International Journal of Uncertainty,
Fuzziness and Knowledge-Based Systems
Vol. 26, No. 2 (2018) 255–259

DOI: 10.1142/S0218488518500137

On Choquet-Pettis Expectation of Banach-Valued Functions: A Counter Example

Hamzeh Agahi*

Department of Mathematics, Faculty of Basic Science,
Babol Noshirvani University of Technology, Shariati Ave.,
Babol, 47148-71167, Iran
h_agahi@nit.ac.ir

Radko Mesiar

Department of Mathematics and Descriptive Geometry,
Faculty of Civil Engineering, Slovak University of Technology,
SK-810 05 Bratislava, Slovakia
Institute of Information Theory and Automation of the Czech Academy of Sciences,
Pod vodárenskou věži 4, 182 08 Praha 8, Czech Republic
mesiar@math.sk

Received 7 July 2016
Revised 25 October 2017

In probability theory, mathematical expectation of a random variable is very important. Choquet expectation (integral), as a generalization of mathematical expectation, is a powerful tool in various areas, mainly in generalized probability theory and decision theory. In vector spaces, combining Choquet expectation and Pettis integral has led to a challenging and an interesting subject for researchers. In this paper, we indicate and discuss a failure in the previous definition of Choquet-Pettis integral of Banach space-valued functions. To obtain a correct definition of Choquet-Pettis integral, an open problem concerning the linearity of the Choquet integral is stated.

Keywords: Choquet expectation; Pettis integral; vector spaces; Choquet-Pettis integral; generalized probability theory.

1. Introduction

During the last decades the concept of Choquet expectation[1–4] started to be applied in various areas of science.[5–13] Let us recall some basic well-known definitions and notations in generalized probability theory that we will use in this paper.

Let $(\Omega, \mathcal{F})$ be a fixed measurable space.

*Corresponding author.

Definition 1.[4] A set function $\mu : \mathcal{F} \to [0, \infty]$ is called a monotone measure whenever $\mu(\emptyset) = 0$, $\mu(\Omega) > 0$ and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$.

Definition 2. A monotone measure μ is called *finite* if $||\mu|| = \mu(\Omega) < \infty$. μ is said an *additive measure* if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$.

Definition 3. A monotone measure μ is called a *monotone probability* (or *capacity*) if $\|\mu\| = 1$.

Definition 4. A capacity with σ-additivity assumption, i.e., μ is additive and continuous from below, is called a *probability measure*.

The class of real-valued measurable functions is denoted by M and the class of nonnegative real-valued measurable functions is denoted by M^+. Given a real monotone measure space $(\Omega, \mathcal{F}, \mu)$, we denote the elements of Ω by ω and we put $\{f \geq t\} = \{\omega : f(\omega) \geq t\}$ for any $t \in \mathbb{R}$.

Definition 5.[1–3] (I) The Choquet expectation (integral) of $f \in M^+$ with respect to a monotone measure μ on $A \in \mathcal{F}$ is defined by

$$(C) \int_A f d\mu = \int_0^{+\infty} \mu(A \cap \{f \geq t\})\, dt\,,$$

where the right-hand side integral is the Riemann integral. If $(C) \int_A f d\mu < \infty$, then we say that f is Choquet integrable on A with respect to μ. Instead of $(C) \int_\Omega f d\mu$, we will write $(C) \int f d\mu$.
(II) Suppose $||\mu|| < \infty$. The (asymmetric) Choquet integral of $f \in M$ with respect to a real monotone measure μ on $A \in \mathcal{F}$ is defined by

$$(C) \int_A f d\mu = (C) \int_A f^+ d\mu - (C) \int_A f^- d\overline{\mu}\,,$$

where $\overline{\mu}$ is a dual (conjugate) to μ given by $\overline{\mu}(B) = \mu(\Omega) - \mu(\Omega \backslash B)$, $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$.When the right-hand side is $\infty - \infty$, the Choquet integral is not defined. If $(C) \int_A f d\mu$ is finite, then we say that f is Choquet integrable on A with respect to μ.

Remark 1. If $\mu = \mathbf{P}$ is the probability measure, then $(C) \int f d\mu = \mathbb{E}[f]$, where $\mathbb{E}[\cdot]$ means the mathematical expectation with respect to $\mathbf{P}$.

Definition 6.[2] Let $f, g \in M$. We say that f and g are comonotonic if $f(\omega) < f(\omega') \Rightarrow g(\omega) \leq g(\omega')$ for $\omega, \omega' \in \Omega$.

Let X be a real Banach space and X^* its dual.

Definition 7.[12] Let $f : \Omega \to X$ and $g : \Omega \to X$ be weakly measurable. f and g are said to be weakly comonotonic if for each $x^* \in X^*$, $x^* f$ and $x^* g$ are comonotonic.

2. On Choquet-Pettis Integral

In 2014, Park[12] introduced the following concept of Choquet-Pettis integral of Banach space-valued functions (in short, Banach-valued functions). He claimed that the Choquet-Pettis integral is an extension of the Choquet integral for Banach-valued functions and this integral is also a generalization of the Pettis integral, since the Choquet integral and the Lebesgue integral coincide when μ is a classical $\sigma-$additive measure.[12] The concept of Pettis integral, as a generalization of the Lebesgue integral, was introduced by Pettis in 1938.[14]

Definition 8.[12] A function $f : \Omega \to X$ is called Choquet-Pettis integrable if for each $x^* \in X^*$ the function $x^* f$ is Choquet integrable and for every $A \in \mathcal{F}$ there exists $x_A \in X$ such that $x^*(x_A) = (C)\int_A x^* f d\mu$ for all $x^* \in X^*$. The vector x_A is called the Choquet-Pettis integral of f on A and is denoted by $(CP)\int_A f d\mu$.

Park[12] also proved two basic properties of Choquet-Pettis integral, we cite them in Proposition 1.

Proposition 1. *Let $f\colon \Omega \to X$ and $g\colon \Omega \to X$ be Choquet-Pettis integrable. Then*

(1) *af is Choquet-Pettis integrable and*

$$(CP)\int_A af d\mu = a\,(CP)\int_A f d\mu$$

for all $A \in \mathcal{F}$ and $a \geqslant 0$ (positive homogeneity);

(2) *if f and g are weakly comonotonic, then $f+g$ is Choquet-Pettis integrable and*

$$(CP)\int_A (f+g)d\mu = (CP)\int_A f d\mu + (CP)\int_A g d\mu$$

for all $A \in \mathcal{F}$ (comonotonicity).

In this paper, we discuss Choquet-Pettis integral of Banach-valued functions introduced by Park.[12]

3. The Failure of Choquet-Pettis Integral: A Counter Example

Now, we give a short remark about the failure of Choquet-Pettis integral introduced by Park,[12] in general. His Choquet integral is the asymmetric version which is comonotone, but not homogenoeus, only positively homogeneous, on the other side, linear operators from X^* form a homegeneous set, i.e., if x^* is from X^*, then also $-x^*$ is from X^*. But the following example shows that the existence of Choquet-Pettis integral forces, in some sense, the homogeneity, and we are in troubles.

Example 1. Let $X = \mathbb{R}^2$ be Banach space (say, equipped by l_2 norm). Then its dual $X^* = X$, and if $x^* = (a, b)$ and $x = (x_1, x_2)$, we have $x^* x = ax_1 + bx_2$. Consider only Choquet-Pettis integral on a (finite) measure space (Ω, μ) and fix $f : \Omega \longrightarrow X$ and $f(\omega) = (x_1(\omega), x_2(\omega))$ (to avoid problems with measurability, we can consider

Ω to be finite, and we can deal with power set). Obviously, integrability means that for this f, and any subset A of Ω, there is an element $x_A = (\alpha_A, \beta_A)$ such that for each x^* we have

$$x^* x_A = a \cdot \alpha_A + b \cdot \beta_B = (C) \int_A x^* f d\mu = (C) \int_A (a x_1(\omega) + b x_2(\omega))\, d\mu(\omega) \,. \quad (1)$$

If $a = 1$ and $b = 0$, we obtain immediately that $\alpha_A = (C) \int_A x_1 d\mu$. If $a = 0$, $b = 1$, we have $\beta_A = (C) \int_A x_2 d\mu$. But, coming back to equality (1), this means that our Choquet integral w.r.t measure μ is linear when considering functions x_1 and x_2. Thus, when looking for Choquet-Pettis integrable functions, we should restrict ourselves (when μ is fixed) to look for functions x_1, x_2 such that for any real a, b it holds

$$(C) \int_A (a \cdot x_1 + b \cdot x_2) d\mu = a \cdot (C) \int_A x_1 d\mu + b \cdot (C) \int_A x_2 d\mu \,.$$

Of course, if μ is additive, then everything holds (as then we are back by the standard Pettis integral), i.e., any pair of integrable functions x_1 and x_2 can be chosen to define Pettis integrable function f defined by $f(\omega) = (x_1(\omega), x_2(\omega))$, however, in general there may be no Choquet-Pettis integrable function up to constant functions. Take e.g. for our finite space Ω containing at least two elements as μ the strongest capacity, $\mu(B) = 1$ whenever $B \neq \emptyset$, the corresponding Choquet integral is max operator, and thus Choquet-Pettis integrability means the validity, for any real a, b of the equality

$$\max\{a \cdot x_1(\omega) + b \cdot x_2(\omega) | \omega \in \Omega\} = a \cdot \max\{x_1(\omega) | \omega \in \Omega\} + b \cdot \max\{x_2(\omega) | \omega \in \Omega\} \,,$$

which clearly holds only for constant functions x_1 and x_2.

4. Concluding Remarks and an Open Problem

In this paper, we have investigated Choquet-Pettis integral of Banach-valued functions introduced by Park.[12] We have also shown that existence of Choquet-Pettis integral forces, in some sense, the homogeneity, which is a contradiction. It would be possible to deal with the symmetric Choquet integral[15] which is homogeneous, however, then the (comonotone) additivity is lost. Therefore, the definition of the Choquet-Pettis integral is still an open problem for further investigations.

We also propose to consider the following problem:

Open Problem: For which functions f, g, we can ensure that for any $a, b \geq 0$,

$$(C) \int (a \cdot f + b \cdot g)\, d\mu = a \cdot (C) \int f d\mu + b \cdot (C) \int g d\mu ? \quad (2)$$

It is not difficult to check that, due to the positive homogeneity of the (asymmetric) Choquet integral, the equality (2) is equivalent to the additivity equality

$$(C) \int (f + g) d\mu = (C) \int f d\mu + (C) \int g d\mu \,.$$

Moreover, if μ is modular on a set system $\mathcal{A}$ which is closed under unions and intersections, $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for any A, B from $\mathcal{A}$, then (2) holds for any $a, b \geq 0$ and f, g such that $(\{f \geq t\})_{t\in\mathbb{R}} \subset \mathcal{A}$, $(\{g \geq t\})_{t\in\mathbb{R}} \subset \mathcal{A}$. In particular, if μ is additive on $\mathcal{F}$ then (2) holds for any f, g. Similarly, if f and g are comonotonic, then $\mathcal{A} = (\{f \geq t\})_{t\in\mathbb{R}} \cup (\{g \geq t\})_{t\in\mathbb{R}}$ is a chain and hence any μ is modular on $\mathcal{A}$, proving that the comonotonicity of f and g is sufficient for the validity of (2), independently of the considered monotone measure μ.

Acknowledgment

The authors are very grateful to the anonymous reviewers for their insightful comments that have led to an improved version of this paper. The second author was supported by grant APVV-14-0013.

References

1. G. Choquet, Theory of capacities, *Ann. Inst. Fourier* **5** (1954) 131–295.
2. D. Denneberg, *Non-Additive Measure and Integral* (Kluwer Academic Publishers, Dordrecht 1994).
3. E. Pap, *Null-Additive Set Functions* (Kluwer, Dordrecht, 1995).
4. Z. Wang and G. Klir, *Generalized Measure Theory* (Springer Verlag, New York, 2008).
5. R. Mesiar, J. Li and E. Pap, The Choquet integral as Lebesgue integral and related inequalities, *Kybernetika* **46** (2010) 1098–1107.
6. I. Gilboa and D. Schmeidler, Additive representations of non-additive measures and the Choquet integral, *Annals of Operations Research* **52**(1) (1994) 43–65.
7. M. Grabisch and M. Roubens, An axiomatic approach to the concept of interaction among players in cooperative games, *Int. J. Game Theory* **28**(4) (1999) 547–565.
8. I. Kojadinovic, Quadratic distances for capacity and bi-capacity approximation and identification, *4OR — A Quarterly Journal of Operations Research* **5** (2007) 117–142.
9. S. Greco and F. Rindone, The bipolar Choquet integral representation, *Theory and Decision* **77**(1) (2014) 1–29.
10. V. Krätschmer, Robust representation of convex risk measures by probability measures, *Finance and Stochastics* **9**(4) (2005) 597–608.
11. F. Meng and Q. Zhang, Induced continuous Choquet integral operators and their application to group decision making, *Computers & Industrial Engineering* **68** (2014) 42–53.
12. C.-K. Park, Convergence theorems for the Choquet-Pettis integral, *Korean J. Math.* **22**(2) (2014) 383–393.
13. D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica* **57** (1989) 571–587.
14. J. Diestel and J. J. Uhl, Vector measures, *Mathematical Surveys*, Vol. 15 (American Mathematical Society, 1977).
15. J. Sipos, Integral with respect to a pre-measure, *Mathematica Slovaca* **29**(2) (1979) 141–155.