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Nash Equilibrium in a Pay-as-bid Electricity Market Part 2 - Best Response of a Producer

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Dedicated to Professor Franco Giannessi on the occasion of his 80th birthday and to Professor Diethard Pallaschke on the occasion of his 75th birthday

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We consider a multi-leader-common-follower model of a pay-as-bid electricity market in which the producers provide the regulator with either linear or quadratic bids. We prove that for a given producer only linear bids can maximise his profit. Such linear bids are referred as the "best response" of the given producer. They are obtained assuming the demand is known and some estimate of the bids of the other producers is available. Nevertheless we also show that whenever no best response exists, the optimal profit can be asymptotically attained by a sequence of quadratic bids converging to the so-called "limiting best response". An explicit formula for such a sequence is provided.

Keywords: electricity market; multi-leader-follower game; Nash equilibrium; best response

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1. Introduction

The deregulation of the electricity markets in the 90s has induced deep changes that could no longer be accounted for in the classic model of the economy. As explained in Part 1 of this couple of articles (see [4]), there has been a need for modelling these new markets. Nevertheless the non-cooperative characteristic of electricity markets led to concentrate on Nash and generalised Nash equilibria and thus leading different works/authors to multi-leader-follower games models. More precisely it is the concept of multi-leader-common-follower games that has been used, where the producers of electricity are viewed as leaders while the regulator of the market, referred as the Independent System Operator (ISO), is viewed as the common follower.

The Nash equilibrium associated with this problem is the equilibrium state in which the market should operate ideally. For more details see for example the recent works of Hu and Ralph [13], Downward, Zakeri and Philpott [9], Escobar and Jofre [10], Williams, Rumiantseva and Weigt [15] or Aussel and his collaborators [5–7]. Due to different evolutions of the electricity markets -influence of renewable energy, introduction of smart-grid, fusion of several European markets into one (PCR project)- those models need to be constantly updated and/or adapted.

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In our model we consider a pay-as-bid market in which each producer (or consumer) provides the ISO with a bid (function) used to derive directly its revenue (or expenses). For the sake of simplicity we aggregate consumers and consider only their aggregated demand of electricity, thereafter referred as demand, and further assume it is given. Moreover the transmission network is not taken into account as transportation thermal losses are neglected. The existence of Nash equilibria for such a multi-leader-common-follower model has been investigated in Part 1 of this couple of articles [4]. In particular it has been shown that both monopolistic and non-monopolistic equilibria can exist depending on the value of the demand. In both cases explicit formulas of these equilibria have been obtained in terms of the real costs of production for the producers.

For a given producer our aim in Part 2 is to determine the bid, if exists, maximising his profit. Such a bid is referred as his "best response" and is obtained assuming the demand is known as well as an estimate of the bids of the other producers. The evaluation of this best response is of great interest for a producer when preparing his bids to the market. But moreover the characterisation of the best response plays a central role in the proof of the existence and in the characterisation result stated in Part 1 (see [4, Theorem 3.1]).

As it will be seen in our final result (Theorem 3.1), the best response of a given producer, if exists, deeply depends on the electricity demand. Hence a complete analysis of the producer's problem is carried out for any demand. Actually the demand is not precisely known in real-world electricity markets and therefore, it is a common practice for a producer to do some "sampling demand" around a reference value to elaborate his bid.

In previous approaches [5, 6, 9, 12, 13, 15] the bids are assumed to be convex quadratic functions of the produced quantity. Actually in most electricity markets, only piecewise linear bids or block orders may be allowed. However quadratic function with non-negative coefficients capture well the typical behaviour of aggregated block offers and, at the same time, it is amenable to further analysis. Note that a classical way to ensure uniqueness of the solution to the ISO's problem, see e.g. [13], is to assume that all producers are bidding true quadratic functions. A key point in our present analysis is to allow both linear and quadratic bids. It is of main importance since, in Theorem 3.1, we actually show that the best response of a producer is a linear bid if it exists. Otherwise, the supremum of profit is reached by a converging sequence of truly quadratic bids. Then this limiting profit cannot be reached and the corresponding limit of bids is called a *limiting best response*.

The article is organised as follows. Notation and setting of both the ISO and the producer problems are described in Section 2. Our main results are stated in Section 3, namely the characterisation of the best response (limiting or not) of a producer as well as the evaluation of the corresponding optimal production quantities and profit. Being somehow quite technical, the proofs are essentially given in the Appendix.

2. Notation and Problem Setting

The basic notation follows: $D > 0$ is the electricity demand, $\mathcal{N} = \{1, \ldots, N\}$ is the set of producers ($N > 1$), $q_i \geq 0$ represents the non-negative production quantity of the i-th producer. Considering $q \in \mathbb{R}^N_+$ we use $q_{-i} \in \mathbb{R}^{N-1}_+$ to denote the vector $(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N)$, and the same convention is used also for other vectors hereinafter. For $i \in \mathcal{N}$ we use $a_i, b_i \geq 0$ to denote the coefficients of the i-th producer's bid $a_i q_i + b_i q_i^2$ and $A_i \geq 0, B_i > 0$ to denote the coefficients of the true production cost function $A_i q_i + B_i q_i^2$. A producer is said to be *active in the market* if his bid has been accepted by the ISO, i.e., the corresponding production quantity is positive.

In the following three subsections we analyse the problem of the ISO in detail, then the problem of a producer, and finally we present non-smooth tools needed to state the main results in Section 3.

2.1. *Problem of the ISO*

Each producer provides the ISO with a quadratic bid $a_i q_i + b_i q_i^2$. Then, knowing the bid vectors $a = (a_1, \cdots, a_N) \in \mathbb{R}_+^N$ and $b = (b_1, \cdots, b_N) \in \mathbb{R}_+^N$, the ISO computes the electricity dispatch $q = (q_1, \ldots, q_N) \in \mathbb{R}_+^N$ dealing with two aims at the same time. Namely to minimise the total expenses of consumers

$$C(a, b, D) = \sum_{i \in \mathcal{N}} a_i q_i + b_i q_i^2, \tag{1}$$

and to satisfy the demand ensuring $\sum_{i \in \mathcal{N}} q_i = D > 0$ with $q_i \geq 0$ for all $i \in \mathcal{N}$. It is a well-known fact that this problem admits at least one solution due to the continuity of the criterion and compactness of the optimisation domain. Nevertheless the model used in this article to describe electricity market can be ill-posed if the solution set of the ISO's problem contains more than one point, see e.g. [5]. In [6, 13] uniqueness of the solution to the ISO's problem is coming from the strict convexity of the objective function, assuming that producers are bidding true quadratic function, that is $b_i > 0$ for all $i \in \mathcal{N}$. Since our aim in the present work is to allow producers to bid linearly, an alternative assumption is needed to guarantee uniqueness of solution. On that account, see Theorem 2.1, we define the *equity property* assumption:

$$\text{(E)} \qquad (a_i, b_i) = (a_j, b_j) \Longrightarrow q_i = q_j, \; \forall i, j \in \mathcal{N}.$$

This assumption actually formalises the non-discriminatory policy (or fairness) of the ISO. Hence, the optimisation problem $\mathrm{ISO}(a, b, D)$ assuming (E) is as follows

$$\mathrm{ISO}(a,b,D) \qquad \min_q \sum_{i \in \mathcal{N}} (a_i q_i + b_i q_i^2)$$
$$\text{s.t.} \begin{cases} q_i \geq 0, \;\; \forall i \in \mathcal{N} \\ \left[(a_i, b_i) = (a_j, b_j) \Rightarrow q_i = q_j\right], \forall i, j \in \mathcal{N} \\ \displaystyle\sum_{i \in \mathcal{N}} q_i = D. \end{cases}$$

To analyse this problem further, we first define

$$\lambda^m(a) = \min_{i \in \mathcal{N}} a_i \tag{2}$$

and several critical parameters of $\mathrm{ISO}(a, b, D)$, namely a critical marginal price $\lambda^c(a, b)$, a critical value of the overall demand $D^c(a, b)$, and a set of producers bidding critical (linear) bids $\mathcal{N}^c(a,b) \subset \mathcal{N}$

$$\begin{cases} \lambda^c(a,b) = \min\limits_{i\in\mathcal{N}:b_i=0} a_i, \\ \mathcal{N}^c(a,b) = \{i \in \mathcal{N} \ : \ a_i = \lambda^c(a,b) \text{ and } b_i = 0\}, \\ D^c(a,b) = \sum\limits_{i\in\mathcal{N}:a_i<\lambda^c(a,b)} \dfrac{\lambda^c(a,b) - a_i}{2b_i}. \end{cases} \tag{3}$$

We note that $a_i < \lambda^c(a,b)$ implies $b_i > 0$ and so $D^c(a,b)$ is well-defined. If there is no $i \in \mathcal{N}$ such that $a_i < \lambda^c(a,b)$, we set $D^c(a,b) = 0$ meaning that all active producers bid linearly. More generally, any sum with respect to an empty set of indexes will be set to zero throughout this article. If there is no producer bidding linear function, i.e., we have $b_i > 0$ for all $i \in \mathcal{N}$, we set $\lambda^c(a,b) = D^c(a,b) = +\infty$. For the cardinality of $\mathcal{N}^c(a,b)$ we use the notation $N^c(a,b)$.

Remark 1 The above defined critical parameters have clear economic meanings:

(a) On the one hands, if the marginal price is strictly below $\lambda^c(a,b)$ then only truly quadratically bidding producers will be active in the market. On the other hand, if marginal price equals $\lambda^c(a,b)$, there is some linearly bidding producer ($b_i = 0$) that can formally provide arbitrary amount of electricity at marginal price $\lambda^c(a,b)$. Hence, $\lambda^c(a,b)$ is also the upper bound for the possible marginal price in the market, see for instance the forthcoming property (9).
(b) $\mathcal{N}^c(a,b)$ is actually the set of all the critical producers - that is, producers bidding linearly and at the critical marginal price - that may possibly be active in the market.
(c) $D^c(a,b)$ will be later identified with the overall amount of electricity produced by sub-critical producers, i.e., those bidding with $b_i > 0$, see the proof of Theorem 2.1. Thus, condition $D^c(a,b) = 0$ means that no sub-critical producer, i.e. producer bidding $b_i > 0$, can be active in the market. This condition may be equivalently stated as $\lambda^m(a) = \lambda^c(a,b)$.

Now, we define the set $\Gamma = \left\{(a,b,\lambda) \in \mathbb{R}_+^{2N+1} \ : \ 0 \le \lambda \le \lambda^c(a,b)\right\}$ (considering strict right inequality for the case of $\lambda^c(a,b) = +\infty$) and the function $F : \Gamma \to \mathbb{R}_+$ by

$$F(a,b,\lambda) = \sum_{i\in\mathcal{N}:a_i<\lambda} \frac{\lambda - a_i}{2b_i}. \tag{4}$$

Note that for $\lambda \le \lambda^m(a)$ the set $\{i \in \mathcal{N} : a_i < \lambda\}$ is empty and so $F(a,b,\lambda) = 0$. Let us also note that for $\lambda > \lambda^c(a,b)$ formula (4) is ill-posed because there exists $i \in \mathcal{N}$ such that $a_i < \lambda$ and $b_i = 0$.

For any $(a,b) \in \mathbb{R}_+^{2N}$ we observe directly from the definitions of $F(a,b,\lambda)$ and $D^c(a,b)$ that

$$\begin{cases} \lim\limits_{\lambda\to+\infty} F(a,b,\lambda) = +\infty & \text{if} \quad \lambda^c(a,b) = +\infty, \\ F(a,b,\lambda^c(a,b)) = D^c(a,b) & \text{if} \quad \lambda^c(a,b) < +\infty. \end{cases} \tag{5}$$

Moreover, for any $(a,b) \in \mathbb{R}_+^{2N}$ function $\lambda \to F(a,b,\lambda)$ is piecewise linear on $[\lambda^m(a), \lambda^c(a,b)[$. The monotonicity property of $F(a,b,\lambda)$ shown in Lemma A.1 and properties (5) justify the following definition of a function $\lambda(a,b,D) : \mathbb{R}_+^{2N+1} \to \mathbb{R}_+$

$$\lambda(a,b,D) = \begin{cases} \lambda \in [\lambda^m(a), \lambda^c(a,b)[, \text{ s.t. } F(a,b,\lambda) = D & \text{if } D \in [0, D^c(a,b)[\\ \lambda^c(a,b), & \text{if } D \geq D^c(a,b). \end{cases} \tag{6}$$

For any $(a,b) \in \mathbb{R}_+^{2N}$, function $\lambda(a,b,D)$ is continuous and piecewise linear in D due to continuity and piecewise linearity of $F(a,b,\lambda)$.

Remark 2 We observe that for any $(a,b) \in \mathbb{R}_+^{2N}$ and $D > 0$ it holds $\lambda^m(a) \leq \lambda(a,b,D) \leq \lambda^c(a,b)$ and $D \geq F(a,b,\lambda(a,b,D))$. The latter formula holds with equality provided $D \leq D^c(a,b)$.

We are now ready to state an implicit formula for the unique solution $q(a,b,D)$ to the convex minimisation problem $\mathrm{ISO}(a,b,D)$.

THEOREM 2.1 (Explicit solution to the ISO's problem) *Let $D > 0$. Then for any $(a,b) \in \mathbb{R}_+^{2N}$, the regulator's problem ISO(a,b,D) admits a unique solution $q(a,b,D)$. Additionally, for any $i \in \mathcal{N}$, the optimal production quantity $q_i(a,b,D)$ is given by*

$$q_i(a,b,D) = \begin{cases} \dfrac{\lambda - a_i}{2b_i} & \textit{if } a_i < \lambda, \\ \dfrac{D - D^c(a,b)}{N^c(a,b)} & \textit{if } a_i = \lambda, b_i = 0, \\ 0 & \textit{if } a_i > \lambda \textit{ or } (a_i = \lambda, b_i > 0), \end{cases} \tag{7}$$

with $\lambda = \lambda(a,b,D)$ determined by (6).

Note that for a fixed configuration of bids of producers $(a,b) \in \mathbb{R}_+^{2N}$, function $\lambda(a,b,D)$ assigns to each demand $D > 0$ the respective marginal price of electricity in the market, see the forthcoming Proposition 2.4.

Proof of Theorem 2.1. The proof will be as follows. First, we find all solutions of the convex optimisation problem $\mathrm{ISO}(a,b,D)$ where we relax constraints stemming from the equity property (E). Based on this solution set, we show that there exists a unique solution to $\mathrm{ISO}(a,b,D)$ satisfying (E).

Let us now state the Karush-Kuhn-Tucker (KKT) conditions which are sufficient conditions for the solution to a convex optimisation problem of $\mathrm{ISO}(a,b,D)$ with condition (E) omitted

$$\begin{cases} 0 = a_i + 2b_i q_i - \mu_i - \lambda, \\ 0 \leq \mu_i \perp q_i \geq 0, \\ D = \sum_{i \in \mathcal{N}} q_i, \end{cases} \tag{8}$$

considering the first two conditions for all $i \in \mathcal{N}$ with λ and $(\mu_i)_i$ being the Lagrange multipliers associated to the demand equality and to the positivity of the production quantities in $\mathrm{ISO}(a,b,D)$, respectively. Since $D > 0$, there has to be some $i \in \mathcal{N}$ such that $q_i > 0$ at any feasible point of $\mathrm{ISO}(a,b,D)$. Then, we may easily verify that Linear Independence Constraint Qualification (LICQ) is satisfied everywhere, and it is well known that KKT conditions (8) are then also sufficient for the solution to $\mathrm{ISO}(a,b,D)$ without (E). To solve (8), let us first show that for the Lagrange multiplier $\lambda \in \mathbb{R}$ we have

$$\lambda \in [0, \lambda^c(a,b)], \tag{9}$$

which should be again understood as $\lambda \geq 0$ if $\lambda^c(a,b) = +\infty$. First, assume for a contradiction that $\lambda < 0$. Since $D > 0$, there has to be some $j \in \mathcal{N}$ such that $q_j > 0$ and thus also $\mu_j = 0$. Then, $a_j + 2b_j q_j = \lambda < 0$ contradicts $a_j, b_j, q_j \geq 0$. Next, for $\lambda^c(a,b) < +\infty$ consider any producer $i \in \mathcal{N}$ with linear bid, that is $b_i = 0$. Then, the first equation of (8) gives $\lambda = a_i - \mu_i \leq a_i$, and so we have $\lambda \leq \lambda^c(a,b)$ by the definition of $\lambda^c(a,b)$.

Let us observe now that

$$\{i \in \mathcal{N} \;:\; \mu_i = 0\} = \{i \in \mathcal{N} \;:\; a_i \leq \lambda\}. \tag{10}$$

Indeed, for all $i \in \mathcal{N}$ such that $\mu_i = 0$ we have

$$\lambda = a_i + 2b_i q_i \geq a_i. \tag{11}$$

On the other hand, $\mu_i > 0$ implies $q_i = 0$ and thus also $\lambda = a_i - \mu_i < a_i$. Based on (10), we see that $a_i > \lambda$ implies $\mu_i > 0$ and finally $q_i = 0$, thus verifying a part of the third formula of (7). Combining (10) with the complementarity constraints of (8), we immediately observe that the last equation of (8) involves only $i \in \mathcal{N}$ such that $a_i \leq \lambda$ and thus we may rewrite it as

$$\sum_{i \in \mathcal{N}: a_i < \lambda} q_i + \sum_{i \in \mathcal{N}: a_i = \lambda, b_i > 0} q_i + \sum_{i \in \mathcal{N}: a_i = \lambda, b_i = 0} q_i = D. \tag{12}$$

If $a_i < \lambda$ then $b_i > 0$ according to (9), and thus using (11) we derived $q_i = \frac{\lambda - a_i}{2b_i}$ and verified the first formula of (7). By the virtue of (11) again, we may omit the second sum in (12) since $a_i = \lambda$, $b_i > 0$ implies $q_i = 0$, proving also the rest of the third formula in (7). To handle with the last sum in (12), we observe that for each $i \in \mathcal{N}$ such that $a_i = \lambda$ and $b_i = 0$ we have $\lambda^c(a,b) \leq a_i$ since such a producer bids linearly. Now using (9) for such a producer i, we obtain $a_i = \lambda^c(a,b)$ or, in other words, $i \in \mathcal{N}^c(a,b)$ and so producer i is a critical bidder. Now, if we treat all critical producers $i \in \mathcal{N}^c(a,b)$ together and use the notation

$$Q^c(a,b) = \sum_{i \in \mathcal{N}^c(a,b)} q_i \geq 0$$

for their overall production, formula (12) reduces to

$$\sum_{i \in \mathcal{N}: a_i < \lambda} \frac{\lambda - a_i}{2b_i} = D - Q^c(a,b). \tag{13}$$

We will solve this equation in a full generality in two steps. The first step corresponds to solution of (13) such that $\lambda < \lambda^c(a,b)$. Consequently, we avoid producer $i \in \mathcal{N}$ such that $a_i = \lambda$ and $b_i = 0$, then $Q^c(a,b) = 0$ and (13) reduces to $F(a,b,\lambda) = D$. Now, referring to Lemma A.1 we deduce that $D < D^c(a,b)$, and so we equivalently obtain $\lambda = \lambda(a,b,D)$ using (6). Hence, we proved (7) since the second statement is avoided having $\lambda < \lambda^c(a,b)$. It is worth noting that, for the moment, we did not consider equity property assumption at all. Moreover, even if we took (E) into account, it will have no effect since constraints (E) are directly implied by the first equation of (7).

The second step corresponds to $\lambda \geq \lambda^c(a,b)$, but regarding (9) we have to deal with $\lambda = \lambda^c(a,b)$ only. Since

$$\sum_{i\in\mathcal{N}:a_i<\lambda^c(a,b)} \frac{\lambda^c(a,b)-a_i}{2b_i} = D^c(a,b),$$

formula (13) is reduced to $Q^c(a,b) = D - D^c(a,b)$ and thus also $D \geq D^c(a,b)$. Hence, we solved $\mathrm{ISO}(a,b,D)$ also for $\lambda = \lambda^c(a,b)$ omitting the additional assumption (E), but the solution with respect to the production quantities of critical producers $i \in \mathcal{N}^c(a,b)$ is not unique. It is unique only with respect to their overall production $Q^c(a,b)$. If $N^c(a,b) > 1$ then there are infinitely many ways to dispatch $Q^c(a,b)$ among the producers $i \in \mathcal{N}^c(a,b)$ bidding $b_i = 0$. However, there is only one solution q satisfying the equity property (E). It is described by the second formula of (7). □

Remark 3 Note that production quantity $q_i(a,b,D) = \frac{D-D^c(a,b)}{N^c(a,b)}$ deduced for $i \in \mathcal{N}^c(a,b)$ in (7) is well-posed provided $a_i = \lambda(a,b,D)$ and $b_i = 0$. Indeed, then $\lambda(a,b,D) \geq \lambda^c(a,b)$ and together with Remark 2 also $a_i = \lambda(a,b,D) = \lambda^c(a,b)$, thus $D \geq D^c(a,b)$ due to (6) and finally also $N^c(a,b) \geq 1$.

2.2. Problem of a Producer

In the rest of this article we stress the point of view of a particular producer denoted by i. We assume that the set of all producers $\mathcal{N}$ is fixed and we suppose that the true production cost function of producer $i \in \mathcal{N}$ is given by $A_i q_i + B_i q_i^2$ with coefficients $A_i \geq 0$ and $B_i > 0$ being known only to producer i (note that $B_i = 0$ is not realistic since the real-world marginal cost of electricity production is increasing in q_i). Now, producer $i \in \mathcal{N}$ aims to maximise his profit $\pi_i(a,b,D)$ given by

$$\pi_i(a,b,D) = (a_i - A_i)\, q_i(a,b,D) + (b_i - B_i)\, q_i(a,b,D)^2 \tag{14}$$

manipulating his own strategic variables $a_i, b_i \geq 0$ with the rest of variables $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ kept fixed. In other words, the i-th producer's problem $P_i(a_{-i}, b_{-i}, D)$ reads

$$P_i(a_{-i}, b_{-i}, D) \qquad\qquad \tilde{\pi}_i = \sup_{a_i,b_i\geq 0} \pi_i(a_i, a_{-i}, b_i, b_{-i}, D).$$

Note that we omit parameters (a_{-i}, b_{-i}, D) when writing $\tilde{\pi}_i$ to keep the notation concise. Now, the solution to $P_i(a_{-i}, b_{-i}, D)$, i.e., the best response of producer $i \in \mathcal{N}$, provides him with a clear instruction how to bid in the market. We consider the overall demand D as a parameter and provide a full discussion of solution to $P_i(a_{-i}, b_{-i}, D)$ with respect to this parameter, see the forthcoming Theorem 3.1. This closely corresponds to the actual needs of producers in the real-world electricity markets. Indeed, generally they have only some expectations on the overall demand D, and so they consider several possible scenarios with various values of D, thus yielding different optimal bids.

We state that we look only for a solution $(\tilde{a}_i, \tilde{b}_i)$ to $P_i(a_{-i}, b_{-i}, D)$ such that $\pi_i(\tilde{a}_i, a_{-i}, \tilde{b}_i, b_{-i}, D) > 0$, that is we assume all bids are profitable. Indeed, since we model only one time period here, it makes no sense to elaborate non-profitable bids. In the real world, producers do sometimes sell electricity below their production cost, but it is only in a situation where the contract spreads over several time periods and the overall profit is still positive. To cope with such a setting it would be necessary to aggregate the profit $\pi_i(a, b, D)$ over the considered time periods. However, this is beyond the scope of this article.

From now on the strategic variables $(a_{-i}, b_{-i}) \in \mathbb{R}^{2N-2}$ of the other producers are supposed to be fixed. Then there are several variables describing the (potential) situation in a market without producer $i \in \mathcal{N}$, i.e., a market consisting only of producers in $\mathcal{N} \setminus \{i\}$. We define

$$\lambda^c(a_{-i}, b_{-i}) = \min_{j \in \mathcal{N} \setminus \{i\}, b_j = 0} a_j,$$

and similarly to (3) also the other critical parameters $\mathcal{N}^c(a_{-i}, b_{-i})$, $D^c(a_{-i}, b_{-i})$ of $\mathrm{ISO}(a_{-i}, b_{-i}, D)$. In the same manner we define function $F(a_{-i}, b_{-i}, \lambda)$ and derive marginal price $\lambda(a_{-i}, b_{-i}, D)$ in analogy to (6). Meaning of all these reduced variables fully corresponds to the case of the full market definitions. Finally, note that also Theorem 2.1 is valid for the setting of $\mathrm{ISO}(a_{-i}, b_{-i}, D)$.

It may occur that there is no maximiser $(\tilde{a}_i, \tilde{b}_i)$ in problem $P_i(a_{-i}, b_{-i}, D)$, i.e., the best response of producer $i \in \mathcal{N}$ does not exist. However, if the supremum of the profit $\tilde{\pi}_i$ defined in $P_i(a_{-i}, b_{-i}, D)$ is positive, a bid $(\tilde{a}_i, \tilde{b}_i)$ is said to be a *limiting best response of producer i if there exists a sequence $(\tilde{a}_i^k, \tilde{b}_i^k)_k$ converging to $(\tilde{a}_i, \tilde{b}_i)$ that yields the optimal profit $\tilde{\pi}_i$, i.e.,*

$$\lim_{(\tilde{a}_i^k, \tilde{b}_i^k) \to (\tilde{a}_i, \tilde{b}_i)} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D) = \tilde{\pi}_i. \tag{15}$$

In such a situation (no existence of maximiser) we will present in the forthcoming Theorem 3.1 a unique limiting best response together with one sequence of bids $(\tilde{a}_i^k, \tilde{b}_i^k)$ yielding the respective optimal profit, thus providing to producer $i \in \mathcal{N}$ a limiting best response strategy. Then, we call $\tilde{\pi}_i$ a limiting profit, and the respective production quantity $\tilde{q}_i$ will be referred to as a limiting production quantity.

Let us first precise the expression of the profit function of producer i and, at the same time, emphasise what values of $(a_i, b_i) \in \mathbb{R}^2_+$ are of potential interest for this producer.

Theorem 2.2 (Explicit formula for the profit function) *Assume $D > 0$ and take $(a_{-i}, b_{-i}) \in \mathbb{R}^{2N-2}_+$. Then, considering the unique solution $q(a, b, D)$ to the regulator's problem ISO(a, b, D), the i-th producer profit $\pi_i(a, b, D)$ is given by one of the following statements:*

(a) for $a_i \le \lambda(a_{-i}, b_{-i}, D)$ and $b_i > 0$,

$$\pi_i(a, b, D) = \frac{\lambda(a, b, D) - a_i}{4b_i^2} \left[a_i b_i - 2A_i b_i + a_i B_i + \lambda(a, b, D)(b_i - B_i)\right], \tag{16}$$

(b) for $a_i < \lambda(a_{-i}, b_{-i}, D)$ and $b_i = 0$ (and so $a_i = \lambda^c(a, b)$ and $\mathcal{N}^c(a, b) = \{i\}$),

$$\pi_i(a, b, D) = (\lambda^c(a, b) - A_i)(D - D^c(a, b)) - B_i(D - D^c(a, b))^2, \tag{17}$$

(c) for $a_i = \lambda(a_{-i}, b_{-i}, D)$ and $b_i = 0$ (and so $a_i = \lambda^c(a,b)$ and $i \in \mathcal{N}^c(a,b)$),

$$\pi_i(a,b,D) = (\lambda^c(a,b) - A_i)\frac{D - D^c(a,b)}{N^c(a,b)} - B_i\left(\frac{D - D^c(a,b)}{N^c(a,b)}\right)^2, \qquad (18)$$

(d) for $a_i > \lambda(a_{-i}, b_{-i}, D)$ it holds $\pi_i(a,b,D) = 0$

Note that the different cases of Theorem 2.2 are described in terms of comparison between a_i and $\lambda(a_{-i}, b_{-i}, D)$ (that is the marginal price without producer i) thus independently of the value of $\lambda(a,b,D)$, which is not known when producer i sets up his bid (a_i, b_i).

Proof of Theorem 2.2. The announced formulas for the i-th producer profit function are deduced from Theorem 2.1 by substituting the formula for $q_i(a,b,D)$ into (14) and the four cases will be considered. For case (a) we have either $a_i < \lambda(a_{-i}, b_{-i}, D)$ implying $a_i < \lambda(a,b,D)$ due to Lemma A.2 (c) since $b_i > 0$, or $a_i = \lambda(a_{-i}, b_{-i}, D) = \lambda(a,b,D)$ due to Lemma A.2 (b) and (c), then the profit is zero which is consistent with (16). For case (b) let us observe that since $a_i < \lambda(a_{-i}, b_{-i}, D) \le \lambda^c(a_{-i}, b_{-i})$ and $b_i = 0$, one immediately has $\mathcal{N}^c(a,b) = \{i\}$ and $\lambda^c(a,b) = \min\{a_j : j \in \mathcal{N}, b_j = 0\} = a_i$ implying $\lambda(a,b,D) \le a_i$. However, this inequality can not be strict because otherwise, by Lemma A.2 (b), we would have $\lambda(a_{-i}, b_{-i}, D) < a_i$, a contradiction. Now in case (c) we may follow the same line as in the proof of case (b) having $a_i = \lambda^c(a,b)$ and $i \in \mathcal{N}^c(a,b)$. Finally if $a_i > \lambda(a_{-i}, b_{-i}, D)$, we may use Lemma A.2 (b) to show statement (d) thus completing the proof. □

2.3. *Non-smooth Tools and Additional Notations*

Due to their piecewise linear structure, functions $F(a,b,\lambda)$ and $\lambda(a,b,D)$ may be non-smooth, but several directional derivatives can be computed. Since these directional derivatives will play an essential role in the forthcoming best response analysis, we state in this subsection definitions and preliminary results on non-smooth tools. For a function $f : \mathbb{R}^n \to \mathbb{R}$ we denote the right-hand side directional derivative of $f(x_1, \dots, x_n)$ with respect to x_i by

$$\partial^+_{x_i} f(x_1, \dots, x_n) = \lim_{t \to 0^+} \frac{f(x_1, \dots, x_i + t, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{t}$$

and analogously $\partial^-_{x_i} f(x_1, \dots, x_n)$ stands for the left-hand side directional derivative. Now, we may compute $\partial^-_D \lambda(a,b,D)$ and $\partial^+_D \lambda(a,b,D)$, which are well-defined since $\lambda(a,b,D)$ is a piecewise linear function in D. First, for any $(a,b) \in \mathbb{R}^{2N}_+$ we define sets $\Gamma^- = \left\{(a,b,\tilde{\lambda}) \in \mathbb{R}^{2N+1}_+ : \tilde{\lambda} \in]\lambda^m(a), \lambda^c(a,b)]\right\}$ and $\Gamma^+ = \left\{(a,b,\tilde{\lambda}) \in \mathbb{R}^{2N+1}_+ : \tilde{\lambda} \in [\lambda^m(a), \lambda^c(a,b)]\right\}$. Then we may define functions $m^\pm : \Gamma^\pm \to \mathbb{R}_+$ as follows

$$\begin{cases} m^-(a,b,\tilde{\lambda}) = \partial^-_D \lambda(a,b,F(a,b,\tilde{\lambda})) \text{ for } \tilde{\lambda} \in]\lambda^m(a), \lambda^c(a,b)], \\ m^+(a,b,\tilde{\lambda}) = \partial^+_D \lambda(a,b,F(a,b,\tilde{\lambda})) \text{ for } \tilde{\lambda} \in [\lambda^m(a), \lambda^c(a,b)[, \\ m^+(a,b,\tilde{\lambda}) = 0 \quad \text{for} \quad \tilde{\lambda} = \lambda^c(a,b), \end{cases} \qquad (19)$$

where $F(a,b,\tilde\lambda)$ corresponds to the total electricity demand given the marginal price $\tilde\lambda$, see (6). Note that $(a,b,\tilde\lambda)\in\Gamma^-$ implies $\lambda^m(a)<\lambda^c(a,b)$, which, regarding Remark 1 (c), has a direct economic interpretation and may be restated as $D^c(a,b)>0$. Therefore $]0,D^c(a,b)]$ is non-empty and $\partial_D^-\lambda(a,b,D)$ is well-defined. Similar argument holds for $m^+(a,b,\tilde\lambda)$, which may be additionally defined at $\tilde\lambda=\lambda^c(a,b)$ since for $D\geq D^c(a,b)$ it holds $\lambda(a,b,D)=\lambda^c(a,b)$ and so $\partial_D^+\lambda(a,b,D)=0$.

Lemma 2.3 *For any $(a,b)\in\mathbb{R}_+^{2N}$, we have*

$$\begin{aligned}\frac{1}{m^-(a,b,\tilde\lambda)}&=\sum_{i\in\mathcal{N}:a_i<\tilde\lambda}\frac{1}{2b_i}>0 && \text{if } \tilde\lambda\in]\lambda^m(a),\lambda^c(a,b)],\\ \frac{1}{m^+(a,b,\tilde\lambda)}&=\sum_{i\in\mathcal{N}:a_i\leq\tilde\lambda}\frac{1}{2b_i}>0 && \text{if } \tilde\lambda\in[\lambda^m(a),\lambda^c(a,b)[,\end{aligned}\tag{20}$$

and thus $m^-(a,b,\tilde\lambda)\geq m^+(a,b,\tilde\lambda)$ for all $\tilde\lambda\in]\lambda^m(a),\lambda^c(a,b)[$.

The proof of this lemma is included in the appendix. It is derived from calculus rules of directional derivatives. Let us now identify the exact meaning of $\lambda(a,b,D)$.

Proposition 2.4 *Consider the setting of Theorem 2.1. Then, for any $(a,b)\in\mathbb{R}_+^{2N}$, function $C(a,b,D)$ representing the total expenses of consumers, see (1), is smooth with respect to D on $]0,+\infty[$ and $\partial_D^-C(a,b,D)=\partial_D^+C(a,b,D)=\lambda(a,b,D)$. Thus $\lambda(a,b,D)$ corresponds to the marginal price of electricity in the market.*

For the sake of simplicity we will use the term "marginal price" for $\lambda(a,b,D)$ from now on. We think that such a relation to describe $\lambda(a,b,D)$ is necessary to give a real economical meaning to $\lambda(a,b,D)$ which originally corresponds only to a Lagrangian multiplier in the problem ISO(a,b,D). The proof of Proposition 2.4 is given in the appendix. In short, we substitute $q_i(a,b,D)$ given by Theorem 2.1 into (1), and then we differentiate $C(a,b,D)$ using Lemma 2.3.

Remark 4 Using (6) we observe that $m^\pm(a,b,\lambda(a,b,\tilde D))=\partial_D^\pm\lambda(a,b,\tilde D)$. Thus, $\lambda(a,b,D)$ is a smooth function with respect to D for $D\leq D^c(a,b)$ if and only if $m^+(a,b,\lambda(a,b,D))=m^-(a,b,\lambda(a,b,D))$, or, equivalently, $\{i\in\mathcal{N}\;:\;a_i=\lambda(a,b,D)\}=\emptyset$ due to Lemma 2.3. This condition has a clear economic meaning. Consider $a_i=\lambda(a,b,D)$ for some $i\in\mathcal{N}$, then the marginal price $\lambda(a,b,D)$ is high enough to allow producer i to be active in the market. Indeed, from (7) we observe that $a_i>\lambda(a,b,D)$ implies $q_i(a,b,D)=0$, and $a_i<\lambda(a,b,D)$ implies $q_i(a,b,D)>0$. Next, using Lemma 2.3 again we observe that the same condition may be equivalently expressed as $m^+(a,b,\lambda(a,b,D))\neq m^-(a,b,\lambda(a,b,D))$, i.e., the slope of the marginal price "jumps" each time a new producer starts to be active in the market. Finally, we note that this condition may be also stated in terms of demand D using (6). Then, function $\lambda(a,b,D)$ is a smooth function with respect to D if and only if $D\notin\{F(a,b,a_i)\;:\;i\in\mathcal{N}\}$. This observation is valid also for the case of $D=D^c(a,b)<+\infty$ since $D^c(a,b)=F(a,b,a_j)$ for some $j\in\mathcal{N}^c(a,b)\subset\mathcal{N}$.

Next, for producer $i\in\mathcal{N}$ we introduce a quantity of production

$$q_i^\star(a_{-i},b_{-i})=\frac{\lambda^c(a_{-i},b_{-i})-A_i}{2B_i}.$$

In the following lemma we observe that $q_i^\star(a_{-i},b_{-i})$ sometimes corresponds to an ideal quantity of production. Indeed, having $(a_i,b_i)=(\lambda^c(a_{-i},b_{-i}),0)$, the additional production cost for $q_i(a,b,D)>q_i^\star(a_{-i},b_{-i})$ is higher than the respective additional gain due to rising marginal production cost (expressed by $B_i>0$).

Lemma 2.5 *Let $(a_{-i},b_{-i})\in\mathbb{R}^{2N-2}$ and $A_i<\lambda^c(a_{-i},b_{-i})$ for some $i\in\mathcal{N}$. Then*

$$\begin{aligned}\pi_i(\lambda^c(a_{-i},b_{-i}),a_{-i},0,b_{-i},D)&\le(\lambda^c(a_{-i},b_{-i})-A_i)\,q_i^\star(a_{-i},b_{-i})-B_i\,(q_i^\star(a_{-i},b_{-i}))^2\\&=(\lambda^c(a_{-i},b_{-i})-A_i)^2/4B_i,\end{aligned}$$

which becomes equality if and only if $D=D^c(a_{-i},b_{-i})+N^c(a,b)\,q_i^\star(a_{-i},b_{-i})$.

Proof. If we consider q_i in (14) as a free variable, the profit of producer i is given by $\theta(q_i):q_i\to(\lambda^c(a_{-i},b_{-i})-A_i)\,q_i-B_i\,q_i^2$, and so $q_i^\star(a_{-i},b_{-i})$ is a strict maximiser of $\theta(q_i)$ with respect to $q_i\ge 0$. Then, the condition for equality stems from $q_i(a,b,D)=q_i^\star(a_{-i},b_{-i})$ using Theorem 2.1. □

Further, we introduce notation for electricity production quantities $q_i^c(a_{-i},b_{-i})$ and $q_i^m(a_{-i},b_{-i})$ being important for producer $i\in\mathcal{N}$. Indeed, $q_i^c(a_{-i},b_{-i})$ and $q_i^m(a_{-i},b_{-i})$ play a significant role when producer i decides his optimal bid, see the forthcoming Theorem 3.1. They are defined as follows:

$$q_i^m(a_{-i},b_{-i})=\frac{\lambda^m(a_{-i})-A_i}{2B_i+m^+(a_{-i},b_{-i},\lambda^m(a_{-i}))},\tag{21}$$

$$q_i^c(a_{-i},b_{-i})=\begin{cases}\dfrac{\lambda^c(a_{-i},b_{-i})-A_i}{2B_i+m^-(a_{-i},b_{-i},\lambda^c(a_{-i},b_{-i}))} & \text{for } \lambda^m(a_{-i})<\lambda^c(a_{-i},b_{-i}),\\ 0 & \text{for } \lambda^m(a_{-i})=\lambda^c(a_{-i},b_{-i}).\end{cases}\tag{22}$$

Note that $q_i^c(a_{-i},b_{-i})$ can not be defined by the first formula in (22) once $\lambda^m(a_{-i})=\lambda^c(a_{-i},b_{-i})$, see (19). The extended definition is to facilitate the formulation of the concluding Theorem 3.1.

Lemma 2.6 *For any $(a_{-i},b_{-i})\in\mathbb{R}_+^{2N-2}$ it holds $q_i^c(a_{-i},b_{-i})<q_i^\star(a_{-i},b_{-i})$ provided $A_i<\lambda^c(a_{-i},b_{-i})$, and one always has $q_i^m(a_{-i},b_{-i})\le q_i^\star(a_{-i},b_{-i})$.*

Proof. A proof of this lemma stems directly from the respective definitions. □

Finally, considering producer $i\in\mathcal{N}$ and bids of other producers $(a_{-i},b_{-i})\in\mathbb{R}_+^{2N-2}$ such that $A_i\le\lambda^c(a_{-i},b_{-i})$, we define

$$q_i^0(a_{-i},b_{-i})=F(a_{-i},b_{-i},A_i).$$

From definition we observe that one always has $q_i^0(a_{-i},b_{-i})\le D^c(a_{-i},b_{-i})$, and for any $(a_{-i},b_{-i})\in\mathbb{R}_+^{2N}$ the three statements $q_i^m(a_{-i},b_{-i})\ge 0$, $\lambda^m(a_{-i})\ge A_i$ and $q_i^0(a_{-i},b_{-i})=0$ are equivalent.

3. Main Results

Now we are in position to state the main results, namely Theorem 3.1 and Corollary 3.3, where we discuss (existence of) the best response of producer $i\in\mathcal{N}$ with respect *to positive values of the overall electricity demand. Note that some partial answers describing the best response of one producer has been given in [2, 3] where the authors provide necessary optimality conditions for a bid to be a local best response. In [2] time dependent bids are considered. Nevertheless, due to the non-convexity of the objective function of the producer in our model, necessary conditions are not sufficient and local best responses are not global best responses that we are looking for.*

Let us observe that in the sequel we will investigate only values of $(a_i, b_i) \in \mathbb{R}^2_+$ *such that assumptions of Theorem 2.2 (a), (b) and (c) are satisfied. Otherwise, the* i*-th producer's profit would be non-positive and we assume that under such conditions the producer will not bid at all. Next, we characterise conditions for the existence of a solution to* $P_i(a_{-i}, b_{-i}, D)$*, determine this solution and show that it is unique.*

THEOREM 3.1 (Best response evaluation) *Let* $D > 0$, $(a_{-i}, b_{-i}) \in \mathbb{R}^{2N-2}_+$ *for some* $i \in \mathcal{N}$ *and consider the problem*

$$P_i(a_{-i}, b_{-i}, D) \qquad\qquad \tilde{\pi}_i = \sup_{a_i, b_i \geq 0} \pi_i(a_i, a_{-i}, b_i, b_{-i}, D). \tag{23}$$

Then either $A_i \geq \lambda^c(a_{-i}, b_{-i})$ *and* $\tilde{\pi}_i = 0$*, or one of the following alternatives holds:*

(a) if $D \in]0, q_i^0(a_{-i}, b_{-i})]$ *then* $\tilde{\pi}_i = 0$*,*
(b) if $D \in]q_i^0(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i})[$ *then* $\tilde{\pi}_i > 0$ *and there is a unique best response* $(\tilde{a}_i, \tilde{b}_i)$ *given by* $\tilde{b}_i = 0$ *and* $\tilde{a}_i \in [\lambda^m(a_{-i}), \lambda^c(a_{-i}, b_{-i})[$ *satisfying*

$$\begin{cases} \tilde{a}_i = \lambda^m(a_{-i}) & \text{if} \quad D \leq q_i^m(a_{-i}, b_{-i}), \\ \dfrac{\tilde{a}_i - A_i}{2B_i + m^-(a_{-i}, b_{-i}, \tilde{a}_i)} \leq D - F(a_{-i}, b_{-i}, \tilde{a}_i) \\ \qquad\qquad \leq \dfrac{\tilde{a}_i - A_i}{2B_i + m^+(a_{-i}, b_{-i}, \tilde{a}_i)} & \text{if} \quad D > q_i^m(a_{-i}, b_{-i}), \end{cases} \tag{24}$$

(c) if $D \in [D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})]$ *then* $\tilde{\pi}_i > 0$ *and* $(\lambda^c(a_{-i}, b_{-i}), 0)$ *is a unique limiting best response. Moreover, for any sequence of linear bids* $(\tilde{a}_i^k)$ *such that* $\tilde{a}_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$ *one has* $\lim_{k\to+\infty} \pi_i(\tilde{a}_i^k, a_{-i}, 0, b_{-i}, D) = \tilde{\pi}_i$*.*
(d) if $D \in]D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i}), +\infty[$ *and* $D \neq D^c(a_{-i}, b_{-i}) + (N^c(a_{-i}, b_{-i}) + 1)\, q_i^\star(a_{-i}, b_{-i})$ *then* $\tilde{\pi}_i > 0$ *and* $(\lambda^c(a_{-i}, b_{-i}), 0)$ *is a unique limiting best response. Moreover, for any sequence* $(\tilde{a}_i^k, \tilde{b}_i^k)$ *such that* $\tilde{b}_i^k \searrow 0$ *and*

$$\tilde{a}_i^k = \frac{A_i \tilde{b}_i^k + B_i\, \lambda^c(a_{-i}, b_{-i})}{\tilde{b}_i^k + B_i}, \tag{25}$$

one has $\lim_{k\to+\infty} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D) = \tilde{\pi}_i$*.*
(e) if $D = D^c(a_{-i}, b_{-i}) + (N^c(a_{-i}, b_{-i}) + 1)\, q_i^\star(a_{-i}, b_{-i})$ *then* $\tilde{\pi}_i > 0$ *and there is a unique best response* $(\tilde{a}_i, \tilde{b}_i) = (\lambda^c(a_{-i}, b_{-i}), 0)$*.*

Note that inequalities in (24) are, actually, very straightforward to solve due to monotonicity if the involved functions, as discussed in the proof of Proposition A.8. Before proving Theorem 3.1, we first show that any best response (limiting or not) in (23) is a linear bid, and we clarify the role of sequences of bids (a_i^k, b_i^k) *such that* $b_i^k \searrow 0$*.*

Proposition 3.2 *Let $D > 0$, $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ and $A_i < \lambda(a_{-i}, b_{-i}, D)$ for some $i \in \mathcal{N}$. Consider the problem $P_i(a_{-i}, b_{-i}, D)$ and supremum of profit $\tilde{\pi}_i$ as defined in Theorem 3.1. Then no quadratic best response (limiting or not) of problem $P_i(a_{-i}, b_{-i}, D)$ exists. Moreover, one of the following alternatives is valid:*

(a) for $D \leq D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})$ it holds $\tilde{\pi}_i = \sup_{a_i \geq 0} \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$,

(b) otherwise, for any sequence of bids $\tilde{a}_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$ and $\tilde{b}_i^k \searrow 0$ satisfying (25), we have

$$\tilde{\pi}_i = \max\left\{\sup_{a_i \geq 0} \pi_i(a_i, a_{-i}, 0, b_{-i}, D), \lim_{k\to+\infty} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D)\right\}.$$

Moreover, the limiting production quantity yielded by sequence $(\tilde{a}_i^k, \tilde{b}_i^k)$ is $q_i^\star(a_{-i}, b_{-i})$.

Proof of this proposition is included in appendix. Now we state the proof of Theorem 3.1.

Proof of Theorem 3.1. First, observe that bidding, e.g., $(a_i, b_i) = (A_i, B_i)$, we obtain $\pi_i(a, b, D) = 0$, thus $\tilde{\pi}_i \geq 0$. Then, since $\pi_i(a_i, a_{-i}, 0, b_{-i}, D) \leq 0$ for $a_i > \lambda(a_{-i}, b_{-i}, D)$ regarding Theorem 2.2 (d), we may further assume $a_i \leq \lambda(a_{-i}, b_{-i}, D)$ without loss of generality. Similarly, for $A_i \geq \lambda^c(a_{-i}, b_{-i}) \geq \lambda(a_{-i}, b_{-i}, D)$ it holds $\tilde{\pi}_i \leq 0$ according to Corollary A.4. Therefore, throughout the rest of the proof we assume $A_i < \lambda^c(a_{-i}, b_{-i})$. Then we note that $q_i^0(a_{-i}, b_{-i})$ is well-defined and $q_i^0(a_{-i}, b_{-i}) \leq D^c(a_{-i}, b_{-i})$. Now we use this inequality to show that the presented statements are indeed alternatives. Observing $q_i^c(a_{-i}, b_{-i}) < q_i^\star(a_{-i}, b_{-i})$ due to Lemma 2.6, the only problematic cases may be (a) and (c) provided $D^c(a_{-i}, b_{-i}) = 0$. Then, however, $q_i^0(a_{-i}, b_{-i}) = 0$ and the case (a) is avoided.

Next, the strategy of the proof is to reduce the analysis of $P_i(a_{-i}, b_{-i}, D)$ to (possibly limiting) bids such that $b_i = 0$ whenever possible using Proposition 3.2. For $D \leq q_i^0(a_{-i}, b_{-i}) = F(a_{-i}, b_{-i}, A_i)$ we have $\lambda(a_{-i}, b_{-i}, D) \leq A_i$ due to Lemma A.1 and definition (6), and so $\tilde{\pi}_i \leq 0$ with regards to Corollary A.4. Thus we obtained the statement (a). Note that further we may assume $A_i < \lambda(a_{-i}, b_{-i}, D)$.

Now, for $D \in]q_i^0(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i})[$ we may consider only linear bids since assumption

$$D \leq D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i}), \tag{26}$$

of Proposition 3.2 (a) is satisfied. Then, using Proposition A.8 (a) there exists a unique best response candidate $(\tilde{a}_i, 0)$ yielding positive profit with $\tilde{a}_i \in [\lambda^m(a_{-i}), \lambda(a_{-i}, b_{-i}, D)[$ given by (24). To show that it is a unique best response with respect to $a_i \in [\lambda^m(a_{-i}), \lambda(a_{-i}, b_{-i}, D)]$, we observe that $\pi_i(a, b, D)$ given by Theorem 2.2 (b) and (c) is continuous in a_i on $[0, \lambda(a_{-i}, b_{-i}, D)]$ due to Corollary A.5 (a), and so

$$\pi_i(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}, D) \leq \pi_i(\tilde{a}_i, a_{-i}, 0, b_{-i}, D). \tag{27}$$

However, equality in (27) contradicts strict quasiconcavity of $\pi_i(a, b, D)$ in variable a_i on the segment $[\lambda^m(a_{-i}), \lambda(a_{-i}, b_{-i}, D)[$ as given by Proposition A.7. Thus $(\tilde{a}_i, 0)$ is indeed a unique best response and so we shown statement (b) for $D \in]q_i^0(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i})[$.

From now on, we deal with variant $D \geq D^c(a_{-i}, b_{-i})$, then $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$, thus we have to face discontinuity of $\pi_i(a, b, D)$ as described by Corollary A.5 (c) and (d). Let us consider first $D \in [D^c(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i})[$ with $q_i^c(a_{-i}, b_{-i}) > 0$ because otherwise this interval is empty. Observe that inequality (26) holds due to $q_i^\star(a_{-i}, b_{-i}) > q_i^c(a_{-i}, b_{-i})$, see Lemma 2.6, and so we may still consider only $\tilde{b}_i = 0$ as discussed in Proposition 3.2 (a). Further, using (22) we deduce $\lambda^m(a_{-i}) < \lambda^c(a_{-i}, b_{-i})$. Thus we have a unique profit maximiser $\tilde{a}_i$ with respect to $a_i \in [0, \lambda^c(a_{-i}, b_{-i})[$ due to Proposition A.8 (b), and we have positive profit at this point. To show that it is indeed a unique best response, we examine the last possible candidate for a best response, a linear bid having $a_i = \lambda^c(a_{-i}, b_{-i})$. This point can be the best response of producer i only in the case when assumptions of Corollary A.5 (b) or (d) are satisfied. Denoting $\xi = \frac{1}{2}\frac{N^c(a_{-i},b_{-i})+2}{N^c(a_{-i},b_{-i})+1}$, we have $\frac{1}{2} < \xi \leq \frac{3}{4}$ due to $N^c(a_{-i}, b_{-i}) \geq 1$, and observe

$$\begin{aligned}\xi(D - D^c(a_{-i}, b_{-i})) \geq q_i^\star(a_{-i}, b_{-i}) > q_i^c(a_{-i}, b_{-i}) &> D - D^c(a_{-i}, b_{-i}) \\ &> \xi(D - D^c(a_{-i}, b_{-i}))\end{aligned}$$

using assumptions of Corollary A.5 (b) and (d), Lemma 2.6, assumption on $q_i^c(a_{-i}, b_{-i})$, and the fact that $\xi < 1$, respectively. We obtained a contradiction, thus $(\tilde{a}_i, 0)$ is indeed a unique best response, and statement (b) is valid assuming $D \in [D^c(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i})[$.

Next, for $D \in [D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})]$ inequality (26) is still valid, and so we may again consider only $\tilde{b}_i = 0$, see Proposition 3.2 (a). Then we observe that $\lambda^c(a_{-i}, b_{-i}) = \lambda(a_{-i}, b_{-i}, D)$ and $q_i^c(a_{-i}, b_{-i}) \leq D - D^c(a_{-i}, b_{-i})$ at the same time. Thus, using Proposition A.8, the profit is strictly increasing in a_i on $[0, \lambda^c(a_{-i}, b_{-i})[$. However, since $\frac{1}{\xi}\, q_i^\star(a_{-i}, b_{-i}) > q_i^\star(a_{-i}, b_{-i}) \geq D - D^c(a_{-i}, b_{-i})$, the profit is not upper-semicontinuous at $(\lambda^c(a_{-i}, b_{-i}), 0)$ due to Corollary A.5 (c). Thus we may conclude that any sequence $(\tilde{a}_i^k, \tilde{b}_i^k)$ such that $\tilde{a}_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$ and $\tilde{b}_i^k = 0$ is converging to the unique limiting best response $(\lambda^c(a_{-i}, b_{-i}), 0)$. Moreover, it yields a positive profit since condition (A7), rewritten in our case as $D < D^c(a_{-i}, b_{-i}) + 2q_i^\star(a_{-i}, b_{-i})$, is satisfied due to our assumptions, thus showing statement (c) of the theorem.

Next, for $D > D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})$, we consider any sequence $(\tilde{a}_i^k, \tilde{b}_i^k) \to (\lambda^c(a_{-i}, b_{-i}), 0)$ satisfying (25). Then, according to Proposition 3.2 (b), one has that $\tilde{\pi}_i$ is actually the maximum of the three values $\sup_{a_i \in [0, \lambda^c(a_{-i}, b_{-i})[} \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$, $\pi_i(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}, D)$, and $\lim_{k \to +\infty} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D)$.

Now, denoting $\pi_i^* = \frac{(\lambda^c(a_{-i},b_{-i}) - A_i)^2}{4B_i}$, using Proposition 3.2 (b) and Lemma 2.5, we deduce that

$$\lim_{k \to +\infty} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D) = \pi_i^\star \geq \pi_i(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}, D).$$

Moreover, even for $D > D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i}$ the profit is strictly increasing in a_i on $[0, \lambda^c(a_{-i}, b_{-i}[$ by the same arguments as in the previous paragraph. Thus

$$\sup_{a_i \in [0, \lambda^c(a_{-i}, b_{-i})[} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) = \sup_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i})} \pi_i(a_i^k, a_{-i}, 0, b_{-i}, D)$$

and consequently, due to equation (A27) of Proposition A.14, we finally get

$$\tilde{\pi}_i = \lim_{k \to +\infty} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D) = \pi_i^\star,$$

thus showing that $(\lambda^c(a_{-i}, b_{-i}), 0)$ is the unique (limiting or not) best response. Finally, if $D = D^c(a_{-i}, b_{-i}) + (N^c(a_{-i}, b_{-i}) + 1)\, q_i^\star(a_{-i}, b_{-i})$ then $\pi_i(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}, D) = \pi_i^\star = \tilde{\pi}_i$ due to Lemma 2.5, proving that $(\lambda^c(a_{-i}, b_{-i}), 0)$ is a (exact) best response in this case, thus finishing the proof of statements (d) and (e). □

Let us end this work by a complete description of the production quantity corresponding to the (limiting or not) best response of a producer.

Corollary 3.3 (Optimal production quantity) *Consider the setting of Theorem 3.1. Then either $A_i \geq \lambda^c(a_{-i}, b_{-i})$ and $\tilde{q}_i = 0$, or one of the following alternatives holds:*

(a) if $D \in]0, q_i^0(a_{-i}, b_{-i})]$ then $\tilde{q}_i = 0$,
(b) if $D \in]q_i^0(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i})[$ and moreover
 (i) $D \leq q_i^m(a_{-i}, b_{-i})$ then $\tilde{q}_i = D$,
 (ii) $D > q_i^m(a_{-i}, b_{-i})$ then $\tilde{q}_i = D - F(a_{-i}, b_{-i}, \tilde{a}_i)$ with $\tilde{a}_i \in [\lambda^m(a_{-i}), \lambda^c(a_{-i}, b_{-i})[$ given by the second part of (24),
(c) if $D \in [D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})]$ then $\tilde{q}_i = D - D^c(a_{-i}, b_{-i})$,
(d) if $D \in]D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i}), +\infty[$ then $\tilde{q}_i = q_i^\star(a_{-i}, b_{-i})$.

Proof. If the linear term of the production cost function is above or equal to the critical marginal price, $A_i \geq \lambda^c(a_{-i}, b_{-i})$, then producer i prefers not to bid because the generated profit $\tilde{\pi}_i$ would be non-positive, see Theorem 3.1, and so $\tilde{q}_i = 0$. Hence we assume throughout the proof that $A_i < \lambda^c(a_{-i}, b_{-i})$. Now for $D \in]0, q_i^0(a_{-i}, b_{-i})]$ we follow the same reasoning as for Theorem 3.1 (a).

Once $D \in]q_i^0(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i})[$ the best response satisfies $\tilde{b}_i = 0$ with $\tilde{a}_i < \lambda^c(a_{-i}, b_{-i})$ given by (24), see Theorem 3.1 (b). Then $\tilde{a}_i = \lambda^c(\tilde{a}_i, a_{-i}, 0, b_{-i}) = \lambda(\tilde{a}_i, a_{-i}, 0, b_{-i}, D)$ and thus $D^c(\tilde{a}_i, a_{-i}, 0, b_{-i}) = F(a_{-i}, b_{-i}, \tilde{a}_i)$. Moreover, one also has $\mathcal{N}^c(\tilde{a}_i, a_{-i}, 0, b_{-i}) = \{i\}$ (thus $N^c(\tilde{a}_i, a_{-i}, 0, b_{-i}) = 1$) and therefore $\tilde{q}_i = D - F(a_{-i}, b_{-i}, \tilde{a}_i)$ by Theorem 2.1. This demonstrates statement (b)-(ii). But if moreover $D \leq q_i^m(a_{-i}, b_{-i})$, condition (24) reduces to $\tilde{a}_i = \lambda^m(a_{-i})$ and so $F(a_{-i}, b_{-i}, \tilde{a}_i) = 0$, thus proving statement (b)-(i).

Now, case $D \in [D^c(a_{-i}, b_{-i}) + q_i^c(a_{-i}, b_{-i}), D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})]$ is a direct consequence of Theorem 3.1 (c) and Corollary A.5. Finally, for $D \in]D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i}), +\infty[$ the conclusion follows from Theorem 3.1 (d) and (e) together with Proposition A.14 (b). □

Let us illustrate on an example the conclusion of Theorem 3.1.

Example 3.4 Consider a market with 5 producers, $\mathcal{N} = \{1, \ldots, 5\}$, having bids given by

$i \in \mathcal{N}$	1	2	3	4	5
(a_i, b_i)	(40, 2)	(41, 4)	(42, 1)	(43, 0)	(44, 0)

Then we will characterise for which value of the demand D a new producer, denoted by $i = 6$, should bid to obtain a positive profit and what values (a_6, b_6) maximise his profit function $\pi_6(a, b, D)$. We assume that his true production cost coefficients are $(A_6, B_6) = (5, 1)$.

Based on these data we provide four figures to illustrate the previous results. In Figure 1 the positive part of profit $\pi_6(a, b, D)$ is depicted for $D = 30$. Then $D > D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})$ and so the optimal limiting profit is yielded by limiting best response described by the case (d) of Theorem 3.1. For readers convenience we draw also the limiting best response having $b_i = 0$ as given by the case (c) of Theorem 3.1. Thus, the discontinuity of the profit at the critical point $(a_6, b_6) = (\lambda^c(a_{-i}, b_{-i}), 0)$ as discussed in the proof of Theorem 3.1 may be well-observed.

To comment the next figures, we first define $D_1 = q_i^m(a_{-6}, b_{-6})$, $D_2 = D^c(a_{-6}, b_{-6}) + q_i^c(a_{-6}, b_{-6})$ and $D_3 = D^c(a_{-6}, b_{-6}) + q_i^*(a_{-6}, b_{-6})$. Then, in Figure 2 the linear coefficient a_6 of the (eventually limiting) best response given by Theorem 3.1 is shown. The flat parts of the graph correspond to a new producer being ready to enter the market. To avoid sharing part of the production with this producer, producer 6 "fixes" the marginal price by bidding at a relatively low level. For example, starting at $D = 0$, producer 6 fixes its bid at $a_6 = 40$ not to share the production with producer 1 having $a_1 = 40$ and $b_1 > 0$. Then, from $D \approx 6$, it is better to bid $a_6 > 40$ and share part of the production with producer 1. The same reasoning holds also for all the other steps depicted in Figure 2 corresponding to higher marginal prices and more producers in the market becoming active.

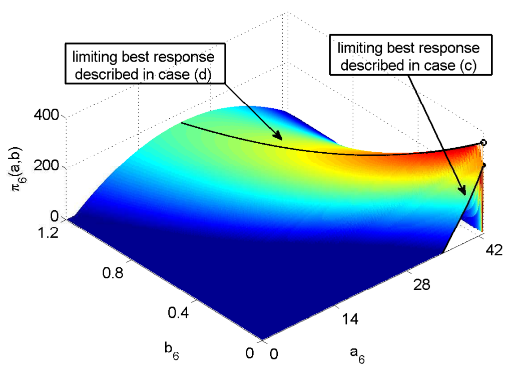

Figure 1. The positive part of profit $\pi_6(a, b, D)$ with $D = 30$, bid of other producers (a_{-6}, b_{-6}) and true production cost coefficients (A_6, B_6).

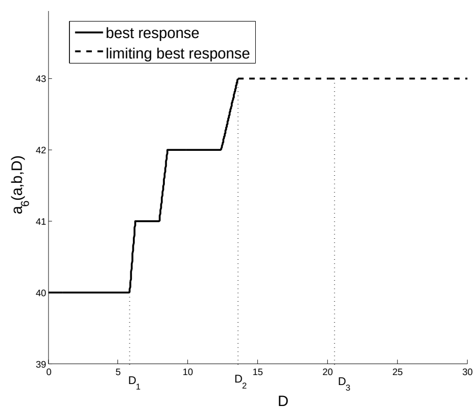

Figure 2. Best response and limiting best response of producer 6 provided $b_6 = 0$ with (a_{-6}, b_{-6}) and true production cost coefficients (A_6, B_6).

Finally based on the above data, Figure 3 shows the production quantity, while Figure 4 shows the obtained profit of producer 6 corresponding to the best response or limiting best response depicted for the complete range of demand D.

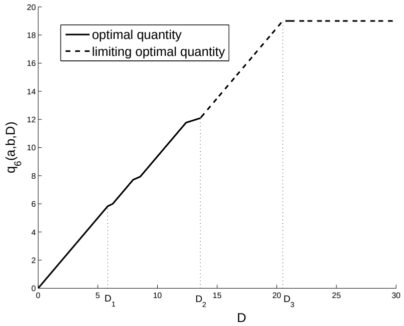

Figure 3. Production quantity yielded by the best response (or limiting best response) of producer 6 with (a_{-6}, b_{-6}) and A_6,B_6.

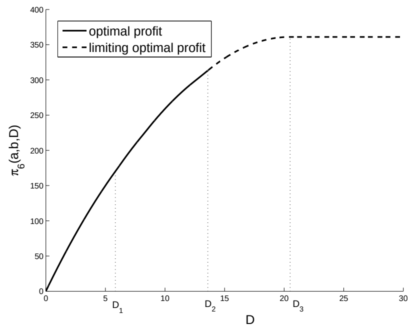

Figure 4. Profit yielded by the best response (or limiting best response) of producer 6 with (a_{-6}, b_{-6}) and A_6,B_6.

4. Conclusion

In this work we provide an analytic solution to the problem of a producer in the electricity market, thus enabling us to find the best response of a producer for all the possible market configurations, see Theorem 3.1. In some cases the best response does not exist, and a limiting best response yielding a positive supremum of profit, was determined instead.

Note also that Proposition A.12 clearly shows, that modelling the electricity market with purely quadratic bids (that is with $b_i > 0$) may not be consistent from the game-theoretical point of view. Indeed considering cases (b) and (e) of Theorem 3.1, the (non-limiting) best response of a producer is reached only by linear bid with $b_i = 0$.

In this work we considered a simplified model of the electricity market with no production bounds. However, as described in Theorem 3.1 and Corollary 3.3, there exists an ideal quantity of production $q_i^\star(a_{-i}, b_{-i})$ which actually corresponds to an implicit production bound, since we always have $\tilde{q}_i \le q_i^\star(a_{-i}, b_{-i})$.

Note that in work - Part 2 of this couple of articles - we assume that bids of the other producers $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ are fixed and in particular not depending on D. However, in order to determine possible Nash equilibrium of the market it is important to consider that the strategic behaviour of the other producers also depends on the demand D or, in other words, that the other producers would also bid their best response, if exists. It is what has been done in Part 1 of this couple of articles (see [4]).

The aim of Part 2 is to characterise an exact best response $(\tilde{a}_i, \tilde{b}_i)$ reaching an attained optimal profit of the considered producer i. But, in some cases (see e.g. (d) of Proposition A.14) only a limiting optimal profit exists and cannot be reached as limits of linear bids. This observation is of practical interest. In such cases the producer can nevertheless evaluate a quadratic bid approximately reaching this limiting optimal profit.

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Appendix A. Proofs of Statements

Let us first show that function $F(a, b, \lambda)$ defined by (4) possesses continuity and monotonicity property playing an important role in the sequel.

LEMMA A.1 *For any $(a, b) \in \mathbb{R}_+^{2N}$, function $\lambda \to F(a, b, \lambda)$ is continuous on its domain. Moreover, it is equal to 0 on $[0, \lambda^m(a)]$, and strictly increasing on $[\lambda^m(a), \lambda^c(a, b)]$ if $\lambda^c(a, b) < +\infty$ and on $[\lambda^m(a), +\infty[$ otherwise.*

Proof of Lemma A.1. Denoting $\mathcal{N}_{sub} = \{i \in \mathcal{N} : a_i < \lambda^c(a,b)\}$ and $(x)_+ = \max\{x, 0\}$ for any $x \in R$, we may rewrite $F(a,b,\lambda)$ as follows

$$F(a,b,\lambda) = \sum_{i \in \mathcal{N}_{sub}} \frac{(\lambda - a_i)_+}{2b_i}$$

immediately observing continuity of $F(a,b,\lambda)$ with respect to λ.

The fact that $F(a,b,\lambda) = 0$ for any $\lambda \in [0, \lambda^m(a)]$ is directly due to definition. Now to enlighten the monotonicity of F let us consider $\lambda^m(a) \le \lambda_1 < \lambda_2 < \lambda^c(a,b)$. Then we have

$$F(a,b,\lambda_1) = \sum_{i \in \mathcal{N}: a_i < \lambda_1} \frac{\lambda_1 - a_i}{2b_i} < \sum_{i \in \mathcal{N}: a_i < \lambda_1} \frac{\lambda_2 - a_i}{2b_i} \le \sum_{i \in \mathcal{N}: a_i < \lambda_2} \frac{\lambda_2 - a_i}{2b_i} = F(a,b,\lambda_2).$$

Note that λ_2 can be taken as $\lambda^c(a,b)$ if $\lambda^c(a,b) < +\infty$. □

LEMMA A.2 *Consider demand $D > 0$ and bid vector $(a,b) \in \mathbb{R}^{2N}_+$. Then*

(a) $\lambda(a,b,D) \le \lambda(a_{-i}, b_{-i}, D)$,
(b) $a_i \le \lambda(a,b,D)$ if and only if $a_i \le \lambda(a_{-i}, b_{-i}, D)$,
(c) if $b_i > 0$, then, $a_i < \lambda(a,b,D)$ if and only if $a_i < \lambda(a_{-i}, b_{-i}, D)$.

Although this lemma can appear to be only a technical issue, it has some straightforward economical interpretations:

(a) *it states that the marginal price in the market including producer i is always less or equal to the marginal price in the market without producer i*
(b) *(respectively (c)) it enlightens that if producer i would have been active with a linear bid (respectively with a quadratic bid) in the market without him then he will be active in the market with him.*

An important consequence of the proof of case (b) above is that if producer i offers a linear bid a_i strictly lower than the marginal price in the market without him, then the marginal price of the market including him will adjust to his bid a_i.

Proof of Lemma A.2. We proof statement (a) directly from the definition of $\lambda(a,b,D)$. There are four different settings to consider. First, for D such that $D < D^c(a,b)$ and $D < D^c(a_{-i}, b_{-i})$ we have $F(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) = D = F(a,b,\lambda(a,b,D)) \ge F(a_{-i}, b_{-i}, \lambda(a,b,D))$ proving the statement with regards to Lemma A.1. Next, if D is such that $D^c(a_{-i}, b_{-i}) > D \ge D^c(a,b)$ and taking into account that one always has $\lambda^c(a_{-i}, b_{-i}) \ge \lambda^c(a,b)$, we necessarily obtain $\lambda^c(a_{-i}, b_{-i}) > \lambda^c(a,b)$. Then we have $a_i = \lambda^c(a,b)$, $b_i = 0$, and $F(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) = D \ge F(a,b,\lambda^c(a,b)) = F(a,b,a_i) = F(a_{-i}, b_{-i}, a_i)$. Finally, since $a_i = \lambda^c(a,b) = \lambda(a,b,D)$ we obtained $F(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) \ge F(a_{-i}, b_{-i}, \lambda(a,b,D))$ proving the statement due to Lemma A.1 again. Now, considering D such that $D^c(a,b) > D \ge D^c(a_{-i}, b_{-i})$, we may write $F(a,b,\lambda^c(a,b)) > F(a_{-i}, b_{-i}, \lambda^c(a_{-i}, b_{-i})) \ge F(a_{-i}, b_{-i}, \lambda^c(a,b))$ implying $\frac{\lambda^c(a,b) - a_i}{2b_i} > 0$ from the definition of F. Then, however, $\lambda^c(a,b) = \lambda^c(a_{-i}, b_{-i})$ and so $\lambda(a,b,D) \le \lambda^c(a,b) = \lambda^c(a_{-i}, b_{-i}) = \lambda(a_{-i}, b_{-i}, D)$. Finally, for the case $D \ge D^c(a,b)$ and $D \ge D^c(a_{-i}, b_{-i})$ we may immediately conclude $\lambda(a,b,D) = \lambda^c(a,b) \le \lambda^c(a_{-i}, b_{-i}) = \lambda(a_{-i}, b_{-i}, D)$.

Next, for both statements (b) and (c) we need to prove only the "if" part of the equivalence, since the other implication is a direct consequence of (a).

For the case (b) we proof the statement by showing that $a_i > \lambda(a,b,D)$ implies $a_i > \lambda(a_{-i},b_{-i},D)$. For the case of $\lambda(a,b,D) = \lambda^c(a,b)$ we observe $D \geq F(a,b,\lambda^c(a,b)) = F(a_{-i},b_{-i},\lambda^c(a,b)) = F(a_{-i},b_{-i},\lambda^c(a_{-i},b_{-i}))$ where $D \geq D^c(a,b)$, the fact that $a_i > \lambda^c(a,b)$, and the observation $\lambda^c(a,b) = \lambda^c(a_{-i},b_{-i})$ were used, respectively. In other words, we have shown $D \geq D^c(a_{-i},b_{-i})$ and thus it also holds $\lambda(a_{-i},b_{-i},D) = \lambda^c(a_{-i},b_{-i})$. Altogether, we have obtained $a_i > \lambda(a,b,D) = \lambda^c(a,b) = \lambda^c(a_{-i},b_{-i}) = \lambda(a_{-i},b_{-i},D)$. For the case of $a_i > \lambda(a,b,D)$ and $\lambda(a,b,D) < \lambda^c(a,b)$ we observe $F(a_{-i},b_{-i},a_i) = F(a,b,a_i) > F(a,b,\lambda(a,b,D)) = D \geq F(a_{-i},b_{-i},\lambda(a_{-i},b_{-i},D))$ using (4), $a_i > \lambda(a,b,D)$ and Remark 2, respectively. Then, however, $F(a_{-i},b_{-i},a_i) > F(a_{-i},b_{-i},\lambda(a_{-i},b_{-i},D))$ implies $a_i > \lambda(a_{-i},b_{-i},D)$ with regards to Lemma A.1.

For case (c) we primarily observe that $\lambda^c(a,b) = \lambda^c(a_{-i},b_{-i})$ and so $D^c(a,b) \geq D^c(a_{-i},b_{-i})$. Then, there are three possibilities. First, considering $D \geq D^c(a,b) \geq D^c(a_{-i},b_{-i})$ and the definition of function λ, we obtain $\lambda(a,b,D) = \lambda^c(a,b) = \lambda^c(a_{-i},b_{-i}) = \lambda(a_{-i},b_{-i},D)$ and the statement is immediate. Next, we assume $D^c(a,b) \geq D \geq D^c(a_{-i},b_{-i})$, $a_i < \lambda(a_{-i},b_{-i},D)$ and observe $F(a,b,\lambda(a,b,D)) = D \geq F(a_{-i},b_{-i},\lambda^c(a_{-i},b_{-i})) > F(a_{-i},b_{-i},a_i) = F(a,b,a_i)$ where Remark 2, definition of $D^c(a_{-i},b_{-i})$, the fact that $\lambda^c(a_{-i},b_{-i}) = \lambda(a_{-i},b_{-i},D) > a_i$, Lemma A.1, and the definition of F were used, respectively. Then, again by Lemma A.1, we have $\lambda(a,b,D) > a_i$. The last variant to examine reads $D^c(a,b) \geq D^c(a_{-i},b_{-i}) > D$. Then, if $a_i < \lambda(a_{-i},b_{-i},D)$ we directly obtain

$$F(a_{-i},b_{-i},a_i) < F(a_{-i},b_{-i},\lambda(a_{-i},b_{-i},D)) = D = F(a,b,\lambda(a,b,D)),$$

and since by the definition of F we have $F(a_{-i},b_{-i},a_i) = F(a,b,a_i)$, we complete the proof by using Lemma A.1 once more. □

Then we may show continuity of the marginal price with respect to bid of one producer.

PROPOSITION A.3 *Let $(a_{-i},b_{-i}) \in \mathbb{R}_+^{2N-2}$, $D > 0$ and $i \in \mathcal{N}$. Then $\lambda(a,b,D)$ is continuous in (a_i,b_i) on $[0,\lambda(a_{-i},b_{-i},D)] \times [0,+\infty[$.*

Proof of Proposition A.3. Consider a sequence $(a_i^k,b_i^k) \to (a_i,b_i)$ such that $(a_i^k,b_i^k) \in [0,\lambda(a_{-i},b_{-i},D)] \times [0,+\infty[$ for all k, and denote $(a^k,b^k) = (a_i^k,a_{-i},b_i^k,b_{-i})$. First we treat a case of $b_i > 0$. Then $b_i^k > 0$ for k large enough, and so $\lambda^c(a^k,b^k) = \lambda^c(a_{-i},b_{-i}) = \lambda^c(a,b)$. Further $D^c(a^k,b^k) = F(a^k,b^k,\lambda^c(a_{-i},b_{-i})) = D^c(a_{-i},b_{-i}) + \frac{\lambda^c(a_{-i},b_{-i})-a_i^k}{2b_i^k}$ and thus $D^c(a^k,b^k) \to D^c(a,b)$. Now we consider D such that $D > D^c(a,b)$ first. Then $\lambda(a,b,D) = \lambda^c(a,b)$ and also $D > D^c(a^k,b^k)$ for k large enough. Thus $\lambda(a^k,b^k,D) = \lambda^c(a^k,b^k) = \lambda^c(a,b)$ and the proof is done. Next, for $D < D^c(a,b)$ we observe that $F(a^k,b^k,\lambda(a^k,b^k,D)) = D = F(a,b,\lambda(a,b,D))$ since for k large enough it holds $D < D^c(a^k,b^k)$. Using the definition of F, we arrive at

$$F(a,b,\lambda(a^k,b^k,D)) - F(a,b,\lambda(a,b,D)) = \frac{\lambda(a,b,D)(b_i^k - b_i) + a_i^k b_i - a_i b_i^k}{2b_i^k b_i}$$

after several technical steps. Since $(\lambda(a^k,b^k,D))_k$ is bounded, there exists a point of accumulation $\tilde{\lambda}$ of $(\lambda(a^k,b^k,D))_k$ satisfying $F(a,b,\tilde{\lambda}) - F(a,b,\lambda(a,b,D)) = 0$. Then, however, $\lambda(a,b,D) = \tilde{\lambda}$ due to Lemma A.1. Considering the last case of $D = D^c(a,b)$, if one can extract from $((a^k,b^k))_k$ a sub-sequence, also denoted by $((a^k,b^k))_k$, such that $D > D^c(a^k,b^k)$ for all k, or $D < D^c(a^k,b^k)$ for all k, the conclusion follows from the respective step above. Otherwise, for k large enough it holds $D = D^c(a^k, b^k) = D^c(a, b)$ and the proof is direct.

Second, we consider $b_i = 0$, i.e., $b_i^k \to 0$. Then $D^c(a,b) = F(a_{-i}, b_{-i}, a_i) \leq F(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) \leq D$, $\lambda^c(a,b) = a_i$ and thus $\lambda(a,b,D) = \lambda^c(a,b) = a_i$. If we can extract a subsequence $(a_i^k, b_i^k)_k$ such that $b_i^k \searrow 0$, then, using Lemma A.2 (b), Remark 2 and the definition of F we obtain

$$0 \leq \frac{\lambda(a^k, b^k, D) - a_i^k}{2b_i^k} \leq D - F(a_{-i}, b_{-i}, \lambda(a^k, b^k, D)) \leq D. \tag{A1}$$

Observing $0 \leq \lambda(a^k, b^k, D) - a_i^k \leq 2b_i^k D$, we have $\lim_{a_i^k \to a_i, b_i^k \searrow 0} \lambda(a^k, b^k, D) = a_i = \lambda(a, b, D)$. Finally, the only case to consider is when $b_i^k = 0$ for k large enough. Then we have $D^c(a^k, b^k) \leq D$ by similar arguments as above, thus $\lambda(a^k, b^k, D) = \lambda^c(a^k, b^k) = a_i^k$ which completes the proof since $a_i^k \to a_i = \lambda(a, b, D)$. $\square$

Let us now state the proof of Lemma 2.3 which provides explicit formulas for the left and right derivatives of $\lambda(a,b,D)$ with respect to D.

Proof of Lemma 2.3. Let $\tilde{\lambda} \in [\lambda^m(a), \lambda^c(a,b)[$ and $\tilde{D} \in [0, D^c(a,b)[$ such that $\lambda(a,b,\tilde{D}) = \tilde{\lambda}$. Then, from (6) we observe $F(a,b,\lambda(a,b,\tilde{D})) = \tilde{D}$, and owing to the piecewise linearity of $F(a,b,\lambda)$ and $\lambda(a,b,D)$ in λ and D, respectively, we have

$$\partial_D^+ F(a,b,\lambda(a,b,\tilde{D})) = \partial_\lambda^+ F(a,b,\lambda(a,b,\tilde{D}))\, \partial_D^+ \lambda(a,b,\tilde{D}) = 1,$$

or in other words $\frac{1}{m^+(a,b,\tilde{\lambda})} = \partial_\lambda^+ F(a,b,\tilde{\lambda})$. Now according to (4) one gets $\partial_\lambda^+ F(a,b,\tilde{\lambda}) = \sum_{i \in \mathcal{N}: a_i \leq \tilde{\lambda}} \frac{1}{2b_i}$. To finish the proof, let us observe that $\{i \in \mathcal{N} : a_i \leq \tilde{\lambda}\}$ is non-empty since $\lambda^m(a) \leq \tilde{\lambda}$, and so $m^+(a,b,\tilde{\lambda}) > 0$. Finally, we can perform similar considerations for $m^-(a,b,\tilde{\lambda})$. $\square$

Using the previous result, we may deduce the meaning of $\lambda(a,b,D)$ as follows.

Proof of Proposition 2.4. When convenient, we use $\lambda = \lambda(a,b,D)$ for brevity. For $D < D^c(a,b)$, we restrict the sum in the definition of $C(a,b,D)$ to $i \in \mathcal{N}$ such that $a_i < \lambda$ due to (7), and we use $q_i(a,b,D) = \frac{\lambda - a_i}{2b_i}$ obtaining

$$C(a,b,D) = \sum_{i \in \mathcal{N}: a_i < \lambda} a_i \frac{\lambda - a_i}{2b_i} + b_i \frac{(\lambda - a_i)^2}{4b_i^2} = \sum_{i \in \mathcal{N}: a_i < \lambda} \frac{\lambda(a,b,D)^2 - a_i^2}{4b_i}.$$

For $D \geq D^c(a,b)$ the way is analogous using formula (7) for $q_i(a,b,D)$ and splitting the sum between linear and non-linear bidders

$$C(a,b,D) = (D - D^c(a,b))\lambda^c(a,b) + \sum_{i \in \mathcal{N}: a_i < \lambda^c(a,b)} \frac{(\lambda^c(a,b))^2 - a_i^2}{4b_i}$$

where we moreover substitute $D^c(a,b) = \sum_{i \in \mathcal{N}: a_i < \lambda^c(a,b)} \frac{\lambda^c(a,b) - a_i}{2b_i}$ obtaining

$$C(a,b,D) = D\,\lambda^c(a,b) - \sum_{i \in \mathcal{N}: a_i < \lambda^c(a,b)} \frac{(\lambda^c(a,b) - a_i)^2}{4b_i}.$$

From these results we observe that $C(a,b,D)$ is continuous with respect to D at $D = D^c(a,b)$. Now, for the left-hand side derivative $\partial_D^- C(a,b,D)$ at D, $D \in]0, D^c(a,b)[$, we have

$$\partial_D^- C(a,b,D) = \sum_{i\in\mathcal{N}:a_i<\lambda(a,b,D)} \frac{2\lambda(a,b,D)\, m^-(a,b,\lambda(a,b,D))}{4b_i} = \lambda(a,b,D)$$

with regards to Lemma 2.3, and the formula $\partial_D^- C(a,b,D) = \lambda^c(a,b) = \lambda(a,b,D)$ for $D \geq D^c(a,b)$ is immediate. Analogously, we may validate also the formula $\partial_D^+ C(a,b,D) = \lambda(a,b,D)$, thus proving smoothness of $\lambda(a,b,D)$ with respect to D. □

Let us now emphasize, through the following corollary, that as soon as the linear coefficient A_i of the production cost function of the i-th producer is greater than the marginal price $\lambda(a_{-i}, b_{-i}, D)$ in the market without producer i, then there is no bid (a_i, b_i) for producer i ensuring him positive profit.

Corollary A.4 *For any $D > 0$, $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ and $A_i \geq \lambda(a_{-i}, b_{-i}, D)$, the i-th producer's profit is non-positive, that is $\pi_i(a,b,D) \leq 0$.*

Proof of Corollary A.4. Since case (d) of Theorem 2.2 concludes to zero profit, we only have to consider cases (a), (b), and (c) of this theorem. Assume first that $a_i < \lambda(a_{-i}, b_{-i}, D)$ and $b_i > 0$. We have $A_i > a_i$ and also $\lambda(a,b,D) \leq \lambda(a_{-i}, b_{-i}, D)$ according to Lemma A.2 (a), then $\lambda(a,b,D) \leq A_i$ and thus

$$a_i b_i - 2A_i b_i + a_i B_i + \lambda(a,b,D)(b_i - B_i) \leq (a_i - A_i)(b_i + B_i) < 0$$

concluding $\pi_i(a,b,D) \leq 0$ according to variant (a) of Theorem 2.2.

Thus, if $a_i \leq \lambda(a_{-i}, b_{-i}, D)$ and $b_i = 0$ then, as in the proof of Theorem 2.2, we can deduce that $\lambda^c(a,b) = a_i$ and therefore $\lambda(a,b,D) \leq a_i \leq \lambda(a_{-i}, b_{-i}, D)$. Now, by Lemma A.2 (b) it follows that actually $a_i = \lambda(a,b,D)$. Then also $D^c(a,b) \leq D$ thanks to $b_i = 0$ and Remark 3. Finally, the non-positiveness of the profit function for cases (b) and (c) of Theorem 2.2 is a direct consequence of (17) and (18), respectively, since $B_i > 0$ and we have shown that $\lambda^c(a,b) \leq A_i$ and $D^c(a,b) \leq D$. □

Now, we state the conditions for upper and lower semi-continuity of profit $\pi_i(a,b,D)$ with respect to a_i approaching $\lambda(a_{-i}, b_{-i}, D)$ from below and $b_i = 0$.

Corollary A.5 *Let $D > 0$, $i \in \mathcal{N}$, $b_i = 0$, $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ and denote $\xi = \frac{1}{2}\frac{N^c(a_{-i},b_{-i})+2}{N^c(a_{-i},b_{-i})+1}$. Then, one of the following alternatives has to be satisfied:*

(a) if $\lambda(a_{-i}, b_{-i}, D) < \lambda^c(a_{-i}, b_{-i})$ then

$$\lim_{a_i \nearrow \lambda(a_{-i},b_{-i},D)} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) = \pi_i(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}, D), \quad \text{(A2)}$$

(b) if $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$ and $q_i^\star(a_{-i}, b_{-i}) = \xi(D - D^c(a_{-i}, b_{-i}))$ then

$$\lim_{a_i \nearrow \lambda^c(a_{-i},b_{-i})} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) = \pi_i(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}, D), \quad \text{(A3)}$$

(c) if $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$ and $q_i^\star(a_{-i}, b_{-i}) > \xi(D - D^c(a_{-i}, b_{-i}))$ then

$$\lim_{a_i \nearrow \lambda^c(a_{-i}, b_{-i})} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) > \pi_i(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}, D), \tag{A4}$$

(d) if $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$ and $q_i^\star(a_{-i}, b_{-i}) < \xi(D - D^c(a_{-i}, b_{-i}))$ then

$$\lim_{a_i \nearrow \lambda^c(a_{-i}, b_{-i})} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) < \pi_i(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}, D). \tag{A5}$$

Moreover, it holds

$$\lim_{a_i \nearrow \lambda(a_{-i}, b_{-i}, D)} q_i(a_i, a_{-i}, 0, b_{-i}, D) = D - D^c(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}), \tag{A6}$$

and condition $\lim_{a_i \nearrow \lambda(a_{-i}, b_{-i}, D)} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) > 0$ may be equivalently restated as

$$D - D^c(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}) < 2\, q_i^\star(a_{-i}, b_{-i}). \tag{A7}$$

From this corollary we observe that for $b_i = 0$ the profit $\pi_i(a, b, D)$ is continuous, upper semi-continuous, and lower semi-continuous with respect to a_i approaching $\lambda(a_{-i}, b_{-i}, D)$ from below in cases described by (a)-(b), (c), and (d), respectively.

Proof of Corollary A.5. First we validate formula (A6) for the limit value of the produced quantity. For $a_i < \lambda(a_{-i}, b_{-i}, D)$ we observe $a_i \leq \lambda(a, b, D) \leq \lambda^c(a, b) = a_i$ due to Lemma A.2 (b) and using $b_i = 0$, and so $a_i = \lambda(a, b, D)$. On that account we have $q_i(a_i, a_{-i}, 0, b_{-i}, D) = D - D^c(a_i, a_{-i}, 0, b_{-i}) = D - F(a_{-i}, b_{-i}, a_i)$ using Theorem 2.1 and definition of $D^c(a_i, a_{-i}, 0, b_{-i})$. Moreover, at the limiting point in (A6) it holds $F(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) = D^c(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i})$ owing to $\lambda^c(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}) = \lambda(a_{-i}, b_{-i}, D)$. Thus (A6) holds directly due to continuity of $F(a, b, \lambda)$ stated in Lemma A.1.

Next, the limit condition $\lim_{a_i \nearrow \lambda(a_{-i}, b_{-i}, D)} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) > 0$ is equivalent to the strict inequality $B_i(D - D^c(a_{-i}, b_{-i})) < (\lambda(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}, D) - A_i)$ yielding (A7) owing to the definition of coefficient $q_i^\star(a_{-i}, b_{-i})$. The fact that $D^c(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}) = D^c(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}) = D^c(a_{-i}, b_{-i})$ provided equality $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$ is directly due to definition of $D^c(a, b)$.

Now, to prove statement (a), we observe $N^c(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}) = 1$ assuming $\lambda(a_{-i}, b_{-i}, D) < \lambda^c(a_{-i}, b_{-i})$. Then formulas (17) and (18) stated in Theorem 2.2 are identical.

Finally, for $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$ the profit at the limit point is given by Theorem 2.2 (c) as follows

$$\begin{aligned}\pi_i(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}, D) = (\lambda^c(a_{-i}, b_{-i}) - A_i) &\tfrac{D - D^c(a_{-i}, b_{-i})}{N^c(a_{-i}, b_{-i}) + 1} \\ &- B_i \left(\tfrac{D - D^c(a_{-i}, b_{-i})}{N^c(a_{-i}, b_{-i}) + 1}\right)^2,\end{aligned}$$

with $\lambda^c(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}) = \lambda^c(a_{-i}, b_{-i})$, $D^c(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}) = D^c(a_{-i}, b_{-i})$ and $N^c(\lambda^c(a_{-i}, b_{-i}), a_{-i}, 0, b_{-i}) = 1 + N^c(a_{-i}, b_{-i})$ were used. In the same way we simplify (A6), calculate the profit yielded by this production quantity, and observe that $\lim_{a_i \nearrow \lambda^c(a_{-i}, b_{-i})} \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$ is equal to

$\pi_i(\lambda^c(a_{-i},b_{-i}),a_{-i},0,b_{-i},D)+2\left[q_i^\star(a_{-i},b_{-i})-\xi(D-D^c(a_{-i},b_{-i}))\right]$, thus showing cases (b), (c), and (d) and finishing the proof. $\square$

The following formulas for partial directional derivatives of $\pi_i(a,b,D)$ are a workhorse for analysis of the i-th producer's problem.

Lemma A.6 *Let $(a_{-i},b_{-i})\in\mathbb{R}_+^{2N-2}$, $D>0$, and for $i\in\mathcal{N}$ consider $a_i\in\,]0,\lambda(a_{-i},b_{-i},D)[$ and $b_i=0$. Then, the partial directional derivatives $\partial_{a_i}^-\pi_i(a,b,D)$ and $\partial_{a_i}^+\pi_i(a,b,D)$ are well-defined and*

(a) for $a_i>\lambda^m(a_{-i})$ both $\partial_{a_i}^-\pi_i(a,b,D)$ and $\partial_{a_i}^+\pi_i(a,b,D)$ are given by

$$\partial_{a_i}^\pm\pi_i(a,b,D)=\left[1+\tfrac{2B_i}{m^\pm(a_{-i},b_{-i},a_i)}\right]\\ \times\left[D-D^c(a,b)-\tfrac{a_i-A_i}{2B_i+m^\pm(a_{-i},b_{-i},a_i)}\right] \tag{A8}$$

(b) for $a_i=\lambda^m(a_{-i})$ it holds $\partial_{a_i}^-\pi_i(a,b,D)=D$ and for $\partial_{a_i}^+\pi_i(a,b,D)$ formula (A8) is still valid,

(c) for $a_i<\lambda^m(a_{-i})$ we have $\partial_{a_i}^-\pi_i(a,b,D)=D$ and $\partial_{a_i}^+\pi_i(a,b,D)=D$.

Proof. Since $a_i<\lambda(a_{-i},b_{-i},D)\le\lambda^c(a_{-i},b_{-i})$ and $b_i=0$, we use formula $\pi_i(a,b,D)$ given by Theorem 2.2 (b) and observe $a_i=\lambda^c(a,b)$. Since $q_i(a,b,D)$ given by (7) for $b_i=0$ is piecewise smooth owing to piecewise linearity of $D^c(a,b)=F(a_{-i},b_{-i},a_i)$, also $\pi_i(a,b,D)$ is piecewise smooth and so directional derivatives exist. Next, we see that for all α close enough to a_i one has $\lambda^c(\alpha,a_{-i},0,b_{-i})=\alpha$, thus $\partial_{a_i}^\pm\lambda^c(a,b)=1$ and we may deduce

$$\partial_{a_i}^\pm\pi_i(a,b,D)=D-D^c(a,b)-\partial_{a_i}^\pm D^c(a,b)\left[a_i-A_i+2B_i(D-D^c(a,b))\right].$$

For $a_i\le\lambda^m(a_{-i})$, it holds from Remark 1 (c) that $D^c(a,b)=0$ and moreover for $\alpha<a_i$ close enough to a_i it holds $D^c(\alpha,a_{-i},0,b_{-i})=0$ implying $\partial_{a_i}^-D^c(a,b)=0$. Thus we obtained (c) and a part of (b) stating formula $\partial_{a_i}^-\pi_i(a,b,D)=D$ at $a_i=\lambda^m(a_{-i})$. Assuming $a_i>\lambda^m(a_{-i})$ next, we have $D^c(a,b)>0$ due to Remark 1 (c) again. Then according to (5) we may write $D^c(\alpha,a_{-i},0,b_{-i})=F(\alpha,a_{-i},0,b_{-i},\alpha)=F(a_{-i},b_{-i},\alpha)$, thus $\partial_{a_i}^\pm D^c(a,b)=\partial_{a_i}^\pm F(a_{-i},b_{-i},a_i)$ and we calculate $\partial_{a_i}^-F(a_{-i},b_{-i},a_i)=\sum_{j\in\mathcal{N}:a_j<a_i}\frac{1}{2b_j}=\frac{1}{m^-(a_{-i},b_{-i},a_i)}$ where the latter equality is due to Lemma 2.3. By symmetrical arguments for $\partial_{a_i}^+F(a_{-i},b_{-i},a_i)$, we have shown $\partial_{a_i}^\pm D^c(a,b)=\frac{1}{m^\pm(a_{-i},b_{-i},a_i)}$, and after a short calculation we may validate (a). To finish the proof, we observe that the previous considerations hold even for $\partial_{a_i}^+\pi_i(a,b,D)$ at $a_i=\lambda^m(a_{-i})$, see Lemma 2.3, and thus statement (b) for $\partial_{a_i}^+\pi_i(a,b,D)$ holds. $\square$

Remark 5 For $D>0$ and $i\in\mathcal{N}$ consider $(a,b)\in\mathbb{R}_+^{2N}$ such that $\lambda(a_{-i},b_{-i},D)>0$ and $a_i=b_i=0$. Then it holds $\partial_{a_i}^+\pi_i(a,b,D)=D$ using the same arguments as in the proof of Lemma A.6.

Remark 6 We recall that a bifunction $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is quasimonotone if $f(x,y-x)>0\implies f(y,x-y)\le 0$ holds for all $x,y\in\mathbb{R}$. Moreover, bifunction $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is strictly quasimonotone if it is quasimonotone and for all $x,y\in\mathbb{R}$, $x\neq y$, there exists $z\in]x,y[$ such that $f(z,y-x)\neq 0$, see [14].

Next, if $\phi:\mathbb{R}\to\mathbb{R}$ admits directional derivatives $\phi'_-(x)=\phi'(x;-1)$ and $\phi'_+(x)=\phi'(x;+1)$ at any $x\in\mathbb{R}$, then quasimonotonicity of the bifunction $(x,d)\to\phi'(x;d)$ is equivalent to condition $\phi'_+(x)>0\implies\phi'_-(y)\le 0$ satisfied for all $y>x$.

If, moreover, for all $x, y \in \mathbb{R}$, $x \neq y$, there exists $z \in]x, y[$ such that $\phi'_+(z) > 0$, then $\phi'(x; d)$ is strictly quasimonotone.

Let us also recall that quasimonotonicity of $-\phi'(x; d)$ is equivalent to quasiconcavity of $\phi(x)$, and strict quasimonotonicity of $-\phi'(x; d)$ is equivalent to strict quasiconcavity of $\phi(x)$.

COROLLARY A.7 *Let $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$, $D > 0$, and for $i \in \mathcal{N}$ consider $b_i = 0$. Then, profit $\pi_i(a, b, D)$ is strictly quasiconcave in a_i on $[0, \lambda(a_{-i}, b_{-i}, D)[$.*

Proof. We first show that function $\phi : [0, \lambda(a_{-i}, b_{-i}, D)[\to \mathbb{R}$ defined by $\phi(a_i) = \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$ is quasiconcave. With regards to Remark 6 this is equivalent to $\phi'_+(x) < 0 \implies \phi'_-(y) \geq 0$ valid for all $x, y \in [0, \lambda(a_{-i}, b_{-i}, D)[$ such that $y > x$. In terms of $\partial^{\pm}_{a_i}\pi_i(\alpha, a_{-i}, 0, b_{-i}, D)$ this condition reads $\partial^{+}_{a_i}\pi_i(\alpha, a_{-i}, 0, b_{-i}, D) < 0 \implies \partial^{-}_{a_i}\pi_i(\tilde{\alpha}, a_{-i}, 0, b_{-i}, D) \leq 0$ valid for all $\tilde{\alpha} > \alpha$. We may consider only $\alpha \geq \lambda^m(a_{-i})$ due to Lemma A.6 (c). We observe that, according to Lemma A.6, the sign of $\partial^{\pm}_{a_i}\pi_i(a, b, D)$ corresponds to the sign of the latter multiplier in (A8). Thus, using $D^c(a, b) = F(a_{-i}, b_{-i}, a_i)$ due to $\lambda(a_{-i}, b_{-i}, D) > a_i = \lambda^c(a, b)$ and (5), and denoting

$$G^{\pm}(a_{-i}, b_{-i}, a_i) = \frac{a_i - A_i}{2B_i + m^{\pm}(a_{-i}, b_{-i}, a_i)} + F(a_{-i}, b_{-i}, a_i). \tag{A9}$$

we observe that $\phi(a_i)$ is quasiconcave if and only if for all $\tilde{\alpha} > \alpha$ we have

$$G^{+}(a_{-i}, b_{-i}, \alpha) > D \implies G^{-}(a_{-i}, b_{-i}, \tilde{\alpha}) \geq D. \tag{A10}$$

Note, that since $m^{\pm}(a_{-i}, b_{-i}, a_i)$ are non-increasing in a_i for $a_i \in]\lambda^m(a_{-i}), \lambda(a_{-i}, b_{-i}, D)[$, see (20), and function $F(a_{-i}, b_{-i}, a_i)$ is increasing in a_i, we observe that functions $G^{\pm}(a_{-i}, b_{-i}, a_i)$ are strictly increasing in a_i.

To verify (A10), we remark that $m^{-}(a_{-i}, b_{-i}, \alpha) \geq m^{+}(a_{-i}, b_{-i}, \alpha)$ for all $\alpha \in]\lambda^m(a_{-i}), \lambda(a_{-i}, b_{-i}, D)[$, see Lemma 2.3. Then, considering $\alpha \leq A_i$ first, we have $G^{-}(a_{-i}, b_{-i}, \alpha) \geq G^{+}(a_{-i}, b_{-i}, \alpha)$, and so $G^{+}(a_{-i}, b_{-i}, \alpha) > D$ implies $G^{-}(a_{-i}, b_{-i}, \alpha) > D$, and finally also $G^{-}(a_{-i}, b_{-i}, \tilde{\alpha}) > D$ for all $\tilde{\alpha} > \alpha$ using monotonicity of $G^{-}(a_{-i}, b_{-i}, a_i)$. For the case that $\alpha > A_i$, we have $G^{-}(a_{-i}, b_{-i}, \alpha) \leq G^{+}(a_{-i}, b_{-i}, \alpha)$. However, $G^{-}(a_{-i}, b_{-i}, a_i) = G^{+}(a_{-i}, b_{-i}, a_i)$ holds whenever $m^{-}(a_{-i}, b_{-i}, a_i) = m^{+}(a_{-i}, b_{-i}, a_i)$, and there are only finitely many points a_i such that $m^{-}(a_{-i}, b_{-i}, a_i) \neq m^{+}(a_{-i}, b_{-i}, a_i)$, cf. Remark 4. Then, having $G^{+}(a_{-i}, b_{-i}, \alpha) > D$ and $\tilde{\alpha} > \alpha$, there exists $\beta \in]\alpha, \tilde{\alpha}[$ such that $G^{-}(a_{-i}, b_{-i}, \beta) = G^{+}(a_{-i}, b_{-i}, \beta)$, and so using monotonicity of $G^{\pm}(a_{-i}, b_{-i}, \alpha)$ again we obtain

$$G^{-}(a_{-i}, b_{-i}, \tilde{\alpha}) > G^{-}(a_{-i}, b_{-i}, \beta) = G^{+}(a_{-i}, b_{-i}, \beta) > G^{+}(a_{-i}, b_{-i}, \alpha) > D.$$

Thus (A10) is indeed valid and so $\phi(a_i)$ is quasiconcave.

To show that $\phi(a_i)$ is strictly quasiconcave, we use criteria from Remark 6 again. For any $x, y \in [0, \lambda(a_{-i}, b_{-i}, D)[$, $x \neq y$, there has to exist $z \in]x, y[$ such that $\phi'_+(z) \neq 0$. Let us assume that $x < y$. For $z < \lambda^m(a_{-i})$ we have $\phi'_+(z) = D > 0$ due to Lemma A.6 (c). For $z \geq \lambda^m(a_{-i})$ we argue that with regards to monotonicity of $G^{+}(a_{-i}, b_{-i}, \alpha)$, there may be at maximum one $\alpha \in [\lambda^m(a_{-i}), \lambda(a_{-i}, b_{-i}, D)[$ such that $G^{+}(a_{-i}, b_{-i}, \alpha) = D$, i.e., $\phi'_+(\alpha) = 0$, and so we may find $z \in]x, y[$ such that $\phi'_+(z) \neq 0$. $\square$

Proposition A.8 *Let $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$, $D > 0$ and $b_i = 0$ be fixed. Then, problem*

$$\hat{P}_i(a_{-i}, b_{-i}, D) \qquad \sup_{a_i \in [0, \lambda(a_{-i}, b_{-i}, D)[} \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$$

admits a solution if and only if one of the following alternatives holds:

(a) $A_i < \lambda(a_{-i}, b_{-i}, D) < \lambda^c(a_{-i}, b_{-i})$ (implying $\lambda^m(a_{-i}) < \lambda(a_{-i}, b_{-i}, D)$),
(b) $\lambda^m(a_{-i}) < \lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$ and $q_i^c(a_{-i}, b_{-i}) > D - D^c(a_{-i}, b_{-i})$.

Moreover, if a solution exists, it is unique. Denoting it by $\tilde{a}_i$ it satisfies $\tilde{a}_i \in [\lambda^m(a_{-i}), \lambda^c(a_{-i}, b_{-i})[$ and is given by

$$\begin{cases} \tilde{a}_i = \lambda^m(a_{-i}) & \text{if } D \le q_i^m(a_{-i}, b_{-i}), \\ \dfrac{\tilde{a}_i - A_i}{2B_i + m^-(a_{-i}, b_{-i}, \tilde{a}_i)} \le D - F(a_{-i}, b_{-i}, \tilde{a}_i) \le \dfrac{\tilde{a}_i - A_i}{2B_i + m^+(a_{-i}, b_{-i}, \tilde{a}_i)} & \text{if } D > q_i^m(a_{-i}, b_{-i}), \end{cases} \tag{A11}$$

and the respective maximal profit is positive, $\pi_i(\tilde{a}_i, a_{-i}, 0, b_{-i}, D) > 0$. Additionally, if a solution does not exist, then $\pi_i(a, b, D)$ is strictly increasing in a_i on $[0, \lambda(a_{-i}, b_{-i}, D)[$.

Proof. Let us consider the function $\phi : [0, \lambda(a_{-i}, b_{-i}, D)] \to \mathbb{R}$ defined for $a_i < \lambda(a_{-i}, b_{-i}, D)$ by $\phi(a_i) = \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$ with profit given by Theorem 2.2 (b), continuously extended to $a_i = \lambda(a_{-i}, b_{-i}, D)$. Then $\hat{P}_i(a_{-i}, b_{-i}, D)$ is equivalent to $\sup_{a_i \in [0, \lambda(a_{-i}, b_{-i}, D)[} \phi(a_i)$. Now, we have $\phi'_+(0) > 0$ due to Remark 5 and we know that ϕ is strictly quasiconcave due to Corollary A.7. We use [1, Proposition 4.9] and observe that there are two alternatives. Either there exists solution to $\hat{P}_i(a_{-i}, b_{-i}, D)$, which is unique due to strict quasiconcavity of ϕ. This is characterized by condition

$$\phi'_-(\lambda(a_{-i}, b_{-i}, D)) > 0. \tag{A12}$$

Or, ϕ is strictly increasing on $[0, \lambda(a_{-i}, b_{-i}, D)]$, and so (A12) is not valid.

Now, we reformulate condition (A12) in terms of our data. For $\lambda(a_{-i}, b_{-i}, D) = \lambda^m(a_{-i})$ there is no solution with regards to Lemma A.6 (c). For $\lambda(a_{-i}, b_{-i}, D) > \lambda^m(a_{-i})$ we use function G^- defined by (A9), and we equivalently rewrite (A12) as $G^-(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) > D$. Regarding the case $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$ first, we rewrite $G^-(a_{-i}, b_{-i}, \lambda^c(a_{-i}, b_{-i})) > D$ as $q_i^c(a_{-i}, b_{-i}) > D - D^c(a_{-i}, b_{-i})$, arriving at part (b) of the statement. Analogously, for $\lambda(a_{-i}, b_{-i}, D) < \lambda^c(a_{-i}, b_{-i})$, we have $F(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) = D$ and so $G^-(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) > D$ simplifies to

$$\frac{\lambda(a_{-i}, b_{-i}, D) - A_i}{2B_i + m^-(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D))} > 0,$$

i.e., finally to $\lambda(a_{-i}, b_{-i}, D) > A_i$. To finish the proof of part (a) of the statement, we observe that $\lambda(a_{-i}, b_{-i}, D) < \lambda^c(a_{-i}, b_{-i})$ actually implies $\lambda^m(a_{-i}) < \lambda(a_{-i}, b_{-i}, D)$, since otherwise we may have $0 = F(a_{-i}, b_{-i}, \lambda^m(a_{-i})) = F(a_{-i}, b_{-i}, \lambda(a_{-i}, b_{-i}, D)) = D > 0$, a contradiction. Once solution of

$\hat{P}_i(a_{-i}, b_{-i}, D)$ exists, the strict quasiconcavity of $\pi_i(a, b, D)$ in a_i, see Corollary A.7, guarantees its uniqueness due to [1, Proposition 4.28].

Further, we denote the unique solution of $\hat{P}_i(a_{-i}, b_{-i}, D)$ by $\tilde{a}_i$. To find it, we may use stationary conditions $\phi'_-(\tilde{a}_i) \geq 0 \geq \phi'_+(\tilde{a}_i)$. With regards to Lemma A.6 (c), we observe that $\tilde{a}_i \geq \lambda^m(a_{-i})$. Now, using Lemma A.6 (b), we see that $\tilde{a}_i = \lambda^m(a_{-i})$ if and only if $\phi'_+(\tilde{a}_i) \leq 0$ which is equivalent to $D \leq q_i^m(a_{-i}, b_{-i})$ observing $q_i^m(a_{-i}, b_{-i}) = G^+(a_{-i}, b_{-i}, \lambda^m(a_{-i}))$ and recalling (A9). Thus we proved the first part of (A11). Next, assuming $D > q_i^m(a_{-i}, b_{-i})$ it has to hold $\tilde{a}_i > \lambda^m(a_{-i})$, and so using Lemma A.6 (a) and (A9), we rewrite $\phi'_-(\tilde{a}_i) \geq 0 \geq \phi'_+(\tilde{a}_i)$ as $G^-(a_{-i}, b_{-i}, \tilde{a}_i) \leq D \leq G^+(a_{-i}, b_{-i}, \tilde{a}_i)$, and thus we obtain the second part of (A11).

Finally, we show that $\pi_i(\tilde{a}_i, a_{-i}, 0, b_{-i}, D) > 0$, or, equivalently

$$\frac{\tilde{a}_i - A_i}{B_i} > D - D^c(\tilde{a}_i, a_{-i}, 0, b_{-i}) = D - F(a_{-i}, b_{-i}, \tilde{a}_i) \tag{A13}$$

due to Theorem 2.2 (b). For the first variant of (A11) we have $\tilde{a}_i = \lambda^m(a_{-i}) = \lambda^c(\tilde{a}_i, a_{-i}, 0, b_{-i})$, thus $D^c(\tilde{a}_i, a_{-i}, 0, b_{-i}) = 0$ due to Remark 1 (c), and moreover $\frac{\tilde{a}_i - A_i}{B_i} > q_i^m(a_{-i}, b_{-i})$ thanks to (21). Thus, (A13) is valid since we assume $D \leq q_i^m(a_{-i}, b_{-i})$. For the latter variant of (A11), we may write

$$\frac{\tilde{a}_i - A_i}{B_i} > \frac{\tilde{a}_i - A_i}{2B_i + m^+(a_{-i}, b_{-i}, \tilde{a}_i)} \geq D - F(a_{-i}, b_{-i}, \tilde{a}_i)$$

where the right-hand side inequality of (A11) is used, and so (A13) is satisfied. □

Next, we find partial directional derivatives of $\lambda(a, b, D)$ with respect to the bid variables of a quadratically bidding producer $i \in \mathcal{N}$, i.e., we assume $b_i > 0$.

Lemma A.9 *Let $D > 0$, $i \in \mathcal{N}$, and $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N}$ be fixed, $a_i < \lambda(a_{-i}, b_{-i}, D)$ and $b_i > 0$. Then $\lambda(a, b, D)$ is piecewise smooth as a function of a_i or b_i, and directional derivatives $\partial_{a_i}^{\pm}\lambda(a, b, D)$ and $\partial_{b_i}^{\pm}\lambda(a, b, D)$ exist. We have*

$$\partial_{a_i}^{\pm}\lambda(a, b, D) = \frac{1}{2b_i}\, m^{\pm}(a, b, \lambda(a, b, D)), \tag{A14}$$

$$\partial_{b_i}^{\pm}\lambda(a, b, D) = \frac{\lambda(a, b, D) - a_i}{2b_i^2}\, m^{\pm}(a, b, \lambda(a, b, D)), \tag{A15}$$

provided $D \leq D^c(a, b)$, and for $D > D^c(a, b)$ it holds $\partial_{a_i}^{\pm}\lambda(a, b, D) = \partial_{b_i}^{\pm}\lambda(a, b, D) = 0$. For the case of $\partial_{a_i}^{-}\lambda(a, b, D)$ we additionally assume $a_i > 0$.

Proof. We observe that $a_i < \lambda(a_{-i}, b_{-i}, D)$ and $b_i > 0$ implies $a_i < \lambda(a, b, D)$ owing to Lemma A.2 (c), and then also $\lambda^m(a) \leq a_i < \lambda(a, b, D)$. Thus $m^-(a, b, \lambda(a, b, D))$ used in (A14) and (A15) is well-defined.

Now we show the formula for $\partial_{a_i}^{+}\lambda(a, b, D)$. For $D \leq D^c(a, b)$ we have $F(a, b, \lambda(a, b, D)) = D$. Based on partial directional derivative calculus for composition of functions we immediately obtain

$$\partial_{a_i}^{+}F(a, b, \lambda(a, b, D)) + \partial_{\lambda}^{+}F(a, b, \lambda(a, b, D))\, \partial_{a_i}^{+}\lambda(a, b, D) = 0.$$

Indeed, $\partial_{a_i}^{+}F(a, b, \lambda)$ exists since $F(a, b, \lambda)$ is piecewise smooth in a_i, as can be seen from formula deduced in the proof of Lemma A.1, and $\partial_{\lambda}^{+}F(a, b, \lambda)$ is well-defined owing to piecewise linearity of $F(a,b,\lambda)$ in λ. Next we note that $\lambda^m(a) < \lambda(a,b,D)$ implies $\partial^+_\lambda F(a,b,\lambda(a,b,D)) > 0$, and so we may write

$$\partial^+_{a_i}\lambda(a,b,D) = -\frac{\partial^+_{a_i}F(a,b,\lambda(a,b,D))}{\partial^+_\lambda F(a,b,\lambda(a,b,D))} = \frac{m^+(a,b,\lambda(a,b,D))}{2b_i}$$

thanks to $\frac{1}{m^+(a,b,\tilde{\lambda})} = \partial^+_\lambda F(a,b,\tilde{\lambda})$ observed in the proof of Lemma 2.3, and with regards to equality $\partial^+_{a_i}F(a,b,\lambda(a,b,D)) = -\frac{1}{2b_i}$ justified by $a_i < \lambda(a,b,D)$.

For $D > D^c(a,b)$ we have $\lambda(a,b,D) = \lambda^c(a,b) = \lambda^c(a_{-i},b_{-i})$ due to $b_i > 0$, and also $\lambda^c(\alpha,a_{-i},\beta,b_{-i}) = \lambda^c(a_{-i},b_{-i})$ for $(\lambda(a_{-i},b_{-i},D),a_{-i},0,b_{-i})$ close enough to (a_i,b_i). For such $(\lambda(a_{-i},b_{-i},D),a_{-i},0,b_{-i})$ it moreover holds $D^c(\alpha,a_{-i},\beta,b_{-i}) = D^c(a_{-i},b_{-i}) + \frac{\lambda^c(a_{-i},b_{-i})-\alpha}{2\beta} < D$, thus, by the definition, we have $\lambda(\alpha,a_{-i},\beta,b_{-i},D) = \lambda^c(\alpha,a_{-i},\beta,b_{-i})$ and so $\partial^+_{a_i}\lambda(a,b,D) = 0$. We note that the proof of other cases is analogical.

Finally, we argue that piecewise smoothness of $\lambda(a,b,D)$ in a_i or b_i is due to (A14) and (A15), as their right-hand sides are piecewise smooth in the respective variables, see Lemma 2.3. □

The following formulas are of high importance for the rest of the Appendix.

Lemma A.10 *Assume $D > 0$, and for $i \in \mathcal{N}$ consider such $(a,b) \in \mathbb{R}^{2N}_+$ that $a_i < \lambda(a_{-i},b_{-i},D)$ and $b_i > 0$. Then, the partial directional derivatives $\partial^\pm_{a_i}\pi_i(a,b,D)$ and $\partial^\pm_{b_i}\pi_i(a,b,D)$ exist and are as follows:*

$$\partial^\pm_{a_i}\pi_i(a,b,D) = \frac{1}{4b_i^3}\Big[(\lambda - A_i)(\mu^\pm b_i - 2b_i^2) - (\lambda - a_i)(\mu^\pm B_i - 2b_iB_i - 2b_i^2)\Big], \quad \text{(A16)}$$

$$\partial^\pm_{b_i}\pi_i(a,b,D) = \frac{\lambda - a_i}{4b_i^4}\Big[(\lambda - A_i)(\mu^\pm b_i - 2b_i^2) - (\lambda - a_i)(\mu^\pm B_i - 2b_iB_i - b_i^2)\Big], \quad \text{(A17)}$$

where $\lambda = \lambda(a,b,D)$ and

$$\mu^\pm = \begin{cases} m^\pm(a,b,\lambda(a,b,D)) & \text{if} \quad D \le D^c(a,b), \\ 0 & \text{if} \quad D > D^c(a,b), \end{cases}$$

and $a_i > 0$ for the case of $\partial^-_{a_i}\pi_i(a,b,D)$. Moreover, for $q_i(a,b,D)$ given by Theorem 2.1 it holds

$$\partial^\pm_{b_i}\pi_i(a,b,D) = q_i(a,b,D)\left(2\partial^\pm_{a_i}\pi_i(a,b,D) - q_i(a,b,D)\right). \quad \text{(A18)}$$

Proof. We assume $a_i < \lambda(a_{-i},b_{-i},D)$ and $b_i > 0$, and so profit $\pi_i(a,b,D)$ is described by Theorem 2.2 (a). Since $q_i(a,b,D)$ given by (7) for $b_i > 0$ is piecewise smooth in a_i or b_i owing to piecewise smoothness $\lambda(a,b,D)$ stated in Lemma A.9, also $\pi_i(a,b,D)$ is piecewise smooth and so directional derivatives exist. Now, substituting for $\partial^\pm_{a_i}\lambda(a,b)$ and $\partial^\pm_{b_i}\lambda(a,b)$ from Lemma A.9, we calculate $\partial^\pm_{a_i}\pi_i(a,b,D)$ and $\partial^\pm_{b_i}\pi_i(a,b,D)$ after several technical steps. Identity (A18) may be shown directly from (A16) and (A17). □

Lemma A.11 *Let $(a_{-i},b_{-i}) \in \mathbb{R}^{2N-2}_+$, $D > D^c(a_{-i},b_{-i})$ and $i \in \mathcal{N}$ such that $A_i < \lambda^c(a_{-i},b_{-i})$. Then, we take $\beta \in \left]0, \frac{1}{2}\frac{\lambda^c(a_{-i},b_{-i})}{D - D^c(a_{-i},b_{-i})}\right]$, denote $K_\beta = [\lambda^c(a_{-i},b_{-i}) -$*

$2\beta(D - D^c(a_{-i}, b_{-i})), \lambda^c(a_{-i}, b_{-i})]$ *and consider the following problem*

$$\max_{\alpha \in K_\beta} \pi_i(\alpha, a_{-i}, \beta, b_{-i}, D). \tag{A19}$$

For such β this problem has a unique solution $\tilde{\alpha}(\beta)$ given by

$$\tilde{\alpha}(\beta) = \lambda^c(a_{-i}, b_{-i}) - \beta \min\left\{2[D - D^c(a_{-i}, b_{-i})], \frac{\lambda^c(a_{-i}, b_{-i}) - A_i}{B_i + \beta}\right\}. \tag{A20}$$

Proof. Existence of solution to (A19) is direct due to continuity of $\pi_i(a, b, D)$, see Proposition A.3, and compactness of the considered interval. Next we observe that $D^c(\alpha, a_{-i}, \beta, b_{-i}, D) = \frac{\lambda^c(a_{-i}, b_{-i}) - \alpha}{2\beta} + D^c(a_{-i}, b_{-i}) \leq D$ since $\alpha \geq \lambda^c(a_{-i}, b_{-i}) - 2\beta(D - D^c(a_{-i}, b_{-i}))$. Thus, in the statement of Lemma A.10 it holds $\mu^+ = 0$ if $D > D^c(\alpha, a_{-i}, \beta, b_{-i}, D)$, and $\mu^+ = m^+(a, b, \lambda^c(\alpha, a_{-i}, \beta, b_{-i}, D)) = 0$ by definition if $D = D^c(\alpha, a_{-i}, \beta, b_{-i}, D)$. Moreover, $\lambda(\alpha, a_{-i}, \beta, b_{-i}, D) = \lambda^c(\alpha, a_{-i}, \beta, b_{-i}) = \lambda^c(a_{-i}, b_{-i})$ due to $\beta > 0$, and so formula (A16) simplifies to

$$\partial^+_{a_i} \pi_i(\alpha, a_{-i}, \beta, b_{-i}, D) = \frac{1}{2\beta^2}\left[(\lambda^c(a_{-i}, b_{-i}) - \alpha)(\beta + B_i) - (\lambda^c(a_{-i}, b_{-i}) - A_i)\beta\right]. \tag{A21}$$

Now, defining functions $\hat{\alpha} : \beta \to \lambda^c(a_{-i}, b_{-i}) - \beta\frac{\lambda^c(a_{-i}, b_{-i}) - A_i}{B_i + \beta}$ and $\phi_\beta : \alpha \to \pi_i(\alpha, a_{-i}, \beta, b_{-i}, D)$, we observe that $\phi_\beta(\alpha)$ is strictly increasing on $\alpha < \hat{\alpha}(\beta)$ and strictly decreasing on $\alpha > \hat{\alpha}(\beta)$ with regards to the sign of (A21). Thus, if $\hat{\alpha}(\beta) \geq \lambda^c(a_{-i}, b_{-i}) - 2\beta(D - D^c(a_{-i}, b_{-i}))$ the optimal solution of (A19) is uniquely given by $\tilde{\alpha}(\beta) = \hat{\alpha}(\beta)$ since $\hat{\alpha}(\beta) < \lambda^c(a_{-i}, b_{-i})$ always holds and so $\hat{\alpha}(\beta) \in K_\beta$. Otherwise it reads $\tilde{\alpha}(\beta) = \lambda^c(a_{-i}, b_{-i}) - 2\beta(D - D^c(a_{-i}, b_{-i}))$ due to monotonicity of $\phi_\beta(\alpha)$ discussed above. Thus we shown (A20). □

Proposition A.12 *Let $(a_{-i}, b_{-i}) \in \mathbb{R}^{2N-2}_+$ and $D > 0$. Then, problem $P_i(a_{-i}, b_{-i})$ does not admit any solution (a_i, b_i) on $[0, \lambda(a_{-i}, b_{-i}, D)[\times]0, +\infty[$ such that $\pi_i(a, b, D) > 0$.*

Proof of Proposition A.12. First, we consider such point (a_i, b_i) that $a_i = 0$. Then we may rewrite the necessary optimality condition $\partial^+_{a_i} \pi_i(a, b, D) \leq 0$ as

$$\frac{\mu^+}{2} \frac{\lambda(0, a_{-i}, b, D)}{A_i b_i + \lambda(0, a_{-i}, b, D) B_i} \leq \frac{\mu^+}{2b_i} - 1 \tag{A22}$$

rearranging (A16) for $a_i = 0$. For $D \geq D^c(a, b)$ we have $\mu^+ = 0$ and so this inequality can not be satisfied. Next, having $D < D^c(a, b)$, we observe that the left-hand side is positive, whereas the right-hand side is non-positive since $\mu^+ = m^+(0, a_{-i}, b, \lambda(0, a_{-i}, b, D))$ and so $2b_i \geq \mu^+$ due to (20). Thus inequality (A22) has no solution and we may further consider $a_i > 0$.

Since our assumptions fit Theorem 2.2 (a), we know that profit $\pi_i(a, b, D)$ is described by (16). First, we assume that either $m^+(a, b, \lambda(a, b, D)) = m^-(a, b, \lambda(a, b, D))$ or $D > D^c(a, b)$. Then, it holds $\mu^+ = \mu^-$ in the statement of Lemma A.10, and thus $\partial^-_{a_i} \pi_i(a, b, D) = \partial^+_{a_i} \pi_i(a, b, D) =: \partial_{a_i} \pi_i(a, b, D)$ and $\partial^-_{b_i} \pi_i(a, b, D) = \partial^+_{b_i} \pi_i(a, b, D) =: \partial_{b_i} \pi_i(a, b, D)$. Now, combining this with the classical stationary condition $\partial_{a_i} \pi_i(a, b, D) = \partial_{b_i} \pi_i(a, b, D) = 0$ we obtain $q_i(a, b, D) = 0$ using (A18). Then, however, $\pi_i(a, b, D) = 0$.

To finish the proof we have to consider the variant $m^+(a, b, \lambda(a, b, D)) \neq m^-(a, b, \lambda(a, b, D))$ and $D \leq D^c(a, b)$ now. Then, there has to be some $j \in \mathcal{N}$ such that $a_j = \lambda(a,b,D)$ with regards to Remark 4. As we assume $a_i < \lambda(a_{-i}, b_{-i}, D)$, we have also $a_i < \lambda(a,b,D)$ owing to Lemma A.2 (c), and so $a_i < a_j$. Then

$$D = F(a,b,\lambda(a,b,D)) = F(a,b,a_j) = F(a_{-i}, b_{-i}, a_j) + \frac{a_j - a_i}{2b_i} \tag{A23}$$

by the definition of $F(a,b,\lambda)$. Next, defining a linear function $\omega(\beta)=a_j - 2\beta(D - F(a_{-i}, b_{-i}, a_j))$ we consider set $\Omega = \{(\alpha,\beta) \in \mathbb{R}^2_+ : \alpha = \omega(\beta)\}$ and restrict function π_i on Ω. Thus we obtain an auxiliary function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned}\phi(\beta) &= \pi_i(\omega(\beta), a_{-i}, \beta, b_{-i}, D)\\ &= (a_j - A_i)[D - F(a_{-i}, b_{-i}, a_j)] - (B_i + \beta)[D - F(a_{-i}, b_{-i}, a_j)]^2.\end{aligned}$$

If (a_i, b_i) satisfies the necessary optimality condition, then b_i has to be a stationary point of $\phi(\beta)$. However, we have $\phi'_-(\beta) = [F(a_{-i}, b_{-i}, a_j) - D]^2 = \frac{(a_i - a_j)^2}{4b_i^2} > 0$ using (A23) and $a_i < a_j$, and so there exists a sequence $b_i^k \nearrow b_i$ such that for some k we have $\phi(b_i^k) > \phi(b_i)$, a contradiction. □

Remark 7 In the proof of Proposition A.12 we observed that non-smoothness of the marginal price $\lambda(a,b,D)$ with respect to D discussed in Remark 4 is of a simple nature and can be analytically identified. Indeed, for any $(a_{-i}, b_{-i}) \in \mathbb{R}^{2N-2}_+$ and $D > 0$ fixed, the marginal price $\lambda(a,b,D)$ is non-smooth at $(a_i, b_i) \in [0, \lambda(a_{-i}, b_{-i}, D)[\times\mathbb{R}_+$ if there exists another producer $j \in \mathcal{N}$ such that $a_j = \lambda(a,b,D)$ and $a_i = a_j - 2b_i(D - F(a_{-i}, b_{-i}, a_j))$.

Then, we extend Corollary A.5 stating continuity of $\pi_i(a,b,D)$ with respect to bid coefficient a_i having $b_i = 0$ to a full domain allowing $b_i > 0$.

Proposition A.13 *Let $D > 0$ and $(a_{-i}, b_{-i}) \in \mathbb{R}^{2N-2}_+$ for some $i \in \mathcal{N}$. Then*

(a) $\lim_{k\to+\infty} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = 0$ provided $a_i^k \in [0, \lambda(a_{-i}, b_{-i}, D)]$ and $b_i^k \to +\infty$,

(b) $\pi_i(a,b,D)$ is continuous in (a_i, b_i) on the subset $[0, \lambda(a_{-i}, b_{-i}, D)] \times [0,+\infty[$ if $\lambda(a_{-i}, b_{-i}, D) < \lambda^c(a_{-i}, b_{-i})$ and on the subset $[0, \lambda^c(a_{-i}, b_{-i})] \times [0,+\infty[\setminus \{(\lambda^c(a_{-i}, b_{-i}), 0)\}$ if $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$.

Note that the formula for profit (14) is continuous in $q_i(a,b,D)$ and the marginal price $\lambda(a,b,D)$ is continuous everywhere, see Proposition A.3. Thus the discontinuity of the profit at point $(\lambda^c(a_{-i}, b_{-i}), 0)$, as indicated in Proposition A.13, comes from switching between the first and second parts of formula (7). This discontinuity, which actually corresponds to sharing with other linearly bidding producers, is examined in detail in the following proposition. We will see that for high enough electricity demand D, producer $i \in \mathcal{N}$ can bid in such a way that he produces his ideal production quantity $q_i^\star(a_{-i}, b_{-i})$ as a limit.

Proof of Proposition A.13. The first part of the statement stems directly from Theorem 2.1 as we observe $\lim_{k\to+\infty} q_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = 0$ for $a_i^k \in [0, \lambda(a_{-i}, b_{-i}, D)]$ and $b_i^k \to +\infty$.

To prove the part (b) of the statement, we have to deal with profit $\pi_i(a,b,D)$ given on three different domains by Theorem 2.2 (a), (b) and (c). In the case of Theorem 2.2 (a) the profit restricted to $[0, \lambda(a_{-i}, b_{-i}, D)]\times]0,+\infty[$ is continuous since $q_i(a,b,D)$ is continuous due to continuity of $\lambda(a,b,D)$, see Theorem 2.1 and Proposition A.3, respectively. Then we observe that the profit function restricted to $[0, \lambda(a_{-i}, b_{-i}, D)[\times\{0\}$, cf. Theorem 2.2 (b), is also continuous. Indeed, for $a_i^k \to a_i < \lambda(a_{-i}, b_{-i}, D)$ and $b_i = 0$ we observe

$$\pi_i(a_i^k, a_{-i}, 0, b_{-i}, D) = (a_i^k - A_i)(D - F(a_{-i}, b_{-i}, a_i^k)) - B_i(D - F(a_{-i}, b_{-i}, a_i^k))^2,$$

and so $\lim_{a_i^k \to a_i} \pi_i(a_i^k, a_{-i}, 0, b_{-i}, D) = \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$.

Moreover, Corollary A.5 (a) give us $\lim_{a_i^k \nearrow \lambda(a_{-i}, b_{-i}, D)} \pi_i(a_i^k, a_{-i}, 0, b_{-i}, D) = \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$ with respect to our assumptions. Thus, the only fact left to prove is the following equality

$$\lim_{a_i^k \to a_i, b_i^k \searrow 0} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$$

corresponding to continuous transition between profit functions given by Theorem 2.2 (a) and (b). We note that from our assumptions we necessarily have $a_i < \lambda^c(a_{-i}, b_{-i})$. Let us denote $(a^k, b^k) = (a_i^k, a_{-i}, b_i^k, b_{-i})$. Since we have $\lambda^c(a^k, b^k) = \lambda^c(a_{-i}, b_{-i}) > a_i$ for all k, it holds $D^c(a^k, b^k) = \frac{\lambda^c(a_{-i}, b_{-i}) - a_i^k}{2b_i^k} + D^c(a_{-i}, b_{-i}) > D$ for k large enough. Then, using Theorem 2.1, Remark 2 and the definition of F we obtain $q_i(a^k, b^k, D) = \frac{\lambda(a^k, b^k, D) - a_i^k}{2b_i^k} = D - F(a_{-i}, b_{-i}, \lambda(a^k, b^k, D))$. Now $\lambda(a^k, b^k, D) \to a_i$ as observed near (A1), we conclude

$$\lim_{a_i^k \to a_i, b_i^k \searrow 0} q_i(a^k, b^k, D) = D - F(a_{-i}, b_{-i}, a_i). \tag{A24}$$

Next, we see that $a_i < \lambda^c(a_{-i}, b_{-i})$ implies $\lambda^c(a_i, a_{-i}, 0, b_{-i}) = a_i$, then $N^c(a_i, a_{-i}, 0, b_{-i}) = \{i\}$ and using Theorem 2.1 also $q_i(a_i, a_{-i}, 0, b_{-i}, D) = D - F(a_{-i}, b_{-i}, a_i) = \lim_{a_i^k \to a_i, b_i^k \searrow 0} q_i(a^k, b^k, D)$. Thus the proof of continuity of $\pi_i(a, b, D)$ is finished considering formula for profit (14). $\square$

Proposition A.14 *Assume $D > 0$, $i \in \mathcal{N}$ and $(a_{-i}, b_{-i}) \in \mathbb{R}_+^{2N-2}$ such that $A_i < \lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$. Then, one of the following alternatives is satisfied:*

(a) for $D \le D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})$ it holds

$$\limsup_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i}), b_i^k \searrow 0} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = \lim_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i})} \pi_i(a_i^k, a_{-i}, 0, b_{-i}, D), \tag{A25}$$

where the upper limit is reached by any sequence $a_i^k \nearrow \lambda^c(a_{-i}, b_{-i}), b_i^k \searrow 0$ such that

$$\lim_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i}), b_i^k \searrow 0} q_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = D - D^c(a_{-i}, b_{-i}), \tag{A26}$$

(b) for $D > D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})$ it reads

$$\limsup_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i}), b_i^k \searrow 0} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) > \lim_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i})} \pi_i(a_i^k, a_{-i}, 0, b_{-i}, D), \tag{A27}$$

where for any given $\tilde{b}_i^k \searrow 0$ this upper limit is reached by $\tilde{a}_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$ satisfying

$$\tilde{a}_i^k = \frac{A_i \tilde{b}_i^k + B_i \, \lambda^c(a_{-i}, b_{-i})}{\tilde{b}_i^k + B_i}, \tag{A28}$$

thus yielding the following limiting profit

$$\lim_{\tilde{a}_i^k \nearrow \lambda^c(a_{-i},b_{-i}), \tilde{b}_i^k \searrow 0} q_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D) = q_i^\star(a_{-i}, b_{-i}). \tag{A29}$$

Proof of Proposition A.14. Let us first concentrate on the limit value of the production quantity q_i such that the profit will be maximised. Without lost of generality we can assume that $a_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$ and $b_i^k \searrow 0$ satisfies either

$$\frac{\lambda^c(a_{-i}, b_{-i}) - a_i^k}{2b_i^k} \geq D - D^c(a_{-i}, b_{-i})$$

or

$$\frac{\lambda^c(a_{-i}, b_{-i}) - a_i^k}{2b_i^k} < D - D^c(a_{-i}, b_{-i}). \tag{A30}$$

In the first case $D^c(a,b) \geq D$ since $\lambda^c(a_{-i}, b_{-i}) = \lambda^c(a,b)$, and then following the same approach as in the proof of Proposition A.13 we arrive at (A24). Now, since $F(a_{-i}, b_{-i}, \lambda^c(a_{-i}, b_{-i})) = D^c(a_{-i}, b_{-i})$ we may write

$$\lim_{a_i^k \nearrow \lambda^c(a_{-i},b_{-i}), b_i^k \searrow 0} q_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = D - D^c(a_{-i}, b_{-i}). \tag{A31}$$

In the second case we note that $D^c(a,b) < D$ by similar arguments as above. Now, since we search for an upper limit of $\pi_i(a^k, b^k, D)$, we may find sequence $\tilde{a}_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$ in such a way that $\pi_i(\tilde{a}_i^k, a_{-i}, b_i^k, b_{-i}, D)$ is maximised for each b_i^k and constraint (A30) is satisfied. For k large enough we may use Lemma A.11 due to $A_i < \lambda^c(a_{-i}, b_{-i})$ and $b_i^k < \frac{1}{2}\frac{\lambda^c(a_{-i},b_{-i})}{D - D^c(a_{-i},b_{-i})}$, thus obtaining $\tilde{a}_i^k = \tilde{\alpha}(b_i^k)$ as given by (A20). We may observe that $\tilde{a}_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$, and so by using Theorem 2.1 we have

$$\lim_{a_i^k \nearrow \lambda^c(a_{-i},b_{-i}), b_i^k \searrow 0} q_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = q_i^\star(a_{-i}, b_{-i}) \tag{A32}$$

if $D - D^c(a_{-i}, b_{-i}) \geq q_i^\star(a_{-i}, b_{-i})$ and

$$\lim_{a_i^k \nearrow \lambda^c(a_{-i},b_{-i}), b_i^k \searrow 0} q_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = D - D^c(a_{-i}, b_{-i}) \tag{A33}$$

if $D - D^c(a_{-i}, b_{-i}) \leq q_i^\star(a_{-i}, b_{-i})$, with conditions derived from (A20) substituting $q_i^\star(a_{-i}, b_{-i})$ in limit $\beta \searrow 0$.

Now, we observe that the limit of the produced quantities associated to sequences $a_i^k \nearrow \lambda^c(a_{-i}, b_{-i})$ and $b_i^k \searrow 0$ is either $D - D^c(a_{-i}, b_{-i})$ or $q_i^\star(a_{-i}, b_{-i})$, see (A31) and (A32)-(A33). Then, we directly obtain (A26) and (A29) recalling Lemma 2.5, since the profit yielded by $q_i^\star(a_{-i}, b_{-i})$ is higher or equal to the profit obtained for $D - D^c(a_{-i}, b_{-i})$. Next, by adapting (A20) from Lemma A.11 we verify formula (A28).

To finish the proof, we use (A6) and observe $D^c(\lambda(a_{-i}, b_{-i}, D), a_{-i}, 0, b_{-i}) = D^c(a_{-i}, b_{-i})$ due to Corollary A.5 and the assumption $\lambda(a_{-i}, b_{-i}, D) = \lambda^c(a_{-i}, b_{-i})$. Then $\lim_{a_i^k \nearrow \lambda^c(a_{-i},b_{-i})} q_i(a_i^k, a_{-i}, 0, b_{-i}, D) = D - D^c(a_{-i}, b_{-i})$ and so (A25) and (A27) are valid due to Lemma 2.5 again. $\square$

We are now in position to further discuss the properties of quadratic limit bids.

Proof of Proposition 3.2. Considering various candidates $a_i, b_i \geq 0$ for the best response of producer $i \in \mathcal{N}$ in $P_i(a_{-i}, b_{-i}, D)$, we first observe that bid given by $a_i \geq 0$ and $b_i > 0$ may be dismissed. Indeed, for $a_i < \lambda(a_{-i}, b_{-i}, D)$ we may refer to Proposition A.12 stating that no solution of $P_i(a_{-i}, b_{-i}, D)$ with $b_i > 0$ yields a positive profit $\tilde{\pi}_i > 0$. Next, for $a_i > \lambda(a_{-i}, b_{-i}, D)$ we immediately obtain $\pi_i(a, b, D) = 0$ due to Theorem 2.2 (d). Now, for $a_i{=}\lambda(a_{-i}, b_{-i}, D)$ we first use Lemma A.2 (c) to infer $a_i \geq \lambda(a, b, D)$, then we have $q_i(a, b, D) = 0$ due to Theorem 2.1 and so $\pi_i(a, b, D) = 0$. Finally, the fact that there is no limiting quadratic best response follows from continuity of the profit function $\pi_i(a, b, D)$ for any $a_i \geq 0$ and $b_i > 0$, see Proposition A.13 (b).

To clarify the role of sequences of quadratic bids in problem $P_i(a_{-i}, b_{-i}, D)$, we first denote $\hat{\pi}_i = \sup_{a_i \geq 0} \pi_i(a_i, a_{-i}, 0, b_{-i}, D)$. Now, we consider $D \leq D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})$, and assume, for a contradiction, that

$$\hat{\pi}_i = \sup_{a_i \geq 0} \pi_i(a_i, a_{-i}, 0, b_{-i}, D) < \sup_{a_i, b_i \geq 0} \pi_i(a_i, a_{-i}, b_i, b_{-i}, D) = \tilde{\pi}_i. \tag{A34}$$

Let (a_i^k, b_i^k) be a sequence such that $\lim_{k\to+\infty} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = \tilde{\pi}_i$. This sequence of bids is bounded by using Proposition A.13 (a), and thus one can extract a subsequence, also denoted by (a_i^k, b_i^k), converging to $(\bar{a}_i, \bar{b}_i)$. Further we observe $\bar{b}_i = 0$ as argued in the previous paragraph. Next, for $\bar{a}_i < \lambda^c(a_{-i}, b_{-i})$ profit is continuous as stated in Proposition A.13 (b), thus

$$\hat{\pi}_i \geq \lim_{k\to+\infty} \pi_i(a_i^k, a_{-i}, 0, b_{-i}, D) = \lim_{k\to+\infty} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = \tilde{\pi}_i,$$

a contradiction with (A34). Now, for $\bar{a}_i = \lambda^c(a_{-i}, b_{-i})$ we may use Proposition A.14 (a) as we assume $A_i < \lambda(a_{-i}, b_{-i}, D)$. Then it holds

$$\begin{aligned}\hat{\pi}_i \geq \lim\nolimits_{k\to+\infty} \pi_i(a_i^k, a_{-i}, 0, b_{-i}, D) &= \operatorname{limsup}_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i}), b_i^k \searrow 0} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) \\ &\geq \lim\nolimits_{k\to+\infty} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) = \tilde{\pi}_i,\end{aligned}$$

a contradiction with (A34) again. Thus we shown statement (a).

Next we deal with D such that $D > D^c(a_{-i}, b_{-i}) + q_i^\star(a_{-i}, b_{-i})$. Using $\hat{\pi}_i$ defined as above, let us assume for a contradiction that

$$\tilde{\pi}_i > \max\left\{\hat{\pi}_i, \lim_{k\to+\infty} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D)\right\}, \tag{A35}$$

where $(\tilde{a}_i^k, \tilde{b}_i^k) \to (\lambda^c(a_{-i}, b_{-i}), 0)$ is given in the statement. By the same arguments as before we consider $(a_i^k, b_i^k) \to (\bar{a}_i, 0)$ yielding the optimal profit $\tilde{\pi}_i$, and we obtain a contradiction for any $\bar{a}_i < \lambda^c(a_{-i}, b_{-i})$ again. Then, using Proposition A.14 (b) we deal with $\bar{a}_i = \lambda^c(a_{-i}, b_{-i})$, obtaining

$$\begin{aligned}\lim\nolimits_{k\to+\infty} \pi_i(\tilde{a}_i^k, a_{-i}, \tilde{b}_i^k, b_{-i}, D) &= \operatorname{limsup}_{a_i^k \nearrow \lambda^c(a_{-i}, b_{-i}), b_i^k \searrow 0} \pi_i(a_i^k, a_{-i}, b_i^k, b_{-i}, D) \\ &\geq \lim\nolimits_{k\to+\infty} \pi_i(\bar{a}_i^k, a_{-i}, \bar{b}_i^k, b_{-i}, D) = \tilde{\pi}_i,\end{aligned}$$

a contradiction with (A35). The proof of statement (b) is done.

□