The Annals of Probability
2017, Vol. 45, No. 5, 3145–3201
DOI: 10.1214/16-AOP1133

INVARIANT MEASURE FOR THE STOCHASTIC NAVIER–STOKES EQUATIONS IN UNBOUNDED 2D DOMAINS

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Building upon a recent work by two of the authors and J. Seidler on bw-Feller property for stochastic nonlinear beam and wave equations, we prove the existence of an invariant measure to stochastic 2-D Navier–Stokes (with multiplicative noise) equations in unbounded domains. This answers an open question left after the first author and Y. Li proved a corresponding result in the case of an additive noise.

1. Introduction. A classical method of proving the existence of an invariant measure for a Markov proceess is the celebrated Krylov–Bogoliubov method. Originally, it was used for Markov processes with values in locally compact state spaces, for example, finite dimensional Euclidean spaces; see, for example, [26] and [33]. In recent years, it has been successfully generalized to Markov processes with nonlocally compact state spaces, for example, infinite dimensional Hilbert and Banach spaces; see, for instance, the books by Da Prato and Zabczyk [18, 19], a paper [5] by the first named authour and Gatarek for stochastic reaction diffusion equations in a Banach space framework and a fundamental paper by Flandoli [20] for the case of 2-dimensional Navier–Stokes equations with additive noise. One should also mention here a somehow reverse problem, found, for instance, in the stochastic quantisation approach of Parisi and Wu [35], of constructing a Markov process with certain properties given an “a priori invariant measure”. In the context of stochastic partial differential equations, this approach has been successfully implemented by Da Prato and Debussche for 2-dimensional Navier–Stokes equations with periodic boundary conditions driven by space time white noise in [16] and for the 2-D stochastic quantization equation in [17].

The latter method is related to the approach by Dirichlet forms as, for instance, in [2]. In the field of deterministic dynamical systems, the so-called Avez method (see [3] and [28]), is also popular. It seems that the first of these methods when used in order to prove the existence of an invariant measure for Markov processes generated by SPDEs one requires the existence of an auxiliary set, which is compactly embedded into the state space and in which the Markov process eventually

Received February 2015; revised June 2016.
[1]Supported in part by the Leverhulme Grant RPG-2012-514.
[2]Supported by the GAČR Grant 15-08819S.
MSC2010 subject classifications. Primary 60H15, 35Q30, 37L40; secondary 76M35, 60J25.
Key words and phrases. Invariant measure, bw-Feller semigroup, stochastic Navier–Stokes equations.

lives. Thus, it has so far been restricted to SPDEs of parabolic type (giving necessary conditions with smoothing effect) and in bounded domains (providing the needed compactness via the Rellich theorem).

On the other hand, as a byproduct of results obtained by Yuhong Li and the first named author in [8], about the existence of a compact absorbing set for stochastic 2-dimensional Navier–Stokes equations with additive noise in a certain class of unbounded domains, there exists an invariant measure for the Markov process generated by such equations. This, to the best of the authors' knowledge, provides the first example of nontrivial SPDEs without the previously required compactness assumption possessing an invariant measure. A posteriori, one can see that behind the proof is the continuity of the corresponding solution flow with respect to the *weak topologies*; see Example 1.1.

It is has been discovered in [29], Proposition 3.1, that a bw-Feller semigroup has an invariant probability measure provided the set

$$\left\{ \frac{1}{T_n} \int_0^{T_n} P_s^* \nu \, ds; n \geq 1 \right\} \tag{1.1}$$

is tight on (H, bw). However, it is far from straightforward to identify stochastic PDEs for which the associated transition semigroups are bw-Feller. This has been recently done for SPDEs of hyperbolic type (i.e., second order in time) such as beam and nonlinear wave equations in [12], but see also [25] for another approach to a similar problem. The aim of this work is to show that the general approach proposed in that paper is also applicable to stochastic Navier–Stokes equations in unbounded domains. In the case of bounded domains, the first such a result has been obtained by Flandoli in the celebrated paper [20]. A similarity between the equations studied in [12] and the current paper is that the linear generator has no compact resolvent. However, in the current situation, the generator is sectorial contrary to the former case. However, the smoothing of the semigroup is rather used to counterweight the nonsmoothness of the nonlinearity.

On the other hand, in [29] Maslowski and Seidler proposed to use the of weak topologies to the proof of the existence of invariant measures but the applications of the proposed theory had limited scope.

These two papers, that is, [29] and [8] have inspired us to investigate this matter further.

Moreover, while working on the existence of solutions to geometric wave equations it has become apparent to us that the methods of using very fine techniques in order to overcome the difficulty arising from having only weak a'priori estimates should also allow one to prove the sequentially weak Feller property required by the Maslowski and Seidler approach. This made it possible to prove the existence of invariant measure for SPDEs of hyperbolic type as, for instance, wave and beam; see the recent paper [12] by the Seidler and the first and third authors.

The aim of the current work is to show that the approach worked out in [12] combined with the method of proving the existence of stochastic Navier–Stokes equations in general domains developed recently by the first and second authors (see, for instance, [9]) indeed can lead to a proof of the existence of an invariant measure for stochastic 2-dimensional Navier–Stokes equations with multiplicative noise (and additive as well) in unbounded domains, and thus generalising the previously mentioned result [8].

Let us stress that the general result proved in Sections 5–10 of [12] does not apply directly to stochastic NSEs. Instead, we propose a scheme which is general enough that it should be applicable to other equations. Let us describe it in more detail. In a domain $\mathcal{O} \subset \mathbb{R}^2$ satisfying the Poincaré inequality, we consider the following stochastic Navier–Stokes equations in the functional form:

$$\begin{cases} du(t) + Au(t)\,dt + B\big(u(t), u(t)\big)\,dt \\ \qquad = f\,dt + G\big(u(t)\big)\,dW(t), \qquad t \in [0, T], \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where A is the Stokes operator, $u_0 \in \mathrm{H}$, $f \in \mathrm{V}'$ and we use the standard notation; see the parts of the paper around equation (3.2). In particular, $W = (W(t))_{t\geq 0}$ is a cylindrical Wiener process on a separable Hilbert space K defined on a ceratin probability space and the nonlinear diffusion coefficient G satisfy some natural assumptions. It is known (but we provide an independent proof of this fact) that the above problem has a unique global solution $u(t; u_0)$, $t \geq 0$. The corresponding semigroup $(P_t)_{t\geq 0}$ is Markov; see Proposition 6.1. This semigroup is defined by the formula [see (6.2)]

$$(P_t\varphi)(u_0) = \mathbb{E}\big[\varphi\big(u(t; u_0)\big)\big], \qquad t \geq 0, u_0 \in \mathrm{H}, \tag{1.3}$$

for any bounded Borel function $\varphi \in \mathcal{B}_b(\mathrm{H})$. Then (see Proposition 6.2) we prove that this semigroup is bw-Feller, that is, for every $t > 0$ and every bounded sequentially weakly continuous function $\phi : \mathrm{H} \to \mathbb{R}$, the function $P_t\phi : \mathrm{H} \to \mathbb{R}$ is also bounded sequentially weakly continuous.

The idea of the proof of the last result can be traced to recent papers by all three of us in which we proved the existence of weak martingale solutions to the stochastic geometric wave and Navier–Stokes and equations developed respectively in [10, 11] and [9].

Finally, our main result, that is, Theorem 6.5 about the existence of an invariant measure for the semigroup $(P_t)_{t\geq 0}$, follows provided some natural assumptions, as inequality (G2) holds with $\lambda_0 = 0$, that is, for some[3] $\rho \geq 0$,

$$\big|G(u)\big|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \leq (2 - \eta)\|u\|^2 + \rho, \qquad u \in \mathrm{V}, \tag{1.4}$$

guaranteeing the uniform boundedness in probability, are satisfied; see Corollary 6.4.

[3]Throughout the entire paper, we use the symbol $\mathcal{T}_2$ to denote the space of Hilbert–Schmidt operators between corresponding Hilbert spaces.

In proving Proposition 6.2, the continuity/stability result contained in Theorem 5.9 plays an essential role.

We will present now the earlier promised example based on the paper [8].

EXAMPLE 1.1. If $\varphi = (\varphi_t)_{t\geq 0}$ is a deterministic dynamical system on a Hilbert space H, then one can define the corresponding Markov semigroup by

$$[P_t(f)](x) := f(\varphi_t(x)), \qquad t \geq 0, x \in \mathrm{H}. \tag{1.5}$$

Suppose that the semiflow is sequentially weakly continuous in the following sense:

(1.6) If $t_n \to t \in \mathbb{R}_+$, $x_n \to x$ weakly in H then $\varphi_{t_n}(x_n) \to \varphi_t(x)$ weakly in H.

Note that the above condition is satisfied for the deterministic 2-d Navier–Stokes equations; see [39] and also [8], Lemma 7.2.

Then the assertion of Theorem 9.4 in [12] holds. Indeed, let us choose and fix a bounded sequentially weakly continuous function $f : \mathrm{H} \to \mathbb{R}$, a sequence $(t_n) \to t$ and a sequence (x_n) such that $x_n \to x$ weakly in H. Then by assumption (1.6) $\varphi_{t_n}(x_n) \to \varphi_t(x)$ weakly in H and since f is sequentially weakly continuous we infer that

$$[P_{t_n}(f)](x_n) = f(\varphi_{t_n}(x_n)) \to f(\varphi_t(x)) = P_t f(x).$$

The condition guaranteeing the existence of an invariant measure (see [12], Theorem 10.1) now reads as follows. There exists $x \in \mathrm{H}$ such for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t 1_{|\varphi_s(x)|_{\mathrm{H}} \geq R}\, ds \leq \varepsilon \tag{1.7}$$

which is obviously satisfied provided the dynamical system $\varphi = (\varphi_t)_{t\geq 0}$ is bounded at infinity, that is, there exists $x \in \mathrm{H}$ and $R > 0$ such that $|\varphi_s(x)|_{\mathrm{H}} \leq R$ for all $s \geq 0$. It is well known that this condition holds for the deterministic 2-d Navier–Stokes equations in a Poincaré domain (as well as for the damped Navier–Stokes equations in the whole space $\mathbb{R}^2$. Thus, we conclude, that in those cases, there exists an invariant measure. Of course, these are known results; the purpose of this example is only to elucidate our paper by showing that it is also applicable to these cases.

Let us point out that [8], Lemma 7.2, played an important role in that paper.

We believe that the result described in this example holds also for the random dynamical system from [8]. In this way, we will get an alternative proof of the result existence of an invariant measure proved in that paper.

The weak continuity property (1.6) has also been investigated [4, 8, 15, 39]. In the first three of these references, the weak-to-weak continuity is an important tool in proving the existence of an attractor for deterministic 2D Navier–Stokes equations in unbounded domains, where, as we pointed out earlier, the compactness of the embedding from the Sobolev space H^1 to L^2 does not hold. A similar type of continuity (weak to strong) is encountered in the proof of the large deviation principle for SPDEs; see, for instance, [6], Lemma 6.3, for the case of stochastic Landau–Lifshitz equations. It might be interesting to understand in the relationship between these two types of continuity.

Let us finish the Introduction with a brief description of the content of the paper. Section 2 is devoted to recalling some basic notation and information. In Section 3, we recall the fundamental facts about Navier–Stokes equations. This section is based on a similar presentation in [9]; however, in the present paper, we make some modifications. In Section 4, we formulate and prove the convergence result for a sequence of martingale solutions of the stochastic NSEs; see, for instance, Theorems 4.9 and 4.11. The results of Section 4 hold both in 2- and 3-dimensional possibly unbounded domains. Let us stress this again; these two results are for sequence of martingale solutions of the stochastic NSEs. In the case when these are replaced by strong solutions of the corresponding Galerkin approximations, the corresponding results have been proved in [9]; see also Theorem 4.8 in the present paper. In Section 5, we recall the main results from [9] in the special case of 2-dimensional domains. Besides, we prove Theorem 5.9, needed in the main section, and being the counterpart of Theorem 4.11 for the 2-dimensional case. Theorems 4.9, 4.11 and 5.9 generalize [8], Lemmata 7.1 and 7.2. In Section 6, we state and prove the main result of this paper, that is, the existence of invariant measures for stochastic Navier–Stokes equations in 2-dimensional Poincaré, possibly unbounded, domains with multiplicative noise.

2. Preliminaries. The following introductory section is for the reader's convenience, and hence relies heavily on paper [9] by the first two named authors.

Let $\mathcal{O} \subset \mathbb{R}^d$, where $d = 2, 3$, be an open connected subset with smooth boundary $\partial\mathcal{O}$. For $p \in [1, \infty)$ by $L^p(\mathcal{O}, \mathbb{R}^d)$, we denote the Banach space of (equivalence classes) of Lebesgue measurable $\mathbb{R}^d$-valued pth power integrable functions on the set $\mathcal{O}$. The norm in $L^p(\mathcal{O}, \mathbb{R}^d)$ is given by

$$|u|_{L^p} := \left(\int_{\mathcal{O}} |u(x)|^p \, dx\right)^{\frac{1}{p}}, \qquad u \in L^p(\mathcal{O}, \mathbb{R}^d).$$

By $L^\infty(\mathcal{O}, \mathbb{R}^d)$, we denote the Banach space of Lebesgue measurable essentially bounded $\mathbb{R}^d$-valued functions defined on $\mathcal{O}$ with the norm defined by

$$|u|_{L^\infty} := \operatorname{esssup}\{|u(x)|, x \in \mathcal{O}\}, \qquad u \in L^\infty(\mathcal{O}, \mathbb{R}^d).$$

If $p = 2$, then $L^2(\mathcal{O}, \mathbb{R}^d)$ is a Hilbert space with the inner product given by

$$(u, \mathrm{v})_{L^2} := \int_{\mathcal{O}} u(x) \cdot \mathrm{v}(x) \, dx, \qquad u, \mathrm{v} \in L^2(\mathcal{O}, \mathbb{R}^d).$$

By $H^1(\mathcal{O},\mathbb{R}^d)=H^{1,2}(\mathcal{O},\mathbb{R}^d)$, we will denote the Sobolev space consisting of all $u\in L^2(\mathcal{O},\mathbb{R}^d)$ for which there exist weak derivatives $D_i u\in L^2(\mathcal{O},\mathbb{R}^d)$, $i=1,\dots,d$. It is a Hilbert space with the inner product given by

$$(u,\mathrm{v})_{H^1}:=(u,\mathrm{v})_{L^2}+(\nabla u,\nabla \mathrm{v})_{L^2},\qquad u,\mathrm{v}\in H^1(\mathcal{O},\mathbb{R}^d),$$

where $(\nabla u,\nabla \mathrm{v})_{L^2}:=\sum_{i=1}^d\int_{\mathcal{O}} D_i u(x)\cdot D_i \mathrm{v}(x)\,dx$. Let $\mathcal{C}_c^\infty(\mathcal{O},\mathbb{R}^d)$ denote the space of all $\mathbb{R}^d$-valued functions of class $\mathcal{C}^\infty$ with compact supports contained in $\mathcal{O}$. We will use the following classical spaces:

$$\mathcal{V}:=\{u\in\mathcal{C}_c^\infty(\mathcal{O},\mathbb{R}^d):\operatorname{div}u=0\},$$
$$\mathrm{H}:=\text{the closure of }\mathcal{V}\text{ in }L^2(\mathcal{O},\mathbb{R}^d),$$
$$\mathrm{V}:=\text{the closure of }\mathcal{V}\text{ in }H^1(\mathcal{O},\mathbb{R}^d).$$

In the space H, we consider the inner product and the norm inherited from $L^2(\mathcal{O},\mathbb{R}^d)$ and denote them by $(\cdot,\cdot)_{\mathrm{H}}$ and $|\cdot|_{\mathrm{H}}$, respectively, that is,

$$(u,\mathrm{v})_{\mathrm{H}}:=(u,\mathrm{v})_{L^2},\qquad |u|_{\mathrm{H}}:=|u|_{L^2(\mathcal{O})},\qquad u,\mathrm{v}\in\mathrm{H}.$$

In the space V, we consider the inner product inherited from $H^1(\mathcal{O},\mathbb{R}^d)$, that is,

$$(u,\mathrm{v})_{\mathrm{V}}:=(u,\mathrm{v})_{L^2}+((u,\mathrm{v})),\tag{2.1}$$

where

$$((u,\mathrm{v})):=(\nabla u,\nabla \mathrm{v})_{L^2},\qquad u,\mathrm{v}\in\mathrm{V}.\tag{2.2}$$

Note that the norm in V satisfies

$$|u|_{\mathrm{V}}^2:=|u|^2+|\nabla u|_{L^2}^2,\qquad \mathrm{v}\in\mathrm{V}.\tag{2.3}$$

We will often use the notation $\|\cdot\|$ for the seminorm

$$\|u\|^2:=((u,u))=(\nabla u,\nabla u)_{L^2},\qquad u\in\mathrm{V}.$$

A domain $\mathcal{O}$ satisfying the Poincaré inequality, that is, there exists a constant $C>0$ such that

$$C\int_{\mathcal{O}}\varphi^2\,d\xi\le\int_{\mathcal{O}}|\nabla\varphi|^2\,d\xi\qquad\text{for all }\varphi\in H_0^1(\mathcal{O})\tag{2.4}$$

will be called a Poincaré domain. It is well known that, in the case when $\mathcal{O}$ is a Poincaré domain, the inner product in the space V inherited from $H^1(\mathcal{O},\mathbb{R}^d)$, that is, $(u,\mathrm{v})_{\mathrm{V}}:=(u,\mathrm{v})_{L^2}+((u,\mathrm{v}))$ is equivalent to the following one:

$$(u,\mathrm{v})_{\mathcal{P}}:=((u,\mathrm{v})),\qquad u,\mathrm{v}\in\mathrm{V}.\tag{2.5}$$

In the sequel, if $\mathcal{O}$ is a Poincaré domain, then in the space V *we consider the inner product* $((\cdot,\cdot))$ *given by* (2.2) *and the corresponding norm* $\|\cdot\|$.

Denoting by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V′, that is, $\langle\cdot,\cdot\rangle := {}_{\mathrm{V}'}\langle\cdot,\cdot\rangle_{\mathrm{V}}$, by the Lax–Milgram theorem, there exists a unique bounded linear operator $\mathcal{A}: \mathrm{V} \to \mathrm{V}'$ such that we have the following equality:

$$\langle \mathcal{A}u, \mathrm{v}\rangle = ((u, \mathrm{v})), \qquad u, \mathrm{v} \in \mathrm{V}. \tag{2.6}$$

The operator $\mathcal{A}$ is closely related to the Stokes operator A defined by

$$\begin{aligned} D(\mathrm{A}) &= \{u \in \mathrm{V} : \mathcal{A}u \in \mathrm{H}\}, \\ \mathrm{A}u &= \mathcal{A}u \qquad \text{if } u \in D(\mathrm{A}). \end{aligned} \tag{2.7}$$

The Stokes operator A is a nonnegative self-adjoint operator in H. Moreover, if $\mathcal{O}$ is a 2D or 3D Poincaré domain [see (4.11) below], then A is strictly positive. We will not use the Stokes operator as in this paper we will be concerned only with the weak solutions to the stochastic Navier–Stokes equations, which in particular do not take values in the domain $D(\mathrm{A})$ of A.

Let us consider the following tri-linear form:

$$b(u, w, \mathrm{v}) = \int_{\mathcal{O}} (u \cdot \nabla w)\mathrm{v}\, dx. \tag{2.8}$$

We will recall fundamental properties of the form b. By the Sobolev embedding theorem (or Gagliardo–Nirenberg inequality), we have (see, for instance, [43], Lemmata III.3.3 and III.3.5)

$$|u|_{L^4(\mathcal{O})} \le 2^{1/4} |u|_{L^2(\mathcal{O})}^{1-\frac{d}{4}} |\nabla u|_{L^2(\mathcal{O})}^{\frac{d}{4}}, \qquad u \in H_0^{1,2}(\mathcal{O}), \text{ for } d = 2, 3, \tag{2.9}$$

by applying the Hölder inequality, we obtain the following estimates:

$$|b(u, w, \mathrm{v})| = |b(u, \mathrm{v}, w)| \le |u|_{L^4} |w|_{L^4} |\nabla \mathrm{v}|_{L^2} \tag{2.10}$$

$$\le c|u|_{\mathrm{V}} \|w\|_{\mathrm{V}} \|\mathrm{v}\|_{\mathrm{V}}, \qquad u, w, \mathrm{v} \in \mathrm{V} \tag{2.11}$$

for some positive constant c. Thus, the form b is continuous on V; see also [43]. Moreover, if we define a bilinear map B by $B(u, w) := b(u, w, \cdot)$, then by inequality (2.11) we infer that $B(u, w) \in \mathrm{V}'$ for all $u, w \in \mathrm{V}$ and, by the Gagliardo–Nirenberg inequality (2.9) that the following inequality holds, for $d = 2, 3$:

$$\begin{aligned} |B(u, w)|_{\mathrm{V}'} &\le c_1 |u|_{L^4} |w|_{L^4} \le c_2 |u|_{L^2}^{1-\frac{d}{4}} |\nabla u|_{L^2}^{\frac{d}{4}} |w|_{L^2}^{1-\frac{d}{4}} |\nabla w|_{L^2}^{\frac{d}{4}}, \\ &\le c_3 \|u\|_{\mathrm{V}} \|w\|_{\mathrm{V}}, \qquad u, w \in \mathrm{V}. \end{aligned}$$

In particular, the mapping $B : \mathrm{V} \times \mathrm{V} \to \mathrm{V}'$ is bilinear and continuous.

Let us also recall the following properties of the form b (see Temam [43], Lemma II.1.3):

$$b(u, w, \mathrm{v}) = -b(u, \mathrm{v}, w), \qquad u, \mathrm{w}, \mathrm{v} \in \mathrm{V}. \tag{2.12}$$

In particular,

$$\langle B(u, \mathrm{v}), \mathrm{v}\rangle = b(u, \mathrm{v}, \mathrm{v}) = 0, \qquad u, \mathrm{v} \in \mathrm{V}. \tag{2.13}$$

We will need the following Fréchet topologies.

DEFINITION 2.1. By $L^2_{\mathrm{loc}}(\mathcal{O},\mathbb{R}^d)=\mathbb{L}^2_{\mathrm{loc}}$, we denote the space of all Lebesgue measurable $\mathbb{R}^d$-valued functions v such that $\int_K |\mathrm{v}(x)|^2\,dx<\infty$ for every compact subset $K\subset\mathcal{O}$. In this space, we consider the Fréchet topology generated by the family of seminorms:

$$p_R := \left(\int_{\mathcal{O}_R} |\mathrm{v}(x)|^2\,dx\right)^{\frac{1}{2}}, \qquad R\in\mathbb{N},$$

where $(\mathcal{O}_R)_{R\in\mathbb{N}}$ is an increasing sequence of open bounded subsets of $\mathcal{O}$ with smooth boundaries and such that $\bigcup_{R\in\mathbb{N}}\mathcal{O}_R=\mathcal{O}$. [4]

By $\mathrm{H}_{\mathrm{loc}}$, we denote the space H endowed with the Fréchet topology inherited from the space $L^2_{\mathrm{loc}}(\mathcal{O},\mathbb{R}^d)$.

Let us, for any $s>0$, define the following standard scale of Hilbert spaces:

$$\mathrm{V}_s := \text{the closure of } \mathcal{V} \text{ in } H^s(\mathcal{O},\mathbb{R}^d).$$

If $s>\frac{d}{2}+1$ then by the Sobolev embedding theorem,

$$H^{s-1}(\mathcal{O},\mathbb{R}^d)\hookrightarrow \mathcal{C}_b(\mathcal{O},\mathbb{R}^d)\hookrightarrow L^\infty(\mathcal{O},\mathbb{R}^d).$$

Here, $\mathcal{C}_b(\mathcal{O},\mathbb{R}^d)$ denotes the space of continuous and bounded $\mathbb{R}^d$-valued functions defined on $\mathcal{O}$. If $u,w\in\mathrm{V}$ and $\mathrm{v}\in\mathrm{V}_s$ with $s>\frac{d}{2}+1$, then for some constant $c>0$,

$$|b(u,w,\mathrm{v})| = |b(u,\mathrm{v},w)| \le |u|_{L^2}|w|_{L^2}|\nabla \mathrm{v}|_{L^\infty} \le c|u|_{L^2}|w|_{L^2}|\mathrm{v}|_{\mathrm{V}_s}.$$

We have the following well-known result used in the proof of [9], Lemma 5.4.

LEMMA 2.2. *Assume that* $s>\frac{d}{2}+1$. *Then there exists a constant* $C>0$ *such that*

$$|B(u,\mathrm{v})|_{\mathrm{V}'_s} \le C|u|_{\mathrm{H}}|\mathrm{v}|_{\mathrm{H}}, \qquad u,\mathrm{v}\in\mathrm{V}. \tag{2.14}$$

Hence, in particular, there exists a unique bilinear and bounded map $\tilde{B}:\mathrm{H}\times\mathrm{H}\to\mathrm{V}'_s$ *such that* $B(u,\mathrm{v})=\tilde{B}(u,\mathrm{v})$ *for all* $u,\mathrm{v}\in\mathrm{V}$.

In what follows, the map $\tilde{B}$ *will be denoted by* B *as well.*

3. Stochastic Navier–Stokes equations. We begin this section with listing all the main assumptions.

ASSUMPTION 3.1. We assume that the following objects are given:

(H.1) A separable Hilbert space K;

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[4] Such sequence $(\mathcal{O}_R)_{R\in\mathbb{N}}$ always exists since it is sufficient to consider as $\mathcal{O}_R$ a smoothed out version of the set $\mathcal{O}\cap B(0,R)$; see, for instance, [40] and references therein.

(H.2) a map $G : \mathrm{V} \to \mathcal{T}_2(\mathrm{K}, \mathrm{H})$ that:

(i) is Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\big|G(u_1) - G(u_2)\big|_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \le L\|u_1 - u_2\|_{\mathrm{V}}, \qquad u_1, u_2 \in \mathrm{V}, \tag{G1}$$

(ii) for some constants λ_0, ρ and $\eta \in (0, 2]$,

$$\big|G(u)\big|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \le (2 - \eta)\|u\|^2 + \lambda_0 |u|^2_{\mathrm{H}} + \rho, \qquad u \in \mathrm{V}, \tag{G2}$$

(iii) extends to a measurable map $G : \mathrm{H} \to \mathcal{T}_2(\mathrm{K}, \mathrm{V}')$ such that for some $C > 0$

$$\|G(u)\|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{V}')} \le C\big(1 + |u|^2_{\mathrm{H}}\big), \qquad u \in \mathrm{H}, \tag{G3}$$

(iv) and, for every $\psi \in \mathcal{V}$ the function

$$\psi^{**}G : \mathrm{H}_{\mathrm{loc}} \ni u \mapsto \big\{\mathrm{K} \ni y \mapsto {}_{\mathrm{V}'}\big\langle G(u)y, \psi\big\rangle_{\mathrm{V}} \in \mathbb{R}\big\} \in \mathrm{K}' \tag{G4}$$

is continuous;

(H.3) a real number p such that

$$p \in \left[2, 2 + \frac{\eta}{2 - \eta}\right), \tag{3.1}$$

where we put $\frac{\eta}{2-\eta} = \infty$ when $\eta = 2$;

(H.4) a Borel probability measure μ_0 on H such that $\int_H |x|^p \mu_0(dx) < \infty$ is given and $f \in L^p_{\mathrm{loc}}([0, \infty); V')$;

(H.5) an linear operator $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$ satisfying equality (2.6).

Now we state definition of a martingale solution of equation (3.2). We really need to consider the infinite time interval, that is, $[0, \infty)$; however, we need also to state some of the results on the interval $[0, T]$, where $T > 0$ is fixed. Thus, in the following definition we distinguish between the two cases of solution on a finite interval $[0, T]$ and on $[0, \infty)$.

DEFINITION 3.2. Let us assume Assumption 3.1. Let $T > 0$ be fixed. We say that there exists a *martingale solution* of the following stochastic Navier–Stokes equations (in an abstract form) on the interval $[0, T]$:

$$\begin{cases} du(t) + \mathcal{A}u(t)\,dt + B\big(u(t), u(t)\big)\,dt \\ \qquad = f(t)\,dt + G\big(u(t)\big)\,dW(t), \qquad t \ge 0, \\ \mathcal{L}\big(u(0)\big) = \mu_0, \end{cases} \tag{3.2}$$

iff there exist:

- a stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$ with a complete filtration $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_t\}_{t\in[0,T]}$,
- a K-cylindrical Wiener process $\hat{W} = (\hat{W})_{t\in[0,T]}$

- and an $\hat{\mathbb{F}}$-progressively measurable process $u : [0,T] \times \hat{\Omega} \to \mathrm{H}$ with $\hat{\mathbb{P}}$-a.e. paths satisfying

$$u(\cdot,\omega) \in \mathcal{C}([0,T],\mathrm{H}_w) \cap L^2(0,T;\mathrm{V}) \tag{3.3}$$

such that

the law on H of $u(0)$ is equal to μ_0

and, for all $t \in [0,T]$ and all $v \in \mathcal{V}$,

$$\begin{aligned} &(u(t),v)_{\mathrm{H}} + \int_0^t \langle \mathcal{A}u(s), v\rangle\, ds + \int_0^t \langle B(u(s)), v\rangle\, ds \\ &\quad = (u(0),v)_{\mathrm{H}} + \int_0^t \langle f(s), v\rangle\, ds + \left\langle \int_0^t G(u(s))\, d\hat{W}(s), v\right\rangle, \qquad \hat{\mathbb{P}}\text{-a.s.} \end{aligned} \tag{3.4}$$

and

$$\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} |u(t)|_{\mathrm{H}}^2 + \int_0^T |\nabla u(t)|^2\, dt\right] < \infty. \tag{3.5}$$

If all the above conditions are satisfied, then the system

$$(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$$

will be called a martingale solution to problem (3.2) on the interval $[0,T]$ with the initial distribution μ_0.

A system $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$ will be called a *martingale solution* to problem (3.2) with the initial distribution μ_0 iff all the above conditions are defined with the interval $[0,T]$ being replaced by $[0,\infty)$ and the condition (3.3) is replaced by

$$u(\cdot,\omega) \in \mathcal{C}([0,\infty),\mathrm{H}_w) \cap L^2_{\mathrm{loc}}([0,\infty);\mathrm{V}), \tag{3.6}$$

and inequality (3.5) holds for every $T > 0$.

Here, H_w denotes the Hilbert space H endowed with the weak topology and $\mathcal{C}([0,T],\mathrm{H}_w)$ and $\mathcal{C}([0,\infty),\mathrm{H}_w)$ denote the spaces of H valued weakly continuous functions defined on $[0,T]$ and $[0,\infty)$, respectively.

In the case when μ_0 is equal to the law on H of a given random variable $u_0 : \Omega \to \mathrm{H}$ then, somehow incorrectly, a martingale solution to problem (3.2) will also be called a martingale solution to problem (3.2) with the initial data u_0. Fully correctly, it should be called a martingale solution to problem (3.2) with the initial data having the same law as u_0. In particular, in this case we require that the laws on H of u_0 and $u(0)$ are equal.

If no confusion seems likely, a system $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$ *from Definition* 3.2 *will be called* martingale solutions.

REMARK 3.3. Let us recall the following observation from [9]. Since $\|u\| := |\nabla u|_{L^2}$ and $\langle \mathcal{A}u, u\rangle = ((u,u)) := (\nabla u, \nabla u)_{L^2}$, we have

$$(2-\eta)\|u\|^2 = 2\langle \mathcal{A}u, u\rangle - \eta\|u\|^2, \qquad u \in \mathrm{V}.$$

Hence, inequality (G2) can be written equivalently in the following form:

$$\text{(G2}'\text{)} \qquad 2\langle \mathcal{A}u, u\rangle - \|G(u)\|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \geq \eta\|u\|^2 - \lambda_0|u|^2_{\mathrm{H}} - \rho, \qquad u \in \mathrm{V}.$$

Inequality (G2′) is the same as considered by Flandoli and Gątarek in [21] for stochastic NSEs in bounded domains. The assumption $\eta = 2$ corresponds to the case when the noise term does not depend on ∇u. We will prove that the set of measures induced on appropriate space by the solutions of the Galerkin equations is tight provided that the map G from part (H.2) of Assumption 3.1 satisfies inequalities (G3) and (G2). Inequality (G3) and condition (G4) from part (H.2) of Assumption 3.1 will be important in passing to the limit as $n \to \infty$ in the Galerkin approximation. Condition (G4) is essential in the case of unbounded domain $\mathcal{O}$. It is worth mentioning that the following example of the noise term, analyzed in details in [9], Section 6, is covered by part (H.2) of Assumption 3.1.

EXAMPLE 3.4. Let us consider the noise term written classically as

$$[G(u)](t,x)\,dW(t) := \sum_{i=1}^{\infty}\big[(b_i(x)\cdot\nabla)u(t,x) + c_i(x)u(t,x)\big]\,d\beta_i(t), \tag{3.7}$$

where

β_i, $i \in \mathbb{N}$, are i.i.d. standard $\mathbb{R}$-valued Brownian motions,

$b_i : \overline{\mathcal{O}} \to \mathbb{R}^d$, $i \in \mathbb{N}$, are functions of class $\mathcal{C}^\infty$ class,

$c_i : \overline{\mathcal{O}} \to \mathbb{R}$, $i \in \mathbb{N}$, are functions of $\mathcal{C}^\infty$—of class,

are given. Assume that

$$C_1 := \sum_{i=1}^{\infty}\big(\|b_i\|^2_{L^\infty} + \|\operatorname{div} b_i\|^2_{L^\infty} + \|c_i\|^2_{L^\infty}\big) < \infty \tag{3.8}$$

and there exists $a \in (0,2]$ such that for all $\zeta = (\zeta_1,\dots,\zeta_d) \in \mathbb{R}^d$ and all $x \in \mathcal{O}$,

$$\sum_{i=1}^{\infty}\sum_{j,k=1}^{d} b_i^j(x)b_i^k(x)\zeta_j\zeta_k \leq 2\sum_{j,k=1}^{d}\delta_{jk}\zeta_j\zeta_k - a|\zeta|^2 = (2-a)|\zeta|^2. \tag{3.9}$$

This noise term can be reformulated in the following manner. Let $\mathrm{K} := l^2(\mathbb{N})$, where $l^2(\mathbb{N})$ denotes the space of all sequences $(h_i)_{i\in\mathbb{N}} \subset \mathbb{R}$ such that $\sum_{i=1}^{\infty} h_i^2 <$

∞. It is a Hilbert space with the scalar product given by $(h,k)_{l^2} := \sum_{i=1}^{\infty} h_i k_i$, where $h = (h_i)$ and $k = (k_i)$ belong to $l^2(\mathbb{N})$. Putting

$$G(u)h = \sum_{i=1}^{\infty} [(b_i \cdot \nabla)u + c_i u] h_i, \qquad u \in \mathrm{V}, h = (h_i) \in l^2(\mathbb{N}), \tag{3.10}$$

we infer that the mapping G fulfils all conditions stated in assumption (H.2); see [9], Section 6, for details.

REMARK 3.5. Let us explain that via the isomorphism between the space V and its dual V′, condition (H.2)(iii) in Assumption 3.1 is understood in the usual sense, that is, for every orthonormal basis $(e_k) \subset \mathrm{K}$

$$\sum_k |G(u)(e_k)|^2_{\mathrm{V}'} \le C(1 + |u|^2_{\mathrm{H}}), \qquad u \in \mathrm{H}.$$

In fact, conditions (H.2)(iii) and (iv) in Assumption 3.1 can be replaced by the following more general:

(iii′) The map $G : \mathrm{V} \to \mathcal{T}_2(\mathrm{K}, \mathrm{H})$ extends to a measurable map $g : \mathrm{H} \to \mathcal{L}(\mathrm{K}, \mathrm{V}')$ such that for some $C > 0$ for every $u \in \mathrm{H}$

$$\sup_{v \in \mathrm{V}, \|v\|_{\mathrm{V}} \le 1} \sup_{k \in \mathrm{K}, \|k\|_{\mathrm{K}} \le 1} \left| {}_{\mathrm{V}'}\langle g(u)(k), v\rangle_{\mathrm{V}} \right|^2 \le C(1 + |u|^2_{\mathrm{H}}). \tag{G3′}$$

(iv′) and, for every $\psi \in \mathcal{V}$ the function

$$\psi^{**} g : \mathrm{H}_{\mathrm{loc}} \ni u \mapsto \{\mathrm{K} \ni y \mapsto {}_{\mathrm{V}'}\langle g(u)y, \psi\rangle_{\mathrm{V}} \in \mathbb{R}\} \in \mathrm{K}' \tag{G4′}$$

is continuous.

REMARK 3.6. Note that by Definition 3.2 every solution to problem (3.2) satisfies equality (3.4) for all $\mathrm{v} \in \mathcal{V}$. However, equality (3.4) holds not only for $\mathrm{v} \in \mathcal{V}$ but also for all $\mathrm{v} \in \mathrm{V}$. Indeed, this follows from the density of $\mathcal{V}$ in the space V and the fact that each term in (3.4) is well defined and continuous with respect to $\mathrm{v} \in \mathrm{V}$. This remark is important while using Itô's formula in the proof of Lemma 5.8.

REMARK 3.7. Let assumptions (H.1)–(H.5) be satisfied. If the system $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$ is a martingale solution of problem (3.2) on the interval $[0, \infty)$, then $\hat{\mathbb{P}}$-a.e. paths of the process $u(t)$, $t \in [0, \infty)$, are V′-valued continuous functions, that is, for $\hat{\mathbb{P}}$-a.e. $\omega \in \hat{\Omega}$

$$u(\cdot, \omega) \in \mathcal{C}([0, \infty), \mathrm{V}'), \tag{3.11}$$

and equality (3.4) can be rewritten as the following one, understood in the space V′:

$$\begin{aligned} u(t) &+ \int_0^t \mathcal{A}u(s)\, ds + \int_0^t B(u(s))\, ds \\ &= u(0) + \int_0^t f(s)\, ds + \int_0^t G(u(s))\, d\hat{W}(s), \qquad t \in [0, \infty). \end{aligned} \tag{3.12}$$

PROOF. Let us fix any $T > 0$. Let us notice that since the map G satisfies inequality (G3) in Assumption 3.1, by inequality (3.5) we infer that

$$\hat{\mathbb{E}}\left[\int_0^T |G(u(s))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{V}')}\,ds\right] \leq C\hat{\mathbb{E}}\left[\int_0^T (1+|u(s)|^2_{\mathrm{H}})\,ds\right] < \infty.$$

Thus, the process μ defined by

$$\mu(t) := \int_0^t G(u(s))\,d\hat{W}(s), \qquad t \in [0,T],$$

is a V'-valued square integrable continuous martingale.

REMARK. The process μ is an H-valued square integrable continuous martingale as well.

PROOF OF REMARK. Since the map G satisfies inequality (G2) in Assumption 3.1, using inequality (3.5) we deduce that

$$\hat{\mathbb{E}}\left[\int_0^T |G(u(s))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,ds\right] \leq \hat{\mathbb{E}}\left[\int_0^T [(2-\eta)\|u(s)\|^2 + \lambda_0|u(s)|^2_{\mathrm{H}} + \rho]\,ds\right] < \infty.$$

Thus, $\mu(t)$, $t \in [0,T]$, is an H-valued square integrable continuous martingale. □

In the framework of Remark 3.7, by the regularity assumption (3.3), we infer that for $\hat{\mathbb{P}}$-a.e. $\omega \in \hat{\Omega}$

$$\mathcal{A}u(\cdot,\omega) \in L^2(0,T;\mathrm{V}'), \qquad B(u(\cdot,\omega),u(\cdot,\omega)) \in L^{4/3}(0,T;\mathrm{V}').$$

By assumption (H.3), in particular, $f \in L^p(0,T;\mathrm{V}')$. Hence, for $\hat{\mathbb{P}}$-a.e. $\omega \in \hat{\Omega}$ the functions

$$[0,T] \ni t \mapsto \int_0^t \mathcal{A}u(s,\omega)\,ds \in \mathrm{V}',$$

$$[0,T] \ni t \mapsto \int_0^t B(u(s,\omega),(u(s,\omega))\,ds \in \mathrm{V}',$$

$$[0,T] \ni t \mapsto \int_0^t f(s)\,ds \in \mathrm{V}'$$

are well defined and continuous. Using (3.4), we infer that for $\hat{\mathbb{P}}$-a.e. $\omega \in \hat{\Omega}$

$$u(\cdot,\omega) \in \mathcal{C}([0,T],\mathrm{V}')$$

and for every $t \in [0,T]$ equality (3.12) holds. Since $T > 0$ has been chosen in an arbitrary way, regularity condition (3.11) and equality (3.12) hold. The proof of the claim is thus complete. □

4. The continuous dependence of the solutions on the initial state and the external forces in 2D and 3D domains. In this section, we will concentrate on martingale solutions to problem (3.2) on a fixed interval $[0, T]$. The main result is Theorem 4.11. We will also need some modification of Theorem 5.1 in [9], contained in Theorem 4.8.

As in [9], in the proofs we will use the following structure. Let us fix $s > \frac{d}{2} + 1$ and notice that the space V_s is dense in V and the natural embedding $\mathrm{V}_s \hookrightarrow \mathrm{V}$ is continuous. By [22], Lemma 2.5 (see also [9], Lemma C.1), there exists a separable Hilbert space U such that U is a dense subset of V_s and

$$\text{(4.1)} \qquad \textit{the natural embedding } \iota_s : U \hookrightarrow \mathrm{V}_s \textit{ is compact}.$$

Then we also have

$$\text{(4.2)} \qquad U \hookrightarrow \mathrm{V}_s \hookrightarrow \mathrm{H} \cong \mathrm{H}' \hookrightarrow \mathrm{V}_s' \hookrightarrow U',$$

where H' and U' are the dual spaces of H and U, respectively, H' being identified with H and the dual embedding $\mathrm{H}' \hookrightarrow U'$ is compact as well.

In the next definition, we will recall definition of a topological space $\mathcal{Z}_T$ that plays an important role in our approach; see page 1629 and Section 3 in [9].

To define the space $\mathcal{Z}_T$, we will need the following four spaces:

$$\begin{aligned}
\mathcal{C}([0,T],U') &:= \text{the space of continuous functions } u:[0,T]\to U' \\
&\quad\ \text{with the topology induced by the norm} \\
&\quad\ |u|_{\mathcal{C}([0,T],U')} := \sup_{t\in[0,T]} |u(t)|_{U'}, \\
L^2_w(0,T;\mathrm{V}) &:= \text{the space } L^2(0,T;\mathrm{V}) \text{ with the weak topology}, \\
L^2(0,T;\mathrm{H}_{\mathrm{loc}}) &:= \text{the space of all measurable functions } u:[0,T]\to \mathrm{H} \\
&\quad\ \text{such that for all } R\in\mathbb{N} \\
&\quad\ p_{T,R}(u) := \Big(\int_0^T \int_{\mathcal{O}_R} |u(t,x)|^2\,dx\,dt\Big)^{\frac{1}{2}} < \infty \\
&\quad\ \text{with the topology generated by the seminorms} \\
&\quad\ (p_{T,R})_{R\in\mathbb{N}}.
\end{aligned}$$

Let H_w denote the Hilbert space H endowed with the weak topology and let us put

$$\begin{aligned}
\mathcal{C}([0,T];\mathrm{H}_w) &:= \text{the space of weakly continuous functions } u:[0,T]\to\mathrm{H} \\
&\quad\ \text{endowed with the weakest topology such that for all } h\in\mathrm{H} \\
&\quad\ \text{the mappings } \mathcal{C}([0,T];\mathrm{H}_w)\ni u\mapsto (u(\cdot),h)_{\mathrm{H}}\in\mathcal{C}([0,T];\mathbb{R}) \\
&\quad\ \text{are continuous.}
\end{aligned}$$

DEFINITION 4.1. For $T > 0$, let us put

$$\mathcal{Z}_T := \mathcal{C}\big([0,T];U'\big) \cap L^2_w(0,T;\mathrm{V}) \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}}) \cap \mathcal{C}\big([0,T];\mathrm{H}_w\big) \tag{4.3}$$

and let $\mathcal{T}_T$ be the supremum of the corresponding four topologies, that is, the smallest topology on $\mathcal{Z}_T$ such that the four natural embeddings from $\mathcal{Z}_T$ are continuous. The space $\mathcal{Z}_T$ will also considered with the Borel σ-algebra, that is, the smallest σ-algebra containing the family $\mathcal{T}_T$.

The following auxiliary result which is needed in the proof of Theorem 4.11, cannot be deduced directly from the Kuratowski theorem [27]; see Counterexample C.4 in Appendix C.

LEMMA 4.2. *Assume that $T > 0$. Then the following fours sets: $\mathcal{C}([0,T];\mathrm{H}) \cap \mathcal{Z}_T$, $\mathcal{C}([0,T];\mathrm{V}) \cap \mathcal{Z}_T$, $L^2(0,T;V) \cap \mathcal{Z}_T$ and $\mathcal{C}([0,T];\mathrm{V}') \cap \mathcal{Z}_T$ are Borel subsets of $\mathcal{Z}_T$ and the corresponding embedding tranforms Borel sets into Borel subsets of $\mathcal{Z}_T$. Moreover, the following $\mathbb{R}_+ \cup \{+\infty\}$-valued functions:*

$$\mathcal{Z}_T \ni u \mapsto \begin{cases} \sup_{s\in[0,T]} |u(s)|^2_{\mathrm{H}}, & \text{if } u \in \mathcal{C}\big([0,T];\mathrm{H}\big) \cap \mathcal{Z}_T, \\ \infty, & \text{otherwise}, \end{cases}$$

$$\mathcal{Z}_T \ni u \mapsto \begin{cases} \int_0^T \|u(s)\|^2\,ds, & \text{if } u \in L^2(0,T;\mathrm{V}) \cap \mathcal{Z}_T, \\ \infty & \text{otherwise}, \end{cases}$$

are Borel.

PROOF. Because $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}})$ is a Polish space, by the Kuratowski theorem $\mathcal{C}([0,T];\mathrm{H})$ is Borel subset of $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}})$. Hence, the intersection $\mathcal{C}([0,T];\mathrm{H}) \cap \mathcal{Z}_T$ is a Borel subset of the intersection $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}}) \cap \mathcal{Z}_T$ which happens to be equal to $\mathcal{Z}_T$.

We can argue in the same way in the case of the spaces $\mathcal{C}([0,T];\mathrm{V}) \cap \mathcal{Z}_T$ and $\mathcal{C}([0,T];\mathrm{V}') \cap \mathcal{Z}_T$.

The proof in case the space $L^2(0,T;\mathrm{V})$ is analogous; one needs to begin with an observation that by the Kuratowski theorem the set $L^2(0,T;\mathrm{V})$ is Borel subset of $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$. We have used a fact that a product of Borel set in $\mathcal{C}([0,T];U') \cap L^2(0,T;\mathrm{H}_{\mathrm{loc}})$ and the set $\mathcal{Z}_T$ is a Borel subset of the latter.

The same argument applies to the proof that i_T and j_T map Borel subsets of their corresponding domains to Borel sets in $\mathcal{Z}_T$. The last part of the lemma is a consequence Proposition C.2. $\square$

4.1. *Tightness criterion and Jakubowski's version of the Skorokhod theorem.* One of the main tools in this section is the tightness criterion in the space $\mathcal{Z}_T$ defined in identity (4.3). We will use a slight generalization of the criterion stated in Corollary 3.9 from [9]; compare with the proof of Lemma 5.4 therein. Namely, we will consider the sequence of stochastic processes defined on their own probability spaces. Let $(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, \mathbb{P}_n)$, $n \in \mathbb{N}$, be a sequence of probability spaces with the filtration $\mathbb{F}_n = (\mathcal{F}_{n,t})_{t\geq 0}$.

COROLLARY 4.3 (Tightness criterion). *Assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of continuous $\mathbb{F}_n$-adapted U'-valued processes defined on Ω_n and such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_n\Big[\sup_{s\in[0,T]} |X_n(s)|_{\mathrm{H}}^2\Big] < \infty, \tag{4.4}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}_n\left[\int_0^T \|X_n(s)\|^2 \, ds\right] < \infty, \tag{4.5}$$

(a) *and for every $\varepsilon > 0$ and for every $\eta > 0$ there exists $\delta > 0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued $\mathbb{F}_n$-stopping times one has*

$$\sup_{n\in\mathbb{N}} \sup_{0\leq\theta\leq\delta} \mathbb{P}_n\big\{|X_n(\tau_n+\theta) - X_n(\tau_n)|_{U'} \geq \eta\big\} \leq \varepsilon. \tag{4.6}$$

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on the Borel σ-field $\mathcal{B}(\mathcal{Z}_T)$. Then for every $\varepsilon > 0$ there exists a compact subset K_ε of $\mathcal{Z}_T$ such that

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{P}}_n(K_\varepsilon) \geq 1 - \varepsilon.$$

The proof of Corollary 4.3 is essentially the same as the proof of [9], Corollary 3.9.

If the sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition (a), then we say that it satisfies the Aldous condition [A] in U' on [0,T]. If it satisfies condition (a) for each $T > 0$, we say that it satisfies the Aldous condition [A] in U'.

Obviously, the class of U'-valued processes satisfying the Aldous condition is a real vector space. Below we will formulate a sufficient condition for the Aldous condition. This idea has been used in the proof of Lemma 5.4 in [9] but it has not been formulated in such a way.

LEMMA 4.4. *Assume that Y is a separable Banach space, $\sigma \in (0,1]$ and that $(u_n)_{n\in\mathbb{N}}$ is a sequence of continuous $\mathbb{F}_n$-adapted Y-valued processes indexed by $[0,T]$ for some $T > 0$, such that*

(a′) *there exists $C > 0$ such that for every $\theta > 0$ and for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $[0,T]$-valued $\mathbb{F}_n$-stopping times with one has*

$$\mathbb{E}_n\big[|u_n(\tau_n+\theta) - u_n(\tau_n)|_Y\big] \leq C\theta^\sigma. \tag{4.7}$$

Then the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition [A] *on* $[0,T]$.

PROOF. Let us fix $\eta > 0$ and $\varepsilon > 0$. By the Chebyshev inequality and the estimate (4.7), we obtain

$$\mathbb{P}_n(\{|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y \geq \eta\}) \leq \frac{1}{\eta}\mathbb{E}_n[|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y] \leq \frac{C\cdot\theta^\sigma}{\eta}, \qquad n\in\mathbb{N}.$$

Let us $\delta := [\frac{\eta\cdot\varepsilon}{C}]^{\frac{1}{\sigma}}$. Then we have

$$\sup_{n\in\mathbb{N}}\sup_{1\leq\theta\leq\delta}\mathbb{P}_n\{|u_n(\tau_n+\theta)-u_n(\tau_n)|_Y \geq \eta\} \leq \varepsilon.$$

This completes the proof of Lemma 4.4. □

REMARK 4.5. As can be seen in (4.3), the space $\mathcal{Z}_T$ is defined as an intersection of four spaces, one of them being the space $\mathcal{C}([0,T];U')$. The latter space plays, in fact, only an auxiliary role. Let us recall that the space U (see (4.1) and [9], Section 2.3) is important in the construction of the solutions to stochastic Navier–Stokes equations via the Galerkin method in the case of an unbounded domain, that is, when the embedding $\mathrm{V}\subset\mathrm{H}$ is not compact. (In the case of a bounded domain, we can take, for example, $U := \mathrm{V}_s$ for sufficiently large s.) In particular, the orthonormal basis of the space H, which we use in the Galerkin method is contained in U, so the Galerkin solutions "live in" the space U.

With the space U in hand, in [9] we prove an appropriate compactness and tightness criteria in the space $\mathcal{Z}_T$; see [9], Lemma 3.3 and Corollary 3.9. Let us emphasize that in order to prove the relative compactness of an appropriate set in the Fréchet space $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$ first we need to prove a certain generalization of the classical Dubinsky theorem; see [9], Lemma 3.1, where the space $\mathcal{C}([0,T];U')$ is used. This result is related to the Aldous condition in the space U' in the tightness criterion, (4.6) in Corollary 4.3 and [9], Corollary 3.9(c).

We will use Corollary 4.3 to prove Theorems 4.9 and 4.11 below. Even though the presence of the space $\mathcal{C}([0,T];U')$ in the definition of the space $\mathcal{Z}_T$ is natural in the context of the Galerkin approximation solutions, its presence in the context of Theorems 4.9 and 4.11 where we consider sequences of the solutions of the Navier–Stokes equations seems to be unnecessary. However, again because of the lack of the compactness of the embedding $\mathrm{V}\subset\mathrm{H}$ to prove tightness in Theorem 4.9 we still use Corollary 4.3 in its original form.

In the proofs of the theorems on the existence of a martingale solution and on the continuous dependence of the data, we use a version of the Skorokhod theorem for nonmetric spaces. For convenience of the reader, let us recall the following Jakubowski's [24] version of the Skorokhod theorem; see also Brzeźniak and Ondreját [11].

THEOREM 4.6 (Theorem 2 in [24]). *Let $(\mathcal{X}, \tau)$ be a topological space such that there exists a sequence (f_m) of continuous functions $f_m : \mathcal{X} \to \mathbb{R}$ that separates points of $\mathcal{X}$. Let (X_n) be a sequence of $\mathcal{X}$-valued Borel random variables. Suppose that for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathcal{X}$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}(\{X_n \in K_\varepsilon\}) > 1 - \varepsilon.$$

Then there exists a subsequence $(X_{n_k})_{k\in\mathbb{N}}$, a sequence $(Y_k)_{k\in\mathbb{N}}$ of $\mathcal{X}$-valued Borel random variables and an $\mathcal{X}$-valued Borel random variable Y defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathcal{L}(X_{n_k}) = \mathcal{L}(Y_k), \qquad k = 1, 2, \dots,$$

and for all $\omega \in \Omega$:

$$Y_k(\omega) \xrightarrow{\tau} Y(\omega) \qquad \text{as } k \to \infty.$$

Note that the sequence (f_m) defines another, weaker topology on $\mathcal{X}$. However, this topology restricted to σ-compact subsets of $\mathcal{X}$ is equivalent to the original topology τ. Let us emphasize that thanks to the assumption on the tightness of the set of laws $\{\mathcal{L}(X_n), n \in \mathbb{N}\}$ on the space $\mathcal{X}$ the maps Y and Y_k, $k \in \mathbb{N}$, in Theorem 4.6 are measurable with respect to the Borel σ-field in the space $\mathcal{X}$.

The following result has been proved in the proof of [9], Corollary 3.12, for the spaces $\mathcal{Z}_T$.

LEMMA 4.7. *The topological space $\mathcal{Z}_T$ satisfies the assumptions of Theorem 4.6.*

4.2. *The existence and properties of martingale solutions on* $[0, T]$. We will concentrate on martingale solutions to problem (3.2) on a fixed interval $[0, T]$. The following result is a slight generalisation of Theorem 5.1 in [9]. In comparison to [9], the deterministic initial state has been replaced by the random one satisfying assumption (H.3). However, our attention will be focused on the estimates satisfied by the solutions of the Navier–Stokes equations. We claim that there exists a solution u satisfying estimate $\hat{\mathbb{E}}[\sup_{t\in[0,T]} |u(t)|_{\mathrm{H}}^q] \le C_1(p, q)$ for every $q \in [2, p]$, (and not only for $q = 2$ as stated in inequality (5.1) in [9]). Moreover, we analyse what is the relation between the constant $C_1(p, q)$ and the initial state u_0 and the external forces f. The same concerns the estimate on $\hat{\mathbb{E}}[\int_0^T \|u(t)\|^2 \, dt]$. These results generalize [9], Theorem 5.1. In the second part of Theorem 4.8, we will prove another estimate on u in the case when $\mathcal{O}$ is a 2D or 3D Poincaré domain; see (4.11) below. This estimate will be of crucial importance in the proof of existence of an invariant measure in 2D case. The proof of Theorem 4.8 is based on the Galerkin method. The analysis of the Galerkin equations is postponed to Appendix A. Recall also that in assumption (H.3) we have put $\frac{\eta}{2-\eta} = \infty$ when $\eta = 2$.

THEOREM 4.8. *Let assumptions* (H.1)–(H.5) *be satisfied. In particular, we assume that* p *satisfies* (3.1), *that is*,

$$p \in \left[2, 2 + \frac{\eta}{2-\eta}\right),$$

where $\eta \in (0, 2]$ *is given in assumption* (H.2).

(1) *For every* $T > 0$ *and* $R_1, R_2 > 0$ *if* μ_0 *is a Borel probability measure on* H, $f \in L^p([0,\infty); \mathrm{V}')$ *satisfy* $\int_{\mathrm{H}} |x|^p \mu_0(dx) \le R_1$ *and* $|f|_{L^p(0,T;\mathrm{V}')} \le R_2$, *then there exists a* martingale solution $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, \hat{W}, u)$ *to problem* (3.2) *with the initial law* μ_0 *which satisfies the following estimates*: *for every* $q \in [1, p]$, *there exist constants* $C_1(p,q)$ *and* $C_2(p)$, *depending also on* T, R_1 *and* R_2, *such that*

$$\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]} |u(s)|_{\mathrm{H}}^q\Big) \le C_1(p,q), \tag{4.8}$$

putting $C_1(p) := C_1(p,p)$, *in particular*,

$$\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]} |u(s)|_{\mathrm{H}}^p\Big) \le C_1(p), \tag{4.9}$$

and

$$\hat{\mathbb{E}}\left[\int_0^T |\nabla u(s)|_{L^2}^2 \, ds\right] \le C_2(p). \tag{4.10}$$

(2) *Moreover, if* $\mathcal{O}$ *is a Poincaré domain and the map* G *satisfies inequality* (G2) *in Assumption* 3.1 *with* $\lambda_0 = 0$, *then there exists a martingale solution* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, u)$ *of problem* (3.2) *satisfying additionally the following inequality for every* $T > 0$:

$$\frac{\eta}{2}\hat{\mathbb{E}}\left[\int_0^T |\nabla u(s)|_{L^2}^2 \, ds\right] \le \hat{\mathbb{E}}\big[|u(0)|_{\mathrm{H}}^2\big] + \frac{2}{\eta}\int_0^T |f(s)|_{\mathrm{V}'}^2 \, ds + \rho T. \tag{4.11}$$

The proof of Theorem 4.8 is postponed to Appendix B.

4.3. *The continuous dependence.* We prove the following results related to the continuous dependence on the deterministic initial condition and deterministic external forces. Roughly speaking, we will show that if $(u_{0,n}) \subset \mathrm{H}$ and $(f_n) \subset L^p(0,T;\mathrm{V}')$ are sequences of initial conditions and external forces approaching $u_0 \in \mathrm{H}$ and $f \in L^p(0,T,\mathrm{V}')$, respectively, then a sequence (u_n) of martingale solutions of the Navier–Stokes equations with the data $(u_{0,n}, f_n)$, satisfying inequalities (4.8)–(4.10), contains a subsequence of solutions, on a changed probability basis, convergent to a martingale solution with the initial condition u_0 and the external force f. Note that existence of such solutions u_n, $n \in \mathbb{N}$, is guaranteed by Theorem 4.8. This result holds both in 2D and 3D possibly unbounded domains with smooth boundaries. Moreover, in the case of $2D$ domains, because of the existence and uniqueness of the strong solutions, the stronger result holds. Namely, the solutions u_n, $n \in \mathbb{N}$, satisfy inequalities (4.8)–(4.10) and not only a subsequence but the whole sequence of solutions (u_n) is convergent to the solution of the Navier–Stokes equation with the data u_0 and f. Their proofs are de facto, modifications of the proofs of corresponding parts of Theorem 5.1 from [9], where Galerkin approximations are substituted by solutions u_n, $n \in \mathbb{N}$. However, the last part of the proof is different. Namely, contrary to the case of the Galerkin aproximations, the martingale $\tilde{M}_n$ defined by (5.16) in [9] is, in general, not square integrable. It would be square integrable, for example, if inequality (4.8) held with some $q > 4$. This holds in the case, when the noise term does not depend on ∇u or if we impose such restriction on η that $\frac{\eta}{2-\eta} > 4$. However, to cover the general case, this part of the proof is different.

In what follows we do not assume that $\mathcal{O}$ is a Poincaré domain.

THEOREM 4.9. *Let assumptions* (H.1)–(H.3) *and* (H.5) *be satisfied and let* $T > 0$. *Assume that* $(u_{0,n})_{n=1}^{\infty}$ *is a bounded* H-*valued sequence and* $(f_n)_{n=1}^{\infty}$ *is a bounded* $L^p(0,T;\mathrm{V}')$-*valued sequence. Let* $R_1 > 0$ *and* $R_2 > 0$ *be such that* $\sup_{n\in\mathbb{N}} |u_{0,n}|_{\mathrm{H}} \le R_1$ *and* $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(0,T;\mathrm{V}')} \le R_2$. *Let*

$$(\hat{\Omega}_n, \hat{\mathcal{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n, \hat{W}_n, u_n)$$

be a martingale solution of problem (3.2) *with the initial data* $u_{0,n}$ *and the external force* f_n *and satisfying inequalities* (4.8)–(4.10). *Then the set of Borel measures* $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$ *is tight on the space* $(\mathcal{Z}_T, \mathcal{T}_T)$.

PROOF. Let us fix $T > 0$ and p satisfying condition (3.1). Let $(u_{0,n})_{n=1}$ and $(f_n)_{n=1}$ be bounded H-valued, respectively, $L^p(0,T;\mathrm{V}')$-valued, sequences. Let

$$(\hat{\Omega}_n, \hat{\mathcal{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n, \hat{W}_n, u_n)$$

be a corresponding martingale solution of problem (3.2) with the initial data u_0^n and the external force f_n, and satisfying inequalities (4.8)–(4.10). Such a solution exists by Theorem 4.8.

To show that the set of measures $\{\mathcal{L}(u_n), n \in \mathbb{N}\}$ are tight on the space $(\mathcal{Z}_T, \mathcal{T}_T)$, where $\mathcal{Z}_T$ is defined in (4.3), we argue as in the proof of Lemma 5.4 in [9] and apply Corollary 4.3. We first observe that due to estimates (4.8) (with $q = 2$) and (4.10), conditions (4.4) and (4.5) of Corollary 4.3 are satisfied. Thus, it is sufficient to prove condition (a), that is, that the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition [A]. By Lemma 4.4, it is sufficient to proof the condition (a′).

We have now to choose our steps very carefully as we no longer treat strong solutions to an SDE in a finite dimensional Hilbert space but instead a weak solution to an SPDE in an infinite dimensional Hilbert space.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times taking values in $[0,T]$. Since each process satisfies equation (3.4), by Remark 3.7 we have

$$\begin{aligned} u_n(t) &= u_{0,n} - \int_0^t \mathcal{A}u_n(s)\,ds - \int_0^t B\big(u_n(s)\big)\,ds + \int_0^t f_n(s)\,ds \\ &\quad + \int_0^t G\big(u_n(s)\big)\,dW(s) \\ &=: J_1^n + J_2^n(t) + J_3^n(t) + J_4^n(t) + J_5^n(t), \qquad t\in[0,T], \end{aligned}$$

where the above equality is understood in the space V'. Let us choose and $\theta>0$. It is sufficient to show that each sequence J_i^n of processes, $i=1,\dots,5$ satisfies the sufficient condition (a′) from Lemma 4.4.

Obviously, the term J_1^n which is constant in time, satisfies whatever we want. We will only deal with the other terms. In fact, we will check that the terms J_2^n, J_4^n, J_5^n satisfy condition (a′) from Lemma 4.4 in the space $Y=\mathrm{V}'$ and the term J_3^n satisfies this condition in $Y=\mathrm{V}_s'$ with $s>\frac{d}{2}+1$. Since the embeddings $\mathrm{V}_s'\subset U'$ and $\mathrm{V}'\subset U'$ are continuous, we infer that (a′) from Lemma 4.4 holds in the space $Y=U'$, as well.

Ad J_2^n. Since the linear operator $\mathcal{A}:\mathrm{V}\to\mathrm{V}'$ is bounded, by the Hölder inequality and (4.10), we have

$$\begin{aligned} &\mathbb{E}_n\big[\big|J_2^n(\tau_n+\theta)-J_2^n(\tau_n)\big|_{\mathrm{V}'}\big] \\ &\quad\le \mathbb{E}_n\bigg[\int_{\tau_n}^{\tau_n+\theta}|\mathcal{A}u_n(s)|_{\mathrm{V}'}\,ds\bigg] \qquad (4.12)\\ &\quad\le \theta^{\frac12}\bigg(\mathbb{E}_n\bigg[\int_0^T\|u_n(s)\|^2\,ds\bigg]\bigg)^{\frac12}\le C_2(p)\cdot\theta^{\frac12}. \end{aligned}$$

Ad J_3^n. Let $s>\frac{d}{2}+1$ Similarly, since $B:\mathrm{H}\times\mathrm{H}\to\mathrm{V}_s'$ is bilinear and continuous (and hence bounded so that the norm $\|B\|$ of $B:\mathrm{H}\times\mathrm{H}\to\mathrm{V}_s'$ is finite), then by (4.8) we have the following estimates:

$$\begin{aligned} &\mathbb{E}_n\big[\big|J_3^n(\tau_n+\theta)-J_3^n(\tau_n)\big|_{\mathrm{V}_s'}\big] \\ &\quad= \mathbb{E}_n\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}B\big(u_n(r)\big)\,dr\bigg|_{\mathrm{V}_s'}\bigg] \qquad (4.13)\\ &\quad\le c\,\mathbb{E}_n\bigg[\int_{\tau_n}^{\tau_n+\theta}\big|B\big(u_n(r)\big)\big|_{\mathrm{V}_s'}\,dr\bigg]\le c\|B\|\,\mathbb{E}_n\bigg[\int_{\tau_n}^{\tau_n+\theta}|u_n(r)|_{\mathrm{H}}^2\,dr\bigg] \\ &\quad\le c\|B\|\cdot\mathbb{E}_n\Big[\sup_{r\in[0,T]}|u_n(r)|_{\mathrm{H}}^2\Big]\cdot\theta\le c\|B\|C_1(p,2)\cdot\theta. \end{aligned}$$

REMARK. The above argument works as well for $d=3$. However for $d=2$ we have the following different proof which exploits inequality (2.12) (which is valid only in the two-dimensional case):

$$
\begin{aligned}
&\mathbb{E}_n\big[\big|J_3^n(\tau_n+\theta)-J_3^n(\tau_n)\big|_{V'}\big]\\
&\quad\le \mathbb{E}_n\bigg[\int_{\tau_n}^{\tau_n+\theta}\big|B\big(u_n(r)\big)\big|_{V'}\,dr\bigg]\\
&\quad\le c_2\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}\big|u_n(r)\big|_{L^2}\big|\nabla u_n(r)\big|_{L^2}\,dr\\
&\quad\le c_2\Big[\mathbb{E}_n\sup_{r\in[\tau_n,\tau_n+\theta]}\big|u_n(r)\big|_H^2\Big]^{\frac12}\bigg[\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}\big|\nabla u_n(r)\big|_{L^2}^2\,dr\bigg]^{\frac12}\theta^{\frac12} \qquad (4.14)\\
&\quad\le c_2\Big[\mathbb{E}_n\sup_{r\in[0,T]}\big|u_n(r)\big|_{H}^2\Big]^{\frac12}\bigg[\mathbb{E}_n\int_0^T\big|\nabla u_n(r)\big|_{L^2}^2\,dr\bigg]^{\frac12}\theta^{\frac12}\\
&\quad\le c_2\big[C_1(p,2)\big]^{\frac12}\big[C_2(p)\big]^{\frac12}\theta^{\frac12}.
\end{aligned}
$$

Ad J_4^n. Since the sequence (f_n) is weakly convergent in $L^p(0,T;V')$, it is, in particular, bounded in $L^p(0,T;V')$. Using the Hölder inequality, we have

$$
\begin{aligned}
&\mathbb{E}_n\big[\big|J_4^n(\tau_n+\theta)-J_4^n(\tau_n)\big|_{V'}\big]\\
&\quad=\mathbb{E}_n\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta}f_n(s)\,ds\bigg|_{V'}\bigg] \qquad (4.15)\\
&\quad\le\theta^{\frac{p-1}{p}}\bigg(\mathbb{E}_n\bigg[\int_0^T\big|f_n(s)\big|_{V'}^p\,ds\bigg]\bigg)^{\frac1p}=\theta^{\frac{p-1}{p}}|f_n|_{L^p(0,T;V')}=c_4\cdot\theta^{\frac{p-1}{p}},
\end{aligned}
$$

where $c_4:=\sup_{n\in\mathbb{N}}|f_n|_{L^p(0,T;V')}$.

Ad J_5^n. By assumption (G3) and inequality (4.8), we obtain the following inequalities:

$$
\begin{aligned}
&\mathbb{E}_n\big[\big|J_5^n(\tau_n+\theta)-J_5^n(\tau_n)\big|_{V'}\big]\\
&\quad\le\big\{\mathbb{E}_n\big[\big|J_5^n(\tau_n+\theta)-J_5^n(\tau_n)\big|_{V'}^2\big]\big\}^{\frac12}\\
&\quad=\bigg[\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}\big\|G\big(u_n(s)\big)\big\|_{\mathcal{T}_2(Y,V')}^2\,ds\bigg]^{\frac12}\\
&\quad\le\bigg[C\cdot\mathbb{E}_n\int_{\tau_n}^{\tau_n+\theta}\big(1+\big|u_n(s)\big|_H^2\big)\,ds\bigg]^{\frac12} \qquad (4.16)\\
&\quad\le\Big[C\Big(1+\Big[\mathbb{E}_n\Big[\sup_{s\in[0,T]}\big|u_n(s)\big|_H^2\Big]\Big)\theta\Big]^{\frac12}\\
&\quad\le\big[C\big(1+C_1(2)\big)\theta\big]^{\frac12}=:c_5\cdot\theta^{\frac12}.
\end{aligned}
$$

Thus, the proof of Theorem 4.9 is complete. □

REMARK 4.10. It is easy to be convinced that u_n take values in $\mathcal{Z}_T$ but it is not so obvious to see that in fact u_n are Borel measurable functions. This is so because our construction of the martingale solution is based on Jakubowski's version of the Skorokhod theorem; see Theorem 4.6 for details.

The main result about the continuous dependence of the solutions of the Navier–Stokes equations on the initial state and deterministic external forces, which covers both cases of 2D and 3D domains, is expressed in the following Theorem 4.11. Stronger version for 2D domains will be formulated in the next section; see Theorem 5.9.

THEOREM 4.11. *Let conditions* (H.1)–(H.3) *and* (H.5) *of Assumption* 3.1 *be satisfied and let* $T > 0$. *Assume that* $(u_{0,n})_{n=1}^{\infty}$ *is an* H*-valued sequence that is convergent weakly in* H *to* $u_0 \in \mathrm{H}$ *and* $(f_n)_{n=1}^{\infty}$ *is an* $L^p(0, T; \mathrm{V}')$*-valued sequence that is weakly convergent in* $L^p(0, T; \mathrm{V}')$ *to* $f \in L^p(0, T; \mathrm{V}')$. *Let* $R_1 > 0$ *and* $R_2 > 0$ *be such that* $\sup_{n\in\mathbb{N}} |u_{0,n}|_{\mathrm{H}} \leq R_1$ *and* $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(0,T;\mathrm{V}')} \leq R_2$. *Let*

$$(\hat{\Omega}_n, \hat{\mathcal{F}}_n, \hat{\mathbb{F}}_n, \hat{\mathbb{P}}_n \hat{W}_n, u_n)$$

be a martingale solution of problem (3.2) *with the initial data* u_0^n *and the external force* f_n *and satisfying inequalities* (4.8)–(4.10). *Then there exist*:

- *a subsequence* $(n_k)_k$,
- *a stochastic basis* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, *where* $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t\geq 0}$,
- *a cylindrical Wiener process* $\tilde{W} = \tilde{W}(t)$, $t \in [0, \infty)$ *defined on this basis*,
- *and progressively measurable processes* $\tilde{u}$, $(\tilde{u}_{n_k})_{k\geq 1}$ (*defined on this basis*) *with laws supported in* $\mathcal{Z}_T$ *such that*

$$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ on } \mathcal{Z}_T \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathcal{Z}_T, \qquad \tilde{\mathbb{P}}\text{-a.s.}, \tag{4.17}$$

for every $q \in [1, p]$

$$\tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}(s)|_{\mathrm{H}}^q\Big] < \infty, \tag{4.18}$$

and the system

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u})$$

is a solution to problem (3.2).

In particular, for all $t \in [0, T]$ *and all* $\mathrm{v} \in \mathrm{V}$

$$\begin{aligned}(\tilde{u}(t), \mathrm{v})_{\mathrm{H}} - (\tilde{u}(0), \mathrm{v})_{\mathrm{H}} + \int_0^t \langle \mathcal{A}\tilde{u}(s), \mathrm{v}\rangle\, ds + \int_0^t \langle B(\tilde{u}(s)), \mathrm{v}\rangle\, ds \\ = \int_0^t \langle f(s), \mathrm{v}\rangle\, ds + \Big\langle \int_0^t G(\tilde{u}(s))\, d\tilde{W}(s), \mathrm{v}\Big\rangle\end{aligned}$$

and

$$\tilde{\mathbb{E}}\left[\int_0^T \|\tilde{u}(s)\|^2\,ds\right] < \infty. \tag{4.19}$$

PROOF. Since the product topological space $\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})$ satisfies the assumptions of Theorem 4.6, by applying it together with Theorem 4.9, there exists a subsequence (n_k), a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})$-valued Borel random variables $(\tilde{u}, \tilde{W})$, $(\tilde{u}_k, \tilde{W}_k)$, $k \in \mathbb{N}$ such that each $\tilde{W}$ and $\tilde{W}_k$, $k \in \mathbb{N}$ is an K-valued Wiener process and such that

(4.20) the laws on $\mathcal{B}(\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K}))$ of (u_{n_k}, W) and $(\tilde{u}_k, \tilde{W}_k)$ are equal,

where $\mathcal{B}(\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K}))$ is the Borel σ-algebra on $\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})$, and, with $\hat{\mathrm{K}}$ being an auxiliary Hilbert space such that $\mathrm{K} \subset \hat{\mathrm{K}}$ and the natural embedding $\mathrm{K} \hookrightarrow \hat{\mathrm{K}}$ is Hilbert–Schmidt,

$$\begin{aligned} &(\tilde{u}_k, \tilde{W}_k) \text{ converges to } (\tilde{u}, \tilde{W}) \text{ in } \mathcal{Z}_T \times \mathcal{C}([0,T],\hat{\mathrm{K}}) \\ &\tilde{\mathbb{P}}\text{-almost surely on } \tilde{\Omega}. \end{aligned} \tag{4.21}$$

Note that since $\mathcal{B}(\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})) \subset \mathcal{B}(\mathcal{Z}_T) \times \mathcal{B}(\mathcal{C}([0,T],\mathrm{K}))$, the function u is $\mathcal{Z}_T$ Borel random variable.

Define a corresponding sequence of filtrations by

$$\begin{aligned} \tilde{\mathbb{F}}_k = (\tilde{\mathcal{F}}_k(t))_{t\ge 0} \qquad &\text{where } \tilde{\mathcal{F}}_k(t) = \sigma(\{(\tilde{u}_k(s), \tilde{W}_k(s)), s \le t\}), \\ &t \in [0,T]. \end{aligned} \tag{4.22}$$

To conclude the proof, we need to show that the random variable $\tilde{u}$ gives rise to a martingale solution. The proof of this claim is very similar to the proof of Theorem 2.3 in [30]. Let us denote the subsequence $(\tilde{u}_{n_k})_k$ again by $(\tilde{u}_n)_n$.

The few differences are:

(i) The finite dimensional space H_n is replaced by the whole space H. But now, by Lemma 4.2 the space $\mathcal{C}([0,T];\mathrm{V}') \cap \mathcal{Z}_T$ is a Borel subset of $\mathcal{Z}_T$ and since by Remark 3.7 $u_n \in \mathcal{C}([0,T];\mathrm{V}')$, $\mathbb{P}$-a.s. and $\tilde{u}_n$ and u_n have the same laws on $\mathcal{Z}_T$, we infer that

$$\tilde{u}_n \in \mathcal{C}([0,T];\mathrm{V}'), \qquad n \ge 1, \tilde{\mathbb{P}}\text{-a.s.}$$

(ii) The operator P_n has to be replaced by the identity. But this is rather a simplification as, for instance, we do not need Lemmas 2.3 and 2.4 from [9].

In addition to point (i) above, we have that for every $q \in [1,p]$, we have

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\Big(\sup_{0\le s\le T} |\tilde{u}_n(s)|_{\mathrm{H}}^q\Big) \le C_1(p,q). \tag{4.23}$$

Similarly,

$$\tilde{u}_n \in L^2(0,T;\mathrm{V}), \qquad n \ge 1, \mathbb{P}\text{-a.s.}$$

and

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\Big[\int_0^T \|\tilde{u}_n(s)\|_{\mathrm{V}}^2\, ds\Big] \le C_2(p). \tag{4.24}$$

By inequality (4.24), we infer that the sequence $(\tilde{u}_n)$ contains a subsequence, still denoted by $(\tilde{u}_n)$, convergent weakly in the space $L^2([0,T]\times\tilde{\Omega};\mathrm{V})$. Since by (4.21) $\tilde{\mathbb{P}}$-a.s. $\tilde{u}_n \to \tilde{u}$ in $\mathcal{Z}_T$, we conclude that $\tilde{u} \in L^2([0,T]\times\tilde{\Omega};\mathrm{V})$, that is,

$$\tilde{\mathbb{E}}\Big[\int_0^T |\tilde{u}(s)|^2\, ds\Big] < \infty. \tag{4.25}$$

Similarly, by inequality (4.23) with $q = p$ we can choose a subsequence of $(\tilde{u}_n)$ convergent weak star in the space $L^p(\tilde{\Omega}; L^\infty(0,T;\mathrm{H}))$ and, using (4.21), infer that

$$\tilde{\mathbb{E}}\Big[\sup_{0\le s\le T} |\tilde{u}(s)|_{\mathrm{H}}^p\Big] < \infty. \tag{4.26}$$

Then, of course, for every $q \in [1,p]$,

$$\tilde{\mathbb{E}}\Big[\sup_{0\le s\le T} |\tilde{u}(s)|_{\mathrm{H}}^q\Big] < \infty. \tag{4.27}$$

The remaining proof will be done in two steps.

Step 1. Let us choose and fix $s > \frac{d}{2}+1$. We will first prove the following lemma.

LEMMA 4.12. *For all* $\varphi \in \mathrm{V}_s$,

(a) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |(\tilde{u}_n(t)-\tilde{u}(t),\varphi)_{\mathrm{H}}|^2\, dt] = 0$,
(b) $\lim_{n\to\infty} \tilde{\mathbb{E}}[|(\tilde{u}_n(0)-\tilde{u}(0),\varphi)_{\mathrm{H}}|^2] = 0$,
(c) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\int_0^t \langle \mathcal{A}\tilde{u}_n(s)-\mathcal{A}\tilde{u}(s),\varphi\rangle\, ds|\, dt] = 0$,
(d) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\int_0^t \langle B(\tilde{u}_n(s))-B(\tilde{u}(s)),\varphi\rangle\, ds|\, dt] = 0$,
(e) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\int_0^t \langle f_n(s)-f(s),\varphi\rangle\, ds|\, dt] = 0$,
(f) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\langle \int_0^t [G(\tilde{u}_n(s))-G(\tilde{u}(s))]\, d\tilde{W}(s),\varphi\rangle|^2\, dt] = 0$.

PROOF OF LEMMA 4.12. Let us fix $\varphi \in \mathrm{V}_s$.

Ad (a). Since by (4.21) $\tilde{u}_n \to \tilde{u}$ in $\mathcal{C}([0,T];\mathrm{H}_w)$ $\tilde{\mathbb{P}}$-a.s., $(\tilde{u}_n(\cdot),\varphi)_{\mathrm{H}} \to (\tilde{u}(\cdot),\varphi)_{\mathrm{H}}$ in $\mathcal{C}([0,T];\mathbb{R})$, $\tilde{\mathbb{P}}$-a.s. Hence, in particular, for all $t \in [0,T]$

$$\lim_{n\to\infty} (\tilde{u}_n(t),\varphi)_{\mathrm{H}} = (\tilde{u}(t),\varphi)_{\mathrm{H}}, \qquad \tilde{\mathbb{P}}\text{-a.s.}$$

Since by (4.23), $\sup_{t\in[0,T]} |\tilde{u}_n(t)|_{\mathrm{H}}^2 < \infty$, $\tilde{\mathbb{P}}$-a.s., using the dominated convergence theorem we infer that

$$\lim_{n\to\infty} \int_0^T |(\tilde{u}_n(t)-\tilde{u}(t),\varphi)_{\mathrm{H}}|^2\, dt = 0, \qquad \tilde{\mathbb{P}}\text{-a.s.} \tag{4.28}$$

By the Hölder inequality and (4.23) for every $n \in \mathbb{N}$ and every $r \in (1, 1+\frac{p}{2}]$,

$$\tilde{\mathbb{E}}\Big[\Big|\int_0^T |\tilde{u}_n(t)-\tilde{u}(t)|_{\mathrm{H}}^2\,dt\Big|^r\Big] \le c\tilde{\mathbb{E}}\Big[\int_0^T \big(|\tilde{u}_n(t)|_{\mathrm{H}}^{2r}+|\tilde{u}(t)|_{\mathrm{H}}^{2r}\big)\,dt\Big] \le \tilde{c}C_1(p,2r), \tag{4.29}$$

where $c, \tilde{c}$ are some positive constants. To conclude the proof of assertion (a), it is sufficient to use (4.28), (4.29) and the Vitali theorem.

Ad (b). Since by (4.21) $\tilde{u}_n \to \tilde{u}$ in $\mathcal{C}(0,T;\mathrm{H}_w)$ $\tilde{\mathbb{P}}$-a.s. and $\tilde{u}$ is continuous at $t=0$, we infer that $(\tilde{u}_n(0),\varphi)_{\mathrm{H}} \to (\tilde{u}(0),\varphi)_{\mathrm{H}}$, $\tilde{\mathbb{P}}$-a.s. Now, assertion (b) follows from (4.23) and the Vitali theorem.

Ad (c). Since by (4.21) $\tilde{u}_n \to \tilde{u}$ in $L_w^2(0,T;\mathrm{V})$, $\tilde{\mathbb{P}}$-a.s., by (2.6) we infer that $\tilde{\mathbb{P}}$-a.s.

$$\begin{aligned}\lim_{n\to\infty}\int_0^t \langle \mathcal{A}\tilde{u}_n(s),\varphi\rangle\,ds &= \lim_{n\to\infty}\int_0^t \big((\tilde{u}_n(s),\varphi)\big)\,ds \\ &= \int_0^t \big((\tilde{u}(s),\varphi)\big)\,ds \\ &= \int_0^t \langle \mathcal{A}\tilde{u}(s),\varphi\rangle\,ds.\end{aligned} \tag{4.30}$$

By (2.6), the Hölder inequality and estimate (4.24) we infer that for all $t \in [0,T]$ and $n \in \mathbb{N}$

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle \mathcal{A}\tilde{u}_n(s),\varphi\rangle\,ds\Big|^2\Big] &= \tilde{\mathbb{E}}\Big[\Big|\int_0^t \big((\tilde{u}_n(s),\varphi)\big)\,ds\Big|^2\Big] \\ &\le c\|\varphi\|_{\mathrm{V}_s}^2\tilde{\mathbb{E}}\Big[\int_0^T \|\tilde{u}_n(s)\|_{\mathrm{V}}^2\,ds\Big] \\ &\le \tilde{c}C_2(p),\end{aligned} \tag{4.31}$$

where $c, \tilde{c} > 0$ are some constants. By (4.30), (4.31) and the Vitali theorem, we conclude that for all $t \in [0,T]$

$$\lim_{n\to\infty}\tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle \mathcal{A}\tilde{u}_n(s)-\mathcal{A}\tilde{u}(s),\varphi\rangle\,ds\Big|\Big]=0.$$

Assertion (c) follows now from (4.24) and the dominated convergence theorem.

Ad (d). Since by (4.24) and (2.3) the sequence $(\tilde{u}_n)$ is bounded in $L^2(0,T;\mathrm{H})$ and by (4.21) $\tilde{u}_n \to \tilde{u}$ in $L^2(0,T;\mathrm{H}_{\mathrm{loc}})$, $\tilde{\mathbb{P}}$-a.s., by Lemma B.1 in [9] we infer that $\tilde{\mathbb{P}}$-a.s. for all $t \in [0,T]$ and $\varphi \in \mathrm{V}_s$

$$\lim_{n\to\infty}\int_0^t \langle B(\tilde{u}_n(s))-B(\tilde{u}(s)),\varphi\rangle\,ds = 0. \tag{4.32}$$

Using the Hölder inequality, Lemma 2.2 and (4.23) we infer that for all $t \in [0, T]$, $r \in (0, \frac{p}{2}]$ and $n \in \mathbb{N}$ the following inequalities hold:

$$
\begin{aligned}
\tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle B(\tilde{u}_n(s)), \varphi\rangle\, ds\Big|^{1+r}\Big] &\le \tilde{\mathbb{E}}\Big[\Big(\int_0^t |B(\tilde{u}_n(s))|_{\mathrm{V}_s'} |\varphi|_{\mathrm{V}_s}\, ds\Big)^{1+r}\Big] \\
&\le (c_2|\varphi|_{\mathrm{V}_s})^{1+r} t^r \mathbb{E}\Big[\int_0^t |\tilde{u}_n(s)|_{\mathrm{H}}^{2+2r}\, ds\Big] \\
&\le \tilde{C}\tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}_n(s)|_{\mathrm{H}}^{2+2r}\Big] \\
&\le \tilde{C} C_1(p, 2+2r).
\end{aligned}
\tag{4.33}
$$

By (4.32), (4.33) and the Vitali theorem, we obtain for all $t \in [0, T]$

$$
\lim_{n\to\infty} \tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle B(\tilde{u}_n(s)) - B(\tilde{u}(s)), \varphi\rangle\, ds\Big|\Big] = 0. \tag{4.34}
$$

Using again Lemma 2.2 and estimate (4.23), we obtain for all $t \in [0, T]$ and $n \in \mathbb{N}$

$$
\tilde{\mathbb{E}}\Big[\Big|\int_0^t \langle B(\tilde{u}_n(s)), \varphi\rangle\, ds\Big|\Big] \le c\tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}_n(s))|_{\mathrm{H}}^2\Big] \le cC_1(p, 2),
$$

where $c > 0$ is a constant. Hence, by (4.34) and the dominated convergence theorem, we infer that assertion (d) holds.

Ad (e). Assertion (e) follows because the sequence (f_n) converges weakly in $L^p(0, T; \mathrm{V}')$ to f and $\mathrm{V}_s \subset \mathrm{V}$.

Ad (f). Let us notice that for all $\varphi \in \mathrm{V}$ we have

$$
\begin{aligned}
&\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2\, ds \\
&\quad = \int_0^t \|\varphi^{**}G(\tilde{u}_n)(s) - \varphi^{**}G(\tilde{u})(s)\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2\, ds \\
&\quad \le \|\varphi^{**}G(\tilde{u}_n) - \varphi^{**}G(\tilde{u})\|_{L^2([0,T];\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R}))}^2,
\end{aligned}
$$

where $\varphi^{**}G$ is the map defined by (G4) in assumption (H.2). Since by (4.21) $\tilde{u}_n \to \tilde{u}$ in $L^2(0, T; \mathrm{H}_{\mathrm{loc}})$, $\tilde{\mathbb{P}}$-a.s., by (G4) we infer that for all $t \in [0, T]$ and $\varphi \in \mathrm{V}$:

$$
\lim_{n\to\infty} \int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2\, ds = 0. \tag{4.35}
$$

By (G3) and (4.23), we obtain the following inequalities for every $t \in [0, T]$, $r \in (1, 1 + \frac{p}{2}]$ and $n \in \mathbb{N}$:

$$
\begin{aligned}
&\tilde{\mathbb{E}}\Big[\Big|\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2\, ds\Big|^r\Big] \\
&\quad \le c\tilde{\mathbb{E}}\Big[|\varphi|_{\mathrm{V}}^{2r} \cdot \int_0^t \{|G(\tilde{u}_n(s))|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathrm{V}')}^{2r} + |G(\tilde{u}(s))|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathrm{V}')}^{2r}\}\, ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
(4.36) \qquad &\le c_1 \tilde{\mathbb{E}}\Big[\int_0^T \big(1 + |\tilde{u}_n(s)|_{\mathrm{H}}^{2r} + |\tilde{u}(s)|_{\mathrm{H}}^{2r}\big)\,ds\Big] \\
&\le \tilde{c}\Big\{1 + \tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}_n(s)|_{\mathrm{H}}^{2r} + \sup_{s\in[0,T]} |\tilde{u}(s)|_{\mathrm{H}}^{2r}\Big]\Big\} \\
&\le \tilde{c}\big(1 + 2C_1(p, 2r)\big),
\end{aligned}
$$

where $c, c_1, \tilde{c}$ are some positive constants. Using the Vitali theorem, by (4.35), (4.36) we infer that for all $\varphi \in \mathrm{V}$,

$$
\lim_{n\to\infty} \tilde{\mathbb{E}}\Big[\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2\,ds\Big] = 0. \tag{4.37}
$$

Hence, by the properties of the Itô integral we infer that for all $t \in [0, T]$ and $\varphi \in \mathrm{V}$,

$$
\lim_{n\to\infty} \tilde{\mathbb{E}}\Big[\Big|\Big\langle \int_0^t [G(\tilde{u}_n(s)) - G(\tilde{u}(s))]\,d\tilde{W}(s), \varphi\Big\rangle\Big|^2\Big] = 0. \tag{4.38}
$$

By the Itô isometry, since the map G satisfies inequality (G3) in part (H.2) of Assumption 3.1, and estimate (4.23) we have for all $\varphi \in \mathrm{V}$, $t \in [0, T]$ and $n \in \mathbb{N}$,

$$
\begin{aligned}
&\tilde{\mathbb{E}}\Big[\Big|\Big\langle \int_0^t [G(\tilde{u}_n(s)) - G(\tilde{u}(s))]\,d\tilde{W}(s), \varphi\Big\rangle\Big|^2\Big] \\
&\quad = \tilde{\mathbb{E}}\Big[\int_0^t \|\langle G(\tilde{u}_n(s)) - G(\tilde{u}(s)), \varphi\rangle\|_{\mathcal{T}_2(\hat{\mathrm{K}};\mathbb{R})}^2\,ds\Big] \\
(4.39) \qquad &\quad \le c\Big\{1 + \tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}_n(s)|_{\mathrm{H}}^2 + \sup_{s\in[0,T]} |\tilde{u}(s)|_{\mathrm{H}}^2\Big]\Big\} \\
&\quad \le c\big(1 + 2C_1(p, 2)\big),
\end{aligned}
$$

where $c > 0$ is some constant. Thus, by (4.38), (4.39) and the Lebesgue dominated convergence theorem, we infer that for all $\varphi \in \mathrm{V}$,

$$
\lim_{n\to\infty} \int_0^T \tilde{\mathbb{E}}\Big[\Big|\Big\langle \int_0^t [G(\tilde{u}_n(s)) - G(\tilde{u}(s))]\,d\tilde{W}(s), \varphi\Big\rangle\Big|^2\Big] = 0. \tag{4.40}
$$

To conclude the proof of assertion (f), it is sufficient to notice that since $s > \frac{d}{2} + 1$, $\mathrm{V}_s \subset \mathrm{V}$, and thus (4.40) holds for all $\varphi \in \mathrm{V}_s$. The proof of Lemma 4.12 is thus complete. □

As a direct consequence of Lemma 4.12, we get the following corollary which we precede by introducing some auxiliary notation. Analogously to [7] and [30], let us denote

$$\begin{aligned}\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t) \\ := \big(\tilde{u}_n(0), \varphi\big)_{\mathrm{H}} - \int_0^t \langle \mathcal{A}\tilde{u}_n(s), \varphi\rangle\, ds - \int_0^t \langle B\big(\tilde{u}_n(s)\big), \varphi\rangle\, ds \\ + \int_0^t \langle f_n(s), \varphi\rangle\, ds + \Big\langle \int_0^t G\big(\tilde{u}_n(s)\big) d\tilde{W}_n(s), \varphi\Big\rangle, \qquad t \in [0, T],\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}\Lambda(\tilde{u}, \tilde{W}, \varphi)(t) \\ := \big(\tilde{u}(0), \varphi\big)_{\mathrm{H}} - \int_0^t \langle \mathcal{A}\tilde{u}(s), \varphi\rangle\, ds - \int_0^t \langle B\big(\tilde{u}(s)\big), \varphi\rangle\, ds \\ + \int_0^t \langle f(s), \varphi\rangle\, ds + \Big\langle \int_0^t G\big(\tilde{u}(s)\big)\, d\tilde{W}(s), \varphi\Big\rangle, \qquad t \in [0, T].\end{aligned} \tag{4.42}$$

COROLLARY 4.13. *For every* $\varphi \in \mathrm{V}_s$,

$$\lim_{n\to\infty} \big|\big(\tilde{u}_n(\cdot), \varphi\big)_{\mathrm{H}} - \big(\tilde{u}(\cdot), \varphi\big)_{\mathrm{H}}\big|_{L^2([0,T]\times\tilde{\Omega})} = 0 \tag{4.43}$$

and

$$\lim_{n\to\infty} \big|\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi) - \Lambda(\tilde{u}, \tilde{W}, \varphi)\big|_{L^1([0,T]\times\tilde{\Omega})} = 0. \tag{4.44}$$

PROOF OF COROLLARY 4.13. Assertion (4.43) follows from the equality

$$\big|\big(\tilde{u}_n(\cdot), \varphi\big)_{\mathrm{H}} - \big(\tilde{u}(\cdot), \varphi\big)_{\mathrm{H}}\big|^2_{L^2([0,T]\times\tilde{\Omega})} = \tilde{\mathbb{E}}\Big[\int_0^T \big|\big(\tilde{u}_n(t) - \tilde{u}(t), \varphi\big)_{\mathrm{H}}\big|^2\, dt\Big]$$

and Lemma 4.12(a). Let us move to the proof of assertion (4.44). Note that by the Fubini theorem, we have

$$\begin{aligned}\big|\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi) - \Lambda(\tilde{u}, \tilde{W}, \varphi)\big|_{L^1([0,T]\times\tilde{\Omega})} \\ = \int_0^T \tilde{\mathbb{E}}\big[\big|\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t) - \Lambda(\tilde{u}, \tilde{W}, \varphi)(t)\big|\big]\, dt.\end{aligned}$$

To conclude the proof of Corollary 4.13, it is sufficient to note that by Lemma 4.12(b)–(f), each term on the right-hand side of (4.41) tends at least in $L^1([0, T] \times \tilde{\Omega})$ to the corresponding term in (4.42). $\square$

Step 2. Since u_n is a solution of the Navier–Stokes equation, for all $t \in [0, T]$ and $\varphi \in \mathcal{V}$,

$$\big(u_n(t), \varphi\big)_{\mathrm{H}} = \Lambda_n(u_n, W, \varphi)(t), \qquad \mathbb{P}\text{-a.s.}$$

In particular,

$$\int_0^T \mathbb{E}\big[\big|\big(u_n(t),\varphi\big)_{\mathrm{H}} - \Lambda_n(u_n, W, \varphi)(t)\big|\big]\,dt = 0.$$

Since $\mathcal{L}(u_n, W) = \mathcal{L}(\tilde{u}_n, \tilde{W}_n)$,

$$\int_0^T \tilde{\mathbb{E}}\big[\big|\big(\tilde{u}_n(t),\varphi\big)_{\mathrm{H}} - \Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t)\big|\big]\,dt = 0.$$

Moreover, by (4.43) and (4.44)

$$\int_0^T \tilde{\mathbb{E}}\big[\big|\big(\tilde{u}(t),\varphi\big)_{\mathrm{H}} - \Lambda(\tilde{u}, \tilde{W}, \varphi)(t)\big|\big]\,dt = 0.$$

Hence, for l-almost all $t \in [0, T]$ and $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$,

$$\big(\tilde{u}(t),\varphi\big)_{\mathrm{H}} - \Lambda(\tilde{u}, \tilde{W}, \varphi)(t) = 0,$$

that is, for l-almost all $t \in [0, T]$ and $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$,

$$\begin{aligned}(4.45)\qquad &\big(\tilde{u}(t),\varphi\big)_{\mathrm{H}} + \int_0^t \langle \mathcal{A}\tilde{u}(s), \varphi\rangle\,ds + \int_0^t \big\langle B\big(\tilde{u}(s)\big), \varphi\big\rangle\,ds \\ &\quad = \big(\tilde{u}(0),\varphi\big)_{\mathrm{H}} + \int_0^t \langle f(s), \varphi\rangle\,ds + \Big\langle \int_0^t G\big(\tilde{u}(s)\big)\,d\tilde{W}(s), \varphi\Big\rangle.\end{aligned}$$

Since a Borel $\tilde{u}$ is $\mathcal{Z}_T$-valued random variable, in particular $\tilde{u} \in \mathcal{C}([0, T]; \mathrm{H}_w)$, that is, $\tilde{u}$ is weakly continuous, we infer that equality (4.45) holds for all $t \in [0, T]$ and all $\varphi \in \mathcal{V}$. Since $\mathcal{V}$ is dense in V, equality (4.45) holds for all $\varphi \in \mathrm{V}$, as well. Putting $\tilde{\mathfrak{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}})$, we infer that the system $(\tilde{\mathfrak{A}}, \tilde{W}, \tilde{u})$ is a martingale solution of equation (3.2). By (4.25) and (4.27), the process $\tilde{u}$ satisfies inequalities (4.19) and (4.18). The proof of Theorem 4.11 is thus complete. □

REMARK 4.14. It seems to us that the same argument works if the space $\mathcal{Z}_T$ defined in (4.3) is replaced by a bigger space $\hat{\mathcal{Z}}_T$ defined by

$$(4.46)\qquad \hat{\mathcal{Z}}_T := L^2_w(0, T; \mathrm{V}) \cap L^2(0, T; \mathrm{H}_{\mathrm{loc}}) \cap \mathcal{C}\big([0, T]; \mathrm{H}_w\big).$$

In particular, to prove that the sequence $(\tilde{u}_n)$ given in (4.20), whose existence follows from the Skorokhod theorem, converges to a solution of the Navier–Stokes equation, it is sufficient to use the convergence of $(\tilde{u}_n)$ in the space $\hat{\mathcal{Z}}_T$.

5. The case of 2D domains. A special result proved recently in [9] is about the existence and uniqueness of strong solutions for 2-D stochastic Navier–Stokes equations in unbounded domains with a general noise.

Let us present the framework and the results. Let us recall Lemma 7.2 from [9].

LEMMA 5.1. *Let $d=2$ and assume that all conditions in parts* (H.1)–(H.3) *and* (H.5) *of Assumption* 3.1 *are satisfied. Assume that $\mu_0=\delta_{u_0}$ for some deterministic $u_0\in\mathrm{H}$. Let $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{F}},\hat{W},\hat{\mathbb{P}},u)$ be a martingale solution of problem* (3.2), *in particular,*

$$\hat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|u(t)|_{\mathrm{H}}^2+\int_0^T|\nabla u(t)|^2\,dt\Big]<\infty. \tag{5.1}$$

Then for $\hat{\mathbb{P}}$-almost all $\omega\in\hat{\Omega}$ the trajectory $u(\cdot,\omega)$ is equal almost everywhere to a continuous H*-valued function defined on* $[0,T]$. *$\hat{\mathbb{P}}$-a.s. and*

$$\begin{aligned} u(t)=u_0&-\int_0^t\big[\mathcal{A}u(s)+B\big(u(s)\big)\big]\,ds+\int_0^t f(s)\,ds\\ &+\int_0^t G\big(u(s)\big)\,d\hat{W}(s),\qquad t\in[0,T]. \end{aligned} \tag{5.2}$$

Let us emphasize that equality (5.2) is understood as the one in the space V′; see Remark 3.7.

The next result is [9], Lemma 7.3.

LEMMA 5.2. *Assume that all conditions in parts* (H.1)–(H.3) *and* (H.5) *of Assumption* 3.1 *are satisfied. In addition, we assume that the Lipschitz constant of G is smaller than $\sqrt{2}$, that is, the map G satisfies condition* (G1) *in part* (H.2) *of Assumption* 3.1 *with $L<\sqrt{2}$. Assume that $u_0\in\mathrm{H}$. If u_1 and u_2 are two solutions of problem* (3.2) *defined on the same filtered probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{F}},\hat{\mathbb{P}})$ and the same Wiener process $\hat{W}$, then $\hat{\mathbb{P}}$-a.s. for all $t\in\mathbb{R}_+$, $u_1(t)=u_2(t)$.*

Because from now we will be dealing with the pathwise uniqueness of solutions, let us formulate the following assumption on the stochastic basis.

ASSUMPTION 5.3. Assume that $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ is a stochastic basis with a filtration $\mathbb{F}=\{\mathcal{F}_t\}_{t\geq 0}$ and $W=(W(t))_{t\geq 0}$ is a cylindrical Wiener process in a separable Hilbert space K defined on this stochastic basis.

We will often consider problem (3.2) with the initial data $\mu_0=\delta_{u_0}$ for some deterministic $u_0\in\mathrm{H}$, and hence we explicitly rewrite that problem in the following way:

$$\begin{cases} du(t)+\mathcal{A}u(t)\,dt+B\big(u(t),u(t)\big)\,dt\\ \qquad =f(t)\,dt+G\big(u(t)\big)\,dW(t),\qquad t\geq 0,\\ u(0)=u_0. \end{cases} \tag{5.3}$$

To avoid any confusion, a martingale solution to problem (5.3) with initial data $u_0\in\mathrm{H}$, is a martingale solution to problem (3.2) with $\mu_0=\delta_{u_0}$.

For the completeness of the exposition, let us also recall a notion of a strong solution.

DEFINITION 5.4. Assume that $u_0 \in \mathrm{H}$ and $f : [0,\infty) \to \mathrm{V}'$. Assume Assumption 5.3. We say that an $\mathbb{F}$-progressively measurable process $u : [0,\infty) \times \Omega \to \mathrm{H}$ with $\mathbb{P}$ - a.e. paths,

$$u(\cdot,\omega) \in \mathcal{C}\big([0,\infty),\mathrm{H}_w\big) \cap L^2_{\mathrm{loc}}\big([0,\infty);\mathrm{V}\big)$$

is a *strong solution* to problem (5.3), that is,

$$\begin{cases} du(t) + \mathcal{A}u(t)\,dt + B\big(u(t),u(t)\big)\,dt \\ \qquad = f(t)\,dt + G\big(u(t)\big)\,dW(t), \qquad t \geq 0, \\ u(0) = u_0, \end{cases}$$

if and only if for all $t \in [0,\infty)$ and all $\mathrm{v} \in \mathcal{V}$ the following identity holds $\mathbb{P}$-a.s.:

$$\begin{aligned} \big(u(t),\mathrm{v}\big)_{\mathrm{H}} &+ \int_0^t \langle \mathcal{A}u(s),\mathrm{v}\rangle\,ds + \int_0^t \big\langle B\big(u(s),u(s)\big),\mathrm{v}\big\rangle\,ds \\ &= (u_0,\mathrm{v})_{\mathrm{H}} + \int_0^t \langle f(s),\mathrm{v}\rangle\,ds + \left\langle \int_0^t G\big(u(s)\big)\,dW(s),\mathrm{v}\right\rangle \end{aligned}$$

and for all $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |u(t)|^2_{\mathrm{H}} + \int_0^T |\nabla u(t)|^2\,dt\right] < \infty. \tag{5.4}$$

Let us recall two basic concepts of uniqueness of the solution, that is, pathwise uniqueness and uniqueness in law; see [23, 32]. Please note the following difference between problems (3.2) and (5.3). In the former, a law of the initial data is prescribed, while in the latter a initial data is given.

DEFINITION 5.5. We say that solutions of problem (5.3) has *pathwise uniqueness property* if and only if for all $u_0 \in \mathrm{H}$ and $f : [0,\infty) \to \mathrm{V}'$ the following condition holds:

$$\begin{aligned} &\textit{if } u^i, i = 1,2, \textit{ are strong solutions of problem } (5.3) \\ &\text{on } (\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W) \textit{ satisfying Assumption } 5.3, \\ &\textit{then } \mathbb{P}\textit{-a.s. for all } t \in [0,\infty), u^1(t) = u^2(t). \end{aligned} \tag{5.5}$$

Assume that $u_0 \in \mathrm{H}$ and $f : [0,\infty) \to \mathrm{V}'$. A solution u to problem (5.3) on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W)$ satisfying Assumption 5.3, is said to be pathwise unique iff for every solution $\tilde{u}$ to problem (5.3) on the same $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},W)$, one has

$$\mathbb{P}\text{-a.s. for all } t \in [0,\infty), \qquad u(t) = \tilde{u}(t).$$

DEFINITION 5.6. We say that problem (3.2) has *uniqueness in law property* iff for every Borel measure μ on H and every $f : [0,\infty) \to \mathrm{V}'$ the following condition holds:

$$
\begin{aligned}
&\textit{if } (\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, W^i, u^i), i = 1, 2, \textit{ are solutions of problem } (3.2) \\
&\textit{then } Law_{\mathbb{P}^1}(u^1) = Law_{\mathbb{P}^2}(u^2) \text{ on } \mathcal{C}([0,\infty), \mathrm{H}_w) \cap L^2_{\mathrm{loc}}([0,\infty); \mathrm{V}),
\end{aligned}
\tag{5.6}
$$

where $Law_{\mathbb{P}^i}(u^i)$, $i = 1, 2$, are by definition probability measures on $\mathcal{C}([0,\infty), \mathrm{H}_w) \cap L^2_{\mathrm{loc}}([0,\infty); \mathrm{V})$.

COROLLARY 5.7. *Assume that conditions* (H.1)–(H.3) *and* (H.5) *of Assumption* 3.1 *are satisfied and that the map G satisfies inequality* (G1) *in part* (H.2) *of Assumption* 3.1 *with a constant L smaller than $\sqrt{2}$. Assume also that* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$ *satisfies Assumption* 5.3. *Then for every $u_0 \in \mathrm{H}$*:

(1) *There exists a pathwise unique strong solution u on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$ of problem* (5.3).
(2) *Moreover, if u is a strong solution of problem* (5.3) *on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$, then for $\mathbb{P}$-almost all $\omega \in \Omega$ the trajectory $u(\cdot, \omega)$ is equal almost everywhere to a continuous* H*-valued function defined on* $[0,\infty)$.
(3) *The martingale solution of problem* (3.2) *with $\mu_0 = \delta_{u_0}$ is unique in law. In particular, if $(\Omega^i, \mathcal{F}^i, \mathbb{F}^i, \mathbb{P}^i, W^i, u^i)$, $i = 1, 2$ are such solutions to problem* (3.2), *then for all $t \geq 0$, the laws on* H *of* H*-valued random variables $u^1(t)$ and $u^2(t)$ coincide.*

PROOF. The proof of part (3) given in [9] yields the uniqueness in law in the trajectory the space $\mathcal{C}([0,\infty), \mathrm{H}_w) \cap L^2_{\mathrm{loc}}([0,\infty); \mathrm{V})$; hence, in $\mathcal{C}([0,T], \mathrm{H}_w) \cap L^2(0,T; \mathrm{V})$ for every $T > 0$. □

Let us emphasize that, by definition, we require a martingale solution of the Navier–Stokes equation to satisfy inequality (3.5), that is,

$$
\hat{\mathbb{E}}\biggl[\sup_{t\in[0,T]} |u(t)|^2_{\mathrm{H}} + \int_0^T |\nabla u(t)|^2\, dt\biggr] < \infty.
$$

In Theorem 4.8, covering both 2D and 3D domains, we proved that there exists a martingale solution satisfying stronger estimates, that is, (4.8)–(4.11). However, in the case when $\mathcal{O}$ is a 2D domain, we can prove that every martingale solution satisfies these inequalities.

LEMMA 5.8. *Assume that $d = 2$ and that conditions* (H.1)–(H.3) *and* (H.5) *from Assumption* 3.1 *are satisfied. Then the following hold*:

(1) *For every* $T > 0$, $R_1 > 0$ *and* $R_2 > 0$ *there exist constants* $C_1(p)$ *and* $C_2(p)$ *depending also on* T, R_1 *and* R_2 *such that if* μ_0 *is a Borel probability measure on* H, $f \in L^p(0,T;\mathrm{V}')$ *satisfy* $\int_H |x|^p \mu_0(dx) \le R_1$ *and* $|f|_{L^p(0,T;\mathrm{V}')} \le R_2$, *then every martingale solution of problem* (3.2) *with the initial data* μ_0 *and the external force* f, *satisfies the following estimates*:

$$\hat{\mathbb{E}}\Big(\sup_{s\in[0,T]} |u(s)|_{\mathrm{H}}^p\Big) \le C_1(p) \tag{5.7}$$

and

$$\hat{\mathbb{E}}\bigg[\int_0^T |u(s)|_{\mathrm{H}}^{p-2} |\nabla u(s)|^2\,ds\bigg] \le C_2(p). \tag{5.8}$$

In particular,

$$\hat{\mathbb{E}}\bigg[\int_0^T |\nabla u(s)|^2\,ds\bigg] \le C_2 := C_2(2). \tag{5.9}$$

(2) *Moreover, if* $\mathcal{O}$ *is a Poincaré domain and the map* G *satisfies inequality* (G2) *in part* (H.2) *of Assumption* 3.1 *with* $\lambda_0 = 0$ (*and with* $\rho \in [0,\infty)$ *and* $\eta \in (0,2]$), *then the process* u *satisfies additionally the following inequality for every* $t \ge 0$:

$$\hat{\mathbb{E}}\big[|u(t)|_{\mathrm{H}}^2\big] + \frac{\eta}{2}\hat{\mathbb{E}}\bigg[\int_0^t |\nabla u(s)|^2\,ds\bigg] \le \hat{\mathbb{E}}\big[|u(0)|_{\mathrm{H}}^2\big] + \frac{2}{\eta}\int_0^t |f(s)|_{\mathrm{V}'}^2\,ds + \rho t. \tag{5.10}$$

The proof of Lemma 5.8 is similar to the proof of estimates (5.4), (5.5) and (5.6) from the Appendix in [9]. The difference is that the solution process u to which the Itô formula (in a classical form; see, for instance, [23]) was applied was taking values in a finite dimensional Hilbert space H_n and u was a solution in the most classical way. Now, u_n is martingale solution to problem (3.2); see Definition 3.2.

If we assume that $d = 2$, by Lemma III.3.4, page 198 in [43], we infer that the regularity assumption (3.3) implies that

$$B\big(u(\cdot,\omega), u(\cdot,\omega)\big) \in L^2_{\mathrm{loc}}\big([0,\infty);\mathrm{V}'\big) \qquad \text{for } \hat{\mathbb{P}}\text{-a.a. } \omega \in \Omega.$$

This, however, does not imply that

$$\hat{\mathbb{E}}\int_0^T \big|B\big(u(t),u(t)\big)\big|_{\mathrm{V}'}^2\,dt < \infty$$

what is necessary in order to apply the infinite dimensional Itô lemma from [34].

Fortunately, we can proceed as in the proof of the uniqueness result, that is, Lemma 7.3 from [9], that is, introduce a family τ_N, $N \in \mathbb{N}$ of the stopping times defined by

$$\tau_N := \inf\big\{t \in [0,\infty) : |u(t)|_{\mathrm{H}} \ge N\big\}, \qquad N \in \mathbb{N}, \tag{5.11}$$

and then consider a stopped process $u(t \wedge \tau_N)$, $t \geq 0$. Note that with this definition of the stopping time τ_N, we have

$$\hat{\mathbb{E}} \int_0^{T \wedge \tau_N} \left|B\big(u(t), u(t)\big)\right|_{\mathrm{V}'}^2 \, dt \leq C N^2 \hat{\mathbb{E}} \int_0^T \|u(t)\|^2 \, dt < \infty.$$

REMARK. If $d = 3$, then

$$B\big(u(\cdot, \omega), u(\cdot, \omega)\big) \in L^{4/3}(0, T; \mathrm{V}') \qquad \text{for } \hat{\mathbb{P}}\text{-a.a. } \omega \in \Omega.$$

Thus, in this case the above procedure with the stopping time τ_N does not help.

PROOF OF LEMMA 5.8. Let us fix p satisfying condition (3.1). As in the proof of Lemma A.1, we apply the Itô formula from [34] to the function F defined by

$$F : \mathrm{H} \ni x \mapsto |x|_{\mathrm{H}}^p \in \mathbb{R}.$$

With the above comments in mind and using Remark 3.6, we have, for $t \in [0, \infty)$,

$$\begin{aligned}
\left|u(t \wedge \tau_N)\right|^p - \left|u(0)\right|^p &= \int_0^{t \wedge \tau_N} \Big[p\left|u(s)\right|^{p-2} \langle u(s), -\mathcal{A}u(s) - B\big(u(s)\big) + f(s)\rangle \\
&\qquad + \frac{1}{2} \operatorname{Tr}\big[F''\big(u(s)\big)\big(G\big(u(s)\big), G\big(u(s)\big)\big)\big] \Big] \, ds \\
&\qquad + p \int_0^{t \wedge \tau_N} \left|u(s)\right|^{p-2} \langle u(s), G\big(u(s)\big) \, d\hat{W}(s)\rangle \\
&= \int_0^{t \wedge \tau_N} \Big[-p\left|u(s)\right|^{p-2} \|u(s)\|^2 + p\left|u(s)\right|^{p-2} \langle u(s), f(s)\rangle \\
&\qquad + \frac{1}{2} \operatorname{Tr}\big[F''\big(u(s)\big)\big(G\big(u(s)\big), G\big(u(s)\big)\big)\big] \Big] \, ds \\
&\qquad + p \int_0^{t \wedge \tau_N} \left|u(s)\right|^{p-2} \langle u(s), G\big(u(s)\big) \, d\hat{W}(s)\rangle.
\end{aligned} \tag{5.12}$$

Proceeding as in the proof of Lemma A.1, we obtain

$$\begin{aligned}
&\left|u(t \wedge \tau_N)\right|^p + \delta \int_0^{t \wedge \tau_N} \left|u(s)\right|^{p-2} \left|\nabla u(s)\right|^2 ds \\
&\quad \leq \left|u(0)\right|^p + K_p(\lambda_0, \rho) \int_0^{t \wedge \tau_N} \left|u(s)\right|^p ds + \frac{2\rho}{p} t \\
&\qquad + \varepsilon^{-p/2} \int_0^{t \wedge \tau_N} \left|f(t)\right|_{\mathrm{V}'}^p ds \\
&\qquad + p \int_0^t \left|u(s)\right|^{p-2} \langle u(s), G\big(u(s)\big) \, d\hat{W}(s)\rangle, \qquad t \in [0, \infty),
\end{aligned} \tag{5.13}$$

where $K_p(\lambda_0, \rho) = \frac{p-1}{2}[\lambda_0 p + 2 + \rho(p - 2)]$.

By the definition of the stopping time τ_N, we infer that the process

$$\mu_N(t) := \int_0^{t\wedge\tau_N} |u(s)|^{p-2}\langle u(s), G(u(s))\,d\hat{W}(s)\rangle, \qquad t \in [0,\infty)$$

is a martingale. Indeed, if we define a map

$$g : \mathrm{V} \ni u \mapsto \{\mathrm{K} \ni k \mapsto \langle u, G(u)k\rangle \in \mathrm{H}\} \in \mathcal{T}_2(\mathrm{K}, \mathbb{R}),$$

then $\mu_N(t) = \int_0^{t\wedge\tau_N} |u(s)|^{p-2} g(u(s))\,dW(s)$ and, since the map G satisfies inequality (G2) in part (H.2) of Assumption 3.1, we infer that, for every $t \geq 0$,

$$\begin{aligned}
&\int_0^{t\wedge\tau_N} \||u(s)|^{p-2} g(u(s))\|^2_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}\,ds \\
&\quad = \int_0^{t\wedge\tau_N} |u(s)|^{p-2} \|g(u(s))\|^2_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}\,ds \\
&\quad \leq \int_0^{t\wedge\tau_N} |u(s)|^{p-2} |u(s)|^2 \|G(u(s))\|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,ds \qquad (5.14)\\
&\quad \leq \int_0^{t\wedge\tau_N} |u(s)|^{p} [(2-\eta)|\nabla u(t)|^2 + \lambda_0 |u(t)|^2 + \rho]\,ds \\
&\quad \leq (2-\eta)N^p \int_0^{t\wedge\tau_N} |\nabla u(t)|^2\,dt + tN^p(\lambda_0 N^2 + \rho).
\end{aligned}$$

Hence, by inequality (3.5) we infer that

$$\hat{\mathbb{E}} \int_0^{t\wedge\tau_N} \||u(s)|^{p-2} g(u(s))\|^2_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}\,ds < \infty, \qquad t \geq 0,$$

and thus we infer, as claimed, that the process μ_N is a martingale. Hence, $\mathbb{E}[\mu_N(t)] = 0$. Let us now fix $T > 0$. By taking expectation in inequality (5.13) we infer that

$$\begin{aligned}
&\hat{\mathbb{E}}[|u(t\wedge\tau_N)|^p] \\
&\quad \leq \hat{\mathbb{E}}[|u(0)|^p] \\
&\qquad + K_p(\lambda_0, \rho) \int_0^{t\wedge\tau_N} \hat{\mathbb{E}}[|u(s)|^p]\,ds + \frac{2\rho}{p}(t\wedge\tau_N) + \varepsilon^{-p/2}(t\wedge\tau_N)|f|^p_{\mathrm{V}'} \\
&\quad \leq \hat{\mathbb{E}}[|u(0)|^p] \\
&\qquad + K_p(\lambda_0, \rho) \int_0^{t\wedge\tau_N} \hat{\mathbb{E}}[|u(s\wedge\tau_N)|^p]\,ds + T\left(\frac{2\rho}{p} + \varepsilon^{-p/2}|f|^p_{\mathrm{V}'}\right), \\
&\qquad\qquad t \in [0,T].
\end{aligned}$$

Hence, by the Gronwall lemma there exists a constant
$C = C_p(T, \eta, \lambda_0, \rho, \hat{\mathbb{E}}[|u(0)|^p], |f|_{L^p(0,T;\mathrm{V}')}) > 0$ such that

$$\hat{\mathbb{E}}[|u(t\wedge\tau_N)|^p] \leq C, \qquad t \in [0,T]. \tag{5.15}$$

Using this bound in (5.13), we also obtain

$$\hat{\mathbb{E}}\left[\int_0^{T\wedge\tau_N} |u(s)|^{p-2}|\nabla u(s)|^2\,ds\right] \le C \tag{5.16}$$

for a new constant $C = \tilde{C}_p(\eta, \hat{\mathbb{E}}|u(0)|^p, \hat{\mathbb{E}}\int_0^T |f(s)|^p_{\mathrm{V}'}\,ds) > 0$. Finally, taking the limit $N \to \infty$ and observing that $T \wedge \tau_N \to T$, by the Lebesgue dominated convergence theorem we infer that for the same constant C we have

$$\sup_{t\in[0,T]} \hat{\mathbb{E}}[|u(t)|^p] \le C, \tag{5.17}$$

$$\hat{\mathbb{E}}\left[\int_0^{T} |u(s)|^{p-2}|\nabla u(s)|^2\,ds\right] \le C. \tag{5.18}$$

This completes the proof of estimates (5.8) and (5.9). The proof of inequality (5.7) is the same as the proof of inequality (A.2), and thus omitted.

To prove inequality (5.10) in the case $\mathcal{O}$ is a Poincaré domain, we use the same arguments as the proof of inequality (A.5). This time, however, the solution to the Galerkin approximating equation is replaced by the stopped process $u(t\wedge\tau_N)$, $t \ge 0$. Let us recall that in the space V we consider the inner product $((\cdot,\cdot))$ given by (2.2).

By identity (5.12) with $p = 2$, we have

$$\begin{aligned}|u(t\wedge\tau_N)|^2 - |u(0)|^2 = \int_0^{t\wedge\tau_N} \Big\{&-2\|u(s)\|^2 + 2\langle u(s), f\rangle \\ &+ \frac{1}{2}\mathrm{Tr}\big[F''(u(s))\big(G(u(s)), G(u(s))\big)\big]\Big\}\,ds \\ &+ 2\int_0^{t\wedge\tau_N} \langle u(s), G(u(s))\,d\hat{W}(s)\rangle, \qquad t \ge 0.\end{aligned}$$

Since $\hat{\mathbb{E}}(\int_0^{t\wedge\tau_N} \langle G(u(s)), u(s)\,d\hat{W}(s)\rangle) = 0$, we infer that

$$\begin{aligned}\hat{\mathbb{E}}|u(t\wedge\tau_N)|^2_{\mathrm{H}} \le \hat{\mathbb{E}}[|u(0)|^2_{\mathrm{H}}] &+ \hat{\mathbb{E}}\int_0^{t\wedge\tau_N} \{-2\|u(s)\|^2 + 2\langle f(s), u(s)\rangle\}\,ds \\ &+ \hat{\mathbb{E}}\int_0^{t\wedge\tau_N} |G(u(s))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\,ds.\end{aligned}$$

Taking next the $N \to \infty$ limit, since the map G satisfies inequality (G2) in part (H.2) of Assumption 3.1 with $\lambda_0 = 0$, that is,
$|G(u(s))|^2_{\mathcal{T}_2(K,\mathrm{H})} \le (2-\eta)\|u(s)\|^2 + \varrho$, we get

$$\begin{aligned}\hat{\mathbb{E}}|u(t)|^2_{\mathrm{H}} \le{}& -\eta\mathbb{E}\int_0^t \|u(s)\|^2\,ds + \hat{\mathbb{E}}[|u(0)|^2_{\mathrm{H}}] \\ &+ 2\hat{\mathbb{E}}\int_0^t \langle f(s), u(s)\rangle\,ds + \varrho t.\end{aligned} \tag{5.19}$$

Since $2\langle f, u(s)\rangle \le \frac{\eta}{2}|\nabla u(s)|^2 + \frac{2}{\eta}|f|^2_{\mathrm{V}'}$ we infer that

$$\hat{\mathbb{E}}|u(t)|^2_{\mathrm{H}} \le -\frac{\eta}{2}\hat{\mathbb{E}}\int_0^t \|u(s)\|^2\,ds + \hat{\mathbb{E}}[|u(0)|^2_{\mathrm{H}}] + \frac{2}{\eta}\int_0^t |f(s)|^2_{\mathrm{V}'} + \varrho t, \qquad t \ge 0. \tag{5.20}$$

The proof of inequality (5.10) is thus complete. This completes the proof of Lemma 5.8. □

Note that if $f : [0,\infty) \to \mathrm{V}'$ is constant, then $f \in L^p(0,T;\mathrm{V}')$ for every $T > 0$ and p satisfying condition (H.3) of Assumption 3.1. In this case, we will write $f \in \mathrm{V}'$.

By Theorem 4.11, Corollary 5.7 and Lemma 5.8, we obtain the following result about the continuous dependence of the solutions to 2D SNSEs with respect to the initial data and the external forces.

THEOREM 5.9. *Let $d = 2$. Let parts* (H.1)–(H.2), (H.5) *and* (G1) *with a constant L smaller than $\sqrt{2}$, of Assumption* 3.1, *be satisfied. Assume that $u_0 \in \mathrm{H}$, $f \in \mathrm{V}'$ and that an* H*-valued sequence $(u_{0,n})_{n=1}^{\infty}$ is weakly convergent in* H *to u_0, and that an* V'*-valued sequence $(f_n)_{n=1}$ is weakly convergent in V' to f. Let*

$$(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, \mathbb{P}_n, W_n, u_n)$$

be a martingale solution of problem (5.3) *on* $[0,\infty)$ *with the initial data $u_{0,n}$ and the external force f_n. Then for every $T > 0$ there exist*:

- *a subsequence $(n_k)_k$,*
- *a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t\}_{t\ge 0}$,*
- *a cylindrical Wiener process $\tilde{W} = \tilde{W}(t)$, $t \in [0,\infty)$ defined on this basis,*
- *and $\mathbb{F}$-progressively measurable processes $\tilde{u}(t)$, $(\tilde{u}_{n_k}(t))_{k\ge 1}$, $t \in [0,T]$ (defined on this basis) with laws supported in $\mathcal{Z}_T$ such that*

$$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ on } \mathcal{Z}_T \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathcal{Z}_T, \qquad \tilde{\mathbb{P}}\text{-a.s.} \tag{5.21}$$

and the system

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}) \tag{5.22}$$

is a martingale solution to problem (5.3) *on the interval* $[0,T]$ *with the initial law δ_{u_0}. In particular, for all $t \in [0,T]$ and* $\mathrm{v} \in \mathrm{V}$

$$\big(\tilde{u}(t), \mathrm{v}\big)_{\mathrm{H}} - \big(\tilde{u}(0), \mathrm{v}\big)_{\mathrm{H}} + \int_0^t \langle \mathcal{A}\tilde{u}(s), \mathrm{v}\rangle\,ds + \int_0^t \langle B\big(\tilde{u}(s)\big), \mathrm{v}\rangle\,ds = \int_0^t \langle f, \mathrm{v}\rangle\,ds + \left\langle \int_0^t G\big(\tilde{u}(s)\big)\,d\tilde{W}(s), \mathrm{v}\right\rangle.$$

Moreover, the process $\tilde{u}$ satisfies the following inequality for every p satisfying condition (3.1) *and $q \in [1, p]$*:

$$\tilde{\mathbb{E}}\Big[\sup_{s\in[0,T]} |\tilde{u}(s)|_{\mathrm{H}}^q\Big] + \tilde{\mathbb{E}}\Big[\int_0^T \|\tilde{u}(s)\|^2\,ds\Big] < \infty. \tag{5.23}$$

PROOF. Let p be any exponent satisfying condition (3.1). Since the sequences $(u_{0,n})_{n=1}^{\infty} \subset \mathrm{H}$ and $(f_n)_{n=1}^{\infty} \subset \mathrm{V}'$ convergent weakly in H and V′, respectively, we infer that there exist $R_1 > 0$ and $R_2 > 0$ such that

$$\sup_{n\in\mathbb{N}} |u_{0,n}|_{\mathrm{H}} \le R_1 \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{V}'} \le R_2.$$

By Lemma 5.8, we infer that the processes u_n, $n \in \mathbb{N}$, satisfy inequalities (4.8)–(4.10). Thus the first part of the assertion follows directly from Theorem 4.11. Inequality (5.23) follows again from Lemma 5.8. The proof of the theorem is thus complete. □

REMARK 5.10. Although this has not been studied in the present paper, we believe that methods developed here can be used to study the continuous dependence of the solutions on other parameters entering our equations, for instance, the linear operator A, the nonlinearity B and the diffusion operator G.

6. Existence of an invariant measure for stochastic NSEs on 2-dimensional domains. In this section, we assume that $d = 2$. Since we are interested in the existence of invariant measures, we assume that the domain $\mathcal{O}$ satisfies the Poincaré condition; see (2.4).[5] However, our results are true for general domains for the stochastic damped Navier–Stokes equations; see, for instance, [14].

Since we assume that $\mathcal{O}$ is a Poincaré domain, by the Poincaré inequality [see (2.4)], the functional given by the formula

$$\|u\| = |\nabla u|_{L^2}, \qquad u \in \mathrm{V}, \tag{6.1}$$

is a norm in the space V equivalent to the norm given by (2.3).

In the sequel, in the space V *we consider the norm given by* (6.1).

We aim in this section to prove that, under some natural assumptions, problem (3.2) has an invariant measure. Let us fix, as in Assumption 5.3, a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0}$; a canonical cylindrical Wiener process $W = W(t)$ in a separable Hilbert space K defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We also fix a function $G : \mathrm{H} \to \mathcal{T}_2(\mathrm{K}, \mathrm{V}')$ satisfying condition (H.2) in Assumption 3.1 and, in addition, the Lipschitz condition (G1) with a constant L smaller than $\sqrt{2}$, and inequality (G2) with $\lambda_0 = 0$. The last assumption on λ_0

[5] It is well known that this condition holds if the domain $\mathcal{O}$ is bounded in some direction, that is, there exists a vector $h \in \mathbb{R}^d$ such that $\mathcal{O} \cap (h + \mathcal{O}) = \varnothing$.

corresponds to the fact that in $\mathcal{O}$ we consider the norm given by (6.1). In what follows, the initial data u_0 will be an element of the space H. By $u(t, u_0)$, $t \geq 0$, we denote the unique solution to the problem (5.3) (defined on the above stochastic basis satisfying Assumption 5.3).

For any bounded Borel function $\varphi \in \mathcal{B}_b(\mathrm{H})$ and $t \geq 0$, we define

$$(P_t\varphi)(u_0) = \mathbb{E}[\varphi(u(t, u_0))], \qquad u_0 \in \mathrm{H}. \tag{6.2}$$

Since by Lemma 5.1 the trajectories $u(\cdot, u_0)$ are continuous, $(P_t)_{t\geq 0}$ is a stochastically continuous semigroup on the Banach space $\mathcal{C}_b(\mathrm{H})$. This means that for every $\varphi \in \mathcal{C}_b(\mathrm{H})$ and $u_0 \in \mathrm{H}$:

$$\lim_{t\to 0} P_t\varphi(u_0) = u_0.$$

As a consequence of Corollary 5.7, we have the following result.

PROPOSITION 6.1. *The family $u(t, u_0)$, $t \geq 0$, $u_0 \in \mathrm{H}$ is Markov. In particular, $P_{t+s} = P_t P_s$ for $t, s \geq 0$.*

The proof of Proposition 6.1 is standard, and thus omitted; see, for example, [1], [18], Section 9.2, [37], Section 9.7.

PROPOSITION 6.2. *The semigroup P_t is bw-Feller, that is, if $\phi : \mathrm{H} \to \mathbb{R}$ is a bounded sequentially weakly continuous function and $t > 0$, then $P_t\phi : \mathrm{H} \to \mathbb{R}$ is also a bounded sequentially weakly continuous function. In particular, if $u_{0n} \to u_0$ weakly in* H *then*

$$P_t\phi(u_{0n}) \to P_t\phi(u_0).$$

PROOF. Let us choose and fix $t > 0$, $u_0 \in \mathrm{H}$ and an H-valued sequence (u_{0n}) that is weakly convergent to u_0 in H. Let also $\phi : \mathrm{H} \to \mathbb{R}$ be a bounded sequentially weakly continuous function. Let us choose an auxiliary time $T \in (t, \infty)$.

Since obviously the function $P_t\phi : \mathrm{H} \to \mathbb{R}$ is bounded, we only need to prove that it is sequentially weakly continuous.

Let $u_n(\cdot) = u(\cdot, u_{0n})$, respectively $u(\cdot) = u(\cdot, u_0)$, be a strong solution of problem (5.3) on $[0, \infty)$ with the initial data u_{0n}, respectively, u_0. We assume that these processes are defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$. By Theorem 5.9, there exist (depending on T):

- a subsequence $(n_k)_k$,
- a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_s\}_{s\in[0,T]}$,
- a cylindrical Wiener process $\tilde{W} = \tilde{W}(s)$, $s \in [0, T]$ defined on this basis,
- and an $\tilde{\mathbb{F}}$-progressively measurable processes $\tilde{u}(s)$, $(\tilde{u}_{n_k}(s))_{k\geq 1}$, $s \in [0, T]$ (defined on this basis) with laws supported in $\mathcal{Z}_T$ such that

$$\tilde{u}_{n_k} \text{ has the same law as } u_{n_k} \text{ on } \mathcal{Z}_T \text{ and } \tilde{u}_{n_k} \to \tilde{u} \text{ in } \mathcal{Z}_T, \qquad \tilde{\mathbb{P}}\text{-a.s.} \tag{6.3}$$

and the system

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u}) \tag{6.4}$$

is a martingale solution to problem (5.3) on the interval $[0, T]$ with the initial data u_0. In particular, by (6.3), $\tilde{\mathbb{P}}$-almost surely

$$\tilde{u}_{n_k}(t) \to \tilde{u}(t) \qquad \text{weakly in H}.$$

Since the function $\phi : \mathrm{H} \to \mathbb{R}$ is sequentially weakly continuous, we infer that $\tilde{\mathbb{P}}$-a.s.,

$$\phi(\tilde{u}_{n_k}(t)) \to \phi(\tilde{u}(t)) \qquad \text{in } \mathbb{R}.$$

Therefore, since the function $\phi : \mathrm{H} \to \mathbb{R}$ is also bounded, by the Lebesgue dominated convergence theorem we infer that

$$\lim_{k\to\infty} \tilde{\mathbb{E}}[\phi(\tilde{u}_{n_k}(t))] = \tilde{\mathbb{E}}[\phi(\tilde{u}(t))]. \tag{6.5}$$

From the equality of laws of $\tilde{u}_{n_k}$ and u_{n_k}, $k \in \mathbb{N}$, on the space $\mathcal{Z}_T$ we infer that

$$\tilde{\mathbb{E}}[\phi(\tilde{u}_{n_k}(t))] = \mathbb{E}[\phi(u_{n_k}(t))] = P_t\phi(u_{0n_k}). \tag{6.6}$$

Since by assumptions $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u)$ is a martingale solution of equation (5.3) with the initial data u_0 and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u})$ is also a martingale solution with the initial of equation (5.3) with the initial data u_0 and since the solution of (5.3) is unique in law, we infer that

the processes u and $\tilde{u}$ have the same law on the space $\mathcal{Z}_t$.

Hence,

$$\tilde{\mathbb{E}}[\phi(\tilde{u}(t))] = \mathbb{E}[\phi(u(t))] = P_t\phi(u_0). \tag{6.7}$$

Thus, by (6.5), (6.6) and (6.7), we infer that

$$\lim_{k\to\infty} P_t\phi(u_{0n_k}) = P_t\phi(u_0).$$

Using the sub-subsequence argument, we infer that the whole sequence $(P_t\phi(u_{0n}))_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty} P_t\phi(u_{0n}) = P_t\phi(u_0),$$

which completes the proof of Proposition 6.2. □

REMARK 6.3. From inequality (5.10) and the Poincaré inequality (2.4), it follows that the following inequality holds for the strong solution u of problem (5.3) defined on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$:

$$\int_0^t \mathbb{E}|u(s)|_{\mathrm{H}}^2\, ds \le \frac{2}{C\eta}|u_0|_{\mathrm{H}}^2 + \frac{2}{C\eta}\left(\frac{2}{\eta}|f|_{\mathrm{V}'}^2 + \varrho\right)t, \qquad t \ge 0. \tag{6.8}$$

PROOF OF INEQUALITY (6.8). Let us fix $t \ge 0$. By the Poincaré inequality (2.4) for almost all $s \in [0, t]$,

$$|u(s)|_{\mathrm{H}}^2 \le \frac{1}{C} |\nabla u(s)|_{L^2}^2.$$

By (5.10), in particular, we obtain

$$\frac{\eta}{2} \mathbb{E} \int_0^t |\nabla u(s)|^2 \, ds \le |u_0|_{\mathrm{H}}^2 + \left(\frac{2}{\eta} |f|_{\mathrm{V}'}^2 + \varrho \right) t.$$

Hence, we infer that

$$\begin{aligned} \int_0^t \mathbb{E} |u(s)|_{\mathrm{H}}^2 \, ds &\le \frac{1}{C} \mathbb{E} \int_0^t |\nabla u(s)|^2 \, ds \\ &\le \frac{2}{C\eta} |u_0|_{\mathrm{H}}^2 + \frac{2}{C\eta} \left(\frac{2}{\eta} |f|_{\mathrm{V}'}^2 + \varrho \right) t, \qquad t \ge 0, \end{aligned}$$

that is, inequality (6.8) holds. □

Using inequality (6.8), we deduce the following result.

COROLLARY 6.4. *Let $u_0 \in \mathrm{H}$ and let $u(t)$, $t \ge 0$, be the unique solution to the problem (5.3) starting from u_0. Then there exists $T_0 \ge 0$ such that for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\sup_{T \ge T_0} \frac{1}{T} \int_0^T (P_s^* \delta_{u_0})(\mathrm{H} \setminus \bar{\mathbb{B}}_R) \, ds \le \varepsilon, \tag{6.9}$$

where $\bar{\mathbb{B}}_R = \{\mathrm{v} \in \mathrm{H} : |\mathrm{v}|_{\mathrm{H}} \le R\}$.

PROOF. Using the Chebyshev inequality and inequality (6.8), we infer that for every $T \ge 0$ and $R > 0$,

$$\begin{aligned} \frac{1}{T} \int_0^T (P_s^* \delta_{u_0})(\mathrm{H} \setminus \bar{\mathbb{B}}_R) \, ds &= \frac{1}{T} \int_0^T \mathbb{P}(\{|u(s)|_{\mathrm{H}} > R\}) \, ds \\ &\le \frac{1}{TR^2} \int_0^T \mathbb{E} |u(s)|_{\mathrm{H}}^2 \, ds \\ &\le \frac{1}{TR^2} \left[\frac{2}{C\eta} |u_0|_{\mathrm{H}}^2 + \frac{2}{C\eta} \left(\frac{2}{\eta} |f|_{\mathrm{V}'}^2 + \varrho \right) T \right] \\ &= \frac{1}{TR^2} \frac{2}{C\eta} |u_0|_{\mathrm{H}}^2 + \frac{1}{R^2} \frac{2}{C\eta} \left(\frac{2}{\eta} |f|_{\mathrm{V}'}^2 + \varrho \right). \end{aligned}$$

Thus, the assertion follows. □

By Proposition 6.2, Corollary 6.4 and the Maslowski–Seidler theorem [29], Proposition 3.1, we deduce the following main result of our paper.

THEOREM 6.5. *Let $\mathcal{O} \subset \mathbb{R}^2$ be a Poincaré domain. Let assumptions* (H.1)–(H.2) *and* (H.5) *be satisfied. In addition, we assume that the function G satisfies condition* (G1) *with $L < \sqrt{2}$ and inequality* (G2) *with $\lambda_0 = 0$. Then there exists an invariant measure of the semigroup $(P_t)_{t \geq 0}$ defined by* (6.2), *that is, a probability measure μ on* H *such that*

$$P_t^* \mu = \mu.$$

REMARK 6.6. In this section, we have used strong solutions. In particular, in order to show a global inequality (6.8) which was a basis for Corollary 6.4. However, we could have easily avoided this. For instance, instead of the global inequality (6.8) we could prove that every martingale solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, u)$ of equation (5.3) with the initial data u_0 on the time interval $[0, T]$ satisfies inequality (6.8) for only $t \in [0, T]$ but with constants C, η and ρ independent of T. Let us also point out that our proof of inequality (6.8) is related to some ideas from the paper [13] by Chow and Khasminskii.

APPENDIX A: UNIFORM ESTIMATES ON THE SOLUTIONS OF GALERKIN APPROXIMATING EQUATIONS

Let us recall that the proof of existence of a martingale solution of the Navier–Stokes equations, given in [9], is based on the Faedo–Galerkin approximation in the space H_n; see (5.2) in the cited paper. In order to continue, we need to choose and fix a stochastic basis, and thus we assume that Assumption 5.3 holds. We also fix an $\mathcal{F}_0$-measurable H-valued random variable. Then the nth equation is the following one in the space H_n:

$$\text{(A.1)} \qquad \begin{cases} du_n(t) = -\big[P_n \mathcal{A} u_n(t) + B_n\big(u_n(t)\big) - P_n f(t)\big]\,dt \\ \qquad\qquad + P_n G\big(u_n(t)\big)\,dW(t), \qquad t > 0, \\ u_n(0) = P_n u_0. \end{cases}$$

Recall that H_n is a finite dimensional subspace spanned by the n first eigenvectors of the operator L given by (2.19) in [9], P_n is defined by [9], (2.25), and B_n is defined on page 1636 in [9]. For details, see [9], Lemmas 2.3 and 2.4. In particular, P_n restricted to H is the orthogonal projection. The existence of a solution of equation (A.1) is guaranteed by Lemma 5.2 in [9].

The following result corresponds to Lemma 5.3 from [9]. The proof of estimates (A.2), (A.3) and (A.5), is similar to the proof of estimates (5.4), (5.5) and (5.6) from Appendix A in [9]. However, we provide the details to indicate the dependence of appropriate constants on the data, which will be important in the proof of continuous dependence of the solutions of the Navier–Stokes equations on the initial state u_0 and the external forces f. Moreover, if $\mathcal{O}$ is the Poincaré domain, we prove a new estimate; see (A.5). This estimate is of crucial importance in the proof of the existence of invariant measure. Recall that we have put $\frac{\eta}{2-\eta} = \infty$ when $\eta = 2$.

LEMMA A.1. *Let Assumption 5.3 and parts* (H.2), (H.3) *and* (H.5) *of Assumption 3.1 be satisfied. In particular, we assume that p satisfies* (3.1), *that is*,

$$p \in \left[2, 2+\frac{\eta}{2-\eta}\right),$$

where $\eta \in (0,2]$ is given in (H.2).

(1) *Then for every $T>0$, ν, R_1 and R_2 there exist constants $C_1(p)$, $\tilde{C}_2(p)$, $C_2(p)$, such that if $u_0 \in L^p(\Omega, \mathcal{F}_0, \mathrm{H})$, $f \in L^p([0,\infty); \mathrm{V}')$ satisfy $\mathbb{E}[|u_0|_{\mathrm{H}}^p] \le R_1$ and $|f|_{L^p(0,T;\mathrm{V}')} \le R_2$, then every solution u_n of Galerkin equation* (A.1) *with the initial data u_0 and the external force f satisfies the following estimates*:

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big(\sup_{s\in[0,T]} |u_n(s)|_{\mathrm{H}}^p\Big) \le C_1(p) \tag{A.2}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T |u_n(s)|_{\mathrm{H}}^{p-2} |\nabla u_n(s)|^2\,ds\right] \le \tilde{C}_2(p) \tag{A.3}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T |\nabla u_n(s)|^2\,ds\right] \le C_2(p). \tag{A.4}$$

(2) *Moreover, if $\mathcal{O}$ is a Poincaré domain and inequality* (G2) *holds with $\lambda_0 = 0$, then for every $t>0$*,

$$\begin{aligned} &\sup_{n\in\mathbb{N}}\left(\mathbb{E}[|u_n(t)|_{\mathrm{H}}^2] + \frac{\eta}{2}\mathbb{E}\left[\int_0^t |\nabla u_n(s)|^2\,ds\right]\right) \\ &\qquad \le \mathbb{E}[|u_0|_{\mathrm{H}}^2] + \frac{2}{\eta}\int_0^t |f(s)|_{\mathrm{V}'}^2\,ds + \rho t. \end{aligned} \tag{A.5}$$

PROOF. Let us fix p satisfying condition (3.1). We apply the Itô formula from [34] to the function F defined by

$$F : \mathrm{H} \ni x \mapsto |x|_{\mathrm{H}}^p \in \mathbb{R}.$$

In the sequel, we will omit the subscript H and write $|\cdot| := |\cdot|_{\mathrm{H}}$. Note that

$$\begin{aligned} F'(x) &= d_x F = p\cdot |x|^{p-2}\cdot x, \\ \|F''(x)\| &= \|d_x^2 F\| \le p(p-1)\cdot |x|^{p-2}, \qquad x \in \mathrm{H}. \end{aligned}$$

With the above comments in mind, we have, for $t \in [0,\infty)$,

$$\begin{aligned} &|u_n(t)|^p - |u_n(0)|^p \\ &\quad = \int_0^t \Big[p|u_n(s)|^{p-2} \langle u_n(s), -\mathcal{A}u_n(s) - B_n(u_n(s)) + P_n f(s)\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}\operatorname{Tr}\big[F''\big(u_n(s)\big)\big(P_n G\big(u_n(s)\big), P_n G\big(u_n(s)\big)\big)\big]\Big] \, ds \\
&+ p\int_0^t \big|u_n(s)\big|^{p-2}\big\langle u_n(s), P_n G\big(u_n(s)\big)\, dW(s)\big\rangle \\
= &\int_0^t \Big[-p\big|u_n(s)\big|^{p-2}\big\|u_n(s)\big\|^2 + p\big|u_n(s)\big|^{p-2}\big\langle u_n(s), P_n f(s)\big\rangle \\
&+ \frac{1}{2}\operatorname{Tr}\big[F''\big(u_n(s)\big)\big(P_n G\big(u_n(s)\big), P_n G\big(u_n(s)\big)\big)\big]\Big] \, ds \\
&+ p\int_0^t \big|u_n(s)\big|^{p-2}\big\langle u_n(s), P_n G\big(u_n(s)\big)\, dW(s)\big\rangle.
\end{aligned}
\tag{A.6}
$$

Since

$$
\operatorname{Tr}\big[F''(u)\big(P_n G(u), P_n G(u)\big)\big] \le p(p-1)|u|^{p-2} \cdot \big|G(u)\big|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}, \qquad u \in \mathrm{V},
$$

and by (G2)

$$
\big|G(u)\big|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})} \le (2-\eta)|\nabla u|^2 + \lambda_0|u|^2 + \rho, \qquad u \in \mathrm{V},
$$

and since by (2.3) and the Young inequality with exponents 2, $\frac{2p}{p-2}$ and p, for $u \in \mathrm{V}$ and $f \in \mathrm{V}'$,

$$
\begin{aligned}
|u|^{p-2}\langle f, u\rangle &\le |u|^{p-2}\|u\|_{\mathrm{V}}|f|_{\mathrm{V}'} = |u|^{p-2}\big(|u|^2 + |\nabla u|^2\big)^{\frac{1}{2}}|f|_{\mathrm{V}'} \\
&\le \frac{\varepsilon}{2}\big(|u|^2 + |\nabla u|^2\big)|u|^{p-2} + \left(\frac{1}{2} - \frac{1}{p}\right)|u|^p + \frac{\varepsilon^{-p/2}}{p}|f|^p_{\mathrm{V}'} \\
&\le \frac{\varepsilon}{2}|\nabla u|^2|u|^{p-2} + \left(\frac{1+\varepsilon}{2} - \frac{1}{p}\right)|u|^p + \frac{\varepsilon^{-p/2}}{p}|f|^p_{\mathrm{V}'},
\end{aligned}
$$

we infer that

$$
\begin{aligned}
&\big|u_n(t)\big|^p + \left[p - p\frac{\varepsilon}{2} - \frac{1}{2}p(p-1)(2-\eta)\right]\int_0^t \big|u_n(s)\big|^{p-2}\big|\nabla u_n(s)\big|^2 \, ds \\
&\quad\le \big|u_n(0)\big| + \int_0^t \left[\left(\frac{p(1+\varepsilon)}{2} - 1\right)\big|u_n(s)\big|^p + \varepsilon^{-p/2}\big|f(s)\big|^p_{\mathrm{V}'}\right. \\
&\qquad\left. + \frac{1}{2}p(p-1)\big|u_n(s)\big|^{p-2} \cdot \big(\lambda_0\big|u_n(s)\big|^2 + \rho\big)\right] ds \\
&\qquad + p\int_0^t \big|u_n(s)\big|^{p-2}\big\langle u_n(s), P_n G\big(u_n(s)\big)\, dW(s)\big\rangle \\
&\quad= \int_0^t \left[\left(\frac{\lambda_0}{2}p(p-1) + \frac{p(1+\varepsilon)}{2} - 1\right)\big|u_n(s)\big|^p\right.
\end{aligned}
$$

$$+\frac{\rho}{2}p(p-1)|u_n(s)|^{p-2}+\varepsilon^{-p/2}|f(s)|_{\mathrm{V}'}^p\Big]\,ds$$
$$+p\int_0^t |u_n(s)|^{p-2}\langle u_n(s), P_nG(u_n(s))\,dW(s)\rangle.$$

Let us choose $\varepsilon \in (0,1)$ such that $\delta = \delta(p,\eta) := p - p\frac{\varepsilon}{2} - \frac{1}{2}p(p-1)(2-\eta) > 0$, or equivalently,

$$\varepsilon < 1 \wedge [2-(p-1)(2-\eta)].$$

Notice that under condition (3.1) such ε exists. Denote also

$$K_p(\lambda_0,\rho) := \frac{\lambda_0}{2}p(p-1)+p-1+\rho p\Big(1-\frac{2}{p}\Big)\frac{p-1}{2}$$
$$= \frac{p-1}{2}[\lambda_0 p + 2 + \rho(p-2)].$$

Thus, since by the Young inequality $x^{p-2} \le (1-\frac{2}{p})x^p + \frac{2}{p}1^{p/2}$ for $x \ge 0$, we obtain

$$\begin{aligned}(A.7)\quad & |u_n(t)|^p + \delta\int_0^t |u_n(s)|^{p-2}|\nabla u_n(s)|^2\,ds \\ &\le |u(0)|^p + K_p(\lambda_0,\rho)\int_0^t |u_n(s)|^p\,ds + \rho(p-1)t \\ &\quad + \varepsilon^{-p/2}\int_0^t |f(s)|_{\mathrm{V}'}^p\,ds \\ &\quad + p\int_0^t |u_n(s)|^{p-2}\langle u_n(s), P_nG(u_n(s))\,dW(s)\rangle, \qquad t\in[0,\infty).\end{aligned}$$

Since u_n is the solutions of the Galerkin equation, we infer that the process

$$\mu_n(t) := \int_0^t |u_n(s)|^{p-2}\langle u_n(s), P_nG(u_n(s))\,dW(s)\rangle, \qquad t\in[0,\infty)$$

is a square integrable martingale. Indeed, if we define a map

$$g : \mathrm{V} \ni u \mapsto \{\mathrm{K} \ni k \mapsto \langle u, P_nG(u)k\rangle \in \mathrm{H}\} \in \mathcal{T}_2(\mathrm{K},\mathbb{R})$$

then $\mu_n(t) = \int_0^t |u_n(s)|^{p-2}g(u_n(s))\,dW(s)$, and hence, by assumption (G2) and the fact that P_n is the orthogonal projection in H we infer that, for every $t \ge 0$,

$$\begin{aligned}(A.8)\quad & \int_0^t \big\||u_n(s)|^{p-2}g(u_n(s))\big\|_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}^2\,ds \\ &= \int_0^t |u_n(s)|^{p-2}\|g(u_n(s))\|_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}^2\,ds \\ &\le \int_0^t |u_n(s)|^{p-2}|u_n(s)|^2\|P_nG(u_n(s))\|_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}^2\,ds \\ &\le \int_0^t |u_n(s)|^p[(2-\eta)|\nabla u_n(t)|^2 + \lambda_0|u_n(t)|^2+\rho]\,ds.\end{aligned}$$

Hence, by the fact that u_n is a Galerkin solution we infer that

$$\mathbb{E}\int_0^t \big\| |u_n(s)|^{p-2} g(u_n(s))\big\|^2_{\mathcal{T}_2(\mathrm{K},\mathbb{R})}\, ds < \infty, \qquad t \geq 0,$$

and thus we infer, as claimed, that the process μ_n is a square integrable martingale. Hence, $\mathbb{E}[\mu_n(t)] = 0$. Let us now fix $T > 0$. By taking expectation in inequality (A.7), we infer that

$$\begin{aligned}\mathbb{E}\big[|u_n(t)|^p\big] &\leq \mathbb{E}\big[|u_0|^p\big] + K_p(\lambda_0,\rho)\int_0^t \mathbb{E}\big[|u_n(s)|^p\big]\, ds \\ &\quad + \rho(p-1)t + \varepsilon^{-p/2}\mathbb{E}\int_0^t |f(s)|^p_{\mathrm{V}'}\, ds \\ &\leq \mathbb{E}\big[|u_0|^p\big] + K_p(\lambda_0,\rho)\int_0^t \mathbb{E}\big[|u_n(s)|^p\big]\, ds + \rho(p-1)T \\ &\quad + \varepsilon^{-p/2}\mathbb{E}\int_0^T |f(s)|^p_{\mathrm{V}'}\, ds, \qquad t \in [0,T].\end{aligned}$$

Hence, by the Gronwall lemma there exists a constant $\tilde{C}_p = \tilde{C}_p(T,\eta,\lambda_0,\rho,\mathbb{E}[|u_0|^p], \|f\|_{L^p(0,T;\mathrm{V}')}) = \tilde{C}_p(T,\eta,\lambda_0,\rho,R_1,R_2) > 0$ such that

$$\mathbb{E}\big[|u_n(t)|^p\big] \leq \tilde{C}_p, \qquad t \in [0,T], n \in \mathbb{N},$$

that is,

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]} \mathbb{E}\big[|u_n(t)|^p\big] \leq \tilde{C}_p. \tag{A.9}$$

Using this bound in (A.7), we also obtain

$$\sup_{n\in\mathbb{N}} \mathbb{E}\bigg[\int_0^T |u_n(s)|^{p-2}|\nabla u_n(s)|^2\, ds\bigg] \leq \tilde{C}_2(p) \tag{A.10}$$

for a new constant $\tilde{C}_2(p) = C_2(p,T,\eta,\lambda_0,\rho,\mathbb{E}[|u_0|^p], \|f\|_{L^p(0,T;\mathrm{V}')}) = \tilde{C}_2(p, T,\eta,\lambda_0,\rho,R_1,R_2)$. This completes the proof of estimates (A.3). Since $\mathbb{E}[|u_0|^2] \leq (\mathbb{E}[|u_0|^p])^{\frac{2}{p}} \leq R_1^{2/p}$, we infer that (A.4) holds with another constant $C_2(p)$.

Let us move to the proof of estimate (A.2). By the Burkholder–Davis–Gundy inequality (see [19]), the Schwarz inequality and inequality (G2), there exists a constant c_p such that, for any $t \geq 0$,

$$\begin{aligned}&\mathbb{E}\bigg[\sup_{0\leq s\leq t}\bigg|\int_0^s p|u_n(\sigma)|^{p-2}\langle u_n(\sigma), P_n G(u_n(\sigma))\, dW(\sigma)\rangle\bigg|\bigg] \\ &\quad \leq c_p \cdot \mathbb{E}\bigg[\bigg(\int_0^t |u_n(\sigma)|^{2p-2} \cdot |P_n G(u_n(\sigma))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\, d\sigma\bigg)^{\frac{1}{2}}\bigg]\end{aligned}$$

$$
\begin{aligned}
&\leq c_p \cdot \mathbb{E}\Big[\sup_{0\leq\sigma\leq t} |u_n(\sigma)|^{\frac{p}{2}} \Big(\int_0^t |u_n(\sigma)|^{p-2} \cdot |G(u_n(\sigma))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\, d\sigma\Big)^{\frac{1}{2}}\Big]\\
&\leq \frac{1}{2}\mathbb{E}\Big[\sup_{0\leq s\leq t} |u_n(s)|^p + \frac{1}{2}c_p^2 \int_0^t |u_n(\sigma)|^{p-2} \cdot |G(u_n(\sigma))|^2_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}\, d\sigma\Big]\\
&\leq \frac{1}{2}\mathbb{E}\Big[\sup_{0\leq s\leq t} |u_n(s)|^p \\
&\quad + \frac{1}{2}c_p^2 \int_0^t |u_n(\sigma)|^{p-2} \cdot [(2-\eta)|u_n(\sigma)|^2 + \lambda_0 |u_n(\sigma)|^2 + \rho]\, d\sigma\Big]\\
&\leq \frac{1}{2}\mathbb{E}\Big[\sup_{0\leq s\leq t} |u_n(s)|^p\Big] + \frac{1}{2}c_p^2 \frac{2\rho}{p} t\\
&\quad + \frac{1}{2}c_p^2 (2-\eta)\mathbb{E}\Big[\int_0^t |u_n(\sigma)|^p \|u_n(\sigma)\|^2\, d\sigma\Big]\\
&\quad + \frac{1}{2}c_p^2 \Big(\lambda_0 + \rho\Big(1-\frac{2}{p}\Big)\Big) \cdot \mathbb{E}\Big[\int_0^t |u_n(\sigma)|^p\, d\sigma\Big].
\end{aligned} \tag{A.11}
$$

Using (A.11) in (A.7), by inequalities (A.9) and (A.10) we infer that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq s\leq t} |u_n(s)|^p\Big]\\
&\quad \leq \mathbb{E}[|u_0|^p]\\
&\qquad + \Big[K_p(\lambda_0,\rho) + \frac{1}{2}c_p^2\Big(\lambda_0 + \rho\Big(1-\frac{2}{p}\Big)\Big)\Big] \int_0^t \mathbb{E}[|u_n(s)|^p]\, ds\\
&\qquad + \Big(\frac{2\rho}{p} + c_p^2 \frac{\rho}{p}\Big) t + \varepsilon^{-p/2} \int_0^t |f(s)|^p_{\mathrm{V}'}\, ds\\
&\qquad + \frac{1}{2}\mathbb{E}\Big[\sup_{0\leq s\leq t} |u_n(s)|^p\Big] + \frac{1}{2}c_p^2(2-\eta)\mathbb{E}\Big[\int_0^t |u_n(\sigma)|^p \|u_n(\sigma)\|^2\, d\sigma\Big]\\
&\quad \leq \mathbb{E}[|u_0|^p] + \Big[K_p(\lambda_0,\rho) + \frac{1}{2}c_p^2\Big(\lambda_0 + \rho\Big(1-\frac{2}{p}\Big)\Big)\Big]\tilde{C}_p t\\
&\qquad + \frac{\rho}{p}(2 + c_p^2) t + \varepsilon^{-p/2}\int_0^t |f(s)|^p_{\mathrm{V}'}\, ds\\
&\qquad + \frac{1}{2}\mathbb{E}\Big[\sup_{0\leq s\leq t} |u_n(s)|^p\Big] + \frac{1}{2}c_p^2(2-\eta) C_2(p), \qquad t \geq 0.
\end{aligned}
$$

Thus, for a fixed $T > 0$,

$$
\mathbb{E}\Big[\sup_{0\leq s\leq T} |u_n(s)|^p\Big] \leq C_1(p),
$$

where

$$\begin{aligned} C_1(p) &= C_1(p, T, \eta, \lambda_0, \rho, R_1, R_2) \\ &:= 2R_1 + 2\Big[K_p(\lambda_0, \rho) + \frac{1}{2}c_p^2\Big(\lambda_0 + \rho\Big(1 - \frac{2}{p}\Big)\Big)\Big]\tilde{C}_p T \\ &\quad + 2\Big(\frac{2\rho}{p} + c_p^2\frac{\rho}{p}\Big)T + 2\varepsilon^{-p/2}R_2 + c_p^2(2-\eta)C_2(p). \end{aligned}$$

This completes the proof of estimate (A.2).

To prove inequality (A.5), let us assume that $\mathcal{O}$ is a Poincaré domain and inequality (G2) holds with $\lambda_0 = 0$. Recall that now in the space V we consider the inner product $((\cdot,\cdot))$ given by (2.2). By identity (A.6) from the previous proof with $p = 2$, we have

$$\begin{aligned} |u_n(t)|^2 - |u(0)|^2 = \int_0^t \Big\{&-2\|u_n(s)\|^2 + 2\langle u_n(s), f(s)\rangle \\ &+ \frac{1}{2}\mathrm{Tr}\big[F''(u_n(s))\big(G(u_n(s)), G(u_n(s))\big)\big]\Big\}\, ds \\ &+ 2\int_0^t \langle u_n(s), P_n G(u_n(s))\, dW(s)\rangle, \qquad t \geq 0. \end{aligned}$$

Since $\mathbb{E}(\int_0^t \langle P_n G(u_n(s)), u_n(s)\, dW(s)\rangle) = 0$, we infer that

$$\begin{aligned} \mathbb{E}|u_n(t)|_{\mathrm{H}}^2 \leq \mathbb{E}[|u_0|_{\mathrm{H}}^2] &+ \mathbb{E}\int_0^t \{-2\|u_n(s)\|^2 + 2\langle f(s), u_n(s)\rangle\}\, ds \\ &+ \mathbb{E}\int_0^t |P_n G(u_n(s))|_{\mathcal{T}_2(\mathrm{K},\mathrm{H})}^2\, ds. \end{aligned}$$

Using assumption (G2) with $\lambda_0 = 0$, i.e., $|G(u_n(s))|_{\mathcal{T}_2(K,\mathrm{H})}^2 \leq (2-\eta)\|u_n(s)\|^2 + \varrho$, we get

$$\begin{aligned} \mathbb{E}|u(t)|_{\mathrm{H}}^2 \leq -\eta\mathbb{E}\int_0^t \|u_n(s)\|^2\, ds &+ \mathbb{E}[|u_0|_{\mathrm{H}}^2] \\ &+ 2\mathbb{E}\int_0^t \langle f(s), u(s)\rangle\, ds + \varrho t. \end{aligned} \tag{A.12}$$

Since $2\langle f, u\rangle \leq \frac{\eta}{2}|\nabla u_n|^2 + \frac{2}{\eta}|f|_{\mathrm{V}'}^2$, for $u \in \mathrm{V}, f \in \mathrm{V}'$, we infer that

$$\begin{aligned} \mathbb{E}|u_n(t)|_{\mathrm{H}}^2 \leq -\frac{\eta}{2}\mathbb{E}\int_0^t \|u_n(s)\|^2\, ds &+ \mathbb{E}[|u_0|_{\mathrm{H}}^2] \\ &+ \frac{2}{\eta}\int_0^t |f(s)|_{\mathrm{V}'}^2 + \varrho t, \qquad t \geq 0. \end{aligned} \tag{A.13}$$

The proof of inequality (A.5) is thus complete. □

APPENDIX B: PROOF OF THEOREM 4.8

Similar to the proof of Theorem 5.1 in [9], the present proof is based on the Galerkin method. We will use the fact the the laws of the Galerkin solutions form a tight set of probability measures on $\mathcal{Z}_T$. We will use the Jakubowski's version of the Skorokhod Theorem 4.6, as well. However, some details are different.

Let us fix positive numbers T, R_1 and R_2. Let us assume that μ is a Borel probability measure on H, $f \in L^p([0,\infty);\mathrm{V}')$ which satisfy $\int_{\mathrm{H}} |x|^p \mu(dx) \le R_1$ and $|f|_{L^p(0,T;\mathrm{V}')} \le R_2$. Similar to the previous section, we choose and fix a stochastic basis, and thus we assume that Assumption 5.3 holds. We also fix an $\mathcal{F}_0$-measurable H-valued random variable whose law is equal to μ.

As in the proof of [9], Theorem 5.1, let $(u_n)_{n\in\mathbb{N}}$ be a sequence of the solutions of the Galerkin equations. Then the set of laws $\{\mathcal{L}(u_n, n \in \mathbb{N}\}$ is tight on the space $(\mathcal{Z}_T, \sigma(\mathcal{T}_T))$, where $\sigma(\mathcal{T}_T)$ denotes the topological σ-field. By Theorem 4.6, there exists a subsequence (n_k), a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and, on this space $\mathcal{Z}_T$-valued random variables u, $\tilde{u}_{n_k}$, $k \in \mathbb{N}$, and a sequence of K-valued Wiener processes $\tilde{W}$, $\tilde{W}_{n_k}$, $k \in \mathbb{N}$ such that

$$\text{the variables } (u_{n_k}, W) \text{ and } (\tilde{u}_{n_k}, \tilde{W}_{n_k}) \text{ have the same laws on the Borel } \sigma\text{-algebra } \mathcal{B}\big(\mathcal{Z}_T \times \mathcal{C}([0,T],\mathrm{K})\big) \tag{B.1}$$

and

$$(\tilde{u}_{n_k}, \tilde{W}_{n_k}) \text{ converges to } (u, \tilde{W}) \text{ in } \mathcal{Z}_T \times \mathcal{C}\big([0,T];\mathrm{K}\big) \text{ almost surely on } \tilde{\Omega}. \tag{B.2}$$

In particular,

$$\tilde{u}_{n_k} \text{ converges to } u \text{ in } \mathcal{Z}_T \text{ almost surely on } \tilde{\Omega}. \tag{B.3}$$

We will denote the subsequence $(\tilde{u}_{n_k}, \tilde{W}_{n_k})$ again by $(\tilde{u}_n, \tilde{W}_n)$. Define a corresponding sequence of filtrations by

$$\tilde{\mathbb{F}}_n = (\tilde{\mathcal{F}}_{n,t})_{t\ge 0} \qquad \text{where } \tilde{\mathcal{F}}_{n,t} = \sigma\big\{\big(\tilde{u}_n(s), \tilde{W}_n(s)\big), s \le t\big\}, t \in [0,T]. \tag{B.4}$$

To obtain (4.8), we modify the proof from [9] at pages 1650–1651. Namely, using Lemma A.1, we infer that the processes $\tilde{u}_n$, $n \in \mathbb{N}$, satisfy the following inequalities:

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\Big(\sup_{s\in[0,T]} \big|\tilde{u}_n(s)\big|_{\mathrm{H}}^p\Big) \le C_1(p) \tag{B.5}$$

and

$$\sup_{n\in\mathbb{N}} \tilde{\mathbb{E}}\bigg[\int_0^T \big|\nabla \tilde{u}_n(s)\big|_{L^2}^2 \, ds\bigg] \le C_2(p). \tag{B.6}$$

Let us emphasize that the constants $C_1(p)$ and $C_2(p)$, being the same as in Lemma A.1, depend on T, R_1 and R_2. Using inequality (B.5), we choose a subsequence, still denoted by $(\tilde{u}_n)$, convergent weak star in the space $L^p(\tilde{\Omega}; L^\infty(0,T;\mathrm{H}))$ and infer that

$$\mathbb{E}\Big[\sup_{s\in[0,T]} |u(s)|_{\mathrm{H}}^p\Big] \le C_1(p) \tag{B.7}$$

and that the limit process u satisfies (B.7), as well. This completes the proof of inequality (4.9). To prove (4.8), let us fix $q \in [1, p)$. Notice that for every $t \in [0,T]$

$$|u(t)|^q = \big(|u(t)|^p\big)^{q/p} \le \Big(\sup_{t\in[0,T]} |u(t)|^p\Big)^{q/p}.$$

Thus, $\sup_{t\in[0,T]} |u(t)|^q \le (\sup_{t\in[0,T]} |u(t)|^p)^{q/p}$, and so by the Hölder inequality

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]} |u(t)|^q\Big] &\le \mathbb{E}\Big[\Big(\sup_{t\in[0,T]} |u(t)|^p\Big)^{q/p}\Big]\\ &\le \Big(\mathbb{E}\Big[\sup_{t\in[0,T]} |u(t)|^p\Big]\Big)^{q/p} \le (C_1(p))^{q/p},\end{aligned}$$

which means that inequality (4.8) holds with the constant $C_1(p,q) := (C_1(p))^{q/p}$.

By inequality (B.6), we infer that the sequence $(\tilde{u}_n)$ contains further subsequence, denoted again by $(\tilde{u}_n)$, convergent weakly in the space $L^2([0,T]\times\tilde{\Omega};\mathrm{V})$ to u. Moreover, it is clear that

$$\tilde{\mathbb{E}}\Big[\int_0^T |\nabla u(s)|_{L^2}^2\,ds\Big] \le C_2(p) \tag{B.8}$$

and the process u satisfies (4.10).

To prove the second part of the theorem, we assume that $\mathcal{O}$ is a Poincaré domain and inequality (G2) holds with $\lambda_0 = 0$. In this case, by Lemma A.1, instead of inequality (B.6) we can use the following one corresponding to the uniform estimates (A.5):

$$\frac{\eta}{2}\sup_{n\in\mathbb{N}}\mathbb{E}\Big[\int_0^T |\nabla\tilde{u}_n(s)|_{L^2}^2\,ds\Big] \le \mathbb{E}\big[|u_0|_{\mathrm{H}}^2\big] + \frac{2}{\eta}\int_0^T |f(s)|_{\mathrm{V}'}^2\,ds + \rho T, \tag{B.9}$$

choose a subseqence convergent weakly in the space $L^2([0,T]\times\tilde{\Omega};\mathrm{V})$ to u and infer that the limit process satisfies the same estimate, which proves estimate (4.11). We will prove that the system $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, u)$ is a martingale solution of problem (3.2).

Step 1. Let us fix $\varphi \in U$. Analogously to [7] and [30], let us denote

$$\begin{aligned}\Lambda_n(\tilde{u}_n, \tilde{W}_n, \varphi)(t) &:= \big(\tilde{u}_n(0),\varphi\big)_{\mathrm{H}} - \int_0^t \langle P_n\mathcal{A}\tilde{u}_n(s),\varphi\rangle\,ds\\ &\quad - \int_0^t \langle B_n(\tilde{u}_n(s)),\varphi\rangle\,ds + \int_0^t \langle f_n(s),\varphi\rangle\,ds\\ &\quad + \Big\langle\int_0^t P_nG(\tilde{u}_n(s))\,d\tilde{W}_n(s),\varphi\Big\rangle, \qquad t\in[0,T],\end{aligned} \tag{B.10}$$

and

$$\Lambda(u,\tilde{W},\varphi)(t) := \big(u(0),\varphi\big)_{\mathrm{H}} - \int_0^t \langle \mathcal{A}u(s),\varphi\rangle\,ds - \int_0^t \langle B\big(u(s)\big),\varphi\rangle\,ds + \int_0^t \langle f(s),\varphi\rangle\,ds + \left\langle \int_0^t G\big(u(s)\big)\,d\tilde{W}(s),\varphi\right\rangle, \qquad t\in[0,T]. \tag{B.11}$$

Using Lemma 2.4(c) from [9] (see also [30], Lemma 5.4), we can prove the following lemma analogous to Lemma 4.12.

LEMMA B.1. *For all $\varphi \in U$:*

(a) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |(\tilde{u}_n(t)-u(t),\varphi)_{\mathrm{H}}|^2\,dt] = 0$,
(b) $\lim_{n\to\infty} \tilde{\mathbb{E}}[|(\tilde{u}_n(0)-u(0),\varphi)_{\mathrm{H}}|^2] = 0$,
(c) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\int_0^t \langle P_n\mathcal{A}\tilde{u}_n(s)-\mathcal{A}u(s),\varphi\rangle\,ds|\,dt] = 0$,
(d) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\int_0^t \langle B_n(\tilde{u}_n(s))-B(u(s)),\varphi\rangle\,ds|\,dt] = 0$,
(e) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\int_0^t \langle P_n f_n(s)-f(s),\varphi\rangle\,ds|\,dt] = 0$,
(f) $\lim_{n\to\infty} \tilde{\mathbb{E}}[\int_0^T |\langle\int_0^t [P_n G(\tilde{u}_n(s))-G(u(s))]d\tilde{W}(s),\varphi\rangle|^2\,dt] = 0$.

Directly from Lemma B.1, we get the following corollary.

COROLLARY B.2. *For every $\varphi \in U$,*

$$\lim_{n\to\infty} \big|\big(\tilde{u}_n(\cdot),\varphi\big)_{\mathrm{H}} - \big(u(\cdot),\varphi\big)_{\mathrm{H}}\big|_{L^2([0,T]\times\tilde{\Omega})} = 0 \tag{B.12}$$

and

$$\lim_{n\to\infty} \big|\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi) - \Lambda(u,\tilde{W},\varphi)\big|_{L^1([0,T]\times\tilde{\Omega})} = 0. \tag{B.13}$$

PROOF. Assertion (B.12) follows from the equality

$$\big|\big(\tilde{u}_n(\cdot),\varphi\big)_{\mathrm{H}} - \big(\tilde{u}(\cdot),\varphi\big)_{\mathrm{H}}\big|^2_{L^2([0,T]\times\tilde{\Omega})} = \tilde{\mathbb{E}}\left[\int_0^T \big|\big(\tilde{u}_n(t)-\tilde{u}(t),\varphi\big)_{\mathrm{H}}\big|^2\,dt\right]$$

and Lemma 4.12(a). To prove (B.13), let us note that, by the Fubini theorem, we have

$$\big|\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi) - \Lambda(u,\tilde{W},\varphi)\big|_{L^1([0,T]\times\tilde{\Omega})} = \int_0^T \tilde{\mathbb{E}}\big[\big|\Lambda_n(\tilde{u}_n,\tilde{W}_n,\varphi)(t) - \Lambda(u,\tilde{W},\varphi)(t)\big|\big]\,dt.$$

To complete the proof of (B.13), it is sufficient to note that by Lemma B.1(b)–(f), each term on the right-hand side of (B.10) tends at least in $L^1([0,T]\times\tilde{\Omega})$ to the corresponding term in (B.11). □

Step 2. Since u_n is a solution of the Galerkin equation, for all $t \in [0, T]$ and $\varphi \in U$

$$\big(u_n(t), \varphi\big)_{\mathrm{H}} = \Lambda_n(u_n, W, \varphi)(t), \qquad \mathbb{P}\text{-a.s.}$$

In particular,

$$\int_0^T \mathbb{E}\big[\big|\big(u_n(t), \varphi\big)_{\mathrm{H}} - \Lambda_n(u_n, W, \varphi)(t)\big|\big]\, dt = 0.$$

Since $\mathcal{L}(u_n, W) = \mathcal{L}(\tilde{u}_n, \tilde{W}_n)$, using (B.12) and (B.13) we infer that

$$\int_0^T \tilde{\mathbb{E}}\big[\big|\big(u(t), \varphi\big)_{\mathrm{H}} - \Lambda(u, \tilde{W}, \varphi)(t)\big|\big]\, dt = 0.$$

Hence, for l-almost all $t \in [0, T]$ and $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$

$$\big(u(t), \varphi\big)_{\mathrm{H}} - \Lambda(u, \tilde{W}, \varphi)(t) = 0. \tag{B.14}$$

Since u is $\mathcal{Z}_T$-valued random variable, in particular $u \in \mathcal{C}([0, T]; \mathrm{H}_w)$, that is, u is weakly continuous, we infer that equality (B.14) holds for all $t \in [0, T]$ and all $\varphi \in U$. Since U is dense in V, equality (B.14) holds for all $\varphi \in \mathrm{V}$, as well. Putting $\tilde{\mathfrak{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}})$, by (B.14) and (B.11) we infer that the system $(\tilde{\mathfrak{A}}, \tilde{W}, u)$ is a martingale solution of equation (3.2). The proof of Theorem 4.8 is thus complete.

APPENDIX C: KURATOWSKI THEOREM

The following is the classical form of the celebrated Kuratowski theorem.

THEOREM C.1 (Kuratowski theorem). *Assume that X_1, X_2 are two Polish spaces with their Borel σ-fields denoted respectively by $\mathcal{B}(X_1), \mathcal{B}(X_2)$. If $\phi : X_1 \longrightarrow X_2$ is an injective Borel measurable map, then for any $E_1 \in \mathcal{B}(X_1)$, $E_2 := \phi(E_1) \in \mathcal{B}(X_2)$.*

Let us formulate a simple corollary to the above result.

PROPOSITION C.2. *Suppose that X_1, X_2 are two topological spaces with their Borel σ-fields denoted respectively by $\mathcal{B}(X_1), \mathcal{B}(X_2)$. Suppose that $\phi : X_1 \longrightarrow X_2$ is an injective Borel measurable map such that for any $E_1 \in \mathcal{B}(X_1)$, $E_2 := \phi(E_1) \in \mathcal{B}(X_2)$. Then if $g : X_1 \to \mathbb{R}$ is a Borel measurable map then a function $f : X_2 \to \mathbb{R}$ defined by*

$$f(x_2) = \begin{cases} g\big(\phi^{-1}(x_2)\big), & \text{if } x_2 \in \phi(X_1), \\ \infty, & \text{if } x_2 \in X_2 \setminus \phi(X_1), \end{cases} \tag{C.1}$$

is also Borel measurable.

PROOF. Note that $g = f \circ \phi$:

$$f^{-1}(A) = \phi[g^{-1}(A)], \qquad A \subset \mathbb{R}.$$

Thus, if $A \in \mathcal{B}(\mathbb{R})$, then by assumptions $g^{-1}(A) \in \mathcal{B}(X_1)$. Hence, by Theorem C.1 we infer that $\phi[g^{-1}(A)] \in \mathcal{B}(X_2)$, and thus by the equality above, we infer that $f^{-1}(A) \in \mathcal{B}(X_2)$. The proof is complete. □

One may wonder if the following a generalisation of the above result to non-Polish spaces is valid.

THEOREM C.3. *Let X_1 and X_2 be a topological spaces such that for each $i = 1, 2$ there exists a sequence $\{f_{i,m}\}$ of continuous functions $f_{i,m} : X_i \to \mathbb{R}$ that separate points of X_i. Let us denote by $\mathscr{S}_i$ the σ-algebra generated by the maps $\{f_{i,m}\}$. If $\phi : X_1 \longrightarrow X_2$ is an injective measurable map, then for any $E_1 \in \mathscr{S}_1$, $E_2 := \phi(E_1) \in \mathscr{S}_2$.*

The following counterexample shows that the answer to the above question is "no".

COUNTEREXAMPLE C.4. (1) Define $f_k(x) = e^{2ikx\pi}$, $x \in [0, 1)$, for every integer k (trigonometric functions).

(2) Let X_1 be a non-Borel subset of [0, 1) equipped with the euclidean metric.
(3) Let X_2 denote [0, 1) with the Euclidean metric.
(4) Denote by f_k^1 the restriction of f_k to X_1.
(5) Then f_k^1 are continuous and separate points in X_1.
(6) Then f_k are continuous and separate points in X_2.
(7) $\sigma(f_k) = Borel(X_2)$ by Stone–Weierstrass.
(8) $\sigma(f_k^1) = \{A \cap X_1 : A \in \sigma(f_k)\} = \{A \cap X_1 : A \in Borel(X_2)\} = Borel(X_1)$.
(9) Let $\varphi : X_1 \to X_2$ be the identity mapping.
(10) φ is a continuous injection.
(11) $\varphi[X_1]$ is not Borel in X_2.

Acknowledgements. The authors would like to thank an anonymous referee for careful reading of the manuscript and useful remarks.

The second named author would like to thank the Department of Mathematics of the University of York, where part of this research started, for the hospitality.

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