Hidden Conflict of Belief Functions

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Abstract

Hidden conflict of belief functions in case where the sum of all multiples of conflicting belief masses being equal to zero was observed. Degrees of hidden conflict and of non-conflictness are defined and analysed including full non-conflictness. Hidden conflict between two belief functions is distinguished from internal hidden conflict(s) of the individual belief function(s). Finally, computational issues of hidden conflict and non-conflictness are presented.

Keywords: belief functions, Dempster-Shafer theory, uncertainty, conflicting belief masses, *internal conflict*, *conflict between belief functions*, *hidden conflict*, *full non-conflictness*.

1 Introduction

When combining belief functions (BFs) by the conjunctive rules of combination, some conflicts often appear (they are assigned either to $\emptyset$ by non-normalised conjunctive rule $\textcircled{\cap}$ or distributed among other belief masses by normalisation in Dempster's rule of combination $\oplus$). Combination of conflicting BFs and interpretation of their conflicts are often questionable in real applications. Thus a series of papers related to conflicts of BFs was published, e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. A new interpretation of conflicts of belief functions was introduced in [11]: important distinction of internal conflicts of individual BFs (due to their inconsistency) from conflicts between BFs (due to conflict/contradiction of evidences represented by the BFs) was introduced there. Note that zero sum of all multiples of conflicting belief masses (denoted by $m_{\textcircled{\cap}}(\emptyset)$) is usually considered as non-conflictness of the belief functions in all these approaches.

When analysing the conflict between belief functions based on their non-conflicting parts[1] [4] a positive value of conflict was observed even in a situation when sum of all multiples of conflicting belief masses equals to zero. This arose a series of new questions: how to interpret the sum of conflicting masses, is the conflict based on non-conflicting parts of belief functions correct? Some of the answers are provided in this text. They are positive in favour of the conflict based on non-conflicting parts. This led us to a definition of a hidden conflict of BFs (Section 3).

Going further, different levels / degrees of hidden conflicts are defined and a maximal degree of hidden conflict is investigated. Analogously to the degrees of hidden conflict, there also exist different degrees of non-conflictness. Full non-conflictness and conditions, under which belief functions are fully non-conflicting, are defined and presented.

In accordance with approach from [11], there are observed and presented not only hidden conflicts between two belief functions, but also internal hidden conflicts of individual BFs (Section 5). Finally, computational aspects of hidden conflict are presented in Section 6.

[1] Conflicting and non-conflicting parts of belief functions originally come from [12].

2 Preliminaries

We assume classic definitions of basic notions from theory of *belief functions* [13] on finite exhaustive frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is a *power-set* of Ω. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. Because there is a unique correspondence between m and corresponding Bel and Pl, we often speak about m as of a belief function.

A *focal element* is a subset of the frame of discernment $X \subseteq \Omega$, such that $m(X) > 0$. If all focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF); in fact, it is a probability distribution on Ω. If there are only focal elements such that $|X| = 1$ or $|X| = n$ we speak about *quasi-Bayesian BF (qBBF)*. In the case of $m(\Omega) = 1$ we speak about *vacuous BF* (VBF) and about *a non-vacuous BF* otherwise. In the case of $m(X) = 1$ for $X \subset \Omega$ we speak about *categorical BF*. If all focal elements have a non-empty intersection, we speak about *a consistent BF*; and if all of them are nested, about *a consonant BF*.

Dempster's (normalized conjunctive) rule of combination $\oplus$: $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X) m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [13]. Putting $K = 1$ and $(m_1 \textcircled{\cap} m_2)(\emptyset) = \kappa$ we obtain the *non-normalized conjunctive rule of combination* $\textcircled{\cap}$, see e. g. [14].

Smets' *pignistic probability* is given by $BetP(\omega_i) = \sum_{\omega_i \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1-m(\emptyset)}$, see e.g. [14]. *Normalized plausibility of singletons*[2] of Bel is a probability distribution Pl_P such that $Pl\_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [15, 16].

A conflict of BFs Bel', Bel'' *based on their non-conflicting parts* Bel'_0, Bel''_0 is defined by the expression $Conf(Bel', Bel'') = (m'_0 \textcircled{\cap} m''_0)(\emptyset)$, where non-conflicting part Bel_0 (of a BF Bel) is unique consonant BF such that $Pl\_P_0 = Pl\_P$ (normalized plausibility of singletons corresponding to Bel_0 is the same as that corresponding to Bel); m_0 is a bba related to Bel_0. For an algorithm to compute Bel_0 see [4].

3 Hidden Conflict

3.1 An Introductory Example

Let us assume two simple consistent belief functions Bel' and Bel'' on $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ given by the bbas $m'(\{\omega_1, \omega_2\}) = 0.6$, $m'(\{\omega_1, \omega_3\}) = 0.4$, and $m''(\{\omega_2, \omega_3\}) = 1.0$. Then $(m' \textcircled{\cap} m'')(\emptyset) = 0$ what seems — and it is usually considered — to be a non-conflictness of m' and m'', but there is positive conflict based on non-conflicting parts $Conf(Bel', Bel'') = (m'_0 \textcircled{\cap} m''_0)(\emptyset) = 0.4 > 0$. (This holds true despite of Theorem 4 from [4] which should be revised in future).

We can easily verify this: the only focal element of m'' has a non-empty intersection with both focal elements of m', thus $\sum_{(X \cap Y) = \emptyset} m'(X) m''(Y)$ is an empty sum; Bel'' is consonant, thus $Bel''_0 = Bel''$, $m''_0 = m''$, $Pl'(\{\omega_1\}) = 1$, $Pl'(\{\omega_2\}) = 0.6$, $Pl'(\{\omega_3\}) = 0.4$, thus $m'_0(\{\omega_1\}) = 0.4$, $m'_0(\{\omega_1, \omega_2\}) = 0.2$, $m'_0(\{\omega_1, \omega_2, \omega_3\}) = 0.4$, hence $Conf(Bel', Bel'') = (m'_0 \textcircled{\cap} m''_0)(\emptyset) = m'_0(\{\omega_1\}) m''_0(\{\omega_2, \omega_3\}) = 0.4 \cdot 1 = 0.4$.

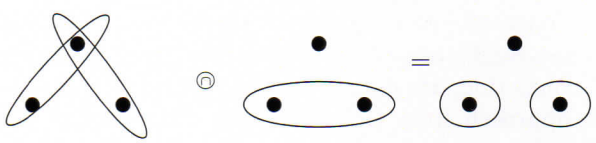

Figure 1: Introductory Example: focal elements of m', m'', and of $m' \textcircled{\cap} m''$.

[2] Plausibility of singletons is called *contour function* by Shafer in [13], thus $Pl\_P(Bel)$ is a normalization of contour function in fact.

Table 1: Hidden conflict in the Introductory Example

X :	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω_3	$\emptyset$
$m'(X)$:	0.0	0.0	0.0	0.60	0.40	0.00	0.00	—
$m''(X)$:	0.0	0.0	0.0	0.00	0.00	1.00	0.00	—
$(m'\textcircled{\cap}m'')(X)$:	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m'\textcircled{\cap}m''\textcircled{\cap}m'')(X)$:	0.00	0.60	0.40	0.00	0.00	0.00	0.00	0.00
$(m'\textcircled{\cap}m''\textcircled{\cap}m')(X)$:	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48
$(m'\textcircled{\cap}m''\textcircled{\cap}m'\textcircled{\cap}m'')(X)$:	0.00	0.36	0.16	0.00	0.00	0.00	0.00	0.48

3.2 Interpretation of the Example – Observation of a Hidden Conflict

The following questions arise: Does $(m'\textcircled{\cap}m'')(\emptyset) = 0$ really represent non-conflictness of respective BFs as it is usually assumed? Is the definition of conflict based on non-conflicting parts correct? Are m' and m'' conflicting? What does $(m'\textcircled{\cap}m'')(\emptyset) = 0$ mean?

Suppose that Bel' and Bel'' are non-conflicting now. Thus their combination should be also non-conflicting with both of them. Does this hold for BFs from our example? This holds true when we combine $m'\textcircled{\cap}m''$ with m'' one more time (assuming m'' comming from two independent belief sources). It follows from the idempotency of categorical m'': $m'\textcircled{\cap}m''\textcircled{\cap}m'' = m'\textcircled{\cap}m''$ and therefore $(m'\textcircled{\cap}m''\textcircled{\cap}m'')(\emptyset) = 0$ again. On the other hand, we obtain positive $(m'\textcircled{\cap}m''\textcircled{\cap}m')(\emptyset) = (m'\textcircled{\cap}m'\textcircled{\cap}m'')(\emptyset) = 0.48$ (assuming m' comming from two independent belief sources). See Table 1 and Figure 2. When m'' and m' are combined once, then we observe $m_{\textcircled{\cap}}(\emptyset) = 0$. When combining m'' with m' twice then $m_{\textcircled{\cap}}(\emptyset) = 0.48$. We observe some kind of *a hidden conflict*. Moreover, both individual BFs are consistent. I.e. there are no internal conflicts. Thus the hidden conflict is *hidden conflict between the BFs* and we have an argument for correctness of positive value of $Conf(Bel', Bel'')$.

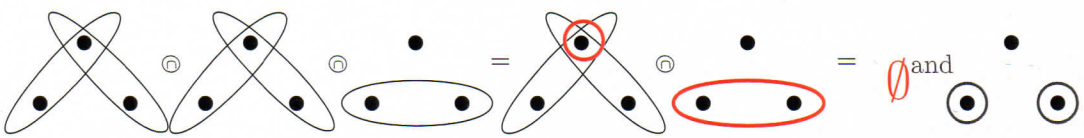

Figure 2: Arising of a hidden conflict between BFs in the Introductory Example: focal elements of m', m', m'' — $m'\textcircled{\cap}m', m''$ and of $(m'\textcircled{\cap}m')\textcircled{\cap}m''$.

What is a decisional interpretation of our BFs? Contours, i.e. plausibilities of singletons are $Pl' = (1.0, 0.6, 0.4)$ and $Pl'' = (0.0, 1.0, 1.0)$, we obtain $Pl\_P' = (0.5, 0.3, 0.2)$ and $Pl\_P'' = (0.0, 0.5, 0.5)$ by normalization; thus at Bel', ω_1 is significantly preferred, whereas at Bel'', one of ω_2, ω_3; this is also an argument for mutual conflictness of the BFs. Considering Smets' pignistic probability we obtain $BetP' = (0.5, 0.3, 0.2)$ and $BetP'' = (0.0, 0.5, 0.5)$, just the same values as in the case when normalized plausibility of singletons (normalized contour) is used for decision. Both the, in general different, probabilistic approximations $BetP$ and Pl_P give highest value to different singletons for Bel' and Bel''. Thus the argument for mutual conflictness of the BFs is strengthened and we obtain the same pair of incompatible decisions based on the BFs in both frequent decisional approaches: using either normalized contour (which is compatible with conjunctive combination of BFs) or pignistic probability (designed for betting).

Hence $(m'\textcircled{\cap}m'')(\emptyset)$ does not mean non-conflictness of the BFs. It means a simple or partial compatibility of their focal elements only.

3.3 Objections Against Our Interpretation

There are several objections against our interpretation of the introductory example:

- *in case of a combination of two identical belief functions some idempotent rule of combination should be used.* Yes, this would be right for belief functions coming from two dependent belief sources. But this is not true for two or more numerically same BFs coming from two or more independent belief sources.

- *The result is not surprising, because conflict is increasing when combining more belief functions.* This is definitely not true:
  - to be correct it should be stated 'non-decreasing' instead of increasing.
  - more precisely, a conflict is non-decreasing when more conflicting BFs are combined. When truly non-conflicting BFs are combined, any positive conflict can never arise there; e.g. $(\textcircled{\scriptsize}_1^k Bel_1 \circledcirc \textcircled{\scriptsize}_1^k Bel_2)(\emptyset) = 0$ for any $k > 0$ and Bel_i on Ω_3 given by $m_1(\{\omega_1\}) = 0.3$, $m_1(\{\omega_1, \omega_2\}) = 0.2$, $m_1(\{\omega_1, \omega_3\}) = 0.1$, $m_1(\{\omega_1, \omega_2, \omega_3\}) = 0.4$, $m_2(\{\omega_1, \omega_3\}) = 0.7$, $m_2(\{\omega_1, \omega_2, \omega_3\}) = 0.3$.
  - Thus an appearance of a positive $(m' \circledcirc m' \circledcirc)(\emptyset)$ implies a conflict between m' and m'' or between m' and m' in case of consonant BFs; there cannot be any conflict between consonant m' and m' thus there must be conflict between m' and m''.
- The result is rather unsurprising. Because one can see clearly that the hidden conflict occurs when the first combination results in disjoint focal sets. Yes, in the very simple Introductory Example, this may by unsurprising for someone; but there are no disjoint sets after the first combination in the following Little Angel example. Moreover, this should be surprising for all who accept the following assumption / axiom: BFs Bel' and Bel'' are non-conflicting whenever $(m' \circledcirc m'')(\emptyset) = 0$, e. g. [1, 5, 7, 8] and the previous Daniel's publications, e. g. [2, 4, 3].
- It is obvious that a combination results in a conflict if a Bayesian BF ($m' \circledcirc m''$ in the Introductory Example) is combined with any other BF. Yes, this is true in the very simple introductory example, but not in a general example, see e. g. the following Little Angel example again.

Analysing these objections we can see, that is was not easy to observe the hidden conflicts, in simple cases the observation seems to be obvious, thus not interesting, in more general examples this seems to be really hidden.

3.4 Definition of Hidden Conflict

Definition 1 *Let us suppose two BFs Bel', Bel'' defined by bbas m', m'', such that $(m' \circledcirc m'')(\emptyset) = 0$. If there further holds $(m' \circledcirc m'' \circledcirc m')(\emptyset) > 0$ or $(m' \circledcirc m'' \circledcirc m'')(\emptyset) > 0$ we say that there is* a hidden conflict *of the BFs.*

Observation 1 *A condition $(m' \circledcirc m'' \circledcirc m')(\emptyset) > 0$ or $(m' \circledcirc m'' \circledcirc m'')(\emptyset) > 0$ from Definition 1 is equivalent to the following condition $(m' \circledcirc m'' \circledcirc m' \circledcirc m'')(\emptyset) > 0$.*

We have to note that a hidden conflict is quite a new phenomenon, qualitatively different from the ideas of all previous Daniel's works on conflict of belief functions and also different from the other referred approaches. Till now, it was supposed that $m_{\textcircled{\scriptsize}}(\emptyset)$ includes both conflict between BFs and also internal conflicts of individual BFs. Thus conflict between BFs was supposed to be less or equal to $m_{\textcircled{\scriptsize}}(\emptyset)$. Here, we deal with a situation of a positive conflict between BFs while $m_{\textcircled{\scriptsize}}(\emptyset) = 0$.

We have already observed that $m_{\textcircled{\scriptsize}}(\emptyset) = 0$ does not mean full non-conflictness of BFs and that the condition $(m' \circledcirc m'' \circledcirc m' \circledcirc m'')(\emptyset) > 0$ together with $(m' \circledcirc m'')(\emptyset) = 0$ defines hidden conflict. What about the condition $(m' \circledcirc m'' \circledcirc m' \circledcirc m'')(\emptyset) = 0$? Is this condition sufficient for full non-conflictness of BFs Bel' and Bel''? May some conflict be still hidden there?

The zero version of the condition seems to imply non-conflictness on Ω_3, the frame of discernment of the Introductory Example. To solve the question in general, we have to consider a larger frame of discernment.

3.5 Little Angel Example

For Ω_5 one can find the following Little Angel Example, see Table 2. Similarly to Introductory Example, we have two consistent BFs Bel^i and Bel^{ii} with disjoint sets of max-plausibility elements while zero condition $(m^i \circledcirc m^{ii})(\emptyset) = 0$ holds true.

Table 2: Hidden Conflict in the Little Angel Example

X :	$A=\{\omega_1,\omega_2,\omega_5\}$	$B=\{\omega_1,\omega_2,\omega_3,\omega_4\}$	$C=\{\omega_1,\omega_3,\omega_4,\omega_5\}$	$X=\{\omega_2,\omega_3,\omega_4,\omega_5\}$	$\emptyset$
$m^i(X)$:	0.1	0.30	0.60	0.00	—
$m^{ii}(X)$:	0.0	0.00	0.00	1.00	—

X :	$A\cap X$	$B\cap X$	$C\cap X$	$A\cap B\cap X$	$A\cap C\cap X$	$B\cap C\cap X$	$\emptyset$
$(m^i\textcircled{\cap}m^{ii})(X)$:	0.1	0.3	0.6	0.0	0.0	0.0	0.000
$(m^i\textcircled{\cap}m^{ii}\textcircled{\cap}m^{ii})(X)$:	0.10	0.30	0.60	0.00	0.00	0.00	0.00
$(m^i\textcircled{\cap}m^i\textcircled{\cap}m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i\textcircled{\cap}m^i\textcircled{\cap}m^{ii}\textcircled{\cap}m^{ii})(X)$:	0.01	0.09	0.36	0.06	0.12	0.36	0.00
$(m^i\textcircled{\cap}m^{ii}\textcircled{\cap}m^{ii}\textcircled{\cap}m^{ii})(X)$:	0.010	0.090	0.360	0.060	0.120	0.360	0.000
$(m^i\textcircled{\cap}m^i\textcircled{\cap}m^i\textcircled{\cap}m^{ii})(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108
$m^i\textcircled{\cap}m^i\textcircled{\cap}m^i\textcircled{\cap}m^{ii}\textcircled{\cap}m^{ii}\textcircled{\cap}m^{ii}(X)$:	0.001	0.027	0.216	0.036	0.126	0.486	0.108

In addition to Introductory Example, $(m^i\textcircled{\cap}m^{ii}\textcircled{\cap}m^i\textcircled{\cap}m^{ii})(\emptyset) = 0$ (see Table 2) while $Conf(Bel^i, Bel^{ii}) = 0.1$ is positive again. Positiveness of the $Conf$ value can be easily seen from the fact that sets of max-plausibility elements are disjoint for Pl^i and Pl^{ii}. Numerically, we have again $Bel_0^{ii} = Bel^{ii}$, and $Pl\_P^i = (\frac{10}{39}, \frac{4}{39}, \frac{9}{39}, \frac{9}{39}, \frac{7}{39})$. We obtain $m_0^i(\{\omega_1\}) = 0.1, m_0^i(\{\omega_1,\omega_3,\omega_4\}) = 0.2, m_0^i(\{\omega_1,\omega_3,\omega_4,\omega_5\}) = 0.3, m_0^i(\{\Omega_5\}) = 0.4$, and $Conf(Bel^i, Bel^{ii}) = m_0^i(\{\omega_1\})m^{ii}(X) = 0.1$. Analogous arguments hold true for the positive $Conf$ and hidden conflict again (of the 2nd degree this time). $BetP^i = (0.2583, 0.1083, 0.2250, 0.2250, 0.1833)$ which is not numerically the same as Pl_P^i, but both prefer ω_1, whereas $BetP^{ii} = Pl\_P^{ii} = (0.00, 0.25, 0.25, 0.25, 0.25)$.

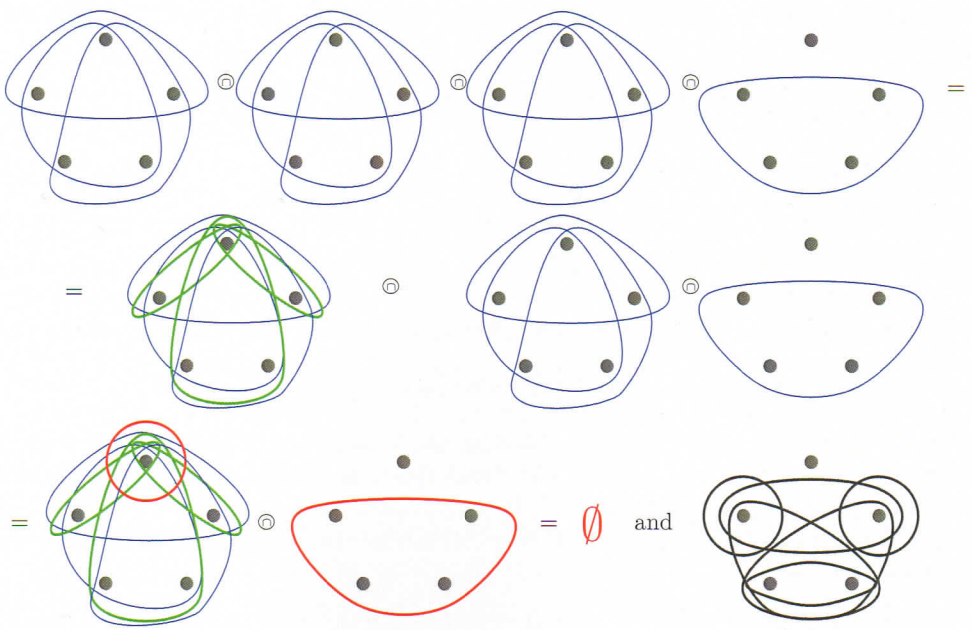

Figure 3: Arrising of a hidden conflict between BFs in the Little Angel Example. Focal elements of m^i, m^{ii}, $m^i\textcircled{\cap}m^i$, $m^i\textcircled{\cap}m^i\textcircled{\cap}m^i$ and of $(m^i\textcircled{\cap}m^i\textcircled{\cap}m^i)\textcircled{\cap}m^{ii}$. Red-colored focal elements are those responsible for creation of the empty-set in the last step.

For an existence of a hidden conflict, it is the structure of focal elements that is important — not their belief masses. Belief masses are important for the size of a conflict. In general, we can take $m^i(A) = a$, $m^i(B) = b$, $m^i(C) = c$, for A, B, C defined in Table 2, and for any $a, b, c > 0$,

such that $a+b+c=1$, we obtain $m(\emptyset)=6abc$ as a hidden conflict of the 2nd degree (in our numeric case there is $6abc=6\cdot 0.1\cdot 0.3\cdot 0.6=0.108$). For graphical presentation of the Little Angel Example see Figure 3.

Degrees of hidden conflict, its maximal value, and the issue of full non-conflictness will be analyzed in the following section.

4 Degrees of Hidden Conflict and Full Non-conflictness

When analyzing examples from the previous section, we have observed different degrees of hidden conflict. We can formalize it in the next definition.

Definition 2 *Assume two BFs Bel^i, Bel^{ii} defined by bbas m^i, m^{ii}, such that for some $k>0$ $(\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset)=0$. If there further holds $(\textcircled{\cap}_{j=1}^{k+1}(m^i \textcircled{\cap} m^{ii}))(\emptyset)>0$ we say that there is* a conflict of BFs *Bel^i and Bel^{ii}* hidden in the k-th degree.

Analogously to particular degrees of hidden conflict, there are degrees of non-conflictness. Particular degrees of non-conflictness are not very important. However, there is an important question whether there is some hidden conflict or not, i.e. whether or not the BFs in question are fully non-conflicting.

Definition 3 *We say that BFs Bel^i and Bel^{ii} are* fully non-conflicting *if $(m^i \textcircled{\cap} m^{ii})(\emptyset)=0$ and, further, if there is no hidden conflict of any degree. I.e. if $(\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset)=0$ for any $k\geq 0$.*

Thus there is a question how many times we have to combine $(m^i \textcircled{\cap} m^{ii})$, i.e., for which k value of $(\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset)$ shows whether there is some hidden conflict of the BFs Bel^i and Bel^{ii} or not. For answers to this question see corollaries of the following two theorems.

Theorem 4 (maximal degree of hidden conflict) *For any non-vacuous BFs Bel^i and Bel^{ii} defined by m^i and m^{ii} on any frame Ω_n it holds that*

$$(\textcircled{\cap}_{j=1}^{n-1}(m^i \textcircled{\cap} m^{ii}))(\emptyset)=0 \quad \textit{iff} \quad (\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset)=0$$

for any $k>n-2$.

Corollary 5 *Hidden conflict of any non-vacuous BFs on any Ω_n has always degree less or equal to $n-2$; i.e., the condition*

$$(\textcircled{\cap}_{j=1}^{n-1}(m^i \textcircled{\cap} m^{ii}))(\emptyset)=0$$

always means full the non-conflictness of any BFs Bel^i and Bel^{ii} on any Ω_n.

Let us present an example of such highly hidden conflict now.

Example 6 *Example of hidden conflict of the $(n-2)$-th degree: Let us suppose n-element frame of discernment $\Omega_n=\{\omega_1,\omega_2,...,\omega_n\}$. Bel^i and Bel^{ii} are given by $m^i(\{\omega_1,\omega_2,...,\omega_{n-1}\})=\frac{1}{n-1}$, $m^i(\{\omega_1,\omega_2,...,\omega_{n-2},\omega_n\})=\frac{1}{n-1}$, $m^i(\{\omega_1,\omega_2,...,\omega_{n-3},\omega_{n-1},\omega_n\})=\frac{1}{n-1}$, . . . $m^i(\{\omega_1,\omega_3,\omega_4,...,\omega_n\})=\frac{1}{n-1}$, $m^{ii}(\{\omega_2,\omega_3,...,\omega_n\})=\frac{1}{1}$. At $m^i \textcircled{\cap} m^i$ $(n-2)$-element focal elements appear, at $m^i \textcircled{\cap} m^i \textcircled{\cap} m^i$ $(n-3)$-element focal elements appear, at $\textcircled{\cap}_{j=1}^{k} m^i$ $(n-k)$-element focal elements appear, at $\textcircled{\cap}_{j=1}^{n-2} m^i$ 2-element focal elements appear, all these focal elements have non-empty intersections with the only focal element of m^{ii}, and finally at $\textcircled{\cap}_{j=1}^{n-1} m^i$ singleton focal element $\{\omega_1\}$ appears which has empty intersection with the only focal element of m^{ii} $\{\omega_2,\omega_3,...,\omega_n\}$.*

What does m^i express? It gives a big support to all elements of the frame, to the entire frame Ω_n and even greater support to ω_1 which is included in all focal elements; ω_1 is preferred and, moreover, it has plausibility 1. We can modify m^i and express this more easily: $\overline{m}^i(\Omega_n)=\frac{n-1}{n}$, $\overline{m}^i(\{\omega_1\})=\frac{1}{n}$, or more generally, $\widetilde{m}^i(\Omega_n)=1-a$, $\widetilde{m}^i(\{\omega_1\})=a$ for some $0<a<1$. We can easily see evident conflict corresponding to positive $\overline{m}(\emptyset)=(\overline{m}^i \textcircled{\cap} m^{ii})(\emptyset)=\frac{1}{n}$, $\widetilde{m}(\emptyset)=a$ for these modifications of m^i. Hence either hidden conflict of the $(n-2)$-th degree of m^i and m^{ii} or positive $Conf(m^i,m^{ii})=Conf(\overline{m}^i,m^{ii})=\frac{1}{n}$ should not be very surprising.

We have to note that the Introductory Example is a special instance of Example 6 for $n=3$.

Theorem 7 *(i) Any non-vacuous BFs Bel^i, Bel^{ii} have a conflict hidden at most in (c–1)-th degree where $c = min(c^i, c^{ii}) + sgn(|c^i - c^{ii}|)$. where c^i, c^{ii} are maximal cardinalities of focal elements of Bel^i, Bel^{ii} different from Ω. In the other words*

$$(\textcircled{\cap}_{j=1}^{c}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0 \quad \textit{iff} \quad (\textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$$

for any $k \geq c = min(c^i, c^{ii}) + sgn(|c^i - c^{ii}|)$.
(ii) There are no hidden conflicts of any non-vacuous BFs on any two-element frame Ω_2.
(iii) There are no hidden conflicts of any non-vacuous quasi-Bayesian BFs on any frame Ω_n.
(iv) For a BF Bel^i and a quasi-Bayesian BF Bel^{ii} there is a hidden conflict of (at most) the first degree; if it appears then it is an internal conflict of Bel^{ii} in fact.

Corollary 8 *(i) Assume two non-vacuous BFs Bel^i, Bel^{ii} on Ω_n. The zero value of the expression $(\textcircled{\cap}_{j=1}^{c}(m^i \textcircled{\cap} m^{ii}))(\emptyset)$, i.e., the condition*

$$(\textcircled{\cap}_{j=1}^{c}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$$

means full non-conflictness of the BFs for $c = min(c^i, c^{ii}) + sgn(|c^i - c^{ii}|)$, where c^i, c^{ii} are maximal cardinalities of focal elements of Bel^i, Bel^{ii} different from Ω_n.
(ii) For any two non-vacuous quasi Bayesian BFs Bel^i, Bel^{ii} on any frame of discernment Ω_n the condition $(m^i \textcircled{\cap} m^{ii})(\emptyset) = 0$ always means full non-conflictness of the BFs.
(iii) For any BF Bel^i and any quasi-Bayesian BF Bel^{ii} the condition $(\textcircled{\cap}_{j=1}^{2}(m^i \textcircled{\cap} m^{ii}))(\emptyset) = 0$ always means full non-conflictness of the BFs.

5 Internal Hidden Conflict

Little Angel Modification

Let us take m^{iii} instead of m^i, such that $m^{iii}(A) = m^i(A)$, $m^{iii}(C) = m^i(C)$, and $m^{iii}(D) = m^{iii}(\{\omega_2, \omega_3, \omega_4\}) = 0.30$ instead of $m^i(B)$. There is $(m^{ii} \textcircled{\cap} m^{ii} \textcircled{\cap} m^{iii} \textcircled{\cap} m^{iii})(\emptyset) = 0$, but $(m^{ii} \textcircled{\cap} m^{iii} \textcircled{\cap} m^{iii} \textcircled{\cap} m^{iii})(\emptyset) > 0$, even $(m^{iii} \textcircled{\cap} m^{iii} \textcircled{\cap} m^{iii})(\emptyset) > 0$, i.e. $(\textcircled{\cap}_1^3 m^{iii})(\emptyset) > 0$, for detail see [17].

We observe a conflict of the belief functions hidden in the 2-nd degree again. Nevertheless, the situation of focal elements is different now: the only focal element X of $m^{ii} = \textcircled{\cap}_1^3 m^{ii}$ has non-empty intersection with any focal element of $\textcircled{\cap}_1^3 m^{iii}$, but $(\textcircled{\cap}_1^3 m^{iii})(\emptyset) > 0$ now. Thus this is not a hidden conflict between m^{ii} and m^{iii}, but *an internal hidden conflict* of m^{iii}.

Computation of $\textcircled{\cap}_1^k(m' \textcircled{\cap} m'')(\emptyset)$ and internal hidden conflict have a relation to Martin's auto-conflict [18, 8], thus we can speak about hidden auto-conflict here. See [19] for more details.

In general, we can observe internal hidden conflict when at least one of BFs in hidden conflict is not consistent. Let us present an example with highly hidden internal hidden conflict of the $(n-2)$-th degree on a general frame of discernment now.

Example 9 *Let us consider the following modification of Example 6 on Ω_n. Instead of m^i we take m^{iii} having all focal elements of cardinality $n-1$, such that $m^{iii}(\Omega_n \setminus \{\omega\}) = \frac{1}{n}$ for any $\omega \in \Omega_n$; m^{ii} same as in Example 6. m^{iii} is not consistent; $Pl^{iii}(\{\omega\}) = \frac{n-1}{n}$ for any $\omega \in \Omega_n$. We observe hidden conflict of the $(n-2)$-th degree again. Because of same plausibilities of all singletons $m''_0(\Omega_n) = 1$ and $Conf(m^{iii}, m^{ii}) = 0$ now.*

There is a positive hidden conflict of BFs Bel^{iii} and Bel^{ii}, but zero conflict between them. We say that there is *an internal hidden conflict*. This corresponds to non-consistency of BF Bel^{iii}; Bel^{ii} is consistent thus there is an internal hidden conflict of BF Bel^{iii} in this case.

A numeric example was computed on Ω_{16}, see Table 3 for a comparison of focal elements and $m_{\textcircled{\cap}}(\emptyset)$ values of Examples 6 and 9. For simplicity, same bbms $m^i(X) = \frac{1}{15}$ and $m^{ii}(X) = \frac{1}{16}$ were used there.

Ω_{16}		$m' = m^i$	$m'' = m^{ii}$		$m' = m^{iii}$	$m'' = m^{ii}$	
Degree	$m_{\textcircled{\cap}}$	no of f.e.	Card. of f.e.	$m_{\textcircled{\cap}}(\emptyset)$	no of f.e.	Card. of f.e.	$m_{\textcircled{\cap}}(\emptyset)$
–	m'	15	15	–	16	15	–
–	m''	1	15	–	1	15	–
0	$m' \textcircled{\cap} m''$:	15	14	0	16	14–15	0
1	$m' \textcircled{\cap} m'' \textcircled{\cap} m' \textcircled{\cap} m''$	120	13–14	0	121	13–15	0
2	$\textcircled{\cap}_{j=1}^{3}(m' \textcircled{\cap} m'')$	575	12–14	0	576	12–15	0
	. . .			0			0
k	$\textcircled{\cap}_{j=1}^{k+1}(m' \textcircled{\cap} m'')$	. . .	(14-k)–14	0	. . .	(14-k)–15	0
	. . .			0			0
13	$\textcircled{\cap}_{j=1}^{14}(m' \textcircled{\cap} m'')$	32766	1–14	0	32767	1–15	0
14	$\textcircled{\cap}_{j=1}^{15}(m' \textcircled{\cap} m'')$	32766	1–14	2.98–06	32767	1–15	1.13–06

Table 3: Hidden conflict between BFs Bel^i and Bel^{ii} from Example 6 and internal hidden conflict of Bel^{ii} and Bel^{iii} from Example 9, both on Ω_{16}.

6 Computational Complexity and Computations of Examples

Based on Definition 2 and Theorem 4, the complexity of computation of the degree of hidden conflict of two BFs Bel^i and Bel^{ii} is — on a general Ω_n — $O(n)$ of $\textcircled{\cap}$ operations. In the case of checking existence of a hidden conflict of the BFs we obtain the complexity $O(log_2(n))$ of $\textcircled{\cap}$ operations utilizing a simplification of computation based on $\textcircled{\cap}_{j=1}^{2k}(m^i \textcircled{\cap} m^{ii}) = \textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii}) \textcircled{\cap} \textcircled{\cap}_{j=1}^{k}(m^i \textcircled{\cap} m^{ii})$. Note that the complexity of $\textcircled{\cap}$ operation depends on the number and structure of focal elements.

During our analysis of hidden conflicts a series of example computations was performed on frames of discernment of cardinality from 5 to 16. A number of focal elements rapidly grows up to $|\mathcal{P}(\Omega)| = 2^{|\Omega|} - 1$ when conjunctive combination $\textcircled{\cap}$ is repeated, see e.g. 32766 and 32767 focal elements in the presented Examples 6 and 9 at Table 3. Because the degree of the hidden conflict and existence of the hidden conflict depends on the number and the structure of focal elements not on their bbms, we have used same bbms for all focal elements of a BF in our computations on frames of cardinality greater than 10.

All our experiments were performed in R [20] using R Studio [21]. We are currently developing an R package for dealing with belief functions on various frames of discernment. It is based on a relational database approach - nicely implemented in R, in package called data.table [22].

7 Several Important Remarks

We have to underline that hidden conflict of belief functions is not a new measure of conflict. It just improves/extends the classic measure defining the conflict by $m(\emptyset)$ in situations where $m(\emptyset) = 0$; it distinguishes fully non-conflicting BFs from those with a positive hidden conflict. This notion serves for deeper understanding of conflictness / non-conflictness, it enables to point out the conflict also in situations where conflicts had not been expected, in situations where $m_{\textcircled{\cap}}(\emptyset) = 0$; hence to point out and to help to understand the conflicts which are hidden due to $m_{\textcircled{\cap}}(\emptyset) = 0$.

Particular numeric values of a hidden conflict have no reasonable interpretation so far. For now, we are interested whether the value is zero (i.e. no conflict) or not.

Repeated applications of the conjunctive combination $\oplus$ of a BF with itself is used here to simulate situations where different independent believers have numerically the same bbm. Thus this has nothing to do with idempotent belief combination (where of course no conflict between two BFs is possible).

There is brand new idea of hidden conflicts in [19] and in this contribution. Brand new interpretation of $m(\emptyset)$ distinguishing fully non-conflicting BFs from those with hidden conflict. The

assumption of non-conflictness when $m_{\textcircled{\cap}}(\emptyset) = 0$ was relaxed, due to observation of a qualitatively new phenomena — observation of hidden conflict even in the cases where $m_{\textcircled{\cap}}(\emptyset) = 0$. Both these studies want to point out the existence of hidden (auto-)conflicts in situations where no conflict was expected till now. Thus the definitions of hidden conflict and hidden auto-conflict[19] are not anything against the previous Daniel's research and results on conflict of belief functions e.g. [11, 2, 4]. Of course, some parts of the previous approaches should be updated to be fully consistent with the newly presented results on hidden conflicts and auto-conflicts.

Our study was motivated by investigation of conflict $Conf$ of BFs based on their non-conflicting parts [4], thus we were interested in independent BFs when hidden conflict was observed. But we have to note that conflictness / non-conflictness of BFs has nothing to do with dependence/independence of the BFs. Repeated computation of several (up to n) numerically identical BFs, when looking for hidden conflict is just a technical tool for computation of $m(\emptyset)$ or more precisely say for computation of $\kappa = \sum_{X \cap Y = \emptyset} m_j(X) m_j(Y)$. We are not interested in entire result of repeated application of $\textcircled{\cap}$, we are interested only in $m(\emptyset)$ or more precisely say in $\kappa = \sum_{X_1 \cap X_2 \cap \ldots \cap X_k = \emptyset} m_j(X_1)$ $m_j(X_2) \ldots m_j(X_K)$. Thus our computation has nothing to do with any idempotent combination of BFs. And we can compute hidden conflict using $\textcircled{\cap}_1^k$ (or κ) in the same way for both dependent and indepentent BFs. It is either not necessary to include any independence assumption to our Definitions 1 and 2.

8 Summary and Conclusion

Hidden conflicts of belief functions in situations where mutual intersections of any focal element of one BF with all focal element of the other BF are non-empty has been presented and analysed. There may be a positive conflict in situations, where sums of conflicting belief masses are empty, i.e. in situations which have been usually considered to be non-conflicting till now.

Several levels — degrees of hidden conflict were observed, maximal degree of hidden conflicts dependent on size of corresponding frame of discernment was found. A variety of hidden conflicts of degrees $1 - (n-2)$ was described for an n-element frame of discernment. A necessary and sufficient condition for full non-conflictness of BFs in dependence on maximal cardinality of their focal elements has been specified and computational aspects analysed. Analogously to the evident conflicts, internal hidden conflicts are distinguished from the hidden conflicts between BFs.

This qualitatively new phenomenon of conflicts of BFs moves us to better understanding of nature of conflicts of belief functions in general and brings a challenge to elaborate and update existing approaches to conflicts of BFs.

This may consequently serve as a basis for better combination of conflicting belief functions and better interpretation of the results of belief combination whenever conflicting belief functions appear in real applications.

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References

[1] Thomas Burger. Geometric views on conflicting mass functions: From distances to angles. *International Journal of Approximate Reasoning*, 70:36–50, 2016.

[2] Milan Daniel. Properties of plausibility conflict of belief functions. In L. Rutkowski, editor, *ICAISC 2013*, volume 7894 of *LNCS*, pages 235–246, Heidelberg, 2013. Springer.

[3] Milan Daniel and Jianbing Ma. Conflicts of belief functions: continuity and frame resizement. In U. Straccia and A. Calì, editors, *SUM 2014*, volume 8720 of *LNCS*, pages 106–119, Cham, 2014. Springer.

[4] Milan Daniel. Conflict between belief functions: a new measure based on their non-conflicting parts. In F. Cuzzolin, editor, *BELIEF 2014*, volume 8764 of *LNCS*, pages 321–330, Cham, 2014. Springer.

[5] Sébastien Destercke and Thomas Burger. Toward an axiomatic definition of conflict between belief functions. *IEEE transactions on cybernetics*, 43(2):585–596, 2013.

[6] Eric Lefèvre and Zied Elouedi. How to preserve the conflict as an alarm in the combination of belief functions? *Decision Support Systems*, 56:326–333, 2013.

[7] Weiru Liu. Analyzing the degree of conflict among belief functions. *Artificial Intelligence*, 170(11):909–924, 2006.

[8] Arnaud Martin. About conflict in the theory of belief functions. In *Belief Functions: Theory and Applications*, pages 161–168. Springer, Heidelberg, 2012.

[9] Johan Schubert. The internal conflict of a belief function. In *Belief Functions: Theory and Applications*, pages 169–177. Springer, Heidelberg, 2012.

[10] Philippe Smets. Analyzing the combination of conflicting belief functions. *Information fusion*, 8(4):387–412, 2007.

[11] Milan Daniel. Conflicts within and between belief functions. In Eyke Hüllermeier, Rudolf Kruse, and Frank Hoffmann, editors, *IPMU 2010*, volume 6178 of *LNCS*, pages 696–705. Springer, Berlin, Heidelberg, 2010.

[12] Milan Daniel. Non-conflicting and conflicting parts of belief functions. In *7th International symposium on Imprecise Probability: Theories and Applications (ISIPTA 2011)*, pages 149–158, Innsbruck, 2011. SIPTA.

[13] Glenn Shafer. *A mathematical theory of evidence*, volume 1. Princeton university press Princeton, 1976.

[14] Philippe Smets. Decision making in the tbm: the necessity of the pignistic transformation. *International Journal of Approximate Reasoning*, 38(2):133–147, 2005.

[15] Barry R Cobb and Prakash P Shenoy. On the plausibility transformation method for translating belief function models to probability models. *International journal of approximate reasoning*, 41(3):314–330, 2006.

[16] Milan Daniel. Probabilistic transformations of belief functions. In L. Godo, editor, *ECSQARU 2005*, volume 3571 of *LNCS (LNAI)*, pages 539–551. Springer, Heidelberg, 2005.

[17] Milan Daniel and Václav Kratochvíl. Hidden conflict of belief functions (full text in preparation). http://www.cs.cas.cz/~milan/for_download/hidden_conflict_ext.pdf, in preparation.

[18] Arnaud Martin, Anne-Laure Jousselme, and Christophe Osswald. Conflict measure for the discounting operation on belief functions. In *Information Fusion, 2008 11th International Conference on*, pages 1–8. IEEE, 2008.

[19] Milan Daniel and Václav Kratochvíl. Hidden auto-conflict in the theory of belief functions. In *Proceedings of the 20th Czech-Japan Seminar on Data Analysis and Decision Making under Uncertainty*, pages 34–45, 2017.

[20] R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2016.

[21] RStudio Team. *RStudio: Integrated Development Environment for R*. RStudio, Inc., Boston, MA, 2015.

[22] Matt Dowle and Arun Srinivasan. *data.table: Extension of 'data.frame'*, 2016. R package version 1.10.0.