decision variable*. Given a *reference pair* $(\bar{p}, \bar{x}) \in \operatorname{gph} S$, one has the following sufficient condition for the Aubin property of S around $(\bar{p}, \bar{x})$.

Theorem 1 (Gfrerer and Outrata 2016, Theorem 4.4, Corollary 4.5). *Assume that*

(i)

$$\{u|0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\} \neq \emptyset \quad \textit{for all } q \in \mathbb{R}^l; \tag{4}$$

(ii) *M is metrically subregular at $(\bar{p}, \bar{x}, 0)$;*
(iii) *For every nonzero $(q, u) \in \mathbb{R}^l \times \mathbb{R}^n$ verifying $0 \in DM(\bar{p}, \bar{x}, 0)(q, u)$ one has the implication*

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*) \Rightarrow q^* = 0. \tag{5}$$

Then S has the Aubin property around $(\bar{p}, \bar{x})$ and for any $q \in \mathbb{R}^l$

$$DS(\bar{p}, \bar{x})(q) = \{u|0 \in DM(\bar{p}, \bar{x}, 0)(q, u)\}. \tag{6}$$

The above assertions remain true provided assumptions (ii), (iii) are replaced by

(iv) *For every nonzero $(q, u) \in \mathbb{R}^l \times \mathbb{R}^n$ verifying $0 \in DM(\bar{p}, \bar{x}, 0)(q, u)$ one has the implication*

$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*) \Rightarrow \begin{cases} q^* = 0 \\ v^* = 0. \end{cases} \tag{7}$$

In this paper we will consider the case of *variational systems* where

$$M(p, x) := H(p, x) + \hat{N}_\Gamma(x), \quad \Gamma = g^{-1}(D). \tag{8}$$

In (8), $H : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, $g : \mathbb{R}^n \to \mathbb{R}^s$ is twice continuously differentiable and $D \subset \mathbb{R}^s$ is a closed set.

Given an optimization problem

$$\begin{array}{ll} \text{minimize} & f(p, x) \\ \text{subject to} & \\ & x \in \Gamma \end{array}$$

with a twice continuously differentiable objective, then the corresponding necessary optimality conditions can be written down in the form (8) with $H(p, x) = \nabla_x f(p, x)$. If the constraint set is defined by a parameter-dependent constraint system

$$d(p, x) \in K$$

with a twice continuously differentiable function $d : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^a$ then, under a suitable constraint qualification, the respective KKT system attains the form (8) with

$$H(p, x, \lambda) = \begin{bmatrix} f(p, x) + \langle \lambda, d(p, x) \rangle \\ -d(p, x) \end{bmatrix} \text{ and } \Gamma = \mathbb{R}^n \times K^\circ,$$

where the *Lagrange multiplier* $\lambda \in \mathbb{R}^s$ is considered as an additional decision variable.

As pointed out in Gfrerer and Outrata (2016) and in the Introduction, Theorem 1 improves the currently available conditions whenever $\nabla_p H(\bar{p}, \bar{x})$ is not surjective, i.e., the considered parameterization is not ample at $(\bar{p}, \bar{x})$.

By the continuous differentiability of H one has that for M given in (8) and any $(q, u) \in \mathbb{R}^l \times \mathbb{R}^n$

$$DM(\bar{p}, \bar{x}, 0)(q, u) = \nabla_p H(\bar{p}, \bar{x})q + \nabla_x H(\bar{p}, \bar{x})u \\ + D\hat{N}_\Gamma(\bar{x}, -H(\bar{p}, \bar{x}))(u, -\nabla_p H(\bar{p}, \bar{x})q - \nabla_x H(\bar{p}, \bar{x})u), \tag{9}$$

cf. Rockafellar and Wets (1998, Exercise 10.43). Likewise, for any $v^* \in \mathbb{R}^n$,

$$D^* M((\bar{p}, \bar{x}, 0); (q, u, 0))(v^*) \\ = \begin{bmatrix} \nabla_p H(\bar{p}, \bar{x})^T v^* \\ \nabla_p H(\bar{p}, \bar{x})^T v^* + D^* \hat{N}_\Gamma((\bar{x}, -H(\bar{p}, \bar{x})); (u, -\nabla_p H(\bar{p}, \bar{x})q - \nabla_x H(\bar{p}, \bar{x})u))(v^*) \end{bmatrix}, \tag{10}$$

cf. Gfrerer and Outrata (2016, Theorem 2.10). The application of Theorem 1 requires thus the computation of $D\hat{N}_\Gamma(\bar{x}, -H(\bar{p}, \bar{x}))(\cdot, \cdot)$ and $D^* \hat{N}_\Gamma((\bar{x}, -H(\bar{p}, \bar{x})); (\cdot, \cdot))(v^*)$ for directions generated by the vectors q, u. This problem will be tackled in the next section.

3 Graphical derivatives and directional limiting coderivatives of $\hat{N}_\Gamma$

Throughout this section we will impose a weakened version of the *reducibility* and the *nondegeneracy* conditions introduced in Bonnans and Shapiro (2000). Concretely, in what follows we will assume that

(A1) There exists a closed set $\Theta \subset \mathbb{R}^d$ along with a twice continuously differentiable mapping $h : \mathbb{R}^s \to \mathbb{R}^d$ and a neighborhood $\mathcal{V}$ of $g(\bar{x})$ such that $\nabla h(g(\bar{x}))$ is surjective and

$$D \cap \mathcal{V} = \{z \in \mathcal{V} | h(z) \in \Theta\};$$

(A2)

$$\operatorname{rge} \nabla g(\bar{x}) + \ker \nabla h(g(\bar{x})) = \mathbb{R}^s. \tag{11}$$

Note that conditions (A1), (A2) amount to the reducibility of D to Θ at $g(\bar{x})$ and the nondegeneracy of $\bar{x}$ with respect to Γ and the mapping h in the sense of Bonnans and Shapiro (2000) provided the sets D, Θ are convex. The assumptions (A1), (A2) have the following important impact on the representation of Γ and $\hat{N}_\Gamma$ near $\bar{x}$.

Proposition 1 *Let $b := h \circ g$. Then there exists neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{W} \supset g(\mathcal{U})$ of $g(\bar{x})$ such that*

$$\Gamma \cap \mathcal{U} = \{x \in \mathcal{U} | b(x) \in \Theta\}, \tag{12}$$

$\nabla b(x)$ is surjective for every $x \in \mathscr{U}$, $\nabla h(y)$ is surjective for every $y \in \mathscr{W}$ and

$$\hat{N}_D(y) = \nabla h(y)^T \hat{N}_\Theta(h(y)), \; y \in \mathscr{W}, \tag{13}$$

$$\hat{N}_\Gamma(x) = \nabla b(x)^T \hat{N}_\Theta(b(x)) = \nabla g(x)^T \hat{N}_D(g(x)), \; x \in \mathscr{U}. \tag{14}$$

Proof First we show that (11) is equivalent with the surjectivity of $\nabla b(\bar{x}) = \nabla h(g(\bar{x}))\nabla g(\bar{x})$. Indeed, $\nabla b(\bar{x})$ is surjective if and only if

$$\{0\} = \ker \nabla b(\bar{x})^T = \ker(\nabla g(\bar{x})^T \nabla h(g(\bar{x}))^T),$$

which, by the assumed surjectivity of $\nabla h(g(\bar{x}))$, in turn holds if and only if

$$\begin{aligned}\{0\} &= \ker \nabla g(\bar{x})^T \cap \operatorname{rge} \nabla h(g(\bar{x}))^T = \Big((\ker \nabla g(\bar{x})^T)^\perp + (\operatorname{rge} \nabla h(g(\bar{x}))^T)^\perp\Big)^\perp \\ &= (\operatorname{rge} \nabla g(\bar{x}) + \ker \nabla h(g(\bar{x})))^\perp\end{aligned}$$

and this is clearly equivalent with (11). Hence $\nabla b(\bar{x})$ is surjective and we can find open neighborhoods $\mathscr{W} \subset \mathscr{V}$ and $\mathscr{U} \subset g^{-1}(\mathscr{W})$ of $\bar{x}$ such that $\nabla b(x)$ is surjective for all $x \in \mathscr{U}$ and $\nabla h(y)$ is surjective for all $y \in \mathscr{W}$, where $\mathscr{V}$ is given by assumption (A1). Hence for every $x \in \mathscr{U}$ we have $g(x) \in \mathscr{V}$ and (12) follows from (A1). The descriptions of the regular normal cones (13), (14) result from Rockafellar and Wets (1998, Exercise 6.7). □

Remark 1 Note that, given a vector $x^* \in \widehat{N}_\Gamma(x)$ with $x \in \Gamma \cap \mathscr{U}$, there is a unique $\lambda \in N_D(g(x))$ satisfying

$$x^* = \nabla g(x)^T \lambda. \tag{15}$$

Indeed, from (14) it follows that there is a unique $\mu \in \widehat{N}_\Theta(b(x))$ such that $x^* = \nabla b(x)^T \mu$ thanks to the surjectivity of $\nabla b(x)$. Since $\lambda = \nabla h(g(x))^T \mu$, we are done.

The rest of this section is divided to two subsections devoted to the graphical derivatives and the directional limiting coderivatives of $\hat{N}_\Gamma$, respectively.

3.1 Graphical derivatives of $\hat{N}_\Gamma$

The computation of graphical derivatives of $\hat{N}_\Gamma$ has been considered in numerous works, see Rockafellar and Wets (1998) and the references therein. Recently, in Mordukhovich et al. (2015a, b) the authors have derived two different formulas for $D\hat{N}_\Gamma$ by using a strengthened variant of (A1), (A2) together with some additional assumptions. They include either the convexity of Γ or a special *projection derivation condition* (PDC) defined next.

Definition 4 A convex set $\Xi \subset \mathbb{R}^s$ satisfies the *projection derivation condition* (PDC) at the point $\bar{z} \in \Xi$ if we have

$$P'_\Xi(\bar{z} + b; h) = P_{K(\bar{z},b)}(h) \quad \text{for all } \; b \in N_\Xi(\bar{z}) \; \text{ and } \; h \in \mathbb{R}^s,$$

where $K(\bar{z}, b) := T_\Xi(\bar{z}) \cap \{b\}^\perp$.

In our case the PDC condition is automatically fulfilled provided D is convex polyhedral. If D is a non-polyhedral convex cone, then PDC is always fulfilled at the vertex (Mordukhovich et al. 2015b, Proposition 4.4) but, typically, not at all other points. In Mordukhovich et al. (2015b) it is further shown that PDC is implied by the *extended polyhedricity* and one finds there also an illustrative example of a non-polyhedral set, satisfying PDC at a point which is not the vertex. Throughout Sects. 3.1 and 3.2 it is enough to assume, however, the weakened reducibility and nondegeneracy assumptions (A1), (A2) and we obtain new workable formulas without any additional requirements.

Theorem 2 *Let assumptions (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying*

$$\bar{\lambda} \in \hat{N}_D(g(\bar{x})), \;\; \nabla g(\bar{x})^T \bar{\lambda} = \bar{x}^*. \tag{16}$$

Then

$$\begin{aligned} T_{\mathrm{gph}\, \hat{N}_\Gamma}(\bar{x}, \bar{x}^*) = \{(u, u^*) | \exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}), \\ u^* = \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g\rangle(\bar{x})u\}. \end{aligned} \tag{17}$$

Proof Let $(u, u^*) \in T_{\mathrm{gph}\, \widehat{N}_\Gamma}(\bar{x}, \bar{x}^*)$ and consider sequences $t_k \searrow 0$ and $(u_k, u_k^*) \to (u, u^*)$ with $x_k^* := \bar{x}^* + t_k u_k^* \in \widehat{N}_\Gamma(x_k)$, where $x_k := \bar{x} + t_k u_k$. We can assume that $x_k \in \mathscr{U}$ and that $\nabla b(x_k)$ is surjective for all k, where b and $\mathscr{U}$ are given by Proposition 1. Hence we can find multipliers $\mu^k \in \widehat{N}_\Theta(b(x_k))$ such that $x_k^* = \nabla b(x_k)^T \mu^k$. The sequence μ^k is bounded and, after passing to some subsequence, converges to some $\bar{\mu} \in \widehat{N}_\Theta(h(g(\bar{x})))$ with $\bar{x}^* = \nabla b(\bar{x})^T \bar{\mu}$. Further, by (13) we have $\bar{\lambda} = \nabla h(g(\bar{x}))^T \mu$ for some $\mu \in \widehat{N}_\Theta(h(g(\bar{x})))$ implying $\bar{x}^* = \nabla b(\bar{x})^T \mu$ and $\bar{\mu} = \mu$ follows from the surjectivity of $\nabla b(\bar{x})$.

Since

$$\begin{aligned} t_k u_k^* = x_k^* - \bar{x}^* &= \nabla b(x_k)^T \mu_k - \nabla b(\bar{x})^T \bar{\mu} \\ &= t_k \nabla^2 \langle \bar{\mu}, b\rangle(\bar{x})u_k + \nabla b(\bar{x})^T (\mu^k - \bar{\mu}) + o(t_k), \end{aligned}$$

we obtain that

$$\nabla b(\bar{x})^T \frac{\mu^k - \bar{\mu}}{t_k} = u^* - \nabla^2 \langle \bar{\mu}, b\rangle(\bar{x})u + o(t_k)/t_k.$$

By the surjectivity of $\nabla b(\bar{x})$ we obtain that the sequence $\eta^k := (\mu^k - \bar{\mu})/t_k$ is bounded and, after passing to some subsequence, η^k converges to some η fulfilling

$$\nabla b(\bar{x})^T \eta = u^* - \nabla^2 \langle \bar{\mu}, b\rangle(\bar{x})u.$$

Denoting $\lambda^k = \nabla h(g(x_k))\mu^k$ we obtain $\lambda^k \in \widehat{N}_D(g(x_k))$ by (13) and

$$\begin{aligned}\lambda^k - \bar{\lambda} &= \nabla h(g(x_k))^T \mu_k - \nabla h(g(\bar{x}))^T \bar{\mu}\\ &= \nabla^2 \langle \bar{\mu}, h\rangle (g(\bar{x}))\nabla g(\bar{x})(t_k u_k) + \nabla h(g(\bar{x}))^T (\mu_k - \bar{\mu}) + o(t_k),\end{aligned}$$

implying that $(\lambda^k - \bar{\lambda})/t_k$ converges to

$$\xi := \nabla^2 \langle \bar{\mu}, h\rangle (g(\bar{x}))\nabla g(\bar{x})u + \nabla h(g(\bar{x}))^T \eta. \tag{18}$$

We conclude $(\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \widehat{N}_D}(g(\bar{x}), \bar{\lambda})$ and

$$\begin{aligned}u^* &= \nabla b(\bar{x})^T \eta + \nabla^2 \langle \bar{\mu}, b\rangle (\bar{x})u\\ &= \nabla g(\bar{x})^T \nabla h(g(\bar{x}))^T \eta + \nabla g(\bar{x})^T \nabla^2 \langle \bar{\mu}, h\rangle (g(\bar{x}))\nabla g(\bar{x})u\\ &\quad + \nabla^2 \langle \nabla h(g(\bar{x})^T \bar{\mu}, g\rangle (\bar{x})u\\ &= \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g\rangle (\bar{x})u\end{aligned}$$

showing

$$\begin{aligned}(u, u^*) \in \mathscr{T} := \{&(u, u^*) \mid \exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \widehat{N}_D}(g(\bar{x}), \lambda),\\ &u^* = \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g\rangle (\bar{x})u\}\end{aligned}$$

Thus $T_{\mathrm{gph}\, \widehat{N}_\Gamma}(\bar{x}, \bar{x}^*) \subset \mathscr{T}$ holds.

In order to show the reverse inclusion $T_{\mathrm{gph}\, \widehat{N}_\Gamma}(\bar{x}, \bar{x}^*) \supset \mathscr{T}$, consider $(u, u^*) \in \mathscr{T}$ together with some corresponding ξ. Then there are sequences $t_k \searrow 0$, $v_k \to \nabla g(\bar{x})u$ and $\xi^k \to \xi$ such that $\bar{\lambda} + t_k \xi^k \in \widehat{N}_D(g(\bar{x}) + t_k v_k)$ and thus $h(g(\bar{x}) + t_k v_k) \in \Theta$ and $\bar{\lambda} + t_k \xi^k = \nabla h(g(\bar{x}) + t_k v_k))^T \mu^k$ with $\mu^k \in \widehat{N}_\Theta(h(g(\bar{x}) + t_k v_k)$ for all k sufficiently large. Further, the sequence μ^k is bounded. Since

$$\begin{aligned}b(\bar{x} + t_k u_k) - h(g(\bar{x}) + t_k v_k) &= \nabla b(\bar{x})(t_k u_k) - \nabla h(g(\bar{x}))(t_k v_k) + o(t_k)\\ &= t_k \nabla h(g(\bar{x}))(\nabla g(\bar{x})u - v_k) + o(t_k) = o(t_k)\end{aligned}$$

and $\nabla b(\bar{x})$ is surjective, we can find for each k sufficiently large some x_k with $b(x_k) = h(g(\bar{x}) + t_k v_k) \in \Theta$ and $x_k - (\bar{x} + t_k u_k) = o(t_k)$. It follows that

$$\nabla b(x_k)^T \mu^k = \nabla g(x_k)^T \nabla h(g(x_k))^T \mu^k \in \widehat{N}_\Gamma(x_k)$$

and

$$\begin{aligned}
\nabla b(x_k)^T\mu^k - \bar{x}^* &= \nabla g(x_k)^T \nabla h(g(x_k))^T\mu^k - \bar{x}^* \\
&= \nabla g(x_k)^T\big(\nabla h(g(x_k))^T\mu^k - \bar{\lambda}\big) + \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})(x_k - \bar{x}) + o(t_k) \\
&= \nabla g(x_k)^T\big(\nabla h(g(\bar{x}) + t_k v_k)^T\mu^k - \bar{\lambda} + o(t_k)\big) \\
&\quad + t_k\nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})u + o(t_k) \\
&= t_k\nabla g(x_k)^T\xi^k + t_k\nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})u + o(t_k) \\
&= t_k\big(\nabla g(\bar{x})^T\xi + \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})u\big) + o(t_k)
\end{aligned}$$

showing $(u, u^*) \in T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{x}, \bar{x}^*)$. □

Remark 2 Everything remains true if we replace $\widehat{N}_\Gamma, \widehat{N}_D, \widehat{N}_\Theta$ by N_Γ, N_D, N_Θ,

Remark 3 Note that to each pair $(u, u^*) \in T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{x}, \bar{x}^*)$ there is a unique ξ satisfying the relations on the right-hand side of (17). Its existence has been shown in the first part of the proof and its uniqueness follows from (18) and the uniqueness of η implied by the surjectivity of $\nabla b(\bar{x})$.

From (17) one can relatively easily derive the formulas from Mordukhovich et al. (2015a, b) by imposing appropriate additional assumptions. Indeed, let us suppose that, in addition to (A1), (A2), D is convex and the (single-valued) operator P_D is directionally differentiable at $g(\bar{x})$. Then one has the relationship

$$\begin{aligned}
T_{\mathrm{gph}\,N_D}(g(\bar{x}), \bar{\lambda}) &= \left\{(v, w) \,\middle|\, \begin{bmatrix} v + w \\ v \end{bmatrix} \in T_{\mathrm{gph}\,P_D}(g(\bar{x}) + \bar{\lambda}, g(\bar{x}))\right\} \\
&= \{(v, w) | v = P'_D(g(\bar{x}) + \bar{\lambda}; v + w)\},
\end{aligned}$$

which implies that under the posed additional assumptions the relation

$$(\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\,N_D}(g(\bar{x}), \bar{\lambda}) \tag{19}$$

amounts to the equation

$$\nabla g(\bar{x})u = P'_D(g(\bar{x}) + \bar{\lambda}; \nabla g(\bar{x})u + \xi). \tag{20}$$

Formula (17) attains thus exactly the form from Mordukhovich et al. (2015a, Theorem 3.3). Note that in this way it was not necessary to assume the convexity of Γ like in Mordukhovich et al. (2015a). Thanks to this, upon imposing the PDC condition on D at $g(\bar{x})$, one gets from (20) that

$$\nabla g(\bar{x})u = P_K(\nabla g(\bar{x})u + \xi), \tag{21}$$

where K stands for the critical cone to D at $g(\bar{x})$ with respect to $\bar{\lambda}$, i.e., $K = T_D(g(\bar{x})) \cap [\bar{\lambda}]^\perp$. From (21) we easily deduce that

$$\xi \in N_K(\nabla g(\bar{x})u)$$

and relation (17) thus simplifies to

$$T_{\operatorname{gph}\hat{N}_\Gamma}(\bar{x},\bar{x}^*) = \{(u,u^*) | u^* \in \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})u + \nabla g(\bar{x})^T N_K(\nabla g(\bar{x})u)\}. \tag{22}$$

We have recovered the formula from Mordukhovich et al. (2015b, Theorem 5.2). This enormous simplification of the way how this result has been derived is due to Theorem 2 and the equivalence of relations (19), (20) (under the posed additional assumptions).

As mentioned above, the PDC condition automatically holds whenever D is a convex polyhedral set. Thus, for instance, in case of variational systems with Γ given by NLP constraints, one can compute $DM(\bar{p},\bar{x},0)(q,u)$ by the workable formula

$$DM(\bar{p},\bar{x},0)(q,u) = \nabla_p H(\bar{p},\bar{x})q + \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})u + \nabla g(\bar{x})^T N_K(\nabla g(\bar{x})u), \tag{23}$$

where

$$\mathscr{L}(p,x,\lambda) := H(p,x) + \nabla g(x)^T\lambda$$

is the *Lagrangian* associated with the considered variational system.

3.2 Regular and directional limiting coderivatives of $\hat{N}_\Gamma$

Theorem 3 *Let assumptions (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying* (16). *Then*

$$\hat{N}_{\operatorname{gph}\hat{N}_\Gamma}(\bar{x},\bar{x}^*) = \left\{ (w^*,w) \,|\, \exists v^* : \begin{array}{l} (v^*, \nabla g(\bar{x})w) \in \hat{N}_{\operatorname{gph}\hat{N}_D}(g(\bar{x}),\bar{\lambda}), \\ w^* = -\nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})w + \nabla g(\bar{x})^T v^* \end{array} \right\}. \tag{24}$$

Proof First we justify (24) in the case when the derivative operator $\nabla g(\bar{x}) : \mathbb{R}^n \to \mathbb{R}^s$ is surjective. By the definition we have $(w^*,w) \in \hat{N}_{\operatorname{gph}\hat{N}_\Gamma}(\bar{x},\bar{x}^*)$ if and only if $\langle w^*, u\rangle + \langle w, u^*\rangle \le 0\ \forall (u,u^*) \in T_{\operatorname{gph}\hat{N}_\Gamma}(\bar{x},\bar{x}^*)$, which by virtue of Theorem 2 is equivalent to the statement that $(0,0)$ is a global solution of the problem

$$\begin{aligned} \max_{u,\xi}\ & \gamma(u,\xi) := \langle w^*, u\rangle + \langle w, \nabla g(\bar{x})^T\xi + \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})u\rangle \\ \text{subject to}\ & (\nabla g(\bar{x})u, \xi) \in T_{\operatorname{gph}\hat{N}_D}(g(\bar{x}),\bar{\lambda}). \end{aligned}$$

Since the objective can be rewritten as $\gamma(u,\xi) = \langle w^* + \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})w, u\rangle + \langle\nabla g(\bar{x})w, \xi\rangle$, this is in turn equivalent to the statement

$$(w^* + \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})w, \nabla g(\bar{x})w) \in C^\circ$$

where $C := \{(u,\xi) \,|\, (\nabla g(\bar{x})u, \xi) \in T_{\operatorname{gph}\hat{N}_D}(g(\bar{x}),\bar{\lambda})\}$. By surjectivity of $\nabla g(\bar{x})$ the linear mapping $(u,\xi) \to (\nabla g(\bar{x})u, \xi)$ is surjective as well and we can apply Rockafellar and Wets (1998, Exercise 6.7) to obtain

$$C^\circ = \hat N_C(0,0) = \{(\nabla g(\bar x)^T v^*, v) \mid (v^*, v) \in \hat N_{T_{\operatorname{gph} \hat N_D}(g(\bar x),\bar\lambda)}(0,0)\}$$
$$= \{(\nabla g(\bar x)^T v^*, v) \mid (v^*, v) \in \hat N_{\operatorname{gph} \hat N_D}(g(\bar x),\bar\lambda)\}.$$

Now formula (24) follows.

It remains to replace the surjectivity of $\nabla g(\bar x)$ by the weaker nondegeneracy assumption from (A2). To proceed, we employ the local representation of D provided by its reducibility at $g(\bar x)$, see assumption (A1). By Proposition 1 we have $\Gamma \cap \mathscr{U} = \{x \in \mathscr{U} \mid b(x) \in \Theta\}$ and by assumption (A1) we have $D \cap \mathscr{V} = \{z \in \mathscr{V} \mid h(z) \in \Theta\}$, where $\mathscr{U}$ and $\mathscr{V}$ denote neighborhoods of $\bar x$ and $g(\bar x)$, respectively. Since both $\nabla b(\bar x)$ and $\nabla h(g(\bar x))$ are surjective, we can apply (24) twice to obtain

$$\hat N_{\operatorname{gph} \hat N_\Gamma}(\bar x, \bar x^*) = \left\{ (w^*, w) \mid \exists z^* : \begin{array}{l} (z^*, \nabla b(\bar x) w) \in \hat N_{\operatorname{gph} \hat N_\Theta}(b(\bar x), \bar\mu), \\ w^* = -\nabla^2 \langle \bar\mu, b\rangle(\bar x) w + \nabla b(\bar x)^T z^* \end{array} \right\} \tag{25}$$

and

$$\hat N_{\operatorname{gph} \hat N_D}(g(\bar x), \bar\lambda) = \left\{ (v^*, v) \mid \exists z^* : \begin{array}{l} (z^*, \nabla h(g(\bar x)) v) \in \hat N_{\operatorname{gph} \hat N_\Theta}(h(g(\bar x)), \bar\mu), \\ v^* = -\nabla^2 \langle \bar\mu, h\rangle(g(\bar x)) v + \nabla h(g(\bar x))^T z^* \end{array} \right\}, \tag{26}$$

where $\bar\mu$ is the unique multiplier satisfying $\bar\lambda = \nabla h(g(\bar x))^T \bar\mu$. By the classical chain rule we have $\nabla b(\bar x) = \nabla h(g(\bar x)) \nabla g(\bar x)$ and

$$\nabla^2 \langle \bar\mu, b\rangle(\bar x) = \nabla g(\bar x)^T \nabla^2 \langle \bar\mu, h\rangle(g(\bar x)) \nabla g(\bar x) + \nabla^2 \langle \nabla h(g(\bar x))^T \bar\mu, g\rangle(\bar x)$$
$$= \nabla g(\bar x)^T \nabla^2 \langle \bar\mu, h\rangle(g(\bar x)) \nabla g(\bar x) + \nabla^2 \langle \bar\lambda, g\rangle(\bar x).$$

Now consider $(w^*, w) \in \hat N_{\operatorname{gph} \hat N_\Gamma}(\bar x, \bar x^*)$ and let z^* be chosen such that $(z^*, \nabla b(\bar x) w) \in \hat N_{\operatorname{gph} \hat N_\Theta}(b(\bar x), \bar\mu)$ and $w^* = -\nabla^2 \langle \bar\mu, b\rangle(\bar x) w + \nabla b(\bar x)^T z^*$. By substituting $v := \nabla g(\bar x) w$, $v^* := -\nabla^2 \langle \bar\mu, h\rangle(g(\bar x)) q + \nabla h(g(\bar x))^T z^*$ we obtain $(z^*, \nabla h(g(\bar x)) v) \in \hat N_{\operatorname{gph} \hat N_\Theta}(h(g(\bar x)), \bar\mu)$ implying $(v^*, v) = (v^*, \nabla g(\bar x) w) \in \hat N_{\operatorname{gph} \hat N_D}(g(\bar x), \bar\lambda)$ by (26) and

$$w^* = -\nabla^2 \langle \bar\lambda, g\rangle(\bar x) w + \nabla g(\bar x)^T \big(-\nabla^2 \langle \bar\mu, h\rangle(g(\bar x)) \nabla g(\bar x) w + \nabla h(g(\bar x))^T z^* \big)$$
$$= -\nabla^2 \langle \bar\lambda, g\rangle(\bar x) w + \nabla g(\bar x)^T v^*.$$

Thus

$$(w^*, w) \in \mathscr{N} := \left\{ (w^*, w) \mid \exists v^* : \begin{array}{l} (v^*, \nabla g(\bar x) w) \in \hat N_{\operatorname{gph} \hat N_D}(g(\bar x), \bar\lambda), \\ w^* = -\nabla^2 \langle \bar\lambda, g\rangle(\bar x) w + \nabla g(\bar x)^T v^* \end{array} \right\}$$

establishing the inclusion $\hat N_{\operatorname{gph} \hat N_\Gamma}(\bar x, \bar x^*) \subset \mathscr{N}$. To establish the reverse inclusion consider $(w^*, w) \in \mathscr{N}$ together with the corresponding element v^*. By (26) we can find some z^* such that $(z^*, \nabla h(g(\bar{x}))\nabla g(\bar{x})w) = (z^*, \nabla b(\bar{x})w) \in \hat{N}_{\mathrm{gph}\, \hat{N}_\Theta}(h(g(\bar{x})), \bar{\mu})$ and $v^* = -\nabla^2\langle\bar{\mu}, h\rangle(g(\bar{x}))\nabla g(\bar{x})w + \nabla h(g(\bar{x}))^T z^*$. Hence

$$\begin{aligned} w^* &= -\nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})w + \nabla g(\bar{x})^T v^* \\ &= -(\nabla^2\langle\bar{\lambda}, g\rangle(\bar{x}) + \nabla g(\bar{x})^T \nabla^2\langle\bar{\mu}, h\rangle(g(\bar{x}))\nabla g(\bar{x}))w + \nabla g(\bar{x})^T \nabla h(g(\bar{x}))^T z^* \\ &= -\nabla^2\langle\bar{\mu}, b\rangle(\bar{x})w + \nabla b(\bar{x})^T z^* \end{aligned}$$

and we conclude $(w^*, w) \in \hat{N}_{\mathrm{gph}\, \hat{N}_\Gamma}(\bar{x}, \bar{x}^*)$ by (25). Hence $\hat{N}_{\mathrm{gph}\, \hat{N}_\Gamma}(\bar{x}, \bar{x}^*) = \mathscr{N}$ and this finishes the proof. $\square$

By the definition of the regular coderivative we obtain the following Corollary.

Corollary 1 *Under the assumptions of Theorem 3 one has*

$$\hat{D}^*\hat{N}_\Gamma(\bar{x}, \bar{x}^*)(w) = \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})w + \nabla g(\bar{x})^T \hat{D}^*\hat{N}_D(g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x})w), \;\; w \in \mathbb{R}^n. \tag{27}$$

In order to show the following result on the directional limiting coderivative note that assumptions (A1) and (A2) hold for all $x \in \Gamma$ near $\bar{x}$. In fact, by taking into account Proposition 1 and its proof, we have that $\nabla h(g(x)$ and $\nabla b(x)$ are surjective for all x near $\bar{x}$ and the latter is equivalent with validity of the condition $\mathrm{rge}\, \nabla g(x) + \ker \nabla h(g(x)) = \mathbb{R}^n$ for those x.

Theorem 4 *Let assumptions (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying (16). Further we are given a pair of directions $(u, u^*) \in T_{\mathrm{gph}\, \hat{N}_\Gamma}(\bar{x}, \bar{x}^*)$. Then for any $w \in \mathbb{R}^n$*

$$\begin{aligned} &D^*\hat{N}_\Gamma((\bar{x}, \bar{x}^*); (u, u^*))(w) \\ &\quad = \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})w + \nabla g(\bar{x})^T D^*\hat{N}_D((g(\bar{x}), \bar{\lambda}); (\nabla g(\bar{x})u, \bar{\xi}))(\nabla g(\bar{x})w), \end{aligned} \tag{28}$$

where $\bar{\xi} \in \mathbb{R}^s$ is the (unique) vector satisfying the relations

$$(\nabla g(\bar{x})u, \bar{\xi}) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}), \;\; u^* = \nabla g(\bar{x})^T\bar{\xi} + \nabla^2\langle\bar{\lambda}, g\rangle(\bar{x})u. \tag{29}$$

Proof In the first step we observe that for arbitrary sequences $\vartheta_k \searrow 0$, $u_k \to u$, $u_k^* \to u^*$ and $w_k \to w$ such that $(x_k, x_k^*) := (\bar{x} + \vartheta_k u_k, \bar{x}_k^* + \vartheta_k u_k^*) \in \mathrm{gph}\, \hat{N}_\Gamma$ and k sufficiently large one has

$$\hat{D}^*\hat{N}_\Gamma(x_k, x_k^*)(w_k) = \nabla^2\langle\lambda_k, g\rangle(x_k)w_k + \nabla g(x_k)^T \hat{D}^*\hat{N}_D(g(x_k), \lambda_k)(\nabla g(x_k)w_k),$$

where λ_k is the (unique) multiplier satisfying the relations

$$\nabla g(x_k)^T\lambda_k = x_k^*, \;\; \lambda_k \in \hat{N}_D(g(x_k)). \tag{30}$$

Indeed, this follows immediately from Corollary 1 due to the mentioned robustness of assumptions (A1), (A2). Moreover, we know that $\lambda_k \to \bar{\lambda}$ which is the unique multiplier satisfying (16).

Next we observe that

$$g(x_k) = g(\bar{x}) + \vartheta_k h_k \text{ with } h_k = \frac{g(x_k) - g(\bar{x})}{\vartheta_k} \to \nabla g(\bar{x})u$$

and

$$\lambda_k = \bar{\lambda} + \vartheta_k \xi_k \text{ with } \xi_k = \frac{\lambda_k - \bar{\lambda}}{\vartheta_k}.$$

It follows that

$$\begin{aligned}&\hat{D}^* \hat{N}_\Gamma(\bar{x} + \vartheta_k u_k, \bar{x}^* + \vartheta_k u_k^*)(w_k)\\&\quad = \nabla^2 \langle \lambda_k, g\rangle(x_k) w_k + \nabla g(x_k)^T \hat{D}^* \hat{N}_D(g(\bar{x}) + \vartheta_k h_k, \bar{\lambda} + \vartheta_k \xi_k)(\nabla g(x_k) w_k).\end{aligned} \tag{31}$$

We may now use the argumentation from the proof of Theorem 2 to show that ξ_k converges to the unique $\bar{\xi}$ satisfying (29). Taking now the outer set limits for $k \to \infty$ on both sides of (31), we obtain that $w^* \in D^* \hat{N}_\Gamma((\bar{x}, \bar{x}^*); (u, u^*))(w)$ if and only if it admits the representation

$$w^* \in \nabla^2 \langle \bar{\lambda}, g\rangle(\bar{x}) w + \nabla g(\bar{x})^T D^* \hat{N}_D((g(\bar{x}), \bar{\lambda}); (\nabla g(\bar{x}) u, \bar{\xi}))(\nabla g(\bar{x}) w)$$

with $\bar{\lambda}$ and $\bar{\xi}$ specified above. □

Remark 4 Setting $(u, u^*) = (0, 0)$, we recover in this way the formula

$$D^* \hat{N}_\Gamma(\bar{x}, \bar{x}^*)(w) = \nabla^2 \langle \bar{\lambda}, g\rangle(\bar{x}) w + \nabla g(\bar{x})^T D^* \hat{N}_D(g(\bar{x}), \bar{\lambda})(\nabla g(\bar{x}) w),$$

which has been derived in Outrata and Ramiréz (2011) under the standard reducibility and nondegeneracy assumptions from Bonnans and Shapiro (2000). This formula thus holds also under the weakened assumptions (A1), (A2).

Under the additional assumptions, mentioned in Sect. 3.1, relations (29) can be simplified. In particular, under the PDC condition at $g(\bar{x})$, the first relation from (29) reduces to (21) (with ξ replaced by $\bar{\xi}$).

4 Main results

On the basis of Theorems 1, 2 and 4 we may now state our main result—a new condition for the Aubin property of the solution map of a variational system, given by (3), (8) around a specified reference point.

Theorem 5 *Let* $0 \in M(\bar{p}, \bar{x})$ *with* M *specified by* (8)*, the assumptions (A1), (A2) be fulfilled and let* $\bar{\lambda}$ *be the (unique) multiplier satisfying* (16) *with* $\bar{x}^* = -H(\bar{p}, \bar{x})$. *Further assume that*

(i) *for any $q \in \mathbb{R}^l$ the variational system*

$$0 = \nabla_p H(\bar{p}, \bar{x})q + \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda})u + \nabla g(\bar{x})^T \xi$$
$$(\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}) \tag{32}$$

has a solution $(u, \xi) \in \mathbb{R}^n \times \mathbb{R}^s$;

(ii) *M is metrically subregular at $(\bar{p}, \bar{x})$, and*

(iii) *for any nonzero (q, u) satisfying (with a corresponding unique ξ) relations (32) one has the implication*

$$0 \in \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda})^T v^* + \nabla g(\bar{x})^T D^* \hat{N}_D((g(\bar{x}), \bar{\lambda}); (\nabla g(\bar{x})u, \bar{\xi}))(\nabla g(\bar{x})v^*)$$
$$\Rightarrow v^* \in \ker \nabla_p H(\bar{p}, \bar{x})^T. \tag{33}$$

Then the respective S has the Aubin property around $(\bar{p}, \bar{x})$ and for any $q \in \mathbb{R}^l$

$$DS(\bar{p}, \bar{x})(q) = \{u | \exists \xi : (\nabla g(\bar{x})u, \xi) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}),$$
$$0 = \nabla_p H(\bar{p}, \bar{x})q + \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda})^T u + \nabla g(\bar{x})^T \xi\}. \tag{34}$$

The above assertions remain true provided assumptions (ii), (iii) are replaced by

(iv) *for any nonzero (q, u) satisfying (with a corresponding unique ξ) relations (32) one has the implication*

$$0 \in \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda})^T v^* + \nabla g(\bar{x})^T D^* \hat{N}_D((g(\bar{x}), \bar{\lambda}); (\nabla g(\bar{x})u, \bar{\xi}))(\nabla g(\bar{x})v^*)$$
$$\Rightarrow v^* = 0. \tag{35}$$

The proof follows easily from Theorems 1, 2 and 4 and relations (9), (10). By imposing the additional assumptions, mentioned in Sect. 3.1, formulas (32) and (34) can be appropriately simplified. In particular, when D is convex polyhedral, then (32) attains the form of the generalized equation (GE)

$$0 = \nabla_p H(\bar{p}, \bar{x})q + \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda})u + \nabla g(\bar{x})^T \xi, \ \xi \in N_K(\nabla g(\bar{x})u). \tag{36}$$

Denoting now $w := (q, u)$ and $\Lambda := \mathbb{R}^l \times (\nabla g(\bar{x}))^{-1} K$, (36) amounts to the homogenous affine variational inequality

$$0 \in \begin{bmatrix} 0 & 0 \\ \nabla_p H(\bar{p}, \bar{x}), & \nabla_x \mathscr{L}(\bar{p}, \bar{x}, \bar{\lambda}) \end{bmatrix} w + N_\Lambda(w). \tag{37}$$

Indeed, thanks to the polyhedrality of D, K is also polyhedral and

$$N_\Lambda(w) = N_{\mathbb{R}^l}(q) \times \nabla g(\bar{x})^T N_K(\nabla g(\bar{x})u)$$

without any qualification conditions. This case will now be illustrated by an academic example.

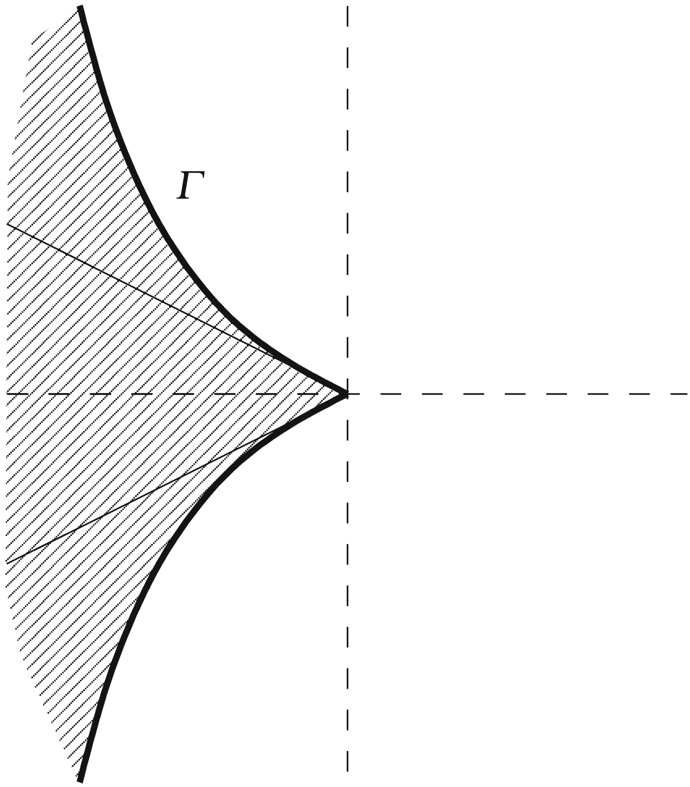

Fig. 1 Set Γ

Example 1 Consider the solution map $S : \mathbb{R} \rightrightarrows \mathbb{R}^2$ of the GE

$$0 \in M(p, x) = \begin{bmatrix} x_1 - p \\ -x_2 + x_2^2 \end{bmatrix} + \hat{N}_\Gamma(x) \tag{38}$$

with Γ given by $D = \mathbb{R}^2_-$ and

$$g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} 0.5x_1 - 0.5x_1^2 - x_2 \\ 0.5x_1 - 0.5x_1^2 + x_2 \end{bmatrix}.$$

Clearly, Γ is a nonconvex set depicted in Fig. 1. Let $(\bar{p}, \bar{x}) = (0, (0, 0))$ be the reference point. Since Γ fulfills LICQ at $\bar{x}$, we conclude that assumptions (A1), (A2) are fulfilled. Clearly, $x^* = -H(\bar{p}, \bar{x}) = (0, 0)$ and $\bar{\lambda} = (0, 0)$ as well. By virtue of the polyhedrality of D the variational system (32) attains the form (36). In our case it amounts to

$$0 = \begin{bmatrix} -q \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix} \xi, \quad \xi \in N_{\mathbb{R}^2_-}\left(\begin{bmatrix} 0.5u_1 - u_2 \\ 0.5u_1 + u_2 \end{bmatrix} \right), \tag{39}$$

because $K = T_D(g(\bar{x})) \cap [\bar{\lambda}]^\perp = D$.

It is not difficult to compute that for $q \le 0$ one has three different solutions (u, ξ) of (39), namely

$$u_1 = q,\ u_2 = 0,\ \xi_1 = 0,\ \xi_2 = 0 \tag{40}$$

$$u_1 = \frac{4}{3}q,\ u_2 = -\frac{2}{3}q,\ \xi_1 = 0,\ \xi_2 = -\frac{2}{3}q \tag{41}$$

$$u_1 = \frac{4}{3}q,\ u_2 = \frac{2}{3}q,\ \xi_1 = -\frac{2}{3}q,\ \xi_2 = 0, \tag{42}$$

and for $q \ge 0$ we have the unique solution

$$u_1 = u_2 = 0,\ \xi_1 = \xi_2 = q. \tag{43}$$

So, assumption (i) of Theorem 5 is fulfilled and we know the critical directions $(q, u) \neq 0$ for which the implication (35) will be examined. Starting with (40), one has $\nabla g(\bar{x})u = (0.5q, 0.5q)$ and

$$D^* N_{\mathbb{R}^2_-}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right); \left(\begin{bmatrix}0.5q\\0.5q\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)\left(\begin{bmatrix}0.5v_1^* - v_2^*\\0.5v_1^* + v_2^*\end{bmatrix}\right)$$

$$= \begin{bmatrix} D^* N_{\mathbb{R}_-}((0,0); (0.5q, 0))(0.5v_1^* - v_2^*) \\ D^* N_{\mathbb{R}_-}((0,0); (0.5q, 0))(0.5v_1^* + v_2^*) \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

by virtue of the definition and Rockafellar and Wets (1998, Proposition 6.41). The left-hand side of (35) reduces to the linear system in variables $(v^*, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$

$$0 = \begin{bmatrix} v_1^* \\ -v_2^* \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix}\eta,\ \eta = 0,$$

verifying the validity of implication (35). In the case (41), $\nabla g(\bar{x})u = (\frac{4}{3}q, 0)$ and

$$D^* N_{\mathbb{R}^2_-}\left(\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right); \left(\begin{bmatrix}\frac{4}{3}q\\0\end{bmatrix}, \begin{bmatrix}0\\-\frac{2}{3}q\end{bmatrix}\right)\right)\left(\begin{bmatrix}0.5v_1^* - v_2^*\\0.5v_1^* + v_2^*\end{bmatrix}\right) = \{0\} \times \mathbb{R}$$

provided $v_2^* = -0.5v_1^*$. The respective linear system in variables (v^*, η) reduces to

$$0 = \begin{bmatrix} v_1^* \\ 0.5v_1^* \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ -1 & 1 \end{bmatrix}\begin{bmatrix}0\\ \eta\end{bmatrix},$$

verifying again the validity of (35). In the same way we compute that in the case (42) one has $\nabla g(\bar{x})u = (0, \frac{4}{3}q)^T$ and

$$D^* N_{\mathbb{R}^2_-}\left(\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right);\left(\begin{bmatrix}0\\\frac{4}{3}q\end{bmatrix},\begin{bmatrix}-\frac{2}{3}q\\0\end{bmatrix}\right)\right)\left(\begin{bmatrix}0.5v_1^*-v_2^*\\0.5v_1^*+v_2^*\end{bmatrix}\right)=\mathbb{R}\times\{0\}$$

provided $v_2^* = 0.5v_1^*$. Taking this into account, we arrive at the linear system

$$0=\begin{bmatrix}v_1^*\\-0.5v_1^*\end{bmatrix}+\begin{bmatrix}0.5 & 0.5\\-1 & 1\end{bmatrix}\begin{bmatrix}\eta\\0\end{bmatrix},$$

showing that $v^* = 0$. Finally, concerning the last case (43), $\nabla g(\bar{x})u = (0, 0)$ and

$$D^* N_{\mathbb{R}^2_-}\left(\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right);\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}q\\q\end{bmatrix}\right)\right)\left(\begin{bmatrix}0.5v_1^*-v_2^*\\0.5v_1^*+v_2^*\end{bmatrix}\right)=\mathbb{R}\times\mathbb{R},$$

provided $v_1^* = 0.5v_2^*$ and, at the same time, $v_1^* = -0.5v_2^*$. This imediately implies that $v^* = 0$ and we are done. On the basis of Theorem 5 we have shown that the implicit set-valued map S generated by (38) has the Aubin property around $(0, 0)$ and, for a given q, $DS(0, 0)(q)$ is the set of solutions to (39).

Next we show that this result cannot be obtained via the Mordukhovich criterion and the standard calculus, which amounts to proving that the "standard" adjoint GE (cf.Mordukhovich 2006, Corollary 4.61) possesses only the trivial solution. Indeed, this GE amounts in our case to

$$0\in\begin{bmatrix}v_1^*\\-v_2^*\end{bmatrix}+\begin{bmatrix}0.5 & 0.5\\-1 & 1\end{bmatrix}D^* N_{\mathbb{R}^2_-}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right)\left(\begin{bmatrix}0.5v_1^*-v_2^*\\0.5v_1^*+v_2^*\end{bmatrix}\right) \tag{44}$$

and it is easy to check that, e.g., $v^* = (-0.5, 1)^T$ is a solution of (44). Consequently, the Aubin property of S cannot be detected in this way. $\triangle$

The preceding example indicates the difficulties which arise at the numerical verification of the conditions of Theorem 5. First, for the solution of (37) one has to use a numerical method which is able to compute *all* critical directions (q, u). Various candidates for such a method can be found, e.g., in Facchinei and Pang (2003). Concerning conditions (iii) or (iv), for $D = \mathbb{R}^s_-$ (Γ given by inequality constraints) the directional normal cones to gph $\hat{N}_D$ for nonzero directions amount to linear subspaces. Therefore, the verification of the validity of implications (33), (35) consists in analysis of linear systems of equations, which is definitely numerically tractable. However, if D amounts to a more complicated set (e.g. the Lorentz cone discussed in Sect. 5), then the verification of (33), (35) could be more demanding.

5 Variational systems with conic constraint sets

In this concluding section we will consider a variant of Theorem 5 under the additional assumption that D is a closed convex cone with vertex at 0 and $P_D(\cdot)$ is directionally differentiable over $\mathbb{R}^s$. As implied by (20), the variational system (32) attains then the form

$$\begin{aligned} 0 &= \nabla_p H(\bar{p},\bar{x})q + \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})u + \nabla g(\bar{x})^T\xi \\ \nabla g(\bar{x})u &= P'_D(g(\bar{x})+\bar{\lambda}; \nabla g(\bar{x})u+\xi) \end{aligned} \tag{45}$$

which, under the PDC condition at $g(\bar{x})$, further simplifies to the form (36). If D is the Carthesian product of Lorentz cones or the Löwner cone (Bonnans and Shapiro 2000), then we dispose with an efficient formula for $P'_D(\cdot;\cdot)$ which depends on the position of $(g(\bar{x}),\bar{\lambda})$ in gph N_D, cf. Outrata and Sun (2008, Lemma 2) and Sun and Sun (2003, Theorem 4.7).

Concerning the GE on the left-hand side of (33) or (35), it is advantageous to rewrite it in terms of P_D (instead of N_D). Let $(\bar{a},\bar{b}) \in \operatorname{gph} N_D$. Since

$$\operatorname{gph} N_D = \left\{(a,b) \in \mathbb{R}^s \times \mathbb{R}^s \,\middle|\, \begin{pmatrix} a+b \\ a \end{pmatrix} \in \operatorname{gph} P_D \right\},$$

one has, by virtue of Mordukhovich (2006, Theorem 1.17) that

$$p \in \hat{D}^* N_D(a,b)(q) \Longleftrightarrow -q \in \hat{D}^* P_D(a+b,a)(-q-p)$$

for any $(p,q) \in \mathbb{R}^s \times \mathbb{R}^s$. It follows that the GE on the left-hand side of (33) can be equivalently written down as the system

$$0 = \nabla_x \mathscr{L}(\bar{p},\bar{x},\bar{\lambda})^T v^* + \nabla g(\bar{x})^T(d - \nabla g(\bar{x})v^*) \tag{46}$$

$$-\nabla g(\bar{x})v^* \in D^* P_D((g(\bar{x})+\bar{\lambda}, g(\bar{x})); (\nabla g(\bar{x})u+\xi, \nabla g(\bar{x})u))(-d) \tag{47}$$

in variables $(v^*, d) \in \mathbb{R}^n \times \mathbb{R}^s$. If D is the Carthesian product of Lorentz cones or the Löwner cone, then the directional limiting coderivative of P_D can be computed by using Definition 2(ii) and the formulas for regular coderivatives of P_D in Outrata and Sun (2008) and Ding et al. (2014), respectively. For illustration consider the case when D amounts to just one Lorentz cone in $\mathbb{R}^s$, i.e.,

$$D = \mathscr{K} := \{(z_0,\bar{z}) \in \mathbb{R} \times \mathbb{R}^{s-1} | z_0 \geq \|\bar{z}\|\}.$$

We will analyze here only the most difficult situation when $g(\bar{x}) = 0$ and $\bar{\lambda} = 0$ and provide formulas for the directional limiting coderivatives of $P_{\mathscr{K}}$ at $(0,0)$ for all possible nonzero directions from

$$T_{\operatorname{gph} P_{\mathscr{K}}}(0,0) = \{(h,k) | k \in P_{\mathscr{K}}(h)\}, \tag{48}$$

see Outrata and Sun (2008, Lemma 2(iv)). We have thus to distinguish among the following five situations:

- $h \in \operatorname{int} \mathscr{K}, \ k = h;$ (49)
- $h \in \operatorname{int} \mathscr{K}^\circ, \ k = 0;$ (50)
- $h \notin \mathscr{K} \cup \mathscr{K}^\circ, k = P_{\mathscr{K}}(h);$ (51)
- $h \in \operatorname{bd} \mathscr{K}, \ k = h;$ (52)
- $h \in \operatorname{bd} \mathscr{K}^\circ, \ k = 0.$ (53)

In the cases (49), (50) we get immediately from Outrata and Sun (2008, Lemma 1(iv)) the formulas

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = u^*, \tag{54}$$

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = 0, \tag{55}$$

respectively. Likewise, in the case (51) one has

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = \{C(w,\alpha)u^* | w \in \mathbb{S}_{n-1}, \alpha \in [0,1]\}, \tag{56}$$

where

$$C(w,\alpha) = \frac{1}{2}\begin{bmatrix} 2\alpha I + (1-2\alpha)ww^T & w \\ w^T & 1 \end{bmatrix}.$$

Concerning the case (52), by passing to subsequences if necessary, one may have sequences $(h_i, k_i) \xrightarrow{\text{gph}\, P_{\mathscr{K}}} (h,k)$, $\lambda_i \searrow 0$ such that for i sufficiently large one of the following three situations occurs:

$*\ h_i \notin \ \mathscr{K} \cup \mathscr{K}^0 \ (k_i = P_{\mathscr{K}}(h_i));$
$*\ h_i \in \operatorname{int} \mathscr{K} \ (k_i = h_i);$
$*\ h_i \in bd\ \mathscr{K} \ (k_i = h_i).$

Correspondingly, we obtain from Outrata and Sun (2008, Lemma 1(iv) and Theorem 4), that

$$D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) = \{C(w,\alpha)u^* | w \in \mathbb{S}_{n-1}, \alpha \in [0,1]\} \cup \bigcup_{A \in \mathscr{A}(u^*)} \operatorname{conv}\{u^*, Au^*\}, \tag{57}$$

where

$$\mathscr{A}(u^*) := \left\{ I + \frac{1}{2}\begin{bmatrix} -ww^T & w \\ w^T & -1 \end{bmatrix} \middle| w \in \mathbb{S}_{n-1}, \left\langle \begin{bmatrix} -w \\ 1 \end{bmatrix}, u^* \right\rangle \geq 0 \right\}.$$

Analogously, in the case (53), by passing to subsequences if necessary, one may have sequences $(h_i, k_i) \xrightarrow{\text{gph}\, P_{\mathscr{K}}} (h,k)$, $\lambda_i \searrow 0$ such that for i sufficiently large one of the following three situations occurs:

* $h_i \notin \mathscr{K} \cup \mathscr{K}^0$ $(k_i = P_{\mathscr{K}}(h_i))$;
* $h_i \in \operatorname{int} \mathscr{K}^0$ $(k_i = 0)$;
* $h_i \in bd\, \mathscr{K}_0$ $(k_i = 0)$.

Correspondingly, we obtain from Outrata and Sun (2008, Lemma 1(iv) and Theorem 4) that

$$\begin{aligned} &D^* P_{\mathscr{K}}((0,0);(h,k))(u^*) \\ &\quad = \{C(w,\alpha)u^* | w \in \mathbb{S}_{n-1}, \alpha \in [0,1]\} \cup \bigcup_{B \in \mathscr{B}(u^*)} \operatorname{conv}\{u^*, Bu^*\}, \end{aligned} \tag{58}$$

where

$$\mathscr{B}(u^*) := \left\{ \frac{1}{2} \begin{bmatrix} ww^T & w \\ w^T & 1 \end{bmatrix} \middle| w \in \mathbb{S}_{n-1}, \left\langle \begin{bmatrix} w \\ 1 \end{bmatrix}, u^* \right\rangle \geq 0 \right\}.$$

Next we illustrate the above described procedure via a conic reformulation of Gfrerer and Outrata (2016, Example 5).

Example 2 Consider the solution map $S : \mathbb{R} \rightrightarrows \mathbb{R}^2$ of the GE given by (3), (8) with

$$H(p,x) = \begin{bmatrix} x_1 - p \\ -x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 2x_2 \\ -x_1 \end{bmatrix}$$

and $D = \mathscr{K}$ being the Lorentz cone in $\mathbb{R}^2$. Let $(\bar{p}, \bar{x}) = (0, (0,0))$ be the reference point so that $\bar{\lambda} = (0,0)$. It is easy to see that assumptions (A1), (A2) are fulfilled and, since the Lorentz cone in $\mathbb{R}^2$ is a polyhedral set, instead of (45) we can compute the "critical" directions via (36). The variational system (36) attains the form

$$0 = \begin{bmatrix} -q \\ 0 \end{bmatrix} + \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \xi, \; \xi \in N_{\mathscr{K}} \left(\begin{bmatrix} 2u_2 \\ -u_1 \end{bmatrix} \right). \tag{59}$$

It is not difficult to compute that for $q \leq 0$ one has three different solutions (u, ξ) of (59), namely

$$u_1 = q, \; u_2 = 0, \; \xi_1 = 0, \; \xi_2 = 0 \tag{60}$$

$$u_1 = \frac{4}{3}q, \; u_2 = -\frac{2}{3}q, \; \xi_1 = -\frac{1}{3}q, \; \xi_2 = \frac{1}{3}q \tag{61}$$

$$u_1 = \frac{4}{3}q, \; u_2 = \frac{2}{3}q, \; \xi_1 = \frac{1}{3}q, \; \xi_2 = \frac{1}{3}q \tag{62}$$

and for $q \geq 0$ one has the unique solution

$$u_1 = u_2 = 0, \quad \xi_1 = 0, \quad \xi_2 = -q. \tag{63}$$

So, assumption (i) of Theorem 5 is fulfilled and we will check assumption (iv). Starting with (60), system (46), (47) attains the form

$$0 = \begin{bmatrix} v_1^* \\ -v_2^* \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} d - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} v^* = \begin{bmatrix} -d_2 \\ -5v_2^* + 2d_1 \end{bmatrix} \tag{64}$$

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} \in D^* P_{\mathscr{K}} \left((0,0); \left(\begin{bmatrix} 0 \\ -q \end{bmatrix}, \begin{bmatrix} 0 \\ -q \end{bmatrix} \right) \right) (-d). \tag{65}$$

By virtue of formula (54) this system reduces to the equations

$$d_2 = 0,\ d_1 = \frac{5}{2} v_2^*,\ v_1^* = 0,\ 2v_2^* = d_1,$$

verifying that $v^* = 0$. In the case (61), one arrives at the Eq. (64) together with the relation

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} \in D^* P_{\mathscr{K}} \left((0,0); \left(\begin{bmatrix} -\frac{5}{3}q \\ -q \end{bmatrix}, \begin{bmatrix} -\frac{4}{3}q \\ -\frac{4}{3}q \end{bmatrix} \right) \right) (-d). \tag{66}$$

Now we have to employ formula (56). For $w = -1$ one obtains from (66) the equation

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} = - \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} d$$

which, together with (64), implies that $v^* = 0$. For $w = 1$ one obtains from (66) the equation

$$\begin{bmatrix} -2v_2^* \\ v_1^* \end{bmatrix} = - \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} d$$

that again implies that $v^* = 0$. Thus the case (61) is completed. Likewise, in the remaining cases (62), (63) we apply the formulas (56) and (55), respectively, and verify again that in all solutions of the respective system (46), (47) one has $v^* = 0$. The examined solution map S has thus the Aubin property around $(\bar{p}, \bar{x})$. Note that, as in Example 1, this conclusion cannot be made on the basis of the standard conditions. △

6 Concluding remarks

The advantage of the sufficient condition stated in Theorem 5 with respect to standard conditions consists in the fact that it takes into account the specific way how the parameters (entering via H) influence the solutions of the considered variational system. The gain is especially dramatic, if the difference between the dimensions l and n is large. So, the main application area of the achieved results lies in the post-optimal analysis of optima or equilibria with just a few unknown problem data (taking the roles of parameters) but a considerable number of decision variables. Moreover, formula (34) can very well be used in continuation methods (Haslinger et al. 2012).

In Dontchev and Rockafellar (1996) the authors have shown that for a variational system given by the GE

$$p \in F(x) + N_\Gamma(x)$$

with Γ being a convex polyhedron the Aubin property of S around a given reference point amounts in fact to the strong regularity (Dontchev and Rockafellar 2014, Chapter 3). This is, however, not true in the case of variational systems considered here, when one admits a general parameterization and Γ is given by (1). To ensure the strong regularity within our approach, one has to impose, in addition to the assumptions of Theorem 5, the local uniqueness of S around $(\bar{p}, \bar{x})$. To this aim one could employ, e.g., a suitable monotonicity assumption,

In general, the metric subregularity of M (assumption (ii) in Theorem 5) is not easy to verify. Apart from the "polyhedral" case, when this assumption holds thanks to Robinson (1981), there are various other sufficient conditions tailored mostly to some specific classes of mappings. In our case one could use, for instance, the first- or the second-order sufficient condition for metric subregularity (Gfrerer 2011), or the conditions concerning subdifferential mappings see, e.g., Artacho and Geoffroy (2008) or Drusvyatskiy et al. (2014). On the other hand, even the variant of Theorem 5, based on assumption (iv), seems to be an efficient new condition for the Aubin property.

Concerning a future research in this area, observe that the formulas, provided in the second part of Sect. 5 for D being the Lorentz cone, could easily be extended to the case when D amounts to the Carthesian product of several Lorentz cones. Further, on the basis of Ding et al. (2014) one could compute the directional limiting coderivatives of the projection mapping onto the Löwner cone which would enable us to apply the presented theory also to parameterized semidefinite programs. Finally, one could think of variational systems, not having the (relatively simple) structure (8). For example, p could arise also in the constraints or one could consider implicit constraints like in quasi-variational inequalities (Mordukhovich and Outrata 2007).

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