SECOND ORDER STOCHASTIC DOMINANCE CONSTRAINTS IN MULTI–OBJECTIVE STOCHASTIC PROGRAMMING PROBLEMS

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ABSTRACT Many economic and financial applications lead to deterministic optimization problems depending on a probability measure. These problems can be either static (one stage) or dynamic with finite (multistage) or infinite horizon, single– objective or multi–objective. Constraints sets can be "deterministic", given by probability constraints or stochastic dominance constraints. We focus on multi–objective problems and second order stochastic dominance constraints. To this end we employ the former results obtained for stochastic (mostly strongly) convex multi–objective problems and results obtained for one objective problems with second order stochastic dominance constraints. The relaxation approach will be included in the case of second order stochastic dominance constraints.

Keywords Stochastic multi–objective optimization problems, efficient solution, Wasserstein metric and $\mathcal{L}_1$ norm, Lipschitz property, second order stochastic dominance constraints, relaxation
AMS classification: 90 C 15
JEL classification: C 44

1 Introduction

Let $(\Omega, \mathcal{S}, P)$ be a probability space, $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ an s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$, $F := F_\xi$ the distribution function of ξ, P_F, and Z_F the probability measure and the support corresponding to F, respectively; E_F denote the operator of mathematical expectation corresponding to F. Let, moreover, $g_i := g_i(x, z)$, $i = 1, \ldots, l$, $l \geq 1$ be real–valued (say, continuous) functions defined on $\mathbb{R}^n \times \mathbb{R}^s$; $X_F \subset X \subset \mathbb{R}^n$ be a nonempty set generally depending on F, and $X \subset \mathbb{R}^n$ be a nonempty deterministic set. If for every $x \in X$ there exist finite $\mathsf{E}_F g_i(x, \xi)$, $i = 1, \ldots, l$, then a rather general (often employed) type of "multi–objective" one–stage optimization problem depending on a probability measure can be introduced in the form:

$$\text{Find} \quad \min \mathsf{E}_F g_i(x, \xi),\ i = 1, \ldots, l \quad \text{subject to} \quad x \in X_F. \tag{1}$$

There are known (from the literature) mainly X_F: "deterministic", given by probability constrains, by mathematical expectation and recently often appear stochastic dominance constraints. To define second order stochastic dominance constraints let $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^1$ be a real-valued function, $Y : \mathbb{R}^s \to \mathbb{R}^1$ random value. If for $x \in X$ there exists finite $\mathsf{E}_F g(x, \xi)$, $\mathsf{E}_F Y(\xi)$ and if

$$F^2_{g(x,\,\xi)}(u) = \int\limits_{-\infty}^{u} F_{g(x,\,\xi)}(y)dy, \quad F^2_Y(u) = \int\limits_{-\infty}^{u} F_Y(y)dy, \quad u \in \mathbb{R}^1,$$

then we can define the second order stochastic dominance constraints X_F by

$$X_F \;=\; \{x \in X : F^2_{g(x,\,\xi)}(u) \leq F^2_Y(u) \quad \text{for every} \quad u \in \mathbb{R}^1\}. \tag{2}$$

Consequently multi–objective stochastic programming problems with second order stochastic dominance constraints can be defined by the relations (1), (2). Employing the results of [9](see also Lemma 1), the multi–objective stochastic programming problem with second order stochastic dominance constraints can be written in a more friendly form:

$$\text{Find} \quad \min \mathsf{E}_F g_i(x, \xi),\ i = 1, \ldots, l \quad \text{subject to} \quad x \in X_F, \tag{3}$$

where

$$X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \mathbb{R}^1\}. \tag{4}$$

Remark 1. Second order stochastic dominance corresponds to an order in the space of non negative concave utility functions.

The paper [6] is focus on the investigation of stability (obtained on the base of Wasserstein metric) and empirical estimates for the multi–objective stochastic problems. However replacing there general convex X_F by second order stochastic dominance constraints (4) we obtain an infentisimal optimization problem for which the Slater's condition is not generally fulfilled. The aim of this contribution is to relax constraints set to obtain problems for which Slater's condition is already fulfilled and to estimate the error of approximation. To this end the stability based on the Wasserstein metric is employed.

The stochastic multi–objective problem defined by (3),(4) is a deterministic multi–objective problem depending on the probability measure; consequently to analyze this problem it is possible to employ classical well known results (see, e.g., [2], [6] and [7]).

2 Some Definitions and Auxiliary Assertions

2.1 Deterministic Problems

First, we recall some results from the deterministic multi–objective optimization problems. To this end let $f_i(x)$, $i = 1, \ldots, l$ be real–valued functions defined on $\mathbb{R}^n$, $\mathcal{K} \subset \mathbb{R}^n$ be a nonempty set. We consider a multi–objective deterministic optimization problem in the form:

$$\text{Find} \quad \min f_i(x),\ i = 1, \ldots, l \quad \text{subject to} \quad x \in \mathcal{K}. \tag{5}$$

Definition 1. The vector x^* is an efficient solution of the problem (5) if and only if there exists no $x \in \mathcal{K}$ such that $f_i(x) \le f_i(x^*)$ for $i = 1, \ldots, l$ and such that for at least one i_0 one has $f_{i_0}(x) < f_{i_0}(x^*)$.

Definition 2. The vector x^* is a properly efficient solution of the multi–objective optimization problem (5) if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $x \in \mathcal{K}$ satisfying $f_i(x) < f_i(x^*)$ there exists at least one j such that $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \le M. \tag{6}$$

Proposition 1. [4] Let $\mathcal{K} \subset R^n$ be a nonempty convex set and let $f_i(x)$, $i = 1, \ldots, l$ be convex functions on $\mathcal{K}$. Then $x^0 \in \mathcal{K}$ is a properly efficient solution of the problem (5) if and only if x^0 is optimal in

$$\min_{x \in \mathcal{K}} \sum_{i=1}^{l} \lambda_i f_i(x) \quad \text{for some} \quad \lambda_1, \ldots, \lambda_l > 0, \quad \sum_{i=1}^{l} \lambda_i = 1.$$

A relationship between efficient and properly efficient points is introduced, e.g., in [3] or [4]. We summarize it in the following Remark.

Remark 2. Let $f(x) = (f_1(x), \ldots, f_l(x))$, $x \in \mathcal{K}$; $\mathcal{K}^{eff}$, $\mathcal{K}^{peff}$ be sets of efficient and properly efficient points of the problem (5). If $\mathcal{K}$ is a convex set, $f_i(x)$, $i = 1, \ldots, l$ are convex functions on $\mathcal{K}$, then

$$f(\mathcal{K}^{peff}) \subset f(\mathcal{K}^{eff}) \subset \bar{f}(\mathcal{K}^{peff}), \tag{7}$$

where $\bar{f}(\mathcal{K}^{peff})$ denotes the closure set of $f(\mathcal{K}^{peff})$.

2.2 Wasserstein Metric in Stochastic Optimization Problems

To recall the Wasserstein metric and its application to single–objective stochastic optimization problem we consider the case $l = 1$. To this end let $\mathcal{P}(\mathbb{R}^s)$ denote the set of all (Borel) probability measures on $\mathbb{R}^s$ and let the system $\mathcal{M}_1^1(\mathbb{R}^s)$ be defined by the relation:

$$\mathcal{M}_1^1(\mathbb{R}^s) := \left\{ \nu \in \mathcal{P}(\mathbb{R}^s) : \int_{\mathbb{R}^s} \|z\|_1 d\nu(z) < \infty \right\}, \quad \|\cdot\|_1^s := \|\cdot\|_1 \text{ denotes } \mathcal{L}_1 \text{ norm in } \mathbb{R}^s. \tag{8}$$

If the assumption A.0, A.1 are defined by

A.0 $g_1(x, z)$ is for $x \in X$ a Lipschitz function of $z \in \mathbb{R}^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x,

A.1 $g_1(x, z)$ is either a uniformly continuous function on $X \times \mathbb{R}^s$ or there exists $\varepsilon > 0$ such that $g_1(x,\, z)$ is a convex bounded function on $X(\varepsilon)$ ($X(\varepsilon)$ denotes the ε–neighborhood of the set$X$),

and if $P_F,\, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$; $F_i,\, G_i,\, i = 1,\, \ldots,\, s$ denote one-dimensional marginal distribution functions corresponding to F and G, then

Proposition 2. [5] Let $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$. If Assumption A.0 is fulfilled, then

$$|\mathsf{E}_F g_1(x,\, \xi) - \mathsf{E}_G g_1(x,\, \xi)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i \quad \text{for} \quad x \in X. \tag{9}$$

If, moreover, X is a compact set and Assumption A.1 is fulfilled, then also

$$|\inf_{x \in X} \mathsf{E}_F g_1(x,\, \xi) - \inf_{x \in X} \mathsf{E}_G g_1(x,\, \xi)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i. \tag{10}$$

To study the constraints set defined by (2) we recall the next Lemma.

Lemma 1. [7] Let $g(x, z)$, $Y(z)$ be for every $x \in X$ Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $x \in X$. Let, moreover, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$. If X_F is defined by the relation (2), then

1. $X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \mathbb{R}^1\}$,
2. $(u - g(x, z))^+,\ (u - Y(z))^+, \quad u \in \mathbb{R}^1,\, x \in \mathbb{R}^n$ are Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $u \in \mathbb{R}^1$, $x \in \mathbb{R}^n$. (See before employed the relation (4.))

If the assumptions of Lemma 1 are fulfilled, $P_F,\, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, $u \in \mathbb{R}^1$, $x \in X$, then it follows from Proposition 2 that

$$\begin{aligned}
|\mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_G(u - g(x, \xi))^+| &\le L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i, \\
|\mathsf{E}_F(u - Y(\xi))^+ - \mathsf{E}_G(u - Y(\xi))^+| &\le L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i.
\end{aligned} \tag{11}$$

Further defining the sets X^{ε} by

$$X_F^{\varepsilon} = \{x \in X : \mathsf{E}_F(u - g(x,\, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \le \varepsilon \quad \text{for every} \quad u \in \mathbb{R}^1\}, \quad \varepsilon \in \mathbb{R}^1, \tag{12}$$

we can obtain

$$x \in X_F \Longrightarrow x \in X_G^{\varepsilon}, \quad x \in X_G \Longrightarrow x \in X_F^{\varepsilon} \quad \text{with} \quad \varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i,$$

$$\text{and generally} \qquad X_G^{\delta-\varepsilon} \subset X_F^{\delta} \subset X_G^{\delta+\varepsilon} \quad \text{for} \quad \delta \in \mathbb{R}^1. \tag{13}$$

2.3 Relaxation

Till now we have considered stochastic multi–objective problems with constraints set (4). According to the well known fact from the infitisimal programming theory, there can be problem with Slater's condition. Consequently to this fact Dentcheva and Ruszczynski [1] suggested to relax one objective problems by the modification of the constraints set; replacing X_F by $X_F^{a,\,b}$:

$$X_F^{a,\,b} = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \langle a,\, b\rangle\},\ a,\, b\, \in \mathbb{R}^1. \tag{14}$$

However they did not specified how to choice a, b. Surely, it is desirable to determine a, b to be a difference between X_F and $X_F^{a,b}$ small. More precisely, it is desirable to be small difference between the corresponding optimal solutions and optimal values. On the other side they have proven the following assertion.

Proposition 3. [1]. Let $\bar{Y} := \bar{Y}(\xi)$ be a random value defined on $(\Omega, \mathcal{S}, P)$. Let, moreover, $Y(\xi)$ has a discrete distribution with realizations y_i, $i = 1, \ldots, m$, where $a \le y_i \le b$, $a,\, b \in \mathbb{R}^1$ for all i. Then the inequality

$$\mathsf{E}_{F_{\bar{Y}}}(u - \bar{Y}(\xi))^+ \le \mathsf{E}_{F_Y}(u - Y(\xi))^+ \quad \text{for all} \quad u \in \langle a,\, b\rangle$$

is equivalent to

$$\mathsf{E}_{F_{\bar{Y}}}(y_i - \bar{Y})^+ \le \mathsf{E}_{F_Y}(y_i - Y)^+, \quad i = 1, \ldots, m.$$

If $Y(\xi)$ is a random value defined on $(\Omega, \mathcal{S}, P)$ $(\xi = \xi(\omega))$ and if $a < b,\ a,\, b \in \mathbb{R}^1$, then we can define a random value $Y^{a,\,b} := Y^{a,\,b}(\xi)$ by

$$\begin{aligned} Y^{a,\,b}(\xi) = \quad & Y(a) \quad \text{if} \quad \xi \le a, \\ & Y(\xi) \quad \text{if} \quad \xi \in (a,\, b), \\ & Y(b) \quad \text{if} \quad \xi \ge b \end{aligned} \tag{15}$$

and to note the corresponding distribution function by $F_Y^{a,\,b}$.

Employing (15), under rather generalize assumptions, we can for constants $a,\, b,\ a < b; a_1\, b_1,\ a_1 < b_1$ define

i. the random value $Y^{a,\,b}(\xi)$ $(a \le Y^{a,\,b}(\xi) \le b)$ with the distribution function $F_Y^{a,\,b}$,

ii. for every $x \in X$ the random value $g^{a,\,b}(x,\, \xi)$ $(a \le g^{a,\,b}(x,\, \xi) \le b)$ with the distribution function $F_{g(x,\,\xi)}^{a,\,b}$.

Evidently, $Y(\xi)$, $g(x,\, \xi)$, for every $x \in X$, are functions of the random vector ξ and, simultaneously, they are functions of the components $\xi_1, \ldots, \xi_s$ of the random vector ξ. Consequently, under rather general assumptions it is possible to choose $a,\ b,\ a_1,\ b_1$ such that

iii. $a_1 < b_1,\, i = 1, \ldots, s \Longrightarrow a < Y(\xi) < b, \quad a < g(x, \xi) < b \quad \text{for every} \quad x \in X.$

The constants a_1, b_1 determine a distribution function $F^{a_1, b_1} := F_{\xi}^{a_1, b_1}$ with a support $\prod\limits_{i=1}^{s} \langle a_1, b_1 \rangle$.

iv there exists (for a natural number m) points $y_1, \ldots, y_m \in \prod\limits_{i=1}^{s} \langle a_1, b_1 \rangle$ those define discrete distribution function $\bar{F}^{a, b} := \bar{F}_Y^{a, b}$ with atoms $y_1., \ldots, y_m$ ($\bar{F}_Y^{a, b}$ approximates $F_Y^{a, b}$).

According to the above recalled assertions we can see that now it is possible to define two constraints sets $X_F^{a, b} := X_F^{a_1, b_1}$ and $\bar{X}_F^{a, b} = \bar{X}^{a_1, b_1}$ by

$$X_{F^{a, b}}^{a, b} = (X_F^{a_1, b_1}) = \{x \in X : \mathsf{E}_{F^{a_1, b_1}}(u - g(x, \xi))^+ \le \mathsf{E}_{F^{a_1, b_1}}(u - Y(\xi))^+ \quad \text{for every} \quad u \in \langle a, b \rangle\}, \tag{16}$$

$$\bar{X}_{\bar{F}}^{a_1, b_1} = \{x \in X : \mathsf{E}_{F^{a_1, b_1}}(y_i - g(x, \xi))^+ \le \mathsf{E}_{\bar{F}^{a, b}}(y_i - Y(\xi))^+ \quad \text{for every} \quad i = i, \ldots, m\}, \tag{17}$$

and to define optimization problems

$$\text{To find} \quad \varphi^{a_1, b_1}(F^{a_1, b_1}, X_{F^{a_1, b_1}}^{a_1, b_1}) = \inf\{\mathsf{E}_{F^{a_1, b_1}} g_0(x, \xi) | x \in X_F^{a_1, b_1}\}, \tag{18}$$

$$\text{To find} \quad \bar{\varphi}^{a_1, b_1}(F^{a_1, b_1}\, \bar{X}_{\bar{F}}^{a_1, b_1}) = \inf\{\mathsf{E}_{F^{a_1, b_1}} g_0(x, \xi) | x \in \bar{X}_{\bar{F}}^{a_1, b_1})\}. \tag{19}$$

It has been proven in [8] that the optimization problems (18), (19) already fulfil the Slater's condition. The next assertion follows also from [8].

Proposition 4. Let X_F, $X_F^{a_1, b_1}$, $\bar{X}_F^{a_1, b_1}$ be compact sets, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, Assumptions A.1, i., ii., iii., iv be fulfilled. If

1. $g(x, z)$ is for every $z \in Z_F$ a Lipschitz function of $x \in X$ with the Lipschitz constant $\bar{L}$ not depending on $z \in Z_F$,
2. there exists a constant $D > 0$ such that

$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \le D\varepsilon \quad \text{for every} \quad \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon \rangle,$$

with $\varepsilon = 2L_g \max[\sum\limits_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i^{a_1, b_1}(z_i) - \bar{F}_i^{a_1, b_1}(z_i)| dz_i, \; \sum\limits_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1, b_1}(z_i)| dz_i]$,

then

1.
$$\begin{aligned} |\varphi(F, X_F) - \varphi(F, X_F^{a_1, b_1})| &\le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1, b_1}(z_i)| dz_i, \\ |\varphi(F, X_F) - \varphi(F, \bar{X}_{\bar{F}}^{a_1, b_1})| &\le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - \bar{F}_i^{a_1, b_1}(z_i)| dz_i, \end{aligned}$$

2. If, moreover, Assumptions A.0 is fulfilled, then

$$|\varphi(F, X_F) - \varphi^{a_1, b_1}(F^{a_1, b_1}, \bar{X}_{\bar{F}}^{a_1, b_1})| \le (2L + 2D\bar{L}L_g)[\sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - \bar{F}_i^{a_1, b_1}(z_i)| dz_i,$$

$$|\varphi(F^{a_1, b_1}, X_F^{a_1, b_1}) - \bar{\varphi}^{a_1, b_1}(\bar{F}^{a_1, b_1}, \bar{X}_{\bar{F}}^{a_1, b_1})| \le$$

$$(2L + 2D\bar{L}L_g)[\sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{a_1, b_1}(z_i) - \bar{F}_i^{a_1, b_1}(z_i)| dz_i. \tag{20}$$

($\Delta[\cdot,\, \cdot] = \Delta_n[\cdot,\, \cdot]$ denotes the Hausdorff distance in the space of subsets of n-dimensional Euclidean space; for more details see, e.g., [10].)

Remark 3. Evidently, it is reasonable to choose a_1, b_1, m to be "small" the values

$$|\varphi(F,\, X_F) - \varphi(F,\, X_F^{a_1,\, b_1})|, \quad |\varphi(F,\, X_F) - \varphi(F,\, \bar{X}_F^{a_1,\, b_1})|.$$

To this end the relation (11) (with $G := F_\xi^{a_1,\ b_1}$), $G := \bar{F}_Y^{a,\, b}$ and Proposition 4 can be employed. The constant a, b, has to be chosen with respect to the function g. (The assumption i. ii. iii. iv. has to be fulfilled.)

3 Application to Multi-Objective Stochastic Programming Problems

First, we recall one very simple assertion.

Lemma 2. Let $X \subset \mathbb{R}^n$ be a compact convex set, $a,\, b \in \mathbb{R}^1,\ a < b$. Let, moreover, $g(x,\, z)$ be for every $z \in \mathbb{R}^s$ a concave function of $x \in X$, then

$$\begin{aligned}
X_F &= \{x \in X :\, \mathsf{E}_F(u - g(x,\xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \mathbb{R}^1\},\\
X_F^{a,\, b} &= \{x \in X :\, \mathsf{E}_{F^{a.,\, b.}}(u - g(x,\xi))^+ \le \mathsf{E}_{F^{a_1,\, b_1}}(u - Y(\xi))^+ \quad \text{for every} \quad u \in \langle a,\, b\rangle\},\\
\bar{X}_{\bar{F}}^{a,\, b} &= \{x \in X :\, \mathsf{E}_{F^{a_1,\, b_1}}(y_i - g(x,\xi))^+ \le \mathsf{E}_{\bar{F}^{a_1,\, b_1}}(y_i - Y(\xi))^+ \quad \text{for every} \quad i = 1,\, \ldots,\, m\},
\end{aligned}$$

are convex sets.

Proof. The assertion follows immediately from the properties of convex functions and convex sets.

Further setting successively

$$\begin{aligned}
f_i(x) &= \mathsf{E}_F g_i(x,\xi),\, i = 1,\, \ldots,\, l & \mathcal{K} &= X_F,\\
f_i(x) &= \mathsf{E}_{F^{a_1,\, b_1}} g_i(x,\xi),\, i = 1,\, \ldots,\, l, & \mathcal{K} &= X_F^{a,\, b},\\
f_i(x) &= \mathsf{E}_{\bar{F}^{a_1,\, b_1}} g_i(x,\xi),\, i = 1,\, \ldots,\, l, & \mathcal{K} &= \bar{X}_F^{a,\, b},
\end{aligned} \tag{21}$$

we obtain three "deterministic" (depending on the probability measures) multi–objective problems. If the "original" functions $g_i(x,\, z)$, $i = 1,\, \ldots,\, l$ are for every $z \in \mathbb{R}^s$ convex functions, $g(x,\, z)$ for every $z \in \mathbb{R}^s$ concave function, then the corresponding problems (5) are convex multi–objective problems. According to Proposition 1 we can obtain the sets of properly efficient points and further according to Remark 2 these sets approximate the corresponding sets of efficient points. Moreover the Slater's condition is fulfilled for the second and the third optimization problem in (21). Of course, the second and the third problem "approximate" the original one. Employing the results of Subsections 2.2, 2.3 we can estimate the errors of approximations.

Furthermore, employing the results of the paper [6] we can study the stability of these problems and their empirical estimates. It means, the case when F is replaced by empirical distribution function or the case when $F^{a,\, b}$, $\bar{F}$ are determined by random sample. However the corresponding results have been obtained under the assumptions of strongly convex $g_i(x,\, z)$, $i = 1,\, \ldots,\, l$. (For the definition of strongly convex functions see, e.g., [6]). The investigation in this direction is surely very interesting and important but it is beyond of the scope of this paper. Moreover it can be obtained on the base of the paper [6].

4 Conclusion

The contribution deals with multi–objective stochastic programming problems with second order stochastic dominance constraints. In particular, the aim of the paper is to show a possibility to generalize the paper [6] with rather general convex constraints set to the special case of second order stochastic dominance constraints. To this end a relaxation approach has been employed.

Acknowledgements: This work was supported by the Czech Science Foundation under Grant No. 18–02739S.

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