Multi–Objective Optimization Problems with Random Elements; Survey of Approaches

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Abstract.

Many economic and financial situations depend simultaneously on a random element and a decision parameter. Mostly, it is possible to influence the above mentioned situation only by an optimization model depending on a probability measure. This optimization problem can be static (one–stage), dynamic with finite or infinite horizon, single–objective or multi–objective. We focus on one–stage multi–objective problems corresponding to applications those are suitable to evaluate simultaneously by a few objectives. The aim of the contribution is to give a survey of different approaches (as they are known from the literature) of the above mentioned applications. To this end we start with well–known mean–risk model and continue with other known approaches. Moreover, we try to complete every model by a suitable application. Except an analysis of a choice of the objective functions type we try to discuss suitable constraints set with respect to the problem base, possible investigation and relaxation. At the end we mention properties of the problem in the case when the theoretical"underlying" probability measure is replaced by its "deterministic" or "stochastic" estimate.

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1 Introduction

Multi–objective optimization problems with a random element correspond to many economic situations in which an economic process is influenced by a random factor say ξ, a decision parameter say x and it is suitable to evaluate it by a few objectives. To recall an exact mathematical definition of the optimization problem depending on a random factor, let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F(z),\, z \in R^s)$, P_F and Z_F denote the distribution function, the probability measure and the support corresponding to ξ; $x = (x_1, \ldots, x_n) \in R^n$. Let, moreover, $g_i := g_i(x, z)$, $i = 1, \ldots, l$, $l \geq 1$, $\quad g^*_j := g^*_j(x, z)$, $j = 1, \ldots, l'$, $\quad l' \geq 1$ be real–valued (say continuous) functions defined on $R^n \times R^s$, $\quad X \subset R^n$ be a nonempty set.

The above mentioned rather general multi–objective problem with a random element (in static setting) can be introduced in the following form:

$$\begin{aligned} &\text{Find} && \min g_i(x, \xi),\ i = 1, \ldots, l \\ &\text{subject to} && g^*_j(x, \xi) \leq 0,\ j = 1, \ldots, l', \quad x \in X. \end{aligned} \tag{1}$$

(1) is (generally) a non linear multi–objective (practically "deterministic") programming problem everywhere when the decision x can depend on the random element ξ, it means when the realization ξ is known in the time of the problem solution. However, it is known that such "nice" situation happen very seldom.

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On the other hand rather often it is possible to assume that the probability measure P_F is known or at least it can be estimated. Consequently, (in such case) a question arises how to determine a suitable problem. Of course, it is possible to assume that the problem will be again multi–objective and that it will be depending on a probability measure. To explain this situation, let us start with very simple and well known one–objective portfolio problem (1) in which $l = l' = 1$, the functions g_1^* is not depending on the random element ξ. The problem (with random element) is the following:

$$\text{Find} \quad \max \sum_{k=1}^{n} \xi_k x_k \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1, \ldots, n, \quad n = s, \tag{2}$$

(x_k is a fraction of the unit wealth invested in the asset k; ξ_k return of the asset k). Evidently, if it is necessary to determine x_k without knowledge the realization ξ_k, $k = 1, \ldots, n$, then it is (very often) reasonable to set to (2) two–objectives "deterministic" optimization problem:

$$\text{Find} \quad \max \sum_{k=1}^{n} \mu_k x_k; \qquad \max[-\sum_{k=1}^{n}\sum_{j=1}^{n} x_k c_{k,\,j} x_j], \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1, \ldots, n, \tag{3}$$

in which $\mu_k = \mathsf{E}_F \xi_k, \quad c_{k,\,j} = \mathsf{E}_F(\xi_k - \mu_k)(\xi_j - \mu_j)$. (Symbol E_F denotes the operator of mathematical expectation corresponding to the distribution function F; of course we suppose that the corresponding final mathematical expectation exists).

Remark 1. The "underlying" problem with random element (2) is single –objective with deterministic constraints, the corresponding problem (3) depending on the probability measure is two–objectives, where one is in a form of mathematical expectation and the other is given by the second moment; constraints set is deterministic.

To find x that maximize simultaneously both objectives is mostly impossible. Markowitz in [13] suggested to replace the problem (3) by one–objective problem:

$$\text{Find} \quad \min[-\sum_{k=1}^{n} \mu_k x_k + K \sum_{k=1}^{n}\sum_{j=1}^{n} x_k c_{k,\,j} x_j] \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1, \ldots, n,\, K \ge 0. \tag{4}$$

The Markowitz approach to the portfolio problem (3) have started the general approach to multi–objective stochastic optimization problems, known (from the literature) as scalarizing. (This approach is well known also from the deterministic multi–objective problems theory.)

2 Scalarizing

To explain this approach we start with more general multi–objective problem with random elements and suppose: the decision vector x has to be determined without knowledge of the random element realization ξ. If it is reasonable to determine the decision with respect to the mathematical expectation of the objectives; the constraints set can be included in "deterministic" constraints depending on P_F, then the corresponding multi–objective stochastic problem can be introduced in the form:

$$\text{Find} \quad \min \mathsf{E}_F g_i(x, \xi), \; i = 1, \ldots, l \quad \text{s.t.} \quad x \in X_F. \tag{5}$$

Remark 2.

a. We assume that the final mathematical expectation $\mathsf{E}_F g_i(x,\, \xi)$, $\mathsf{E}_F g_j^*(x,\, \xi)$, $i1, \ldots, l$, $j = 1, \ldots, l'$ exist for all $x \in X$.

b. The mathematical expectation in (5) can be replaced by another functional (see, e.g., the relation (4)); we denote it generally by a symbol $\mathcal{F}$.

c. Considering (1) with g_j^* omitted, the following problem in [4] is considered:

$$\text{Find} \quad \min_{x,\,u}(u_1, \ldots, u_l) \quad \text{s.t.} \quad P_F\{g_i(x, \xi) \le u_i\} \ge \beta_i, \quad \beta_i \in \langle 0,\, 1\rangle, \quad i = 1, \ldots, l, \quad x \in X.$$

Simultaneously, a comparison with problems mean, mean − variance, mean − standard deviation is introduced in [4].

A few types $X_F \subset R^n$ are known from the stochastic programming literature, they are "deterministic" ($X_F = X$); given by mathematical expectation $\mathsf{E}_F g_j^*$, $j = 1, \ldots, l'$; determined by probability constraints (for more details see see, e.g., [1]); recently rather often stochastic dominance constraints appear. The results obtained for deterministic multi–objective problems enable to apply these cases.

2.1 Deterministic Problems

To recall the suitable results obtained for deterministic problems, let $f_i(x)$, $i = 1, \ldots, l$ be real–valued functions defined on R^n; $\mathcal{K} \subset R^n$ be a nonempty set. The multi–objective problem can be defined by:

$$\text{Find} \quad \min f_i(x),\, i = 1, \ldots, l \quad \text{subject to} \quad x \in \mathcal{K}. \tag{6}$$

Definition 1. The vector x^* is an efficient solution of the problem (6) if and only if there exists no $x \in \mathcal{K}$ such that $f_i(x) \le f_i(x^*)$ for $i = 1, \ldots, l$ and such that for at least one i_0 one has $f_{i_0}(x) < f_{i_0}(x^*)$.

Definition 2. The vector x^* is a properly efficient solution of the multi–objective optimization problem (6) if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $x \in \mathcal{K}$ satisfying $f_i(x) < f_i(x^*)$ there exists at least one j such that $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \le M. \tag{7}$$

Proposition 1. [7] Let $\mathcal{K} \subset R^n$ be a nonempty convex set and let $f_i(x)$, $i = 1, \ldots, l$ be convex functions on $\mathcal{K}$. Then $x^0 \in \mathcal{K}$ is a properly efficient solution of the problem (6) if and only if x^0 is optimal in

$$\min_{x \in \mathcal{K}} \sum_{i=1}^{l} \lambda_i f_i(x) \quad \text{for some} \quad \lambda_1, \ldots, \lambda_l > 0, \quad \sum_{i=1}^{l} \lambda_i = 1.$$

A relationship between efficient and properly efficient points is introduced, e.g., in [6] or [7]. We summarize it in the following Remark.

Remark 3. Let $f(x) = (f_1(x), \ldots, f_l(x))$, $x \in \mathcal{K}$; $\mathcal{K}^{eff}$, $\mathcal{K}^{peff}$ be sets of efficient and properly efficient points of the problem (6). If $\mathcal{K}$ is a convex set, $f_i(x)$, $i = 1, \ldots, l$ are convex functions on $\mathcal{K}$, then

$$f(\mathcal{K}^{peff}) \subset f(\mathcal{K}^{eff}) \subset \bar{f}(\mathcal{K}^{peff})$$

($\bar{f}(\mathcal{K}^{peff})$ denotes the closure set of $f(\mathcal{K}^{peff})$.

2.2 Multi–Objective Stochastic Optimization Problems

Setting

$$f_i(x) = \mathsf{E}_F g_i(x, \xi),\, i = 1, \ldots, l, \quad \mathcal{K} = X_F, \tag{8}$$

then evidently, under assumptions of convex functions $g_i(x, \xi)$, $i = 1, \ldots, l$ on convex, nonempty set X_F, we can (employing Proposition 1) to obtain the set of properly efficient points of the problem (5). According to Remark 2 this set approximate the set of efficient points of (5). Consequently it is suitable, first, to suppose X be nonempty convex set and to analyze properties of the sets type X_F separately:

1. $X_F = X$. In this case $X_F = X$ is a convex set.
2. $X_F = \{x \in X : \mathsf{E}_F g_j^*(x, \xi) \le 0,\ j = 1, \ldots, l'\}$. Evidently if $g_j^*(x, \xi)$, $j = 1, \ldots, l'$ are convex functions on convex nonempty set X, then X_F is a convex set.
3. X_F given by individual probability constraints. In particular, in this case we assume $l' = s$, there exist functions $\bar{g}_j(x)$, $j = 1, \ldots, s$ defined for $x \in X$ such that $g_j^*(x, z) = \bar{g}_j(x) - z_j$, $j = 1, \ldots, l$ and
$$X_F := X_F(\alpha) := \bigcap_{j=1}^{s} \{x \in X :\, P[\omega : \bar{g}_j(x) \le \xi_j] \ge \alpha_j\},$$

where $\alpha_j \in (0, 1)$, $j = 1, \ldots, s$, $\alpha = (\alpha_1, \ldots, \alpha_s)$, $z = (z_1, \ldots, z_s)$.

If $\bar{g}_j(x)$, $j = 1, \ldots, s$ are convex functions, P_F absolutely continuous with respect to the Lebesque measure on R^s, then X_F is a convex set (for more details see, e.g., [9]).

The situation with general probability constraints is rather complicated; see, e.g., [16].

4. To deal with the last case, let $l' = 1$, $g_1^*(x, \xi)$ be a real-valued function, $Y(\xi)$ a random value. If for $x \in X$ there exists finite $\mathsf{E}_F g_1^*(x, \xi)$, $\mathsf{E}_F Y(\xi)$ and if

$$F^2_{g_1^*(x,\,\xi)}(u) = \int\limits_{-\infty}^{u} F_{g_1^*(x,\,\xi)}(y)dy, \quad F^2_Y(u) = \int\limits_{-\infty}^{u} F_Y(y)dy, \quad u \in R^1,$$

then we can define the second order stochastic dominance constraints X_F by

$$X_F = \{x \in X : F^2_{g_1^*(x,\,\xi)}(u) \le F^2_Y(u) \quad \text{for every} \quad u \in R^1\}. \tag{9}$$

Employing the results [14], the stochastic second order dominance constraints (9) can be rewritten in a more friendly form:

$$X_F = \{x \in X : \mathsf{E}_F(u - g_1^*(x, \xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in R^1\}. \tag{10}$$

If $g_1^*(x, \xi)$ is a concave function on X, then X_F is a convex set.

Since for the optimization problem given by (8), (10) the Slater's condition is not fulfilled generally, it is necessary to relax constraints set (for more details see, e.g., [2], [11]).

The multi–objective problem given by (8) (with X_F fulfilling first and third case, $g_i(x, \xi)$, $i = 1, \ldots, l$ be strongly convex function) has been investigated in [10], where also the definition of a strongly convex function can be found. However, [10] is mainly focus on the case when the theoretical measure P_F is replaced by empirical one given by independent random sample. To obtained these results the stability with respect to the Wasserstein metric has been there investigated.

The special case of the functions f_i has been considered in [5]. In particular, there were considered the following multi–objective two-stage stochastic problem:

$$\begin{aligned} &\text{Find} \quad \min f_i(x) \; = \; g_i'(x) + \mathsf{E}_F \min q'y, \; i = 1, \ldots, l \\ &\text{s.t.} \quad\;\; Ax = b, \\ &\qquad\quad\;\; Dx + Wy = \xi, \quad x \ge 0, \quad y \ge 0. \end{aligned}$$

A, D, W, b, q are deterministic matrix of the corresponding dimensions, g_i', $i = 1, \ldots, l$ suppose to be linear deterministic.

The stability of the last problem considered with respect P_F and based on the bounded Lipschitz metric, has been investigated in [5]. We recall this work because it has been first one dealing with the stability of multi–objective stochastic problem.

Remark 4. An idea of scalarization has been also employed in [8]. However there the approach is combined with utility function approach. Consequently, there the linear dependence objectives on the probability measure is a very suitable property. In [10] this property can be replaced by more general assumptions.

3 Multi–Objective Stochastic Objectives via Stochastic Dominance

The relations (9), (10) recalled the constraints set given by the second order stochastic dominance. Considering in (1) the case $l = 1$ and g_1^*, $j = 1, \ldots, l'$ being omitted, we can evaluate the solution x by the stochastic dominance. To this end, first, we generalize the problem (3) and consider the problem:

$$\text{Find} \quad \max \mathsf{E}_F g_1(x, \xi), \quad \min \rho(g_1(x, \xi)) \quad \text{s.t.} \quad x \in X, \quad \rho(\cdot) \quad \text{denotes a risk measure.} \tag{11}$$

(11) is two objectives optimization problem. Evidently to optimize simultaneously both objectives is very often impossible (see also a comment to the problem (3)). Following the approach of Markowitz we can obtain the problem:

$$\text{Find} \quad \max\{(1-\lambda)\mathsf{E}_F[g_1(x,\,\xi)] - \lambda\rho(g_1(x,\,\xi)\} \quad \text{s.t.} \quad x \in X; \quad \lambda \in \langle 0,\,1\rangle. \tag{12}$$

To recall examples of the risk measure ρ considered in [8] we set $V := g_1(x,\,\xi)$. They are

1. variance – $\rho(V)(:= \mathrm{var}(V)) = \mathsf{E}_F[V - \mathsf{E}_F V]^2$,
2. absolute semi–deviation – $\rho(V)\,(:= \bar{\delta}(V)) \;=\; \mathsf{E}_F[\max\,(\mathsf{E_F}[V] - V),\,0]$,
3. the standard semi–deviation – $\rho(V)\,(:= \;(\delta(V)) \;=\; (\mathsf{E}_F[(\max(\mathsf{E}_F[V] - V,\,0))^2])^{1/2}$,
4. $\rho(V)\,(:= \text{Average Value} - \text{at} - \text{Risk}) = AV@R_\alpha(V)$ for some fixed $\alpha \in [0,\,1]$.
   (For the definition of $AV@R_\alpha(V)$ see, e.g., [8]), [19].)

It is well known that the second order stochastic dominance corresponds to order in the space of non-negative nondecreasing concave utility functions. If we denote by the symbol $\succeq_2$ second order stochastic dominance, then the following definition can be found in [8], [15].

Definition 3. The mean–risk model (11) is called consistent with the second order ($\succeq_2$) stochastic dominance if for every $x \in X$ and $y \in X$,

$$g_1(x,\,\xi) \succeq_2 g_1(y,\,\xi) \quad\Longrightarrow\quad \mathsf{E}g_1(x,\,\xi) \geq \mathsf{E}g_1(y,\,\xi) \quad \text{and} \quad \rho(g_1(x,\,\xi)) \leq \rho(g_1(y,\,\xi)).$$

It has been recalled in [8] that the mean–risk model using Average Value–at–Risk at some level α is consistent with second order stochastic dominance relation.

Remark 5. The Markowitz mean-variance model is not consistent with second order stochastic dominance relation ($\succeq_2$), so it is not perfectly suited as a decision aid for rational, risk averse decision marker. In the case of absolute semi–deviation and standard semi–deviation the situation is a little bit more complicated.

Till now we have employed univariate second order stochastic dominance order. However, there are known also (from the literature) the definitions of multivariate stochastic orders. Evidently they are defined on sets of vector random variables. There are known approaches those generalize univariate stochastic order to multivariate case. To recall them let $\mathbf{X}$, $\mathbf{Y}$ be two m–dimensional vectors with components $(X_1,\,\ldots,\,X_m)$, $(Y_1,\,\ldots,\,Y_m)$.

Definition 4. $\mathbf{X}$ dominates $\mathbf{Y}$ in sense of components in second order if $X_j \succeq_2 Y_j$ for every $j = 1,\,\ldots,\,m$.

Definition 5. $\mathbf{X}$ dominates $\mathbf{Y}$ in sense of positive linear second order if $\mathbf{a^T X} \succeq_2 \mathbf{a^T Y}$ for all $\mathbf{a} \geq 0$.

The others generalized models of the multivariate stochastic dominance have appeared in the last time. We recall a work [17] devoted very carefully to this topic.

Applications of multivariate stochastic orders to multi–objective stochastic programming problems can be found in [8].

4 Special Approach

At the end we mention one special approach in which an objective and constraints are determined by one decision parameter (see [12]). The problem can be introduced in the form:

$$\begin{aligned} &\text{Find} && \max q^T x + \eta(p) \\ &\text{s.t.} && Ax \geq b, \\ & && P\{Tx \leq d\} \geq p, \quad p \geq \underline{\mathrm{p}}, \quad 0 \leq x \leq u. \end{aligned} \tag{13}$$

$T(:= T(t_{i,j}))$ is a matrix with rows $T_1^T, \ldots . T_s^T$ discretely distributed random vectors not necessary independent), moreover, each component of $T_i = t_{i,j}\xi_j$, where $t_{i,j}$ is a scalar and ξ_j random variable; q, A, b, u, d, $\underline{\mathrm{p}}$ are given deterministic with suitable dimension; $\eta(p)$ is monotone increasing function of p. Evidently the problem (13) can find applications in production planing; $q^T x$ can be considered as a profit of production, $\underline{\mathrm{p}}$ can be interpreted as the lowest acceptable reliability level of quality control process or the ready rate service level provided to customers.

5 Conclusion

We have tried to give a brief survey of the approaches to multi–objective problems with a random element. These problems arise in applications, some of them can be found in [3], [10], [18]. However, to deal with them is beyond of the scop of this contribution .

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