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Bi-cooperative games in bipolar fuzzy settings

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ABSTRACT

In this paper, we introduce the notion of a bi-cooperative game with *Bipolar Fuzzy Bi-coalitions* and discuss the related properties. In many decision-making situations, players show bipolar motives while cooperating among themselves. This is modelled in both crisp and fuzzy environments. Bi-cooperative games with fuzzy bi-coalitions have already been proposed under the product order of bi-coalitions where one had memberships in $[0, 1]$. In the present paper, we adopt the alternative ordering: ordering by monotonicity and account for players' memberships in $[-1, 1]$, a break from the previous formulation. This simplifies the model to a great extent. The corresponding Shapley axioms are proposed. An explicit form of the Shapley value to a particular class of such games is also obtained. Our study is supplemented with an illustrative example.

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1. Introduction

In this paper, we introduce the notion of bi-cooperative games with bipolar fuzzy bi-coalitions (BFB).[1] Since its inception by von Neumann and Morgenstern (1944) cooperative game theory has been playing a pivotal role in decision-making situations where a group of people(players) join their hands to get more than what they would have accrued individually. However, it may happen that a second group of players opposes the formation of the first group and the remaining players abstain from taking any alignment to either of the two groups. Such issues usually occur in different socio-economic situations, viz. passing a bill in the House of Legislature, digging a canal across the fields that shares water among its land owners, etc. (see Bilbao et al. 2008; Labreuche and Grabisch 2008). Such situations are modelled using bi-cooperative games techniques in the literature. Mathematically, given a finite set N of n players, and $\mathcal{Q}(N)$, the set of all pairs (S, T) with $S, T \subseteq N$ and $S \cap T = \emptyset$, a bi-cooperative game is defined by a function $v : \mathcal{Q}(N) \to \mathbb{R}$ such that $v(\emptyset, \emptyset) = 0$. For each $(S, T) \in \mathcal{Q}(N)$, $v(S, T)$ represents the worth of the set S containing the players that support an issue with members of T in opposition while the players in $N \setminus (S \cup T)$ remain indifferent. The pair $(S, T) \in \mathcal{Q}(N)$ is called a bi-coalition. Let us denote the cardinalities of sets S, T, etc. by the respective small letters s, t, etc. Felsenthal and Machover (1997) introduced the notion of bipolarity in ternary voting

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games Bilbao (2000) further extended this idea and introduced a bi-cooperative game. It is interesting enough that $\mathcal{Q}(N)$ is a finite distributive lattice under the order $\sqsubseteq$ given by $(S, T) \sqsubseteq (A, B)$ iff $S \subseteq A$ and $T \supseteq B$. An alternative order $\sqsubseteq_1$ is also found in the literature in which $(S, T) \sqsubseteq_1 (A, B)$ if and only if $S \subseteq A$ and $T \subseteq B$ for every $S, T, A, B \in \mathcal{Q}(N)$. Under this approach $\mathcal{Q}(N)$ is an inf-semi lattice, see Bilbao (2000), Bilbao et al. (2008), Grabisch and Labreuche (2005a, 2005b, 2008), Fujimoto and Murofushi (2005), Labreuche and Grabisch (2008), etc. for more details.

A solution is a function, which assigns to every cooperative game an n-dimensional real vector. The ith component of the vector represents the payoff to the ith player. It is an assessment of the players on their cooperative endeavours. Similarly a solution in a bi-cooperative game evaluates the role changing power of a player between supporting and opposing groups. Bilbao et al. (2008) defined the Shapley value for the class of bi-cooperative games. In their approach, it is assumed that the maximum bi-coalition $(N, \emptyset)$ evolves after some sequential process from the minimum bi-coalition $(\emptyset, N)$. Labreuche and Grabisch (2008) introduced a value for crisp bi-cooperative games which we call the LG value. It is specific to a particular bi-coalition and given by the Shapley value of a suitably selected associate cooperative game.

Choquet integral (Choquet 1995) as a generalization of the weighted arithmetic mean is efficiently used in decision-making problems. In Grabisch and Labreuche (2005b) it is shown that Choquet integral is the best linear interpolator in a binary situation. Bipolarity is a common phenomenon in decision-making problems where the scale of the scores (bipolar scale) goes from negative to positive values or conversely. In Grabisch and Labreuche (2005b), bipolar Choquet integrals are introduced to incorporate such bipolarity in the aggregation process.

In crisp cooperative game the membership of a player is either 1(for full participation) or 0(for no participation). But there arise real life situations where it is not possible for a player to provide full participation in the coalition. This may be the case when a player involves in more than one project simultaneously. We assume that she provides only a partial participation in the coalition that ranges in $[0, 1]$. When players join a coalition partially we call it a fuzzy coalition. Similarly under the bi-cooperative set-up when the players in a bi-coalition participate partially in each of the opposite roles we call it a fuzzy bi-coalition. Bi-cooperative games with fuzzy bi-coalitions under this set-up is discussed in Borkotokey and Sarmah (2012). A set of axioms for characterizing the LG value is proposed. However, the Shapley value due to Bilbao et al. (2008) is not extended in fuzzy environment so far. Moreover, players with opposite polarities have never been represented by a single membership function that ranges in $[-1, 1]$ which would otherwise simplify the mathematical model to a great extent.

In this paper, we introduce the notion of a bi-cooperative game with bipolar fuzzy bi-coalitions (BFB in short). As the name suggests, we assume that the participation level of each player in a bi-coalition ranges in $[-1, 1]$. This means that we treat in a single membership function the positive and negative contributions of a player. The benefit of taking a single membership function for players in both positive and negative roles is that it simplifies the mathematical formulation to a great extent. Moreover, we will show that this representation is more general than the one proposed in Borkotokey and Sarmah (2012). To the best of our knowledge such approach has not yet been adopted in the literature. The Shapley value as a possible solution concept to a bi-cooperative game with BFB is proposed.

Furthermore, a class of bi-cooperative games with BFB in *Bipolar Choquet Integral* form is proposed. An explicit form of the Shapley value is found for this class of games and an illustrative example is provided to show the robustness of our findings.

The rest of the paper is organized as follows. In Section 2, we discuss the preliminary notions of bi-cooperative games and a corresponding solution concept under crisp settings. Section 3 presents the main results of the paper pertaining to bi-cooperative games with BFB. A class of bi-cooperative games in *Bipolar Choquet Integral* form is proposed in Section 4 followed by an illustrative example in Section 5 and finally some concluding remarks are added.

2. Preliminaries

In this section, we discuss the basic definitions related to bi-cooperative games, results of bi-cooperative games and some aspects of fuzzy sets. To a large extent this section is based on Bilbao et al. (2008). Throughout the paper $N = \{1, 2, \ldots, n\}$ denotes the player set and the set of bi-coalitions of N is given by $\mathcal{Q}(N) = \{(S, T) \mid S,\ T \in N; S \cap T = \emptyset\}$. We alternatively use i for the singleton set $\{i\}$. Denote by small letters the cardinalities of sets, e.g. s for S. Note that $\mathcal{Q}(N)$ is a distributive lattice under the maximum and minimum operators namely $\vee$ and $\wedge$, respectively. A bi-cooperative game is a pair (N, v) where N is the set of players and $v : \mathcal{Q}(N) \to \mathbb{R}$ is such that $v(\emptyset, \emptyset) = 0$. Let $\mathcal{BG}^N$ denote the class of bi-cooperative games with N players. Given $(\emptyset, \emptyset) \neq (S, T) \in \mathcal{Q}(N)$, the identity game $\delta_{(S,T)} : \mathcal{Q}(N) \to \mathbb{R}$ is defined by:

$$\delta_{(S,T)}(A, B) = \begin{cases} 1 & \text{if } (A, B) = (S, T) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that the set $\{\delta_{(S,T)} \mid (S, T) \in \mathcal{Q}(N)\}$ is the standard basis for $\mathcal{BG}^N$. A special bi-cooperative game, namely the bicapacity due to Grabisch and Labreuche (2005a) is of importance to the development of our model and is defined as follows.

Definition 1: A bicapacity is a function $\mu_b \colon \mathcal{Q}(N) \to [-1, 1]$ such that,

(i) for all $A \subseteq C \subseteq N$ and $D \subseteq B \subseteq N$ such that $(A, B), (C, D) \in \mathcal{Q}(N)$, $\mu_b(A, B) \leq \mu_b(C, D)$.
(ii) $\mu_b(\emptyset, \emptyset) = 0$.
(iii) $\mu_b(N, \emptyset) = 1$ and $\mu_b(\emptyset, N) = -1$.

In the above *(i)* is just the monotonicity of μ_b with respect to the partial ordering on $\mathcal{Q}(N)$, and *(ii)* and *(iii)* are the boundary conditions for bicapacities.

Following Definition 1, the *Bipolar Choquet Integral* (Ch_b) of $\mathbf{x} = (x_1, \ldots, x_n) \in (\alpha, \beta)^N$ ($(\alpha, \beta) \subseteq R$) with respect to the bicapacity μ_b is uniquely defined as,

$$Ch_b(\mathbf{x}, \mu_b) = \sum_{i=1}^{n} \left(\left|x_{[i]}\right| - \left|x_{[i-1]}\right| \right) \mu_b(A^+_{[i]}, A^-_{[i]})$$

where $[.]$ indicates a permutation of N such that $\left|x_{[1]}\right| \leq \cdots \leq \left|x_{[n]}\right|$, $\left|x_{[0]}\right| = 0$ and $A^+_{[i]} = \{j \in N \colon x_j \geq |x_i|\}$, $A^-_{[i]} = \{j \in N \colon x_j < 0, -x_j \geq |x_i|\}$.

In what follows next we define the Shapley value as a suitable one point solution concept for $\mathcal{BG}^N$. As an *a priori* requirement, following definitions are presented.

Definition 2: A player $i \in N$ is a dummy player in $v \in \mathcal{BG}^N$ for $(S,T) \in \mathcal{Q}(N \setminus i)$, if it holds

$$v(S \cup i, T) - v(S,T) = v(\{i\}, \emptyset)$$
$$v(S,T) - v(S, T \cup i) = -v(\emptyset, \{i\})$$

Note that the notion of dummy player in Def. 2 is specific to a particular bi-coalition (S,T). Thus, it is a weaker version of the dummy player given in Bilbao et al. (2008).

Definition 3: A function $\Phi' : \mathcal{BG}^N \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$ defines the Shapley value if it satisfies the following axioms.

Axiom b1 (Efficiency): If $v \in \mathcal{BG}^N$ and for every $(S,T) \in \mathcal{Q}(N)$, it holds that,

$$\sum_{i \in S \cup T} \Phi'_i(v)(S,T) = v(S \cup T, \emptyset) - v(\emptyset, S \cup T)$$

Axiom b2 (Linearity): For all $\alpha, \beta \in \mathbb{R}$, (N,w), $(N,w') \in \mathcal{BG}^N$ and for every $(S,T) \in \mathcal{Q}(N)$

$$\Phi'_i(N, \alpha w + \beta w')(S,T) = \alpha\ \Phi'_i(N,w)(S,T) + \beta\ \Phi'_i(N,w')(S,T).$$

Axiom b3 (Dummy): If player $i \in N$ is dummy in $v \in \mathcal{BG}^N$ for $(\emptyset, \emptyset) \neq (S,T) \in \mathcal{Q}(N \setminus i)$, then

$$\Phi'_i(v)(S,T) = v(\{i\}, \emptyset) - v(\emptyset, \{i\}).$$

Axiom b4 (Symmetry): If $(N,v) \in \mathcal{BG}^N$ and a permutation π is defined on N, then it holds that, for all $i \in N$, for every $(S,T) \in \mathcal{Q}(N)$

$$\Phi'_{\pi i}(N, v \circ \pi^{-1})(S,T) = \Phi'_i(N,v)(S,T)$$

where $\pi v(\pi S, \pi T) = v(S,T)$ and $\pi S = \{\pi i : i \in S\}$.

Axiom b5 (Structural): Let $(S,T) \in \mathcal{Q}(N)$ with $S \neq \emptyset$ and $T \neq \emptyset$ and $s + t \geq 2$, then for every $(S',T') \in \mathcal{Q}((S \cup T) \setminus i)$, $j \in S'$ and $k \in T'$, it holds,

$$\frac{c([(\emptyset, S \cup T), (S' \setminus j, T')])}{c([(\emptyset, S \cup T), (S', T' \cup i)])} = -\frac{\Phi'_j(\delta_{(S',T')})}{\Phi'_i(\delta_{(S',T' \cup i)})}, \frac{c([(S', T' \setminus k), (S \cup T, \emptyset)])}{c([(S' \cup i, T'), (S \cup T, \emptyset)])}$$
$$= -\frac{\Phi'_k(\delta_{(S',T')})}{\Phi'_i(\delta_{(S' \cup i, T')})}$$

where c([(A, B), (C, D)]) denotes the number of maximal chains in the sub lattice $[(A,B),(C,D)]$ given as follows.

$$c([(\emptyset, S \cup T), (S', T' \cup i)]) \frac{(s + t - s' - t' - 1)!}{2^{s'}}$$
$$([(\emptyset, S \cup T), (S' \setminus j, T')]) \frac{(s + t - s' - 1 - t')!}{2^{s'-1}}$$

Note that the Shapley axioms given in Def. 3 differ from the ones given in Bilbao et al. (2008) in the sense that the Shapley value to a player i arising out of these axioms in Bilbao et al. (2008) computes her mean prospect when she moves along the maximal chains of the bi-coalitions whose end points are $(\emptyset, N)$ and $(N, \emptyset)$, passing through some arbitrary bi-coalitions (S, T), $(S \cup i, T)$, $(S, T \cup i)$ and (S, T). The aforementioned Shapley axioms are specific to a particular bi-coalition (S, T). They replace the greatest and the least element in $\mathcal{Q}(N)$ in the corresponding maximal chain by $(S \cup T, \emptyset)$ and $(\emptyset, S \cup T)$. In particular if $N = S \cup T$, then these axioms are identical with their counterparts in Bilbao et al. (2008). Thus our axioms can be seen as $2^n - 1$ Bilbao's axiom settings, each valid on a non-empty subset $H = S \cup T \subseteq N$. This is substantiated with a resulting extension of the model to its bi-polar fuzzy counterpart. Following theorem ensures the existence and the uniqueness of the Shapley value on a specific bi-coalition.

Theorem 1: *For $(S, T) \in \mathcal{Q}(N)$, define a function $\Phi' : \mathcal{BG}^N \to (\mathbb{R}^n)^{\mathcal{Q}(N)}$, by:*

$$\Phi_i^{'}(v)(S,T) = \begin{cases} \displaystyle\sum_{(S',T') \in \mathcal{Q}((S \cup T) \setminus i)} \Big[\bar{p}_{(s',t')}(v(S' \cup i, T') - v(S', T')) \\ \qquad + \underline{p}_{(s',t')}(v(S', T') - v(S', T' \cup i)) \Big] & \text{if } i \in S \cup T \\ 0 & \text{otherwise} \end{cases}$$

where,

$$\bar{p}_{(s',t')} = \frac{(s+t+s'-t')!(s+t+t'-s'-1)!}{2(s+t)!} 2^{s+t-s'-t'} \tag{2}$$

$$\underline{p}_{(s',t')} = \frac{(s+t+t'-s')!(s+t+s'-t'-1)!}{2(s+t)!} 2^{s+t-s'-t'} \tag{3}$$

Then the function Φ' is the unique Shapley function on $\mathcal{BG}^N$.

Proof: The proof is immediate from Bilbao et al. (2008) □

3. Bi-cooperative games with BFB

Definition 4: Let $N = \{1, 2, \ldots, n\}$ be given. A bipolar fuzzy bi-coalition (BFB) A is an expression on N given by:

$$A = \{< i, \mu_A(i) >: i \in N\}$$

where, $\mu_A : N \to [-1, 1]$ represents the membership function representing the rates of participation of the players in A. If no ambiguity arises we simply represent the BFB, A itself as the membership function $A : N \to [-1, 1]$.

Recall that in Borkotokey and Sarmah (2012) a fuzzy bi-coalition A of N is defined by the expression:

$$A = \{< i, \mu_A^N(i), \nu_A^N(i) > \mid i \in N, \min_{i \in N} (\mu_A^N(i), \nu_A^N(i)) = 0\}$$

where, $\mu_A^N : N \to [0, 1]$, $\nu_A^N : N \to [0, 1]$ represent, respectively, the membership functions over N of the fuzzy sets of positive and negative contributors of A. Thus, we have the following analogy between the existing notions of a fuzzy bi-coalition and a bi-polar fuzzy bi-coalition. Every fuzzy bi-coalition $\{< i, \mu_A^N(i), \nu_A^N(i) > \mid i \in N, \min_{i \in N}(\mu_A^N(i), \nu_A^N(i)) = 0\}$ can be represented simply by the bipolar fuzzy bi-coalition $\mu_A^N - \nu_A^N$. Conversely, for every bipolar fuzzy bi-coalition A we have two membership functions over N viz. $\mu : N \to [0,1]$, $\mu \equiv \max(A, 0)$ and $\nu : N \to [0,1]$, $\nu \equiv \max(-A, 0)$, such that $A \equiv \mu - \nu$. This representation, however, is not unique and therefore, the notion of BFB is more general in defining a fuzzy bi-coalition. Let $\mathcal{F}_B(N)$ denote the set of all bipolar fuzzy bi-coalitions.

Remark 1: Since every crisp coalition $S \subseteq N$ can be represented by its characteristic function form as $1_S : N \to \{0,1\}$ where,

$$1_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Definition 4 suggests that every crisp bi-coalition (S, T) with $S \cap T = \emptyset$ can be represented by a single characteristic function $A : N \to \{-1, 0, 1\}$ given by $A(i) = 1_S(i) - 1_T(i)$. Therefore, with an abuse of notations we take $\mathcal{Q}(N) \subseteq \mathcal{F}_B(N)$.

Let us denote by $\vee$ and $\wedge$, respectively, the maximum and minimum operators on real numbers.

A partial order "$\preceq$" on $\mathcal{F}_B(N)$ is defined as follows. For $A, B \in \mathcal{F}_B(N)$ we have,

$$A \preceq B \Leftrightarrow \begin{cases} A(i) \leq B(i) & \forall i \in N, \text{ whenever } B(i) > 0 \\ A(i) \geq B(i) & \forall i \in N, \text{ whenever } A(i) \vee B(i) \leq 0 \end{cases} \tag{5}$$

Moreover, $A = B \Leftrightarrow A(i) = B(i), \ \forall i \in N$. For any $A \in \mathcal{F}_B(N)$, denote by $\mathcal{F}_B(A)$, the set of all BFB's B such that $B \preceq A$.

Note that the ordering introduced in (5) is an extension of the ordering on $\mathcal{Q}(N)$ to $\mathcal{F}_B(N)$ given by Grabisch and Labreuche (2005b). Moreover, the standard ordering on $\mathcal{Q}(N)$ due to intuitionistic sets when extended to $\mathcal{F}_B(N)$ is just $A \leq B$ for bipolar membership functions.

The union of two BFB's A and B is defined as follows.

$$(A \cup B)(i) = \begin{cases} A(i) \vee B(i) & \text{if } A(i) \wedge B(i) > 0 \\ A(i) \wedge B(i) & \text{if } A(i) \vee B(i) \leq 0 \end{cases} \tag{6}$$

The Support of a BFB A, denoted by $Supp(A)$ is given by:

$$Supp(A) = (\{i \in N | A(i) > 0\}, \{j \in N | A(j) < 0\}) \tag{7}$$

Given $A, B \in \mathcal{F}_B(N)$, difference $A \setminus B$ is given as follows.

$$(A \setminus B)(k) = \begin{cases} 0 & \text{if } k \in Supp(B) \\ A(j) & \text{otherwise} \end{cases} \tag{8}$$

The null BFB denoted by $\emptyset$ is defined by $\emptyset(i) = 0 \ \ \forall i \in N$. In the following, we define a bi-cooperative game with BFB .

Definition 5: A bi-cooperative game with BFB is a function $w : \mathcal{F}_B(N) \to \mathbb{R}$ with $w(\emptyset) = 0$, where the value $w(A)$ represents the worth generated due to the partial participations of the players in two opposite roles in A.

Let $\mathcal{G}_{\mathcal{FB}}(N)$ denote the class of all bi-cooperative games with BFB.

Definition 6: Let $A, B \in \mathcal{F}_B(N)$ and $A \neq \emptyset$. The identity game $\delta_A : \mathcal{F}_B(N) \to \mathbb{R}$ is defined by:

$$\delta_A(B) = \begin{cases} 1 & \text{if } B = A \\ 0 & \text{otherwise} \end{cases}$$

Note that when $A \in \mathcal{F}_B(N)$ has all players with memberships in $\{-1, 0, 1\}$, δ_A is the standard identity game in $\mathcal{BG}^N$, see Bilbao et al. (2008), Labreuche and Grabisch (2008), etc.

Let $A \in \mathcal{F}_B(N)$. For any permutation π on N, we define $\pi A \in \mathcal{F}_B(N)$ as follows.

$$\pi A(i) = A(\pi^{-1} i) \tag{9}$$

In order to define the dummy player in bi-polar fuzzy setting, we need the following two special types of BFBs. Let $i \in N$ and $A \in \mathcal{F}_B(N)$ so that $A(i) = 0$. Let $\gamma \in (0, 1]$ be such that $\gamma \geq \left|A(j)\right| \ \forall j \in N$. We define the following bipolar fuzzy sets:

$$A_i^\gamma(j) = \begin{cases} \gamma & j = i \\ A(j) & j \neq i \end{cases} \tag{10}$$

$$I_i^\gamma(j) = \begin{cases} \gamma & j = i \\ 0 & j \neq i \end{cases} \tag{11}$$

Definition 7: Given A and γ as above, player $i \in N$ is an f^γ-dummy player in $w \in \mathcal{G}_{\mathcal{FB}}(N)$ for A, if it holds that,

$$w(A_i^\gamma) - w(A) = w(I_i^\gamma) \tag{12}$$

$$w(A) - w(A_i^{-\gamma}) = -w(I_i^{-\gamma}) \tag{13}$$

Note that the f^γ-dummy player is one who cannot contribute to the coalition value further if her membership exceeds a certain rate γ either in a positive or a negative role. Moreover, when restricted to $\mathcal{BG}^N$ the f^γ-dummy player becomes synonymous with a crisp dummy player.

Given $A \in \mathcal{F}_B(N)$, define the BFB's A^+ and A^- as follows. $A^+(i) = |A(i)|$ and $A^-(i) = -|A(i)|$ for all $i \in N$. In what follows, we define the solution concept of bi-cooperative games with BFB following the approach of Bilbao et al. (2008).

Definition 8: A function $\Phi : \mathcal{G}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{F}_B(N)}$ is said to be a Shapley value on $\mathcal{G}_{\mathcal{FB}}(N)$ if it satisfies the following five axioms:

Axiom $f1$ (Efficiency): If $w \in \mathcal{G}_{\mathcal{FB}}(N)$ and $\emptyset \neq A \in \mathcal{F}_B(N)$, then

$$\sum_{i \in Supp(A)} \Phi_i(w)(A) = w(A^+) - w(A^-).$$

Axiom $f2$ (Linearity): For $\alpha, \beta \in \mathbb{R}$ and $w, w' \in \mathcal{G}_{\mathcal{FB}}(N)$ we must have

$$\Phi(\alpha w + \beta w') = \alpha\Phi(w) + \beta\Phi(w'),$$

Axiom $f3$ (f^γ-Dummy): Given $i \in N$, $\emptyset \neq A \in \mathcal{F}_B(N)$ such that $A(i) = 0$ and γ such that $\gamma \geq |A(j)|$ for all $j \in N$, if player i is f^γ-dummy in $w \in \mathcal{G}_{\mathcal{FB}}(N)$ for A, then

$$\Phi_i(w)(A) = w(I_i^\gamma) - w(I_i^{-\gamma}) \tag{14}$$

Axiom $f4$ (Symmetry): For any $w \in \mathcal{G}_{\mathcal{FB}}(N)$, a BFB $A \neq \emptyset$, and a permutation π defined on N such that $\pi A = A$, it holds for all $i \in N$ that,

$$\Phi_i(w)(A) = \Phi_{\pi i}(\pi w)(\pi A) \tag{15}$$

where $\pi w \in \mathcal{G}_{\mathcal{FB}}(N)$ is defined by $\pi w(\pi A) = w(A)$, with πA defined in Definition 9.

Axiom $f5$ (Structural): Let $i, j, k \in N$, $\emptyset \neq A \in \mathcal{F}_B(N)$ and $\emptyset \neq B \in \mathcal{F}_B(A)$ such that $i \notin Supp(B)$, $B(j) > 0$ and $B(k) < 0$ and for each $\gamma > 0$, it holds that

$$\frac{c(Supp(A^-), Supp(B \setminus I_j^\gamma))}{c(Supp(A^-), Supp(B \cup I_i^{-\gamma}))} = -\frac{\Phi_j(\delta_B)(A)}{\Phi_i(\delta_{(B \cup I_i^{-\gamma})})(A)},$$
$$\frac{c(Supp(B \setminus I_k^{-\gamma}), Supp(A^+))}{c(Supp(B \cup I_i^\gamma), Supp(A^+))} = -\frac{\Phi_k(\delta_B)(A)}{\Phi_i(\delta_{(B \cup I_i^\gamma)})(A)}$$

Note that the structural axiom follows exactly the same principle as that of its crisp counterpart. The only exception here is that we consider the lattice of bi-coalitions formed by the players even if they have partial memberships in it whereas in the crisp case we consider only players with ternary memberships viz. 1, 0 or -1. Thus, this axiom also says that beginning from the BFB A^- the probability of forming a bi-coalition with player j shifting to the positive membership from no-membership (i.e. from $B \setminus I_j^\gamma$ to B) differs from the probability of forming the same bi-coalition from negative membership (i.e. B from $B \cup I_i^{-\gamma}$), see Bilbao et al. (2008). The payoff to the player accordingly will differ in these two cases and are proportional to the number of steps that all the players require to shift their orientations. Similarly, we can interpret the second expression also.

Remark 2: It is easy to see that if Φ satisfies Axiom $f1 - f5$ then $\Phi|_{\mathcal{BG}^N}$ (restriction of Φ to the class of crisp bi-cooperative games) satisfies Axioms b1-b6. Thus, the crisp value can be recovered from its fuzzy counterpart under restriction of its domain. Indeed, all the above axioms are intuitive of their crisp analogues. Furthermore, the above definition adapts to any class of bi-cooperative games with BFB. Moreover, Axiom $f5$ and Axiom b5 are structurally same however their uses in the characterization of the Shapley value, respectively, in fuzzy and crisp settings are different as can be seen in the following section.

4. Bi-cooperative games with BFB in *Bipolar Choquet Integral* form

Choquet integrals are reasonable means to fuzzify crisp games as the corresponding games are continuous and monotone non-decreasing Tsurumi and Tanino (2001). Here, we propose a new family of bi-cooperative games with BFB and discuss the corresponding properties.

Definition 9: Given $A \in \mathcal{F}_B(N)$, let $Q(A) = \{|A(i)| \ : \ A(i) \neq 0, i \in N\}$ and let $q(A)$ be the cardinality of $Q(A)$. We write the elements of $Q(A)$ in the increasing order as $h_1 < \cdots < h_{q(A)}$ and let $h_0 = 0$. Then corresponding to a given $v \in \mathcal{BG}^N$, a game $w \in \mathcal{G}_{\mathcal{FB}}(N)$ is said to be a bi-cooperative game with BFB in *Bipolar Choquet Integral* form over $\mathcal{F}_B(N)$ if it is given by,

$$w(A) = \sum_{l=1}^{q(A)} (h_l - h_{l-1}) v(A^+_{[l]}, A^-_{[l]}) \tag{16}$$

where for each $l \in N$ we define,

$$A^+_{[l]} = \{j \in N \colon A(j) \geq |A(l)|\} \tag{17}$$
$$A^-_{[l]} = \{j \in N \colon -A(j) \geq |A(l)|\} \tag{18}$$

Denote by $\mathcal{G}^{\text{cb}}_{\mathcal{FB}}(N)$ the class of all bi-cooperative games with BFB in *Bipolar Choquet Integral* form over $\mathcal{BG}^N$. For each $v \in \mathcal{BG}^N$, there always is a $w \in \mathcal{G}^{\text{cb}}_{\mathcal{FB}}(N)$ and we call v the associated game of w.

4.1. A Shapley Function on $\mathcal{G}^{cb}_{\mathcal{FB}}(N)$

Prior to defining a Shapley value we state and prove few important results as follows.

Lemma 1: *Let $i \in N$ and $A \in \mathcal{F}_B(N)$ such that $A(i) = 0$. Suppose that $w \in \mathcal{G}^{cb}_{\mathcal{FB}}(N)$ with $v \in \mathcal{BG}^N$ being the associated bi-cooperative game of w. For $\gamma > 0$, player $i \in N$ is an f^γ-dummy player in w for A iff i is dummy in v for $(A^+_{[l]}, A^-_{[l]})$, $\forall l \in N$ satisfying $\gamma > |A(l)|$.*

***Proof*:** Let $i \in N$ be a dummy player in v for $(A^+_{[l]}, A^-_{[l]})$ $\forall l$ such that $\gamma > |A(l)|$. It follows from the definition of w,

$$w(A^\gamma_i) - w(A) = \sum_{l=1}^{q(A^\gamma_i)} (h_l - h_{l-1}) v([A^\gamma_i]^+_{[l]}, [A^\gamma_i]^-_{[l]}) - \sum_{l=1}^{q(A)} (h_l - h_{l-1}) v(A^+_{[l]}, A^-_{[l]})$$

Since $\gamma > |A(l)|$, it follows that,

$$\begin{aligned} &w(A^\gamma_i) - w(A) \\ &\quad = (h_1 - h_0)[v([A^\gamma_i]^+_{[1]}, [A^\gamma_i]^-_{[1]}) - v(A^+_{[1]}, A^-_{[1]})] + (h_2 - h_1)[v([A^\gamma_i]^+_{[2]}, [A^\gamma_i]^-_{[2]}) \\ &\qquad - v(A^+_{[2]}, A^-_{[2]})] + \cdots + (h_{q(I^\gamma_i)} - h_i) v([A^\gamma_i]^+_{[q(A^\gamma_i)]}, [A^\gamma_i]^-_{[q(A^\gamma_i)]}) \\ &\quad = h_{q(A^\gamma_i)} v([A^\gamma_i]^+_{[q(A^\gamma_i)]}, [A^\gamma_i]^-_{[q(A^\gamma_i)]}) \\ &\quad = h_{q(I^\gamma_i)} v(i, \emptyset) \\ &\quad = w(I^\gamma_i) \end{aligned}$$

Similarly we have, $w(A) - w(A_i^{-\gamma}) = -w(I_i^{-\gamma})$.

Conversely, let $i \in N$ be f^γ-dummy in w for $\emptyset \neq A \in \mathcal{F}_B(N)$ and $\gamma > |A(l)|, \forall l \in N$. Then we have,

$$
\begin{aligned}
& w(A_i^\gamma) - w(A) = w(I_i^\gamma) \\
& \quad \Rightarrow \sum_{l=1}^{q(A_i^\gamma)} (h_l - h_{l-1}) v([A_i^\gamma]_{[l]}^+, [A_i^\gamma]_{[l]}^-) - \sum_{l=1}^{q(A)} (h_l - h_{l-1}) v(A_{[l]}^+, A_{[l]}^-) = h_{q(I_i^\gamma)} v([I_i^\gamma]_{[l]}^+, \emptyset) \\
& \quad \Rightarrow \sum_{l=1}^{q(A)} (h_l - h_{l-1}) \left[v([A_i^\gamma]_{[l]}^+, [A_i^\gamma]_{[l]}^-) - v(A_{[l]}^+, A_{[l]}^-) \right] \\
& \qquad + (h_{q(I_i^\gamma)} - h_{q(A)}) v([A_i^\gamma]_{[q(I_i^\gamma)]}^+, [A_i^\gamma]_{[q(I_i^\gamma)]}^-) = h_{q(I_i^\gamma)} v(i, \emptyset) \\
& \quad \Rightarrow \sum_{l=1}^{q(A)} (h_l - h_{l-1}) \left[v([A_i^\gamma]_{[l]}^+, [A_i^\gamma]_{[l]}^-) - v(A_{[l]}^+, A_{[l]}^-) \right] + (h_{q(I_i^\gamma)} - h_{q(A)}) v(i, \emptyset)) \\
& \quad = h_{q(I_i^\gamma)} v(i, \emptyset) \Rightarrow v([A_i^\gamma]_{[k]}^+, [A_i^\gamma]_{[k]}^-) - v(A_{[k]}^+, A_{[k]}^-) = v(i, \emptyset). \;\; k = 1, 2, \ldots, q(A).
\end{aligned}
$$

Following the fact that $h_l - h_{l-1} > 0$. In a similar manner we can show that,

$$
v(A_{[k]}^+, A_{[k]}^-) - v([A_i^{-\gamma}]_{[k]}^+, [A_i^{-\gamma}]_{[k]}^-) = -v(i, \emptyset)
$$

This completes the proof. □

Lemma 2: *Given $w \in \mathcal{G}_{\mathcal{FB}}^{cb}(N)$ with the associated game $v \in \mathcal{BG}^N$, a permutation π on N, define the permutation game $\pi w \in \mathcal{G}_{\mathcal{FB}}^{cb}(N)$ with BFB of w such that $\pi w(\pi A) = w(A)$. Then,*

$$
\pi w(\pi A) = \sum_{l=1}^{q(\pi A)} (h_l - h_{l-1}) \pi v([\pi A]_{[l]}^+, [\pi A]_{[l]}^-)
$$

Proof: We have $[\pi A]_{[l]}^+ = \pi [A]_{[l]}^+$, $[\pi A]_{[l]}^- = \pi [A]_{[l]}^-$. Moreover, $\mathcal{Q}(N) = \mathcal{Q}(\pi N)$. It follows that,

$$
\begin{aligned}
\pi w(\pi A) &= \pi \sum_{l=1}^{q(\pi A)} (h_l - h_{l-1}) v(\pi [A]_{[l]}^+, \pi [A]_{[l]}^-) \\
&= \pi \sum_{l=1}^{q(\pi A)} (h_l - h_{l-1}) v([\pi A]_{[l]}^+, [\pi A]_{[l]}^-) \\
&= \sum_{l=1}^{q(\pi A)} (h_l - h_{l-1}) \pi v([\pi A]_{[l]}^+, [\pi A]_{[l]}^-)
\end{aligned}
$$

This completes the proof. □

For each $w \in \mathcal{G}^{\text{cb}}_{\mathcal{FB}}(N)$, define the function $\Phi : \mathcal{G}^{\text{cb}}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{F}_B(N)}$ by

$$\Phi_i(w)(A) = \sum_{l=1}^{q(A)} (h_l - h_{l-1})\Phi'_i(v)(A^+_{[l]}, A^-_{[l]}) \tag{19}$$

where $v \in \mathcal{BG}^N$ is the associated game of w and Φ' the corresponding Shapley function defined on $\mathcal{BG}^N$. In the following, we show that Φ given by Equation (19) is the Shapley function on $\mathcal{G}^{\text{cb}}_{\mathcal{FB}}(N)$.

Theorem 2: *The function $\Phi : \mathcal{G}^{cb}_{\mathcal{FB}}(N) \to (\mathbb{R}^n)^{\mathcal{F}_B(N)}$ given by Equation (19) is the Shapley function on $\mathcal{G}^{cb}_{\mathcal{FB}}(N)$.*

***Proof*:** It is enough to show that Φ satisfies Axioms $f1 - f5$.

Axiom $f1$: Let $w \in \mathcal{G}^{\text{cb}}_{\mathcal{FB}}(N)$ and $A \in \mathcal{F}_B(N)$. Since $\sum_{i\in N} \Phi'_i(N, v)(S, T) = v(S \cup T, \emptyset) - v(\emptyset, S \cup T)$ holds for any $l = 1, 2, \ldots, q(A)$, from b1, it follows that,

$$\begin{aligned}
\sum_{i\in N} \Phi_i(w)(A) &= \sum_{i\in N}\sum_{l=1}^{q(A)} (h_l - h_{l-1})\Phi'_i(v)(A^+_{[l]}, A^-_{[l]}) \\
&= \sum_{l=1}^{q(A)} (h_l - h_{l-1}) \sum_{i\in N} \Phi'_i(v)(A^+_{[l]}, A^-_{[l]}) \\
&= \sum_{l=1}^{q(A)} (h_l - h_{l-1})[v(A^+_{[l]} \cup A^-_{[l]}, \emptyset) - v(\emptyset, A^+_{[l]} \cup A^-_{[l]})] \\
&= w(A^+) - w(A^-)
\end{aligned}$$

Axiom $f2$: Follows from linearity of Φ'.

Axiom $f3$: Let $i \in N$ and $A \in \mathcal{F}_B(N)$ such that $A(i) = 0$. Take $\gamma \in (0, 1]$ so that $\gamma \geq |A(j)|$ for all $j \in N$. Let i be f^γ-dummy in $w \in \mathcal{G}_{\mathcal{FB}}(N)$ for A then by Lemma (1) i is a dummy player in v for each subset $(A^+_{[l]}, A^-_{[l]})$, $l \in N$ such that $\gamma \geq |A(l)|$. The result follows immediately from (14).

Axiom 41: Following Lemma (2) and the symmetry of Φ' for $v \in \mathcal{BG}^N$, we have for every $A \in \mathcal{F}_B(N)$,

$$\begin{aligned}
\Phi_{\pi i}(\pi w)(\pi A) &= \sum_{l=1}^{q(\pi A)} (h_l - h_{l-1})\Phi'_{\pi i}(\pi v)([\pi A]^+_{[l]}, [\pi A]^-_{[l]}) \\
&= \sum_{l=1}^{q(\pi A)} (h_l - h_{l-1})\Phi'_{\pi i}(\pi v)(\pi [A]^+_{[l]}, \pi [A]^-_{[l]}) \\
&= \Phi_i(w)(A)
\end{aligned}$$

Axiom $f5$: Let $w \in \mathcal{G}^{\text{cb}}_{\mathcal{FB}}(N)$ and $v \in \mathcal{BG}^N$ be its associated game. Let $\emptyset \neq A \in \mathcal{F}_B(N)$ and $B \in \mathcal{F}_B(A)$ such that $i \notin Supp(B)$, $B(j) > 0$ and $B(k) < 0$

It suffices to prove that for any $\gamma > 0$,

$$\frac{c(Supp(A^-), Supp(B \setminus I_j^{\gamma}))}{c(Supp(A^-), Supp(B \cup I_i^{-\gamma}))} = 2 = -\frac{\Phi_j(\delta_B)(A)}{\Phi_i(\delta_{(B\cup I_i^{-\gamma})})(A)} \tag{20}$$

$$\frac{c(Supp(B \setminus I_k^{-\gamma}), Supp(A^+))}{c(Supp(B \cup I_i^{\gamma}), Supp(A^+))} = 2 = -\frac{\Phi_k(\delta_B)(A)}{\Phi_i(\delta_{(B\cup I_i^{\gamma})})(A)} \tag{21}$$

Here, we prove only (20) as (21) follows from symmetry. Consider,

$$\delta_B(A) = \sum_{l=1}^{q(A)} (h_l - h_{l-1})\delta_{Supp(B)}(A^+_{[l]}, A^-_{[l]}) \tag{22}$$

$$\Phi_j(\delta_B)(A) = \sum_{l=1}^{q(A)} (h_l - h_{l-1})\Phi'_j(\delta_{Supp(B)})(A^+_{[l]}, A^-_{[l]}) \tag{23}$$

Similarly,

$$\delta_{(B\cup I_i^{-\gamma})}(A) = \sum_{l=1}^{q(A)} (h_l - h_{l-1})\delta_{Supp(B\cup I_i^{-\gamma})}(A^+_{[l]}, A^-_{[l]}) \tag{24}$$

$$\Phi_i(\delta_{(B\cup I_i^{-\gamma})})(A) = \sum_{l=1}^{q(A)} (h_l - h_{l-1})\Phi'_i(\delta_{Supp(B\cup I_i^{-\gamma})})(A^+_{[l]}, A^-_{[l]}) \tag{25}$$

It follows from the definition of the Shapley value on $\mathcal{BG}^N$ due to Bilbao et al. (2008),

$$\begin{aligned}\Phi'_j(\delta_{Supp(B)})(A^+_{[l]}, A^-_{[l]}) &= \sum_{(S,T)\in\mathcal{Q}(A^+_{[l]}\cup A^-_{[l]})\setminus j} [\bar{p}_{(s,t)}(\delta_{Supp(B)}(S\cup j, T) - \delta_{Supp(B)}(S,T)) \\ &\quad + \underline{p}_{(s,t)}(\delta_{Supp(B)}(S,T) - \delta_{Supp(B)}(S, T\cup j))]\end{aligned} \tag{26}$$

From $B(j) > 0$ we have $\delta_{Supp(B)}(S,T) = \delta_{Supp(B)}(S, T\cup j) = 0$, for any $(S,T) \in \mathcal{Q}(A^+_{[l]} \cup A^-_{[l]}) \setminus j$. Thus,

$$\begin{aligned}\Phi'_j(\delta_{Supp(B)})(A^+_{[l]}, A^-_{[l]}) &= \sum_{(S,T)\in\mathcal{Q}(A^+_{[l]}\cup A^-_{[l]})\setminus j} \bar{p}_{(s,t)}(\delta_{Supp(B)}(S\cup j, T) \\ &= \bar{p}_{(s,t)}\end{aligned} \tag{27}$$

Furthermore,

$$\begin{aligned}\Phi_i'(\delta_{Supp(B\cup I_i^{-\gamma})})(A^+_{[l]},A^-_{[l]}) \\ = \sum_{(S',T')\in\mathcal{Q}(A^+_{[l]}\cup A^-_{[l]})\setminus i} [\bar{p}_{(s',t')}(\delta_{Supp(B\cup I_i^{-\gamma})}(S'\cup i,T') - \delta_{Supp(B\cup I_i^{-\gamma})}(S',T')) \\ + \underline{p}_{(s',t')}(\delta_{Supp(B\cup I_i^{-\gamma})}(S',T') - \delta_{Supp(B\cup I_i^{-\gamma})}(S',T'\cup i))] \quad (28)\end{aligned}$$

Since $\mu_{(B\cup I_i^{-\gamma})}(i) < 0$ we have $\delta_{Supp(B\cup I_i^{-\gamma})}(S'\cup i,T') = \delta_{Supp(B\cup I_i^{-\gamma})}(S',T') = 0$, for any $(S,T)\in\mathcal{Q}(A^+_{[l]}\cup A^-_{[l]})\setminus i$. Therefore,

$$\begin{aligned}\Phi_i'(\delta_{Supp(B\cup I_i^{-\gamma})})(A^+_{[l]},A^-_{[l]}) &= -\sum_{(S',T')\in\mathcal{Q}(A^+_{[l]}\cup A^-_{[l]})\setminus i} \underline{p}_{(s',t')}\delta_{Supp(B\cup I_i^{-\gamma})}(S',T'\cup i) \\ &= -\underline{p}_{(s',t')} \quad (29)\end{aligned}$$

From (27) and (29) we obtain the following.

$$\frac{\Phi_j'(\delta_{Supp(B)})(A^+_{[l]},A^-_{[l]})}{\Phi_i'(\delta_{Supp(B\cup I_i^{-\gamma})})(A^+_{[l]},A^-_{[l]})} = -\frac{\bar{p}_{(s,t)}}{\underline{p}_{(s',t')}} \quad (30)$$

Since the number of players in S for which $\delta_{Supp(B)}(S\cup j,T) = 1$ is one less than that in S' such that $\delta_{Supp(B\cup I_i^{-\gamma})}(S',T'\cup i)) = 1$ it follows from (2) and (3) that,

$$\frac{\bar{p}_{(s,t)}}{\underline{p}_{(s',t')}} = 2 \quad (31)$$

Therefore,

$$\frac{c(Supp(A^-),Supp(B\setminus I_j^{\gamma}))}{c(Supp(A^-),Supp(B\cup I_i^{-\gamma}))} = -\frac{\Phi_j(\delta_{Supp(B)})(A^+_{[l]},A^-_{[l]})}{\Phi_i(\delta_{Supp(B\cup I_i^{-\gamma})})(A^+_{[l]},A^-_{[l]})} = \frac{\bar{p}_{(s,t)}}{\underline{p}_{(s',t')}} = 2.$$

Uniqueness of Φ follows from the uniqueness of the Shapley value of the associated bi-cooperative game. This completes the proof. □

5. Example

Consider an example involving three players 1, 2 and 3 who represent three political parties. Suppose they participate in a debate on whether a populist bill should be passed or not. The worth of a bi-coalition $(S,T)\in\mathcal{Q}(\{1,2,3\})$ quantifies winning capacity of the members in S. Accepting the bill without amendment would fetch maximum worth, while a rejection has the minimum worth. Any amendments in between result in worths bounded by the maximum and the minimum values. Thus, modelling this situation we define the crisp bi-cooperative game $v:\mathcal{Q}(\{1,2,3\})\to\mathbb{R}$ as shown in Table 1.

Let us assume now that the members of each political party (taken as player here) have only vague ideas about their participation in the voting process. This may be the

Table 1. The bill passing game.

(S,T)	$(\emptyset,\emptyset)$	$(1,\emptyset)$	$(2,\emptyset)$	$(3,\emptyset)$	$(\emptyset,1)$
$v(S,T)$	0	12	9	10	7
(S,T)	$(\emptyset,2)$	$(\emptyset,3)$	$(1,2)$	$(2,1)$	$(1,3)$
$v(S,T)$	8	8	10	8	9
(S,T)	$(3,1)$	$(2,3)$	$(3,2)$	$(2,\{1,3\})$	$(3,\{1,2\})$
$v(S,T)$	9	8	9	7	8
(S,T)	$(1,\{2,3\})$	$(\{2,3\},1)$	$(\{1,3\},2)$	$(\{1,2\},3)$	$(\{1,2,3\},\emptyset)$
$v(S,T)$	8	10	10	10	14
(S,T)	$(\{1,2\},\emptyset)$	$(\{1,3\},\emptyset)$	$(\emptyset,\{1,2\})$	$(\emptyset,\{1,2,3\})$	$(\{2,3\},\emptyset)$
$v(S,T)$	14	13	7	-6	11
(S,T)	$(\emptyset,\{1,3\})$	$(\emptyset,\{2,3\})$	–	–	–
$v(S,T)$	7	7	–	–	–

case when some players envisage more benefits while some other are not so sure about it. Such behaviour of uncertainty is common in decision-making problems. Thus, the voting patterns of the three political parties involve uncertainty that can be well modelled by a bi-cooperative game in fuzzy environment. Let A be a bi-coalition with BFB over N given by:

$$A(1) = -0.1, A(2) = 0.7, A(3) = 0.9$$

A can be interpreted as the bi-coalition with BFB where player 1 (i.e. members in party 1) opposes the bill with membership 0.1 and the remaining two players support with respective memberships 0.7 and 0.9. We compute the worth $w(A)$ of A in Bipolar Choquet Integral form as follows.

$$\begin{aligned} w(A) &= \sum_{l=1}^{q(A)} (h_l - h_{l-1}) v(A^+_{[l]}, A^-_{[l]}) \\ &= (h_1 - h_0) v(A^+_{[1]}, A^-_{[1]}) + (h_2 - h_1) v(A^+_{[2]}, A^-_{[2]}) + (h_3 - h_2) v(A^+_{[3]}, A^-_{[3]}) \\ &= 0.1 \times v((2,3),1) + 0.6 \times v((2,3),\emptyset) + 0.2 \times v(3,\emptyset) \\ &= 0.1 \times 10 + 0.6 \times 11 + 0.2 \times 10 \\ &= 9.6 \end{aligned} \tag{32}$$

Thus, $w(A)$ represents the capacity of winning when uncertainty in memberships of the players are incorporated.

Table 1 shows that individually each player in an opposite role generates the same worth viz. 10. Thus, it is their memberships in a bi-coalition that determines their capabilities of amending the bill. After some calculations we obtain the Shapley value of w for A as $(0.4, 1.08, 2.1)$. Evidently player 3 has the maximum capacity to influence the outcome of the game. This is substantiated by the fact that she provides maximum memberships to A in support of the bill. On the other hand Player 1 is the least influential as she opposes the bill vaguely.

6. Conclusion

In this paper, we have defined a Bi-cooperative game in bipolar fuzzy settings. We have defined a bipolar fuzzy bi-coalition (BFB) where players assume memberships of cooperations from the interval $[-1, 1]$ where a positive membership indicates partial support to an issue and a negative membership indicates partial opposition of the issue. A third group of people who have memberships zero are the absentees. In Borkotokey and Sarmah (2012) a similar model was proposed where the memberships range in $[0, 1]$ and therefore, are represented by ordinary fuzzy sets. We have shown that the BFB model is more general and simple than that of Borkotokey and Sarmah (2012). Moreover, the order relation in Borkotokey and Sarmah (2012) extends that of Labreuche and Grabisch (2008). In the present paper we have proposed an extension of the ordering due to Grabisch and Labreuche (2005b). Note that each member A from $\mathcal{F}_B(N)$ can be represented alternatively in the form (S_α, T_α) for $\alpha \in]0, 1]$, where S_α is the α-cut of the positive part of A and T_α is the α-cut of the negative part of A, i.e. α-cut of A is a pair (S_α, T_α) from $\mathcal{Q}(N)$ and the system of these α-cuts is decreasing in the sense of Grabisch and Labreuche. This alternative look may serve to develop an alternative theory of bi-cooperative games in bipolar fuzzy settings.

Note

1. BFB: Bipolar Fuzzy Bi-coalition.

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