On Choquet-Pettis Expectation of Banach-Valued Functions: 
A Counter Example

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In probability theory, mathematical expectation of a random variable is very important. Choquet expectation (integral), as a generalization of mathematical expectation, is a powerful tool in various areas, mainly in generalized probability theory and decision theory. In vector spaces, combining Choquet expectation and Pettis integral has led to a challenging and an interesting subject for researchers. In this paper, we indicate and discuss a failure in the previous definition of Choquet-Pettis integral of Banach space-valued functions. To obtain a correct definition of Choquet-Pettis integral, an open problem concerning the linearity of the Choquet integral is stated.

Keywords: Choquet expectation; Pettis integral; vector spaces; Choquet-Pettis integral; generalized probability theory.

1. Introduction

During the last decades the concept of Choquet expectation\(^1-4\) started to be applied in various areas of science.\(^5-13\) Let us recall some basic well-known definitions and notations in generalized probability theory that we will use in this paper.

Let \((\Omega, \mathcal{F})\) be a fixed measurable space.

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Definition 1. A set function \( \mu : \mathcal{F} \to [0, \infty] \) is called a monotone measure whenever \( \mu(\emptyset) = 0 \), \( \mu(\Omega) > 0 \) and \( \mu(A) \leq \mu(B) \) whenever \( A \subseteq B \).

Definition 2. A monotone measure \( \mu \) is called finite if \( \|\mu\| = \mu(\Omega) < \infty \). \( \mu \) is said an additive measure if \( \mu(A \cup B) = \mu(A) + \mu(B) \), whenever \( A \cap B = \emptyset \).

Definition 3. A monotone measure \( \mu \) is called a monotone probability (or capacity) if \( \|\mu\| = 1 \).

Definition 4. A capacity with \( \sigma \)-additivity assumption, i.e., \( \mu \) is additive and continuous from below, is called a probability measure.

The class of real-valued measurable functions is denoted by \( M \) and the class of nonnegative real-valued measurable functions is denoted by \( M^+ \). Given a real monotone measure space \( (\Omega, \mathcal{F}, \mu) \), we denote the elements of \( \Omega \) by \( \omega \) and we put \( \{f \geq t\} = \{\omega : f(\omega) \geq t\} \) for any \( t \in \mathbb{R} \).

Definition 5. (I) The Choquet expectation (integral) of \( f \in M^+ \) with respect to a monotone measure \( \mu \) on \( A \in \mathcal{F} \) is defined by

\[
(C) \int_A f d\mu = \int_0^{+\infty} \mu(A \cap \{f \geq t\}) dt,
\]

where the right-hand side integral is the Riemann integral. If \( (C) \int_A f d\mu < \infty \), then we say that \( f \) is Choquet integrable on \( A \) with respect to \( \mu \). Instead of \( (C) \int_\Omega f d\mu \), we will write \( (C) \int f d\mu \).

(II) Suppose \( \|\mu\| < \infty \). The (asymmetric) Choquet integral of \( f \in M \) with respect to a real monotone measure \( \mu \) on \( A \in \mathcal{F} \) is defined by

\[
(C) \int_A f d\mu = (C) \int_A f^+ d\mu - (C) \int_A f^- d\mu,
\]

where \( \overline{\mu} \) is a dual (conjugate) to \( \mu \) given by \( \overline{\mu}(B) = \mu(\Omega) - \mu(\Omega \setminus B) \), \( f^+ = f \lor 0 \) and \( f^- = -(f \land 0) \). When the right-hand side is \( \infty - \infty \), the Choquet integral is not defined. If \( (C) \int_A f d\mu \) is finite, then we say that \( f \) is Choquet integrable on \( A \) with respect to \( \mu \).

Remark 1. If \( \mu = P \) is the probability measure, then \( (C) \int f d\mu = E[f] \), where \( E[\cdot] \) means the mathematical expectation with respect to \( P \).

Definition 6. Let \( f, g \in M \). We say that \( f \) and \( g \) are comonotonic if \( f(\omega) < f(\omega') \Rightarrow g(\omega) \leq g(\omega') \) for \( \omega, \omega' \in \Omega \).

Let \( X \) be a real Banach space and \( X^* \) its dual.

Definition 7. Let \( f : \Omega \to X \) and \( g : \Omega \to X \) be weakly measurable. \( f \) and \( g \) are said to be weakly comonotonic if for each \( x^* \in X^* \), \( x^* f \) and \( x^* g \) are comonotonic.
2. On Choquet-Pettis Integral

In 2014, Park\textsuperscript{12} introduced the following concept of Choquet-Pettis integral of Banach space-valued functions (in short, Banach-valued functions). He claimed that the Choquet-Pettis integral is an extension of the Choquet integral for Banach-valued functions and this integral is also a generalization of the Pettis integral, since the Choquet integral and the Lebesgue integral coincide when $\mu$ is a classical $\sigma-$additive measure.\footnote{The concept of Pettis integral, as a generalization of the Lebesgue integral, was introduced by Pettis in 1938.\textsuperscript{14}} The concept of Pettis integral, as a generalization of the Lebesgue integral, was introduced by Pettis in 1938.\textsuperscript{14}

Definition 8.\textsuperscript{12} A function $f: \Omega \to X$ is called Choquet-Pettis integrable if for each $x^* \in X^*$ the function $x^*f$ is Choquet integrable and for every $A \in \mathcal{F}$ there exists $x_A \in X$ such that $x^*(x_A) = \int_A x^*f d\mu$ for all $x^* \in X^*$. The vector $x_A$ is called the Choquet-Pettis integral of $f$ on $A$ and is denoted by $(CP) \int_A f d\mu$.

Park\textsuperscript{12} also proved two basic properties of Choquet-Pettis integral, we cite them in Proposition 1.

Proposition 1. Let $f: \Omega \to X$ and $g: \Omega \to X$ be Choquet-Pettis integrable. Then

1. $af$ is Choquet-Pettis integrable and

$$\int_A af d\mu = a \int_A f d\mu$$

for all $A \in \mathcal{F}$ and $a \geq 0$ (positive homogeneity);

2. if $f$ and $g$ are weakly comonotonic, then $f + g$ is Choquet-Pettis integrable and

$$\int_A (f + g) d\mu = \int_A f d\mu + \int_A g d\mu$$

for all $A \in \mathcal{F}$ (comonotonicity).

In this paper, we discuss Choquet-Pettis integral of Banach-valued functions introduced by Park.\textsuperscript{12}

3. The Failure of Choquet-Pettis Integral: A Counter Example

Now, we give a short remark about the failure of Choquet-Pettis integral introduced by Park\textsuperscript{12} in general. His Choquet integral is the asymmetric version which is comonotone, but not homogeneous, only positively homogeneous, on the other side, linear operators from $X^*$ form a homogeneous set, i.e., if $x^*$ is from $X^*$, then also $-x^*$ is from $X^*$. But the following example shows that the existence of Choquet-Pettis integral forces, in some sense, the homogeneity, and we are in troubles.

Example 1. Let $X = \mathbb{R}^2$ be Banach space (say, equipped by $l_2$ norm). Then its dual $X^* = X$, and if $x^* = (a, b)$ and $x = (x_1, x_2)$, we have $x^*x = ax_1 + bx_2$. Consider only Choquet-Pettis integral on a (finite) measure space $(\Omega, \mu)$ and fix $f: \Omega \to X$ and $f(\omega) = (x_1(\omega), x_2(\omega))$ (to avoid problems with measurability, we can consider...
Ω to be finite, and we can deal with power set). Obviously, integrability means that for this f, and any subset A of Ω, there is an element x_A = (α_A, β_A) such that for each x^∗ we have

\[ x^* x_A = a \cdot \alpha_A + b \cdot \beta_A = (C) \int_A x^* f d\mu = (C) \int_A (ax_1(\omega) + bx_2(\omega)) d\mu(\omega). \quad (1) \]

If a = 1 and b = 0, we obtain immediately that \( \alpha_A = (C) \int_A x_1 d\mu \). If a = 0, b = 1, we have \( \beta_A = (C) \int_A x_2 d\mu \). But, coming back to equality (1), this means that our Choquet integral w.r.t measure \( \mu \) is linear when considering functions \( x_1 \) and \( x_2 \). Thus, when looking for Choquet-Pettis integrable functions, we should restrict ourselves (when \( \mu \) is fixed) to look for functions \( x_1, x_2 \) such that for any real a, b it holds

\[ (C) \int_A (a \cdot x_1 + b \cdot x_2) d\mu = a \cdot (C) \int_A x_1 d\mu + b \cdot (C) \int_A x_2 d\mu. \]

Of course, if \( \mu \) is additive, then everything holds (as then we are back by the standard Pettis integral), i.e., any pair of integrable functions \( x_1 \) and \( x_2 \) can be chosen to define Pettis integrable function \( f \) defined by \( f(\omega) = (x_1(\omega), x_2(\omega)) \), however, in general there may be no Choquet-Pettis integrable function up to constant functions. Take e.g. for our finite space \( \Omega \) containing at least two elements as the strongest capacity, \( \mu(B) = 1 \) whenever \( B \neq \emptyset \), the corresponding Choquet integral is max operator, and thus Choquet-Pettis integrability means the validity, for any real \( a, b \) of the equality

\[ \max\{a \cdot x_1(\omega) + b \cdot x_2(\omega) | \omega \in \Omega\} = a \cdot \max\{x_1(\omega) | \omega \in \Omega\} \]

\[ + b \cdot \max\{x_2(\omega) | \omega \in \Omega\}, \]

which clearly holds only for constant functions \( x_1 \) and \( x_2 \).

4. Concluding Remarks and an Open Problem

In this paper, we have investigated Choquet-Pettis integral of Banach-valued functions introduced by Park.\textsuperscript{14} We have also shown that existence of Choquet-Pettis integral forces, in some sense, the homogeneity, which is a contradiction. It would be possible to deal with the symmetric Choquet integral\textsuperscript{15} which is homogeneous, however, then the (comonotone) additivity is lost. Therefore, the definition of the Choquet-Pettis integral is still an open problem for further investigations.

We also propose to consider the following problem:

**Open Problem**: For which functions \( f, g \), we can ensure that for any \( a, b \geq 0 \),

\[ (C) \int (a \cdot f + b \cdot g) d\mu = a \cdot (C) \int f d\mu + b \cdot (C) \int g d\mu? \quad (2) \]

It is not difficult to check that, due to the positive homogeneity of the (asymmetric) Choquet integral, the equality (2) is equivalent to the additivity equality

\[ (C) \int (f + g) d\mu = (C) \int f d\mu + (C) \int g d\mu. \]
Moreover, if $\mu$ is modular on a set system $\mathcal{A}$ which is closed under unions and intersections, $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for any $A, B$ from $\mathcal{A}$, then (2) holds for any $a, b \geq 0$ and $f, g$ such that $(\{f \geq t\})_{t \in \mathbb{R}} \subset \mathcal{A}$, $(\{g \geq t\})_{t \in \mathbb{R}} \subset \mathcal{A}$. In particular, if $\mu$ is additive on $\mathcal{F}$ then (2) holds for any $f, g$. Similarly, if $f$ and $g$ are comonotonic, then $\mathcal{A} = (\{f \geq t\})_{t \in \mathbb{R}} \cup (\{g \geq t\})_{t \in \mathbb{R}}$ is a chain and hence any $\mu$ is modular on $\mathcal{A}$, proving that the comonotonicity of $f$ and $g$ is sufficient for the validity of (2), independently of the considered monotone measure $\mu$.

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References