Tomáš Lávička
Carles Noguera

Extension Properties and Subdirect Representation in Abstract Algebraic Logic

Abstract. This paper continues the investigation, started in Lávička and Noguera (Stud Log 105(3): 521–551, 2017), of infinitary propositional logics from the perspective of their algebraic completeness and filter extension properties in abstract algebraic logic. If follows from the Lindenbaum Lemma used in standard proofs of algebraic completeness that, in every finitary logic, (completely) intersection-prime theories form a basis of the closure system of all theories. In this article we consider the open problem of whether these properties can be transferred to lattices of filters over arbitrary algebras of the logic. We show that in general the answer is negative, obtaining a richer hierarchy of pairwise different classes of infinitary logics that we separate with natural examples. As by-products we obtain a characterization of subdirect representation for arbitrary logics, develop a fruitful new notion of natural expansion, and contribute to the understanding of semilinear logics.

Keywords: Abstract algebraic logic, Infinitary logics, Natural extensions, Natural expansions, Semilinear logics, Subdirect representation.

1. Introduction

Abstract algebraic logic (AAL) studies the nature of the connection between propositional logical systems and semantical counterparts based on algebras. At the heart of the theory lies the generalization of the standard Linbendaum–Tarski proof of completeness of classical logic. Recall that when proving completeness w.r.t. the two-element Boolean algebra, we first prove the **classical Lindenbaum Lemma**, which can be formulated in the following two ways:

1. Syntactical version: Let $\Gamma \cup \{\varphi\}$ be a set of classical formulas. If $\Gamma \nvdash_{\mathrm{CL}} \varphi$, then there is a *maximally consistent theory* $T \in \mathrm{Th}(\mathrm{CL})$ such that $\Gamma \subseteq T$ and $\varphi \notin T$.
2. Semantical version: Let $\boldsymbol{A}$ be an algebra for the language of classical logic, $a \in A$, and $F \in \mathcal{F}i_{\mathrm{CL}}(\boldsymbol{A})$ a logical filter. If $a \notin F$, then there is a *maximally consistent filter* $G \in \mathcal{F}i_{\mathrm{CL}}(\boldsymbol{A})$ such that $F \subseteq G$ and $a \notin G$.

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When proving the algebraic completeness for an arbitrary non-classical logic we often apply a convenient version of this method. In most cases, instead of maximally consistent theories or filters (recently studied in general in [22]), we meet other significant kinds of theories and filters such as *prime* (in case of logics with a form of disjunction [6]) and *linear* (in case of logics with implication [8]). The abstract viewpoint of AAL allows to identify the underlying abstract notions: *completely intersection-prime* (which subsumes maximally consistent) and *intersection-prime* (subsumes prime and linear).[1] They give rise (as well as their instances) to corresponding extension properties, which are in fact a **general Lindenbaum lemma** for an arbitrary logic L:

1. Syntactical version: Let $\Gamma \cup \{\varphi\}$ be a set of formulas in the language of L. If $\Gamma \nvdash_{\mathrm{L}} \varphi$, then there is a (completely) intersection-prime theory $T \in \mathrm{Th}(\mathrm{L})$ such that $\Gamma \subseteq T$ and $\varphi \notin T$.
2. Semantical version: Let $\boldsymbol{A}$ be an algebra for the language of L, $a \in A$, and $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ a logical filter. If $a \notin F$, then there is a (completely) intersection-prime filter $G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ such that $F \subseteq G$ and $a \notin G$.

The extension properties corresponding to the two abstract types of theories were introduced in [6] as CIPEP and IPEP. Extension properties allow us to prove completeness w.r.t. refined classes of models; e.g. prime, linear, or simple[2] models. CIPEP and IPEP entail completeness w.r.t. abstract versions of such models: *(finitely) relatively subdirectly irreducible* models, R(F)SI-models for short. Whereas it is easy to prove that each finitary logic[3] enjoys the IPEP and CIPEP, this is no longer true for infinitary logics. This was the topic of [21], where all these properties were studied and shown to provide a new hierarchy of propositional logics (see Figure 1).

We thus obtained a classification of infinitary logics, i.e. systems where a proposition may follow from an infinite set of premises, but not from any of its finite subsets.[4] The literature of non-classical logics provides

[1] (Completely) intersection-prime theories are those that cannot be written as an intersection of any (finite, nonempty) set of different theories.

[2] These models are closely related to semisimplicity in universal algebra; cf. [22].

[3] That is, a logic such that whenever a proposition follows from a set of premises, it must also follows from a *finite* subset of these premises; hence, any such logic can be given by a proof system that generates only finite proofs.

[4] Consequently, any proof system for these logics necessarily generates some infinitely long proofs. Infinitary proofs can be modelled by infinitely branching well-founded trees.

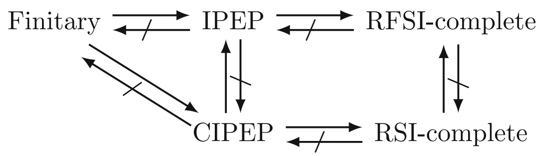


Figure 1. Hierarchy of infinitary logics

several important examples of such logics, including the infinitely-valued Łukasiewicz logic $Ł_\infty$ [23] or the infinitely-valued product logic Π_∞ [20].

Observe that the semantical version of the Lindebaum lemma has the same form as the syntactical one, only that it has been generalized from the algebra of formulas to arbitrary algebras; it is its *transferred* version. But are they equivalent? That is, can the properties CIPEP and IPEP be always transferred from theories to filters over arbitrary algebras? In fact, finitary logics satisfy both versions of the Lindenbaum lemma, but is this also true for infinitary logics?

The present paper is devoted to solving the transfer problem for CIPEP and IPEP and showing that, far from being a marginal technicality in AAL as one might be tempted to think, it is actually an important problem that brings several valuable by-products. Let us substantiate it more precisely in the following points:

1. A finer hierarchy: We will solve the question in the negative way, that is, by showing that these properties do not transfer in general. This means that their transferred versions are distinct properties that define new classes of logics that had not been considered in [21]. We will provide natural examples to separate all the resulting classes and, hence, obtain a finer classification of infinitary logics.

2. Subdirect representation: A central result of universal algebra allows to represent algebras from a variety (resp. quasivariety) as subdirect products of (resp. relatively) subdirectly irreducible ones. A natural question is whether such result can be extended from algebras to matrices in order to obtain a corresponding subdirect representation for the models of propositional logic. It is well known that all finitary logics have this property. We will give a conclusive answer to this question by proving that subdirect representation is equivalent to the conjunction of protoalgebraicity and the transferred CIPEP.

3. Preservation in expansions: Given any relevant property of a logic it is customary to ask whether it is preserved under extensions of the logic (obtained by adding axioms or rules) or, even, expansions (obtained by

adding new connectives to the language and, usually, axioms and rules to describe their behavior). We will prove that all the extension properties considered in this paper and in [21] are preserved under axiomatic extensions, but in general can be lost when adding rules. Moreover, we will obtain reasonable conditions for their preservation under axiomatic expansions.

4. Natural expansions: The study of subdirect representation and preservation in expansions in the two previous points have a common technical requirement: first one needs to understand how can a logic be expanded to a richer language where either the set of variables or the set of connectives have been augmented while, essentially, leaving all logical properties untouched. The former notion (i.e. increasing the number of variables) has been commonly used in AAL known as *natural extensions*. In order to deal with the latter, we will elaborate a new notion, which we will call *natural expansion*, as a kind of dual operation that keeps the set of variables but adds new connectives. This will give us another interesting addition to the general theory of AAL prompted by the main problem considered in the paper.

5. Semilinear logics: An important family of non-classical logics are those that enjoy a semantics of linearly ordered models, usually known as *fuzzy logics* [4] or, in AAL, as *semilinear logics* [5]. The two infinitary logics mentioned above, $Ł_\infty$ and Π_∞, are actually semilinear. We will use them and some variations thereof to provide the necessary examples that we need in this paper to illustrate notions and separate classes of logics. In this way, as another unintended consequence of our investigation, we will also obtain some new knowledge about these two logics and about semilinear logics in general regarding their standing in the infinitary hierarchy and their subdirect representation properties.

The paper is organized as follows. Section 2 gives some necessary preliminaries from the general AAL theory and regarding filter extension properties. In Section 3 we recall the notion of natural extension and develop a new dual theory of natural expansions. Building on the methods of the previous section, Section 4 studies the preservation of the transferred and non-transferred extension properties in (axiomatic) expansions. Then Section 5 concludes the theoretical development of the paper by characterizing subdirect representation in propositional logics and providing the necessary natural examples to solve the transfer problem and separate all the classes in the hierarchy. We end the paper with some concluding remarks in Section 6.

2. Preliminaries

2.1. Abstract Algebraic Logic

In this subsection we briefly recall the definitions and fix the notations of some basic notions of abstract algebraic logic that will be needed in the paper (for comprehensive monographs and a survey see [10,14,16–18,29]); we assume some familiarity with basic notions of universal algebra (see e.g. [1]).

A *propositional language* $\boldsymbol{\mathcal{L}}$ is a pair $\langle \mathcal{L}, \mathit{Var}_{\boldsymbol{\mathcal{L}}}\rangle$, where $\mathcal{L}$ is an algebraic type and $\mathit{Var}_{\boldsymbol{\mathcal{L}}}$ an infinite set of variables. We say that $\boldsymbol{\mathcal{L}}'$ is an *extension* of $\boldsymbol{\mathcal{L}}$ (and denote it as $\boldsymbol{\mathcal{L}} \preceq \boldsymbol{\mathcal{L}}'$) whenever $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathit{Var}_{\boldsymbol{\mathcal{L}}} \subseteq \mathit{Var}_{\boldsymbol{\mathcal{L}}'}$. Furthermore, we say $\boldsymbol{\mathcal{L}}'$ is a *variable (resp. type) extension* of $\boldsymbol{\mathcal{L}}$ whenever $\boldsymbol{\mathcal{L}} \preceq \boldsymbol{\mathcal{L}}'$ and $\mathcal{L} = \mathcal{L}'$ (resp. $\mathit{Var}_{\boldsymbol{\mathcal{L}}} = \mathit{Var}_{\boldsymbol{\mathcal{L}}'}$). Although usual practice in algebraic logic does not need such level of precision in the treatment of propositional languages, the present paper will require it for reasons apparent later.

By $\boldsymbol{Fm}_{\mathcal{L}}(X)$ (resp. $Fm_{\mathcal{L}}(X)$) we denote the absolutely free term algebra of type $\mathcal{L}$ with the set X as generators (resp. its universe). We call $\boldsymbol{Fm}_{\mathcal{L}}(\mathit{Var}_{\boldsymbol{\mathcal{L}}})$ the *algebra of $\boldsymbol{\mathcal{L}}$-formulas* and we denote it simply by $\boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}}$ (we write $Fm_{\boldsymbol{\mathcal{L}}}$ for its universe, i.e. for the set of all $\boldsymbol{\mathcal{L}}$-formulas).

An *$\boldsymbol{\mathcal{L}}$-consecution* is a pair $\Gamma \rhd \varphi$. Given a set of $\boldsymbol{\mathcal{L}}$-consecutions L, we write $\Gamma \vdash_{\mathrm{L}} \varphi$ rather than $\Gamma \rhd \varphi \in \mathrm{L}$. A *logic* L in the language $\boldsymbol{\mathcal{L}}$ is a set of $\boldsymbol{\mathcal{L}}$-consecutions (i.e. $\mathrm{L} \subseteq P(Fm_{\boldsymbol{\mathcal{L}}}) \times Fm_{\boldsymbol{\mathcal{L}}}$) satisfying:

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- $\Delta \vdash_{\mathrm{L}} \varphi$ and $\Delta \subseteq \Gamma$ then $\Gamma \vdash_{\mathrm{L}} \varphi$ (Monotonicity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\boldsymbol{\mathcal{L}}$-substitution σ. (Structurality)

Finally, logic L is *finitary* if it satisfies the following condition:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$. (Finitarity)

We write $\Gamma \vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \varphi$ for every $\varphi \in \Delta$. A *theory* of a logic L is a set of formulas closed under the consequence relation. The set of all theories of L is a closure system, denoted as Th(L). By $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ we denote the theory generated by Γ.

An $\mathcal{L}$-matrix is a pair $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra (the *algebraic reduct* of the matrix) and $F \subseteq A$ is a subset called the *filter* of the matrix. Given a class $\mathbb{K}$ of $\mathcal{L}$-matrices and a language $\boldsymbol{\mathcal{L}} = \langle \mathcal{L}, \mathit{Var}_{\boldsymbol{\mathcal{L}}}\rangle$, the corresponding *semantical consequence relation* is defined as: $\Gamma \models_{\mathbb{K}} \varphi$ iff for each

$\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e (i.e. a homomorphism $e\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{A}$) such that $e[\Gamma] \subseteq F$, we have $e(\varphi) \in F$. Clearly, $\models_{\mathbb{K}}$ is a logic in $\boldsymbol{\mathcal{L}}$.

Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$, we say that a congruence θ of $\boldsymbol{A}$ is *compatible* with F iff for each $a, b \in A$, if $\langle a, b\rangle \in \theta$ and $a \in F$, then $b \in F$. Compatible congruences with F form a complete sublattice of the lattice of all congruences of $\boldsymbol{A}$, and thus there exists the maximum congruence compatible with F, which is called the *Leibniz congruence* of $\mathbf{A}$ and denoted as $\Omega_{\boldsymbol{A}}(F)$. We say that $\mathbf{A}$ is a *reduced matrix* if $\Omega_{\boldsymbol{A}}(F) = Id_{\boldsymbol{A}}$.

A matrix $\mathbf{A}$ is a *model* of L if $\vdash_{\mathrm{L}} \subseteq \models_{\{\mathbf{A}\}}$. The class of models (resp. reduced models) of a logic L is denoted as $\mathbf{MOD}(\mathrm{L})$ (resp. $\mathbf{MOD}^*(\mathrm{L})$). It is well-known that, for any logic L, both of these classes give a complete semantics (in symbols: $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}(\mathrm{L})} = \models_{\mathbf{MOD}^*(\mathrm{L})}$); however it is common to consider meaningful subclasses of reduced models, such as relatively (finitely) subdirectly irreducible matrices, which may provide stronger completeness theorems. A matrix $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ is *relatively (finitely) subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$, in symbols $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ ($\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$), if it cannot be decomposed as a non-trivial subdirect product of an arbitrary (finite non-empty) family of matrices from $\mathbf{MOD}^*(\mathrm{L})$. The class of algebraic reducts of $\mathbf{MOD}^*(\mathrm{L})$ is denoted as $\mathbf{ALG}^*(\mathrm{L})$.

Given a matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$, we say that F is an L-*filter* provided that $\mathbf{A}$ is a model of L. By $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we denote the set of all L-filters over $\boldsymbol{A}$; $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is also a closure system (and, consequently, a complete lattice) and hence it also induces a closure operator.

In this paper we will consider some logics belonging to the following implication-based class introduced in [5] (which generalizes implicative logics in sense of Rasiowa [27]). Let $\Rightarrow(p, q, \bar{r}) \subseteq Fm_{\boldsymbol{\mathcal{L}}}$ be a set of formulas in two variables and, possibly, parameters $\bar{r}$. Then, given formulas $\varphi, \psi \in Fm_{\boldsymbol{\mathcal{L}}}$, we define $\varphi \Rightarrow \psi$ as $\bigcup\{\Rightarrow(\varphi, \psi, \bar{\alpha}) \mid \bar{\alpha} \in Fm_{\boldsymbol{\mathcal{L}}}\}$, i.e. the union of the sets obtained by substituting in $\Rightarrow(p, q, \bar{r})$ p by φ, q by ψ and allowing all possible substitutions for the parameters. Moreover, we denote by $\varphi \Leftrightarrow \psi$ the set $(\varphi \Rightarrow \psi) \cup (\psi \Rightarrow \varphi)$. We say that $\Rightarrow$ is a *weak p-implication* (or just *weak implication* if there are no parameters $\bar{r}$) in L if the following conditions are satisfied:

(R) $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$,
(MP) $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$,
(T) $\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$,
(sCng) $\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \Rightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$
for each $\langle c, n\rangle \in \boldsymbol{\mathcal{L}}$ and each $i < n$.

A logic is called *weakly (p-)implicational* if it has a weak (p-)implication. If $\Rightarrow$ is given by just one formula in two variables (without parameters), then we denote it as $\to$ and call the logic *weakly implicative*. Weakly p-implicational logics are actually an alternative presentation of *protoalgebraic logics*, which can be described in terms of equivalences $\Leftrightarrow$.

An important property of such logics is that their models can be given a preorder relation induced by the implication. Indeed, given a weakly implicative logic L, a matrix $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$, and elements $a, b \in A$, we define: $a \leq b$ iff $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$. This binary relation is always a preorder, it is an order iff the model is reduced, and moreover it allows to characterize the Leibniz congruence on each model in the following way: $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ iff $(a \Rightarrow^{\boldsymbol{A}} b) \cup (b \Rightarrow^{\boldsymbol{A}} a) \subseteq F$. A reduced model is called *linear* if its corresponding order is total; in this case F is called a *linear filter*. The class of all linear models is denoted as $\mathbf{MOD}^{\ell}(\mathrm{L})$. A logic L is *semilinear* (w.r.t. a weak p-implication $\Rightarrow$) if it is complete with respect to the class $\mathbf{MOD}^{\ell}(\mathrm{L})$.

2.2. Intersection-Prime Filters

In this subsection we first recall (from [6, 10]) the definitions of the two kinds of filters that we will use in the rest of the paper and their corresponding extension properties; secondly we recall how they entail completeness with respect to subclasses of reduced matrix models mentioned above.

In general, given a closure system $\mathcal{C}$ on a set A, a set $X \in \mathcal{C}$ is called *intersection-prime in* $\mathcal{C}$ if it is finitely $\cap$-irreducible, i.e. there are no closed sets $X_1, X_2 \in \mathcal{C}$ such that $X = X_1 \cap X_2$ and $X \subsetneq X_1, X_2$. Similarly, X is *completely intersection-prime in* $\mathcal{C}$ if it is $\cap$-irreducible, i.e. whenever $X = \bigcap_{i\in I} X_i$ for a family of closed sets $\{X_i \mid i \in I\} \subseteq \mathcal{C}$, there is $i_0 \in I$ such that $X = X_{i_0}$. Given a logic L, an algebra $\boldsymbol{A}$, and a filter F, we say that F is *(completely) intersection-prime*[5] if it is (completely) intersection-prime in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$; it is analogously defined for theories and Th(L). It is well-known [10, Proposition 1.3.4.] that

- $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ iff F is *intersection-prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$,
- $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ iff F is *completely intersection-prime* in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

The following lemma will be useful later and, interestingly, it implies that in semilinear logics $\mathbf{MOD}^{\ell}(\mathrm{L}) = \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$.

Lemma 2.1. ([8, Lemma 4]) *If a logic* L *is semilinear, then intersection-prime and linear filters coincide.*

[5] We follow the terminology used in [10], p. 147.

Recall that, given a closure system $\mathcal{C}$, a family $\mathcal{B} \subseteq \mathcal{C}$ is a *basis* if for every $X \in \mathcal{C}$ there is a $\mathcal{D} \subseteq \mathcal{B}$ such that $X = \bigcap \mathcal{D}$ (which can be equivalently formulated as an extension property: for every $X \in \mathcal{C}$ and every $a \in A \setminus X$ there is $Y \in \mathcal{B}$ such that $X \subseteq Y$ and $a \notin Y$). Using these notions one can define the following properties for closure systems and for logics.

Definition 2.2. We say that a closure system $\mathcal{C}$ has the *(completely) intersection-prime extension property*, (C)IPEP for short, if the (completely) intersection-prime closed sets form a basis of $\mathcal{C}$. A logic L has the (C)IPEP if Th(L) does. We say that it has the *transferred*-(C)IPEP, τ-(C)IPEP for short, if for every $\boldsymbol{\mathcal{L}}$-algebra $\boldsymbol{A}$ the closure system $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ has the (C)IPEP. A logic L is R(F)SI-*complete* if $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}}$.

In [6] it was shown that all semilinear logics have the IPEP, though in general they do not have the CIPEP (example in [21, Sect. 3.2]). The non-transferred versions of these properties have been studied in [21] where their relationships have been described as depicted in Figure 1. We now present the two best-known examples of infinitary semilinear logics:

1. *The infinitary Łukasiewicz logic* $\mathrm{Ł}_\infty$ [23]. Consider a language with a denumerable set of variables, a unary connective $\neg$ and a binary connective $\to$. Let $[0,1]_{\mathrm{Ł}}$ be the algebra defined over the real interval $[0,1]$ with the operations:

$$a \to^{[0,1]_{\mathrm{Ł}}} b = \begin{cases} 1, & \text{if } a \leq b, \\ 1-a+b, & \text{otherwise.} \end{cases} \qquad \neg^{[0,1]_{\mathrm{Ł}}} a = 1 - a.$$

   The logic $\mathrm{Ł}_\infty$ is defined as $\models_{\langle [0,1]_{\mathrm{Ł}}, \{1\} \rangle}$.

2. *The infinitary product logic* Π_∞ [20]. Consider now a language with a denumerable set of variables, binary connectives & and $\to$, and a constant $\overline{0}$. Let $[0,1]_\Pi$ be the algebra defined over the real interval $[0,1]$ with the operations:

$$a \mathbin{\&}^{[0,1]_\Pi} b = a \cdot b \qquad a \to^{[0,1]_\Pi} b = \begin{cases} 1, & \text{if } a \leq b, \\ \frac{b}{a}, & \text{otherwise,} \end{cases} \qquad \overline{0}^{[0,1]_\Pi} = 0$$

   The logic Π_∞ is defined as $\models_{\langle [0,1]_\Pi, \{1\} \rangle}$.

Both logics are infinitary, weakly implicative, and semilinear (w.r.t. $\to$). They both validate the following proper infinitary rule:

$$\{x \to \underbrace{y \mathbin{\&} y \cdots \mathbin{\&} y}_{\text{n times}} \mid n \in \mathbb{N}\} \vdash \neg x \vee y \tag{A}$$

Also, $Ł_\infty$ and Π_∞ have been proved to have the CIPEP in [21, Sect. 3.1]. Their associated classes of algebras, $\mathbf{ALG}^*(Ł_\infty)$ and $\mathbf{ALG}^*(\Pi_\infty)$, are certain subclasses of, respectively, MV-algebras and product algebras (see e.g. [4]). Of course, the order induced by $\to$ is the usual lattice order on these algebras. In fact, since the logics are infinitary and have countable languages and sets of variables, we obtain the following description of their classes of algebras:

$$\mathbf{ALG}^*(Ł_\infty) = \mathbf{ISP}_{\mathbf{R}_{\aleph_1}}([0,1]_Ł) \quad \text{and} \quad \mathbf{ALG}^*(\Pi_\infty) = \mathbf{ISP}_{\mathbf{R}_{\aleph_1}}([0,1]_\Pi),$$

where $\mathbf{P}_{\mathbf{R}_{\aleph_1}}$ is the class operator for reduced products over $\aleph_1$-complete filters (i.e. filters closed under countable intersections). Moreover, linear models (and consequently all RSI models) of these logics are particularly well-behaved:

Proposition 2.3. *The linear models of* $Ł_\infty$ *(resp.* Π_∞*) are embeddable into* $[0,1]_Ł$ *(resp. into* $[0,1]_\Pi$*). That is*

$$\mathbf{MOD}^*(Ł_\infty)_{\mathrm{RSI}} \subseteq \mathbf{MOD}^*(Ł_\infty)_{\mathrm{RFSI}} = \mathbf{MOD}^\ell(Ł_\infty) \subseteq \mathbf{S}(\langle [0,1]_Ł, \{1\}\rangle),$$

and analogously for Π_∞.

Proof. First recall that a linear MV-(resp. product) algebra $\boldsymbol{A}$ is called *Archimedean*, if for every $a, b \in A$ such that $0 \neq a < b < 1$, there is a natural number n such that $\underbrace{b \,\&^{\boldsymbol{A}}\, b \cdots \&^{\boldsymbol{A}}\, b}_{n \text{ times}} < a$. It is easy to observe that the rule (A) implies that every linear model of both logics is, in fact, Archimedean. Every Archimedean MV- (resp. product) algebra is well-known to be embeddable into the standard one. In both cases it is proven using relations between the categories of corresponding algebras and lattice-ordered Abelian groups (for MV-algebras see e.g. [24] or [4, Chapter 5], and for product algebras see e.g. [2]). Then the result is a consequence of Hölder's theorem which says that every Archimedean linear lattice-ordered Abelian group can be embedded into the additive group of reals. ■

3. Natural Expansions

Natural extensions are a standard tool, in abstract algebraic logic, to prove *transfer theorems*, that is, to show, for a given logic L, that a property of Th(L) remains true in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ for any algebra $\boldsymbol{A}$. They are often obtained by enlarging the set of variables while essentially keeping all the properties of the logic untouched (see [7, 10, 26]). We first recall the precise definition of natural extension and then introduce a dual notion that we call *natural expansion*. The main motivation for us to introduce natural expansions lies in the fact that they will prove useful when arguing about expansions in general.

3.1. Extensions

Throughout this subsection we fix a logic L in $\boldsymbol{\mathcal{L}}$ and a variable extension $\boldsymbol{\mathcal{L}'} = \langle \mathcal{L}, \mathit{Var}_{\boldsymbol{\mathcal{L}'}}\rangle$. Then the *natural extension* of L to variables $\mathit{Var}_{\boldsymbol{\mathcal{L}'}}$ is a logic in language $\boldsymbol{\mathcal{L}'}$, denoted as $\mathrm{L}^{\boldsymbol{\mathcal{L}'}}$, which can be defined in a syntactical way by using an axiomatization of L or, alternatively, semantically by means of the class $\mathbf{MOD}(\mathrm{L})$, i.e. $\mathrm{L}^{\boldsymbol{\mathcal{L}'}} = \models_{\mathbf{MOD}(\mathrm{L})}$.

Recall that the *cardinality of a logic* L, denoted as card(L), is the least infinite cardinal κ such that, whenever $\Gamma \vdash_{\mathrm{L}} \varphi$, there is $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathrm{L}} \varphi$, where $|\Gamma'| < \kappa$. For example L is finitary if and only if $\mathrm{card}(\mathrm{L}) = \omega$.

As shown in [7], under the assumption that

$$\mathrm{card}(\mathrm{L}) \leq |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|^{+} \text{ or } |\mathit{Var}_{\boldsymbol{\mathcal{L}}}| = |\mathit{Var}_{\boldsymbol{\mathcal{L}'}}|, \tag{As1}$$

we obtain the following useful characterization (given in [28]):

$$\begin{aligned} \Gamma \vdash_{\mathrm{L}^{\boldsymbol{\mathcal{L}'}}} \varphi \quad \text{iff} \quad & \text{there is a homomorphism } \sigma\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}'}}, \\ & \text{and a set of formulas } \Delta \cup \{\psi\} \subseteq \mathit{Fm}_{\boldsymbol{\mathcal{L}}} \text{ such that} \\ & \Delta \vdash_{\mathrm{L}} \psi, \sigma[\Delta] \subseteq \Gamma, \text{ and } \sigma(\psi) = \varphi. \end{aligned} \tag{N}$$

Moreover, as proved in the same paper, the same assumption in fact guarantees that $\mathrm{L}^{\boldsymbol{\mathcal{L}'}}$ is the unique conservative extension of L with the same cardinality. In symbols, it is the only logic with the following properties:

1. $\mathrm{L} \subseteq \mathrm{L}^{\boldsymbol{\mathcal{L}'}}$ and $\mathrm{L} = \mathrm{L}^{\boldsymbol{\mathcal{L}'}} \restriction \mathit{Fm}_{\boldsymbol{\mathcal{L}}}$
2. $\mathrm{card}(\mathrm{L}) = \mathrm{card}(\mathrm{L}^{\boldsymbol{\mathcal{L}'}})$

Observation 3.1. L and $\mathrm{L}^{\boldsymbol{\mathcal{L}'}}$ have the same matrix models.

We are usually only interested in the cardinality of the set of variables. Thus we also define *the natural extension of* L *to* κ*-many variables*, denoted as L^{κ}, to be an arbitrary natural extension of L to $\mathit{Var}_{\boldsymbol{\mathcal{L}'}}$ of size κ.

3.2. Expansions

Now, instead of adding variables, we consider logics with additional connectives. Let us fix a logic L in a language $\boldsymbol{\mathcal{L}} = \langle \mathcal{L}, \mathit{Var}_{\boldsymbol{\mathcal{L}}}\rangle$ and its type extension $\boldsymbol{\mathcal{L}'} = \langle \mathcal{L}', \mathit{Var}_{\boldsymbol{\mathcal{L}}}\rangle$.

Definition 3.2. The *natural expansion*[6] *of* L *to* $\mathcal{L}'$ is the logic axiomatized by taking all $\mathcal{L}'$-substitutions of an arbitrary presentation of L. We denote it as $\mathrm{L}_{\mathcal{L}'}$.

Arguing as in [26, Proposition 7], we can describe some of the fundamental properties of $\mathrm{L}_{\mathcal{L}'}$ by means of its semantical characterization (*):

Proposition 3.3. $\mathrm{L}_{\mathcal{L}'}$ *is the smallest conservative expansion of* L *to the language* $\mathcal{L}'$ *with the same cardinality.*

Proof. Define S in the language $\mathcal{L}'$ semantically as the logic of the following class of matrices

$$\{\langle \boldsymbol{A}, F\rangle \mid \boldsymbol{A} \text{ an } \mathcal{L}'\text{-algebra and } \langle \boldsymbol{A}{\restriction}\mathcal{L}, F\rangle \in \mathbf{MOD}(\mathrm{L})\}. \qquad (*)$$

We now show that S has all the properties mentioned in the statement of the proposition. By definition, S is a conservative expansion of L to $\mathcal{L}'$. Moreover, it is the smallest expansion: To this end first observe that $\mathbf{MOD}(\mathrm{L}) = \mathbf{MOD}(\mathrm{S}){\restriction}\mathcal{L}$; the inclusion from left to right is by definition and the converse one is true because $\mathrm{L} \subseteq \mathrm{S}$. Thus, if L' is any expansion of L to $\mathcal{L}'$, then $\mathbf{MOD}(\mathrm{L}'){\restriction}\mathcal{L} \subseteq \mathbf{MOD}(\mathrm{L}) = \mathbf{MOD}(\mathrm{S}){\restriction}\mathcal{L}$. It easily follows that $\mathrm{S} \subseteq \mathrm{L}'$. Let S' denote the restriction of S to consecutions with less than card(L) premises. Then, since it is obviously an expansion of L, we obtain $\mathrm{S} \subseteq \mathrm{S}'$; the other direction is clear. In particular L and S have the same cardinality.

Finally, since $\mathrm{L}_{\mathcal{L}'}$ is clearly the smallest expansion of L to $\mathcal{L}'$, we obtain $\mathrm{S} = \mathrm{L}_{\mathcal{L}'}$. In particular, $\mathrm{L}_{\mathcal{L}'}$ has all the desired properties. ■

Now we aim at developing a link between natural extensions and expansions (Proposition 3.6). Recall that the assumption (As1) on L entails a useful characterization for its natural extensions by means of (N); we will later see an analogous characterization for natural expansions (Proposition 3.7).

To this end, define the following cardinal

$$\mathfrak{c} = \begin{cases} |Fm_{\mathcal{L}'}| = \max\{|\mathit{Var}_{\mathcal{L}}|, |\mathcal{L}'|\} & \text{if } \mathcal{L}'\backslash\mathcal{L} \text{ has a non-nullary connective,} \\ |\mathcal{L}'\backslash\mathcal{L}| & \text{otherwise.} \end{cases}$$

and define the following set of $\mathcal{L}'$-formulas:

$$X_{\mathcal{L}}^{\mathcal{L}'} = \{c(\varphi_1, \ldots, \varphi_n) \in Fm_{\mathcal{L}'} \mid \varphi_1, \ldots, \varphi_n \in Fm_{\mathcal{L}'} \text{ and } c \in \mathcal{L}'\backslash\mathcal{L}\}$$

[6] We choose the terminology "natural expansion" because it aptly captures the meaning of the notion and its resemblance to natural extensions, despite the fact that it was already used in the literature for different purposes (cf. [3]).

and observe that $\mathfrak{c} = |X_{\mathcal{L}}^{\mathcal{L}'}|$. Finally, define the following set of variables:

$$Var_{\boldsymbol{\mathcal{S}}} = Var_{\boldsymbol{\mathcal{L}}} \cup \{x_\varphi \mid \varphi \in X_{\mathcal{L}}^{\mathcal{L}'}\} \tag{1}$$

Thus, $Var_{\boldsymbol{\mathcal{S}}}$ has a new variable for every formula of the new language starting with a new connective. Now we show that $\mathrm{L}_{\mathcal{L}'}$, the natural expansion of L to $\mathcal{L}'$, and $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$, the natural extension of L to variables $Var_{\boldsymbol{\mathcal{S}}}$ (i.e. to the language $\boldsymbol{\mathcal{S}} = \langle \mathcal{L}, Var_{\boldsymbol{\mathcal{S}}} \rangle$) are actually the same logics modulo a certain translation $\boldsymbol{\tau}$.

A map $h : \boldsymbol{A} \to \boldsymbol{B}$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and $\boldsymbol{B}$ an $\mathcal{L}'$-algebra, is called an *$\mathcal{L}$-homomorphism*, if it is a homomorphism between $\boldsymbol{A}$ and the $\mathcal{L}$-reduct of $\boldsymbol{B}$. Then the translation $\boldsymbol{\tau}$ is defined as an $\mathcal{L}$-homomorphism $\boldsymbol{\tau}\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{S}}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}'}}$ by

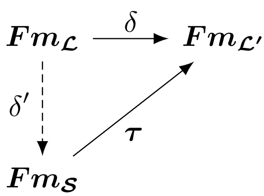
$$\boldsymbol{\tau}(x) = \begin{cases} x & \text{if } x \in Var_{\boldsymbol{\mathcal{L}}} \\ \varphi & x = x_\varphi \text{ for } \varphi \in X_{\mathcal{L}}^{\mathcal{L}'} \end{cases} \tag{2}$$

Moreover, define recursively a map $\boldsymbol{\tau'}\colon Fm_{\boldsymbol{\mathcal{L}'}} \to Fm_{\boldsymbol{\mathcal{S}}}$ as follows: $\boldsymbol{\tau'}(x) = x$, $\boldsymbol{\tau'}(c) = c$ for each constant of $\mathcal{L}$ and $\boldsymbol{\tau'}(c) = x_c$ for each new constant. If c is an n-ary connective of $\mathcal{L}$ and $\varphi = c(\varphi_1, \dots, \varphi_n)$, then $\boldsymbol{\tau'}(\varphi) = c(\boldsymbol{\tau'}(\varphi_1), \dots, \boldsymbol{\tau'}(\varphi_n))$. If c is a new n-ary connective and $\varphi = c(\varphi_1, \dots, \varphi_n)$, then $\boldsymbol{\tau'}(\varphi) = x_\varphi$. Using induction it is easy to prove:

LEMMA 3.4. *$\boldsymbol{\tau}$ is a bijection from $Fm_{\boldsymbol{\mathcal{S}}}$ onto $Fm_{\boldsymbol{\mathcal{L}'}}$ with inverse $\boldsymbol{\tau'}$.*

Therefore, the formulas of $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ and $\mathrm{L}_{\mathcal{L}'}$ are in a bijective correspondence.

LEMMA 3.5. *For every $\mathcal{L}$-homomorphism $\delta\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}'}}$, there is a homomorphism $\delta'\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{S}}}$ such that $\delta = \boldsymbol{\tau} \circ \delta'$, i.e. the following diagram commutes:*



PROOF. By the previous lemma it is enough to set $\delta'(x) = \boldsymbol{\tau'}\delta(x)$. ■

PROPOSITION 3.6. *For any fomulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\boldsymbol{\mathcal{S}}}$, we have*

$$\Gamma \vdash_{\mathrm{L}^{\boldsymbol{\mathcal{S}}}} \varphi \quad \textit{if and only if} \quad \boldsymbol{\tau}[\Gamma] \vdash_{\mathrm{L}_{\mathcal{L}'}} \boldsymbol{\tau}(\varphi). \tag{3}$$

PROOF. By Lemma 3.4, it is enough to show that the translations $\boldsymbol{\tau}$ and $\boldsymbol{\tau'}$ preserve proofs. First, suppose $\Delta \rhd \psi$ is a rule of $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$. By definition of the logic $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$, there is a rule $\Delta' \rhd \psi'$ of L and a homomorphism $h\colon \boldsymbol{Fm}_{\boldsymbol{\mathcal{L}}} \to \boldsymbol{Fm}_{\boldsymbol{\mathcal{S}}}$ such that $h[\Delta'] = \Delta$ and $h(\psi') = \psi$. Then $\boldsymbol{\tau}h$ witnesses that $\boldsymbol{\tau}[\Delta] \rhd \boldsymbol{\tau}(\psi)$ is a rule of $\mathrm{L}_{\mathcal{L}'}$. Conversely, let $\Delta \rhd \psi$ be a rule of $\mathrm{L}_{\mathcal{L}'}$. By definition of the logic $\mathrm{L}_{\mathcal{L}'}$ there is a rule $\Delta' \rhd \psi'$ of L and $\mathcal{L}$-homomorphism $\delta\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\mathcal{L}'}$ such that $\delta[\Delta'] = \Delta$ and $\delta(\psi') = \psi$. Let δ' be as in Lemma 3.5; then obviously $\delta'[\Delta'] \rhd \delta'(\psi')$ is a rule of $\mathrm{L}^{\mathcal{S}}$ equal to $\boldsymbol{\tau}'[\Delta] \rhd \boldsymbol{\tau}'(\psi)$. ■

PROPOSITION 3.7. *Suppose that either* $\mathrm{card}(\mathrm{L}) \leq |\mathit{Var}_{\mathcal{L}}|^{+}$ *or* $\mathfrak{e} \leq |\mathit{Var}_{\mathcal{L}}|$. *Then the natural expansion of* L *to the language* $\mathcal{L}'$ *can be characterized as:*

$$\Gamma \vdash_{\mathrm{L}_{\mathcal{L}'}} \varphi \quad \textit{iff} \quad \begin{array}{l}\textit{there is an } \mathcal{L}\textit{-homomorphism } \sigma\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\mathcal{L}'},\\ \textit{and a set of formulas } \Delta \cup \{\psi\} \subseteq Fm_{\mathcal{L}} \textit{ such that}\\ \Delta \vdash_{\mathrm{L}} \psi, \sigma[\Delta] \subseteq \Gamma, \textit{ and } \sigma(\psi) = \varphi.\end{array} \tag{M}$$

PROOF. Take $\mathit{Var}_{\mathcal{S}}$ as in (1). Then the logic $\mathrm{L}^{\mathcal{S}}$ satisfies the assumptions (As1), since if $\mathfrak{e} \leq |\mathit{Var}_{\mathcal{L}}|$ then $|\mathit{Var}_{\mathcal{S}}| = |\mathit{Var}_{\mathcal{L}}|$. Then using the fact that $\mathrm{L}^{\mathcal{S}}$ is characterized by (N), one can, similarly as in Proposition 3.6, obtain the desired characterization of $\mathrm{L}_{\mathcal{L}'}$. ■

Note that, as in the case of natural extensions, the conditions of the previous proposition are there to ensure that the relation defined by the right side of (M) satisfies (Cut). The conditions are necessary: indeed, thanks to Proposition 3.6 (extending by variables is basically the same as expanding by constants), we can use the same counterexample as in [7]. On the other hand, not even under the assumptions of the previous proposition, we can guarantee that $\mathrm{L}_{\mathcal{L}'}$ is *the unique* conservative natural expansion with the same cardinality. Indeed, let L be the least logic in $\mathcal{L}$. Then L with an additional new constant c which is also added as an axiom c has all the properties mentioned above (and it is different from $\mathrm{L}_{\mathcal{L}'}$).

We can capture the translatability between natural extensions and expansions by means of the following notion.

DEFINITION 3.8. Let L and L′ be logics in languages with $\mathcal{L} \subseteq \mathcal{L}'$, with variables $\mathit{Var}_{\mathcal{L}}$ and $\mathit{Var}_{\mathcal{L}'}$, respectively. We say that L *isomorphically embeds* into L′, in symbols $\mathrm{L} \precsim \mathrm{L}'$, if there is an isomorphism $\boldsymbol{\tau}\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{Fm}_{\mathcal{L}'} \upharpoonright \mathcal{L}$ and for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \text{if and only if} \quad \boldsymbol{\tau}[\Gamma] \vdash_{\mathrm{L}'} \boldsymbol{\tau}(\varphi). \tag{4}$$

In the conditions of the previous definition, we denote $V = \boldsymbol{\tau}^{-1}[\mathit{Var}_{\mathcal{L}'}]$; obviously $V \subseteq \mathit{Var}_{\mathcal{L}}$. It is easy to see that L′ is an expansion of $\mathrm{S} = \mathrm{L}' \upharpoonright \mathcal{L}$ and L is a conservative extension of S, obtained by extending the set of variables V to $\mathit{Var}_{\mathcal{L}}$. Moreover, if L and S have the same cardinality and S satisfies (As1), then L′ is the natural expansion of S to $\mathcal{L}'$.

In particular, $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ isomorphically embeds into $\mathrm{L}_{\mathcal{L}'}$, that is $\mathrm{L}^{\boldsymbol{\mathcal{S}}} \precsim \mathrm{L}_{\mathcal{L}'}$, where, of course, $\mathrm{L}_{\mathcal{L}'}$ is the natural expansion of L to $\boldsymbol{\mathcal{L}'}$ and $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ the corresponding natural extension to variables $Var_{\boldsymbol{\mathcal{S}}}$ described in this section.

Proposition 3.9. *If* $\mathrm{L} \precsim \mathrm{L}'$, *then* $\mathrm{Th}(\mathrm{L})$ *and* $\mathrm{Th}(\mathrm{L}')$ *are isomorphic lattices. Consequently,* L *has the* (C)IPEP *if and only if* L′ *does.*

Proof. Let τ witness $\mathrm{L} \precsim \mathrm{L}'$ and let τ' be its inverse. Lift these functions in the obvious way to $\boldsymbol{\tau} : P(Fm_{\mathcal{L}}) \leftrightarrows P(Fm_{\mathcal{L}'}) : \boldsymbol{\tau'}$. These lifted mappings are as well inverse to each other and monotonous. Moreover, (4) ensures that $\boldsymbol{\tau}[\mathrm{Th}(\mathrm{L})] \subseteq \mathrm{Th}(\mathrm{L}')$ and $\boldsymbol{\tau'}[\mathrm{Th}(\mathrm{L}')] \subseteq \mathrm{Th}(\mathrm{L})$. Thus, $\boldsymbol{\tau}$ restricted to theories is the desired lattice isomorphism. Finally, the IPEP and the CIPEP are clearly properties preserved by isomorphism between complete lattices. ■

4. (C)IPEP and Expansions

In this section we investigate the preservation of the (C)IPEP under expansions. In Section 4.1 we see that these properties are in general not preserved when adding rules (even finitary rules in the same language). Then, in Section 4.2, we show that the IPEP and the CIPEP are always preserved by axiomatic extensions and we specify a condition under which they are also preserved by axiomatic expansions. Moreover, we show that their transferred variants are preserved by axiomatic expansions of protoalgebraic logics. To obtain the results about preservation under expansions we use the notion of natural expansion developed in the previous section.

4.1. Finitary Extensions

We define a logic L in a language $\boldsymbol{\mathcal{L}}$ with three unary connectives l, r, o and countable set of variables. We use metavariables $s, s', \dots$ for finite *nonempty* sequences of $\{l, r\}$. We denote the set of all of them as Seq, which is naturally ordered by: $s < s'$ iff s is a strictly initial sequence of s'. Therefore $\langle \mathrm{Seq}, < \rangle$ can be seen as the full binary tree of height ω without root. So we can see ls as the extension of the node s to the left, and rs as the extension to the right in $\langle \mathrm{Seq}, < \rangle$. Recall that $B \subseteq \mathrm{Seq}$ is a branch in $\langle \mathrm{Seq}, < \rangle$ if it is a maximal chain. The logic L is axiomatized by taking the following infinitary rule for each branch B:

$$\{s(\varphi) \mid s \in B\} \vdash o(\varphi). \tag{B}$$

Let us show that L has the CIPEP. Indeed, let T be a theory and φ a formula and suppose that $\varphi \notin T$; then, if φ is not of the form $o(\psi)$

for some formula ψ, we can take the completely intersection-prime theory $T' = Fm_{\boldsymbol{\mathcal{L}}} \setminus \{\varphi\}$ (the formula φ simply cannot be proven from any premises). If $\varphi = o(\psi)$ then define

$$C = \{s \in \mathrm{Seq} \mid s(\psi) \notin T \text{ and whenever } s' < s \text{ then } s'(\psi) \in T\}$$

and let $T' = Fm_{\boldsymbol{\mathcal{L}}} \setminus (\{s(\psi) \mid s \in C\} \cup \{\varphi\})$. First observe that for every branch B, there is a unique $s \in B$ such that $s(\psi) \notin T'$. Indeed, such s always exists, since otherwise one application of (B) would yield $T \vdash_{\mathrm{L}} \varphi$. If there were two, let us say, $s < s'$, then by the definition of C both $s(\psi) \in T$ and $s(\psi) \notin T$.

Let us prove now that T' is an L-theory. Since φ is the only formula starting with o which is not in T', then, by the definition of L, $\mathrm{Th}_{\mathrm{L}}(T') = T'$ or $\mathrm{Th}_{\mathrm{L}}(T') = T' \cup \{\varphi\}$. If it was the second case, then for some branch B, $\{s(\psi) \mid s \in B\} \subseteq T'$, however, as argued above, this is not possible. Moreover T' clearly extends T. T' is a maximal theory w.r.t. φ: If $s(\psi)$ is not in T' and B is any branch containing s, then, by the uniqueness part of the observation above, for every other $s' \in B$ we have $s'(\psi) \in T'$ and consequently $T', s(\psi) \vdash_{\mathrm{L}} \varphi$, as witnessed by one application of (B); thus L has the CIPEP.

Define L$'$ as the extension of L by the following finitary rules:

$$l(\varphi) \vdash \varphi \quad \text{and} \quad r(\varphi) \vdash \varphi, \tag{5}$$

that can be interpreted as: If a theory contains the node $s(\varphi)$, then it contains all of its predecessors. We show that L$'$ does not have the IPEP. Obviously $l(p), r(p) \nvdash_{\mathrm{L}'} o(p)$. Let T be any theory containing $l(p)$ and $r(p)$ such that $T \nvdash_{\mathrm{L}'} o(p)$. It follows that there must be some sequence s_0 such that $s_0(p) \in T$ and there is no succeeding node s' above s_0 such that $s'(p) \in T$ (otherwise the rules (5) and (B) would give that $o(p) \in T$). It is now a simple observation that $\{ls_0(p)\} \cup T$ and $\{rs_0(p)\} \cup T$ are L$'$-theories (it is obviously closed under (5), and any infinitary rules that was not applicable in T would have to be of the form $\{s(\chi) \mid s \in B\} \vdash o(\chi)$ for some branch B, but any such a rule lacks infinitely many premises in T). It is obvious that $T = (\{ls_0(p)\} \cup T) \cap (\{rs_0(p)\} \cup T)$, and hence L$'$ does not have the IPEP.

4.2. Axiomatic Expansions

In this section we study the preservation of IPEP and the CIPEP under axiomatic expansions. Let us fix language $\boldsymbol{\mathcal{L}}$ and its type extension $\boldsymbol{\mathcal{L}'}$. Clearly, axiomatic expansions can be seen as axiomatic extensions of the corresponding natural expansions; thus we can divide accordingly the preservation theorem into two parts.

PROPOSITION 4.1. IPEP, CIPEP, τ-IPEP, *and* τ-CIPEP *are preserved under axiomatic extensions.*[7]

PROOF. Let L' be an axiomatic extension of L. Assume, for instance, that L has the τ-IPEP (the proof for the other cases is analogous). Take $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L}')$ and $a \in A\backslash F$. Since clearly also $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, there is an intersection-prime filter $F' \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ such that $a \notin F'$. However we also have that $F' \in \mathcal{F}i_{\mathrm{L}'}(\boldsymbol{A})$, because F' it is closed under all rules of L' and moreover, since $F \subseteq F'$, it is also closed under the new axioms. ∎

Capitalizing on the results of the previous section, we can also prove the preservation under natural expansions. To this end, we need the following auxiliary result; recall the cardinal $\mathfrak{e}$ defined on page 11.

PROPOSITION 4.2. *Let* L *be a logic in* $\boldsymbol{\mathcal{L}}$ *and take* $\kappa = \max\{|\mathit{Var}_{\boldsymbol{\mathcal{L}}}|, \mathfrak{e}\}$. *Then the following are equivalent:*

(i) L^{κ} *has the* (C)IPEP,

(ii) $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$ *has the* (C)IPEP.

PROOF. Take $\mathit{Var}_{\boldsymbol{\mathcal{S}}}$ as in (1). The assumptions ensure that $|\mathit{Var}_{\boldsymbol{\mathcal{S}}}| = \kappa$, thus we can identify $\mathrm{L}^{\boldsymbol{\mathcal{S}}}$ with L^{κ}. By Proposition 3.6 and comments below Definition 3.8, we obtain $\mathrm{L}^{\kappa} \precsim \mathrm{L}_{\boldsymbol{\mathcal{L}}'}$. Then the result follows from Proposition 3.9. ∎

THEOREM 4.3. *Let* L' *in* $\boldsymbol{\mathcal{L}}'$ *be an axiomatic expansion of* L *in* $\boldsymbol{\mathcal{L}}$ *and assume that* $\mathfrak{e} \leq |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$. *If* L *has the* (C)IPEP, *then so does* L'.

PROOF. The assumption $\mathfrak{e} \leq |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$ says that Proposition 4.2 applies for $\kappa = |\mathit{Var}_{\boldsymbol{\mathcal{L}}}|$, thus L^{κ} can be identified with L and we can conclude that $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$ has the (C)IPEP. Further, since L' is clearly an axiomatic extension of $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$, it has the (C)IPEP by Proposition 4.1. ∎

In particular the theorem always applies if $\mathcal{L}'$ and $\mathit{Var}_{\boldsymbol{\mathcal{L}}}$ are countable. Also observe that axiomatic expansions by countably many constants always preserve both IPEP and CIPEP (this kind of expansions have been deeply studied in the field of fuzzy logics, see e.g. [11,13]). Moreover, the cardinal restriction in Theorem 4.3 is necessary (even for protoalgebraic logics); indeed, the infinitary product logic Π_∞ does have the CIPEP (see [21]), but there exists a cardinal κ such that the logic Π_∞^{κ} does not have the IPEP (see Theorem 5.13), thus by Proposition 4.2 the natural expansion of Π_∞

[7] In the case of the IPEP this result was already proved in [6, Lemma 2.8].

to a language with additional κ-many constants enjoys neither the CIPEP nor the IPEP.

THEOREM 4.4. *For protoalgebraic logics both* τ-IPEP *and* τ-CIPEP *are preserved under arbitrary axiomatic expansions.*

PROOF. Assume L is a protoalgebraic logic in $\boldsymbol{\mathcal{L}}$ and it has the τ-(C)IPEP. Since protoalgebraic logics are closed under arbitrary axiomatic expansions, by Corollary 5.3 and Proposition 4.1, it is enough to argue that any arbitrarily large natural extension $(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^{\kappa}$ (which is a logic in language $\bar{\boldsymbol{\mathcal{L}}} = \langle \mathcal{L}', \kappa\rangle$) of the natural expansion $\mathrm{L}_{\boldsymbol{\mathcal{L}}'}$ has the (C)IPEP.

This can be proven by a slight modification of the reasoning seen in Section 3. We can again prove a variant of Proposition 3.6 for $(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^{\kappa}$, i.e.

$$\Gamma \vdash_{\mathrm{L}\boldsymbol{\mathcal{S}}} \varphi \quad \text{if and only if} \quad \boldsymbol{\tau}[\Gamma] \vdash_{(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^{\kappa}} \boldsymbol{\tau}(\varphi),$$

which can be done completely analogously, with the difference that instead of the set $X_{\boldsymbol{\mathcal{L}}}^{\boldsymbol{\mathcal{L}}'}$ we use:

$$X_{\boldsymbol{\mathcal{L}}}^{\boldsymbol{\mathcal{L}}'(\boldsymbol{\kappa})} = \{c(\varphi_1, .., \varphi_n) \in Fm_{\mathcal{L}'}(\kappa) \mid \varphi_1, .., \varphi_n \in Fm_{\mathcal{L}'}(\kappa), \text{ and } c \in \mathcal{L}'\backslash\mathcal{L}\}$$

and, of course, as the set $Var_{\boldsymbol{\mathcal{S}}}$ we choose $Var_{\boldsymbol{\mathcal{L}}} \cup \{x_\varphi \mid \varphi \in X_{\boldsymbol{\mathcal{L}}}^{\boldsymbol{\mathcal{L}}'(\boldsymbol{\kappa})}\}$.

Similarly as in Section 3, the translation $\boldsymbol{\tau}$ is an isomorphism from $\boldsymbol{Fm_{\mathcal{S}}}$ onto $\boldsymbol{Fm}_{\bar{\boldsymbol{\mathcal{L}}}} \upharpoonright \mathcal{L}$. Therefore for $\lambda = |Var_{\boldsymbol{\mathcal{S}}}|$ we have $\mathrm{L}^{\lambda} \precsim (\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^{\kappa}$. Consequently, since L^{λ} has the (C)IPEP, so does $(\mathrm{L}_{\boldsymbol{\mathcal{L}}'})^{\kappa}$ (Proposition 3.9). ∎

5. Subdirect Representation and Examples

In this section we consider the notion of subdirect representation, a cornerstone of universal algebra, in the framework of abstract algebraic logic. Namely, we say that a logic L is *(finitely) subdirectly representable* if its reduced models are representable as subdirect product of (finitely) subdirectly irreducibles, in symbols: $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P_{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}})$. We will prove that such property is equivalent with protoalgebraicity plus τ-(C)IPEP. As an example, we will show that the infinitary Łukasiewicz logic has the τ-CIPEP and so do its axiomatic expansions (by Theorem 4.4); as a consequence, we will obtain that every such an expansion is subdirectly representable. The rest of the section is devoted to the presentation of other examples of semilinear logics that separate all the classes in the hierarchy.

5.1. Subdirect Representation in Abstract Algebraic Logic

Let us prove first that, in protoalgebraic logics, surjective homomorphisms preserve the (C)IPEP.

Proposition 5.1. *Let* L *be a protoalgebraic logic in a language* $\boldsymbol{\mathcal{L}}$*. Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be* $\mathcal{L}$*-algebras and* $h : \boldsymbol{A} \twoheadrightarrow \boldsymbol{B}$ *be a surjective homomorphism. Then,* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$ *has the* (C)IPEP *whenever* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *does.*

Proof. Assume $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ has the (C)IPEP and take $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$. Consider h as a strict surjective homomorphism between matrices

$$h : \langle \boldsymbol{A}, h^{-1}[F] \rangle \twoheadrightarrow \langle \boldsymbol{B}, F \rangle$$

The correspondence theorem of protoalgebraic logics (see e.g. [14, Theorem 6.20]) ensures that if $h^{-1}[F]$ can be decomposed as an intersection of (completely) intersection-prime filters, so can F. ■

Corollary 5.2. *Let* L *be a protoalgebraic logic in a language* $\boldsymbol{\mathcal{L}}$*,* $|Var_{\boldsymbol{\mathcal{L}}}| \leq \kappa$ *an infinite cardinal, and suppose* L^{κ} *has the* (C)IPEP*. Then* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ *has the* (C)IPEP *for every* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *with* $|A| \leq \kappa$*.*

Proof. Observation 3.1 clearly implies that $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}(\kappa))$ has the (C)IPEP if and only if $\mathrm{Th}(\mathrm{L}^{\kappa})$ does (i.e. if and only if L^{κ} has the (C)IPEP). But there is a surjective $h : \boldsymbol{Fm}_{\mathcal{L}}(\kappa) \twoheadrightarrow \boldsymbol{A}$. The rest follows from the previous proposition. ■

As an easy consequence, we can obtain a useful characterization of the τ-IPEP and the τ-CIPEP in terms of natural extensions:

Corollary 5.3. *Let* L *be a protoalgebraic logic. Then the following are equivalent:*

(i) L^{κ} *has the* (C)IPEP *for every* $\kappa \geq |Var_{\boldsymbol{\mathcal{L}}}|$*,*

(ii) L *has* τ-(C)IPEP*.*

Moreover, the implication from bottom to top holds for each logic L*.*

Proof. The implication from (i) to (ii) simply follows from Corollary 5.2. The other one: τ-(C)IPEP implies that $\mathcal{F}i_{\mathrm{L}}(Fm_{\mathcal{L}}(\kappa))$ has the (C)IPEP, but then so does $\mathrm{Th}(\mathrm{L}^{\kappa})$ (by Observation 3.1 they are in fact the same lattices). ■

We still need another auxiliary result connecting the transferred extension properties with a decomposition of filters in reduced models:

Proposition 5.4. *A protoalgebraic logic* L *has* τ-(C)IPEP *if, and only if, for each* $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$*,* F *is an intersection of (completely) intersection-prime filters.*

Proof. The direction from left to right is obvious. For the other one consider $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ and let h be the reduction map:

$$h\colon \langle \boldsymbol{A}, F\rangle \twoheadrightarrow \langle \boldsymbol{A}, F\rangle^* = \langle \boldsymbol{A}^*, F^*\rangle.$$

Since L is protoalgebraic, h is strict and surjective, and $F^* = \bigcap_{i\in I} G_i$, where every G_i is intersection-prime, we obtain $F = \bigcap_{i\in I} h^{-1}[G_i]$, as we wanted. The correspondence theorem ensures that $h^{-1}[G_i]$ are (completely) intersection-prime. ■

Using all these elements now we are ready to prove the main result of this subsection.

Theorem 5.5. *For any logic* L*, the following are equivalent:*

i. L *is protoalgebraic and has the* τ-IPEP *(resp.* τ-CIPEP*).*

ii. $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P_{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}})$
(resp. $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P_{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}})$*).*

Proof. We prove the case of the τ-IPEP (for the τ-CIPEP it is analogous). (i) implies (ii): Take $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$. Then, by the τ-IPEP and the previous proposition, we have $F = \bigcap_{i\in I} F_i$, where each F_i is an intersection-prime filter. Therefore there is a natural subdirect representation:

$$h : \langle \boldsymbol{A}, F\rangle \hookrightarrow_{\mathbf{SD}} \prod_{i\in I} \langle \boldsymbol{A}, F_i\rangle^*.$$

The fact that $\langle A, F_i\rangle^* \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ easily follows from the assumption that every F_i is intersection-prime (recall the equivalence before Lemma 2.1). The other inclusion is due to protoalgebraicity.

(ii) implies (i): Since $\mathbf{MOD}^*(\mathrm{L})$ is closed under formation of subdirect products, L is protoalgebraic. By Proposition 5.4, it is sufficient to prove that every filter F from a reduced model $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$ can be decomposed as an intersection of intersection-prime filters. By the assumption, there is a set of RFSI reduced models $\{\langle \boldsymbol{B}_i, G_i\rangle\}_{i\in I}$ and an embedding

$$h\colon \langle \boldsymbol{A}, F\rangle \hookrightarrow_{\mathbf{SD}} \prod_{i\in I} \langle \boldsymbol{B}_i, G_i\rangle.$$

It is easy to verify that F is the intersection of all filters $F_i = (\pi_i \circ h)^{-1}[G_i]$, where π_i is the i-th projection of the product $\prod_{i\in I} \langle \boldsymbol{B}_i, G_i\rangle$. Moreover every filter F_i is intersection-prime, because G_i is intersection-prime, L is protoalgebraic, and $\pi_i \circ h$ is strict and surjective. ∎

5.2. Subdirect Representation in Universal Algebra

In universal algebra, it is well-known (Birkhoff's theorem; see e.g. [1]) that any variety $\mathbb{V}$ and quasivariety $\mathbb{Q}$ of algebras can be described in terms of subdirect products of their (relatively) subdirectly irreducible members:

$$\mathbb{V} = \mathbf{P_{SD}}(\mathbb{V}_{\mathrm{SI}}) \quad \text{and} \quad \mathbb{Q} = \mathbf{P_{SD}}(\mathbb{Q}_{\mathrm{RSI}}).$$

Notice the clear formal analogy with subdirect representation for models of logics. It is well-known that finitary logics are representable and, of course, there is an obvious connection between finitary of logics and (quasi)varieties: namely, both can be syntactically presented by expressions with finitely-many premises (finite rules in the case of logics, quasiequations in the case of quasivarieties). A natural question is whether the subdirect representations can be extended to more general classes of algebras and to infinitary logics. The example in Section 5.3.2 will give a negative answer to this question.

The equational counterpart of infinitary rules are *generalized quasiequations*, which are expressions of the form:

$$\{\alpha_i \approx \beta_i \mid i \in I\} \rhd \alpha \approx \beta, \tag{6}$$

for possibly infinite sets I. A class of algebras $\mathbb{K}$ is called a *prevariety* if it is the class of models of a collection of generalized quasiequations. Moreover, $\mathbb{K}$ is a prevariety precisely when it is closed under isomorphisms, subalgebras, and products, i.e. $\mathbb{K} = \mathbf{ISP}(\mathbb{K})$. Thus, every prevariety contains free algebras. Prevarieties are often simply called **ISP**-*classes*.

In analogy with the development of the previous subsection, we can characterize when a given prevariety is *(finitely) subdirectly representable*, that is, when $\mathbb{K} = \mathbf{P_{SD}}(\mathbb{K}_{\mathrm{RSI}})$ (resp. $\mathbb{K} = \mathbf{P_{SD}}(\mathbb{K}_{\mathrm{RFSI}})$).

THEOREM 5.6. *Let $\mathbb{K}$ be a prevariety. Then the following are equivalent:*

i. $\mathbb{K}$ *is (finitely) subdirectly representable,*
ii. $\mathrm{Con}_{\mathbb{K}}(\boldsymbol{A})$[8] *has the* CIPEP *(resp.* IPEP*) for every $\boldsymbol{\mathcal{L}}$-algebra $\boldsymbol{A}$,*
iii. $\mathrm{Con}_{\mathbb{K}}(\boldsymbol{Fm}_{\mathcal{L}}(\kappa))$ *has the* CIPEP *(resp.* IPEP*) for every cardinal κ.*

Note that, similarly to Proposition 5.4, the condition (ii) is equivalent to

iv. For every $\boldsymbol{A} \in \mathbb{K}$, the identity congruence is an intersection of (completely) intersection-prime $\mathbb{K}$-congruences.

[8] $\mathrm{Con}_{\mathbb{K}}(\boldsymbol{A})$ is the set of all $\mathbb{K}$-congruences, i.e. congruences Θ on $\boldsymbol{A}$ such that $\boldsymbol{A}/\Theta \in \mathbb{K}$.

5.3. Examples

In this subsection we show three examples of semilinear logics that, based on previous results of the paper, allow to separate all the classes in our hierarchy of infinitary logics.

5.3.1. Infinitary Łukasiewicz Logic For each infinite cardinal κ, let $\text{Ł}_{\infty,\kappa}$ be the logic of the standard Łukasiewicz matrix, $\langle [0,1]_{\text{Ł}}, \{1\}\rangle$, in a language with κ-many variables. We want to prove the transfer result using Corollary 5.3; to this end we show that $\text{Ł}_{\infty,\kappa}$ is in fact the natural extension of Ł_{∞}.

PROPOSITION 5.7. *Let κ be an infinite cardinal. Then $\text{Ł}_{\infty,\kappa}$ has cardinality $\aleph_1$. In particular, $\text{Ł}_{\infty,\kappa}$ is the natural extension of Ł_{∞} to κ-many variables, i.e. $\text{Ł}_{\infty,\kappa} = \text{Ł}^{\kappa}_{\infty}$.*

PROOF. Define for each formula φ and each rational $q \in (0,1)$ the following sets of evaluations:

$$\begin{aligned} \mathrm{NSAT}(\varphi) &= \{v\colon \kappa \to [0,1] \mid v(\varphi) \neq 1\} \\ \mathrm{SAT}(\varphi) &= \{v\colon \kappa \to [0,1] \mid v(\varphi) = 1\} \\ \mathrm{SAT}_q(\varphi) &= \{v\colon \kappa \to [0,1] \mid v(\varphi) > q\} \end{aligned}$$

Since the operations of Łukasiewicz logic are all continuous w.r.t. the standard interval topology on $[0,1]$, we obtain that for each φ and q the sets $\mathrm{NSAT}(\varphi)$ and $\mathrm{SAT}_q(\varphi)$ are open in $[0,1]^\kappa$, the topological product of κ-many copies of $[0,1]$. This follows from the fact that we can see every formula φ as a *continuous* mapping $\boldsymbol{\varphi} : [0,1]^\kappa \to [0,1]$ such that $\boldsymbol{\varphi}(v) = v(\varphi)$, thus for example $\mathrm{SAT}_q(\varphi) = \boldsymbol{\varphi}^{-1}[\Uparrow q]$, where $\Uparrow q = \{r \in [0,1] \mid q < r\}$, which is, of course, an open set.

Moreover, for every set of formulas Δ and every formula χ, we have the following equivalence:

$$\Delta \vdash_{\text{Ł}_{\infty,\kappa}} \chi \iff \bigcup_{\psi\in\Delta} \mathrm{NSAT}(\psi) \cup \mathrm{SAT}(\chi) = [0,1]^\kappa \tag{7}$$

Furthermore, since the filter $\{1\}$ can be obtained as the intersection of countably many sets of the form $\Uparrow q$, for each rational $q \in (0,1)$, it follows that $\mathrm{SAT}(\varphi)$ is an intersection of countably many open sets:

$$\mathrm{SAT}(\varphi) = \bigcap_{q\in(0,1)} \mathrm{SAT}_q(\varphi) \tag{8}$$

Clearly, since $\text{Ł}_{\infty,\kappa}$ is a conservative extension of Ł_{∞}, it has cardinality at least $\aleph_1$. To prove the other inequality assume $\Gamma \vdash_{\text{Ł}_{\infty,\kappa}} \varphi$. We need to show that there is a countable $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\text{Ł}^\kappa_\infty} \varphi$. From (7) we obtain

$$\bigcup_{\gamma\in\Gamma} \text{NSAT}(\gamma) \cup \text{SAT}(\varphi) = [0,1]^\kappa.$$

Then for any rational q, since obviously $\text{SAT}(\varphi) \subseteq \text{SAT}_q(\varphi)$, we obtain

$$\bigcup_{\gamma\in\Gamma} \text{NSAT}(\gamma) \cup \text{SAT}_q(\varphi) = [0,1]^\kappa.$$

Thus we have an open cover of $[0,1]^\kappa$. Therefore, by compactness, we obtain a finite $\Gamma_q \subseteq \Gamma$ that generates a subcover. We define $\Gamma' = \bigcup_{q\in[0,1]\cap\mathbb{Q}} \Gamma_q$. Using (8), it is easy to see that

$$\bigcup_{\gamma\in\Gamma'} \text{NSAT}(\gamma) \cup \text{SAT}(\varphi) = [0,1]^\kappa,$$

which, by (7), implies $\Gamma' \vdash_{\text{Ł}_{\infty,\kappa}} \varphi$ and, moreover, Γ' is clearly countable.

It follows that $\text{Ł}_{\infty,\kappa}$ is the unique natural extension of Ł_∞ (cf. Section 3.1). ■

In order to obtain that Ł_∞ has the τ-CIPEP we recall [21, Corollary 3.9]:

Lemma 5.8. *For every protoalgebraic* L *and every class of matrices* $\mathbb{K}$ *such that* $\text{L} = \models_{\mathbb{K}}$*, the following hold:*

1. *If* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{MOD}^*(\text{L})_{\text{RFSI}}$*, then* L *has the* IPEP*.*
2. *If* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{MOD}^*(\text{L})_{\text{RSI}}$*, then* L *has the* CIPEP*.*

Theorem 5.9. Ł_∞ *has the* τ-CIPEP *and, consequently, so does each of its axiomatic expansions.*

Proof. By Corollary 5.3 it is enough to show that Ł^κ_∞ has the CIPEP for every infinite κ. By Proposition 5.7 we have $\text{Ł}^\kappa_\infty = \models_{\langle[0,1]_{\text{Ł}},\{1\}\rangle}$ and moreover it is easy to show that $\mathbf{S}(\langle[0,1]_{\text{Ł}},\{1\}\rangle) \subseteq \mathbf{MOD}^*(\text{Ł}_\infty)_{\text{RSI}}$ – see e.g. [21, Example 10]. Thus the previous lemma applies. The part about axiomatic expansions is due to Theorem 4.4. ■

From the previous theorem and Theorem 5.5 we obtain:

Corollary 5.10. *Any axiomatic expansion* L *of* Ł_∞ *is (finitely) subdirectly representable; in symbols:*

$$\mathbf{MOD}^*(\text{L}) = \mathbf{P_{SD}}(\mathbf{MOD}^*(\text{L})_{\text{R(F)SI}}).$$

Consequently, we obtain an analogous result for the equivalent algebraic semantics of Ł_∞:

$$\mathbf{ALG}^*(\text{Ł}_\infty) = \mathbf{P_{SD}}(\mathbf{ALG}^*(\text{Ł}_\infty)_{\text{R(F)SI}}). \tag{9}$$

This shows that $\mathbf{ALG}^*(Ł_\infty)$ is an example of a subdirectly representable class of algebras, where the representation theorem is not a consequence of Birhoff's theorem (recall that $\mathbf{ALG}^*(Ł_\infty)$ is not a quasivariety). The same is true for every infinitary axiomatic expansion of $Ł_\infty$.

Moreover, as another consequence of Proposition 5.7, we can obtain a nice description of the class $\mathbf{ALG}^*(Ł_\infty)$.

PROPOSITION 5.11. *The class of algebras of $Ł_\infty$ is the prevariety generated by the algebra $[0,1]_Ł$; in symbols:*

$$\mathbf{ALG}^*(Ł_\infty) = \mathbf{ISP}([0,1]_Ł).$$

PROOF. The inclusion from right to left clearly holds (recall, for instance, that $\mathbf{ALG}^*(Ł_\infty) = \mathbf{ISP}_{\mathbf{R}_{\aleph_1}}([0,1]_Ł))$. For the other inclusion we can prove

$$\mathbf{ALG}^*(Ł_\infty) = \mathbf{P_{SD}}(\mathbf{ALG}^*(Ł_\infty)_{\mathrm{RSI}}) \subseteq \mathbf{ISP}([0,1]_Ł),$$

where the equality is (9) and the inclusion follows from Proposition 2.3, which says that $\mathbf{ALG}^*(Ł_\infty)_{\mathrm{RSI}} \subseteq \mathbf{IS}([0,1]_Ł)$. ■

The proof of Proposition 5.7 suggests a general methodology to obtain an upper bound for the cardinality of a logic in κ variables defined by a class of matrices $\mathbb{K}$, call it $\mathrm{L}_{\mathbb{K},\kappa}$. The proof of the theorem is omitted, since it can be easily abstracted from that of Proposition 5.7.

THEOREM 5.12. *Suppose λ is a regular cardinal and $\mathbb{K}$ is a class of matrices, such that $|\mathbb{K}| < \lambda$. Then for every cardinal κ the logic $\mathrm{L}_{\mathbb{K},\kappa}$ has cardinality at most λ whenever the following conditions hold for every $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$:*

1. *There is a compact topology τ on A such that all connectives are interpreted by continuous functions w.r.t. τ,*
2. *F can be written as an intersection of strictly less than λ open sets in τ,*
3. *$A\backslash F$ is open in τ.*

5.3.2. Infinitary Product Logic We will now show that in general neither IPEP nor CIPEP transfer. Indeed, we will see that Π_∞ does not have τ-IPEP, but as proved in [21], it has the CIPEP. The crucial difference between $Ł_\infty$ and Π_∞ is in the fact that the connectives of the latter are not continuous on the unit interval topology.

THEOREM 5.13. *Π_∞ does not have the τ-IPEP.*

PROOF. By virtue of Corollary 5.3, it is enough to prove that one natural extension of Π_∞ does not have the IPEP. Let κ be an arbitrary cardinal strictly larger than the continuum $\mathfrak{c}$ and consider the natural extension Π_∞^κ.

Since Π_∞ is obviously semilinear w.r.t. $\to$, by Lemma 2.1 any filter F is intersection-prime if, and only if, it is linear. Furthermore, since models of Π_∞ and Π_∞^κ coincide (Observation 3.1), it follows that every theory T of Π_∞^κ is intersection-prime exactly if it is linear. Define

$$\Gamma = \{(x_\alpha \to x_\beta) \to x_0 \mid 0 < \alpha < \beta < \kappa\}.$$

Let us show that $\Gamma \nvdash_{\Pi_\infty^\kappa} x_0$. Indeed, otherwise there would be a countable subset Γ' of Γ such that $\Gamma' \vdash_{\Pi_\infty^\kappa} x_0$ (Π_∞^κ is a natural extension of Π_∞, therefore it has cardinality $\aleph_1$). Then we could take a substitution σ such that $\sigma[\mathit{Var}[\Gamma']] \subseteq \mathit{Var}$. Then $\sigma[\Gamma] \vdash_{\Pi_\infty^\kappa} \sigma(x_0)$ and hence, $\sigma[\Gamma] \vdash_{\Pi_\infty} \sigma(x_0)$. Let us define $v_0 = \sigma(x_0)$ and enumerate the remaining variables of Γ' as $\{v_1, v_2, \ldots\}$ in such a way that $\{(v_n \to v_m) \to v_0 \mid 0 < n < m < \omega\} \vdash_{\Pi_\infty} v_0$. This derivation can be falsified by taking the evaluation $e(v_0) = \frac{1}{2}$ and $e(v_n) = \frac{1}{2^{n+1}}$ for each $n \geq 1$.[9]

If Π_∞^κ had the IPEP, there would be a linear theory $T \supseteq \Gamma$, such that $T \nvdash_{\Pi_\infty^\kappa} x_0$. Consider the Lindenbaum–Tarski model given by T, that is $\langle \boldsymbol{Fm}_{\mathcal{L}}(\kappa)^*, T^* \rangle$. It is easy to see that $\boldsymbol{Fm}_{\mathcal{L}}(\kappa)^*$ is an Archimedean (see proof of Proposition 2.3) linear product algebra, thus by Proposition 2.3 it embeds into $[0,1]_\Pi$.

Furthermore observe that for any $0 < \alpha < \beta < \kappa$ we have $x_\alpha \to x_\beta \notin T$, which implies that $|Fm_{\mathcal{L}}(\kappa)^*| = \kappa$ (because $\langle \varphi, \psi \rangle \in \Omega(T)$ iff both $\varphi \to \psi \in T$ and $\psi \to \varphi \in T$). However, since $\kappa > \mathfrak{c}$, we have a contradiction. ∎

By Theorem 5.5, Π_∞ is not finitely subdirectly representable, thus the class $\mathbf{MOD}^*(\Pi_\infty)$ is not generated as subdirect products of chains. In fact, in the proof of the theorem, we have constructed an algebra, namely $\boldsymbol{Fm}_{\mathcal{L}}(\kappa)^*$, which is not a subdirect product of algebras from $\mathbf{ALG}^*(\Pi_\infty)_{\mathrm{RSI}}$, that is

$$\mathbf{ALG}^*(\Pi_\infty) \neq \mathbf{P_{SD}}(\mathbf{ALG}^*(\Pi_\infty)_{\mathrm{RFSI}}).$$

Another consequence of the theorem is that the logic with κ variables given semantically by the standard product chain need not have cardinality $\aleph_1$ (e.g. whenever $\kappa > \mathfrak{c}$). Moreover, as mentioned in Section 4.2, not every axiomatic expansion of Π_∞ has the IPEP (CIPEP); e.g., by Proposition 4.2 and the previous theorem, adding more than continuum many constants does not preserve any of them.

5.3.3. A Non-RSI-Complete Logic with τ-IPEP In this final subsection we introduce an infinitary version of the *degree-preserving Łukasiewicz logic*

[9] Note that the same argument for $Ł_\infty$ would fail here, i.e. it is the case that $\{(x_i \to x_j) \to x_0 \mid i \leq j \text{ in } \mathbb{N}\} \vdash_{Ł_\infty} x_0$.

with rational constants, which can be seen as a natural combination of the degree-preserving Łukasiewicz logic studied in [15] and Łukasiewicz logic with rational constants studied in [19, 25]. After an elementary presentation of the logic we shall show that, as desired, it has τ-IPEP but it is not RSI-complete.

Take the language of Łukasiewicz logic with rational constants, that is:

$$\mathcal{L} = \{\to, \&, \overline{0}\} \cup \{\overline{q} \mid q \in (0,1] \cap \mathbb{Q}\} \text{ and } Var_{\mathcal{L}} = \omega.$$

Let $[0,1]^{\mathbb{Q}}_{\text{Ł}}$ denote $[0,1]_{\text{Ł}}$ expanded with the natural interpretations of the constants. Let $\uparrow q$ denote the lattice filter generated by a rational number q in $[0,1]$, i.e. $\uparrow q = \{r \in [0,1] \mid r \geq q\}$. Define $\mathbf{Ł}_q$ as the matrix $\langle [0,1]^{\mathbb{Q}}_{\text{Ł}}, \uparrow q\rangle$ and define the set $\mathbb{K} = \{\mathbf{Ł}_q \mid q \in (0,1] \cap \mathbb{Q}\}$. Then we define

$$\text{L} = \models_{\mathbb{K}}.$$

Note that, since rationals are a dense subset of the reals, L is indeed the degree-preserving logic over the algebra $[0,1]^{\mathbb{Q}}_{\text{Ł}}$ (i.e. in fact every lattice filter on $[0,1]^{\mathbb{Q}}_{\text{Ł}}$ is a filter of L). Indeed, we could define L as

$$\Gamma \vdash_{\text{L}} \varphi \quad \iff \quad \bigwedge v[\Gamma] \leq v(\varphi), \text{ for all } v \in \text{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, [0,1]^{\mathbb{Q}}_{\text{Ł}}).$$

However, as we will see later, it is of significance that the logic can be defined over countably many matrices.

Define the following generalized implication connective:

$$x \Rightarrow y = \{(x \to y)^n \mid n \in \mathbb{N}\},$$

where $(-)^n$ denotes the n times iterated conjunction. It is easy to check that $\Rightarrow$ is a weak implication in L. Indeed, observe that for every $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and every $a, b \in A$ it holds that $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$ if and only if $a \leq b$, where $\leq$ is the standard order on reals. Clearly, $\mathbb{K} \subseteq \mathbf{MOD}^{\ell}(\text{L})$ and hence L is semilinear.

Therefore, on every reduced model $\langle \boldsymbol{A}, F\rangle$, we have the induced order relation $\leq^{\mathbf{A}}$ given by $a \leq^{\mathbf{A}} b$ iff $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$. Moreover, it is easy to see that L proves:

$$x \Rightarrow y \vdash_{\text{L}} (x \to y) \Rightarrow \overline{1} \text{ and } x \Rightarrow y \vdash_{\text{L}} \overline{1} \Rightarrow (x \to y),$$

which implies the left to right implication in the following characterization of the order on reduced models (the other implication is obvious):

$$a \leq^{\mathbf{A}} b \quad \iff \quad a \to b = \overline{1}^{\boldsymbol{A}}, \tag{10}$$

Now we can easily see that L is infinitary (in fact, card(L) $= \aleph_1$); indeed:

$$\bigcup_{q \in (0,1) \cap \mathbb{Q}} \bar{q} \Rightarrow x \vdash_{\mathrm{L}} x$$

but no finite subset of premises would entail x.

Note that L is equivalential, but not algebraizable, which is a consequence of the fact that all the matrices $\mathbf{Ł}_q$ are reduced (on every algebra in algebraizable logics there is at most one filter making the corresponding matrix reduced).

PROPOSITION 5.14. L *has the* τ-IPEP.

PROOF. Let $\mathrm{L}_{\mathbb{K},\kappa}$ be the logic in κ-many variables semantically given by the class $\mathbb{K}$. We can apply Theorem 5.12 to prove that $\mathrm{L}_{\mathbb{K},\kappa}$ has cardinality $\aleph_1$ and thus show that $\mathrm{L}^\kappa = \mathrm{L}_{\mathbb{K},\kappa}$ (the condition necessary to ensure the uniqueness of natural extensions is fulfilled; see Section 3.1). The theorem applies, because $\mathbb{K}$ is countable, the standard interval topology on $[0, 1]$ is compact and all the connectives are continuous w.r.t. it and finally, all the filters $\uparrow q$ can obviously be approximated by countably many open subsets of $[0, 1]$. So L^κ is complete w.r.t. $\mathbb{K}$ and, consequently, it is semilinear w.r.t. $\Rightarrow$ (because $\mathbb{K} \subseteq \mathbf{MOD}^\ell(\mathrm{L})$) and has the IPEP (in fact, every semilinear logic has the IPEP; see [8, Theorem 3]). By Corollary 5.3, since L is protoalgebraic, it has the τ-IPEP. ■

PROPOSITION 5.15. L *is not* RSI-*complete.*

PROOF. First observe that L satisfies for every real number $r \in (0, 1]$ the following *density* rule:

$$\{\bar{q} \Rightarrow x \mid q < r\} \cup \{\bar{q} \mid q > r\} \vdash_{\mathrm{L}} x \tag{11}$$

We show that L has no RSI-models which implies that it cannot be RSI-complete because it is not the inconsistent logic. In pursuit of contradiction suppose there is a reduced model $\langle \boldsymbol{A}, F\rangle$ in which F is completely intersection-prime. In particular, $F \neq A$ and, since L is semilinear, F is linear (by Lemma 2.1), which implies that $\leq^{\mathbf{A}}$ is a linear order. Consider the following set of rationals

$$C_F = \{q \in (0, 1] \mid \bar{q}^{\boldsymbol{A}} \in F\}.$$

Claim 1: For each $a \in A \backslash F$, there is q such that $a <^{\mathbf{A}} \bar{q}^{\boldsymbol{A}}$ and $q \notin C_F$.
Proof: Take $a \in A\backslash F$ and assume that there is no such q. By linearity of $\leq^{\mathbf{A}}$, we have $\bar{q}^{\boldsymbol{A}} \leq^{\mathbf{A}} a$ for every $q \notin C_F$. Thus, applying (11) for $r = \inf C_F$, we obtain that $a \in F$; a contradiction.

Let us define $\uparrow \bar{q}^{\boldsymbol{A}} = \{a \in A \mid a \geq^{\boldsymbol{A}} \bar{q}^{\boldsymbol{A}}\}$.

Claim 2: For every $q \notin C_F$, $F \subseteq \uparrow \bar{q}^{\boldsymbol{A}}$.

Proof: Take $a \in F$ and suppose $q \nleq^{\boldsymbol{A}} a$. By linearity $a <^{\boldsymbol{A}} q$, thus modus ponens of $\Rightarrow$ implies that $q \in F$ –a contradiction.

Claim 3: For every $\boldsymbol{B} \in \mathbf{ALG}^*(\mathrm{L})$ and every $q \in (0,1] \cap \mathbb{Q}$, the set $\uparrow \bar{q}^{\boldsymbol{B}}$ is an L-filter.

Proof: Since L is equivalential in a countable language with a countable set of variables we have $\mathbf{ALG}^*(\mathrm{L}) = \mathbf{ISP}_{\mathbf{R}_{\aleph 1}}([0,1]_{\mathrm{L}}^{\mathbb{Q}})$, and hence $\boldsymbol{B}$ is embeddable into $\boldsymbol{C} = \prod^{\kappa}[0,1]_{\mathrm{L}}^{\mathbb{Q}}/\mathcal{F}$ for some $\mathcal{F}$, an $\aleph_1$-complete filter on κ (see e.g. [9, Theorem 1]). Take $G = (\prod^{\kappa} \uparrow q)/\mathcal{F}$; it follows that $\langle \boldsymbol{C}, G\rangle$ is a reduced model of L (because $\mathbf{MOD}^*(\mathrm{L})$ is closed under $\mathbf{P}_{\mathbf{R}_{\aleph 1}}$). We show that $G = \uparrow \bar{q}^{\boldsymbol{C}}$:

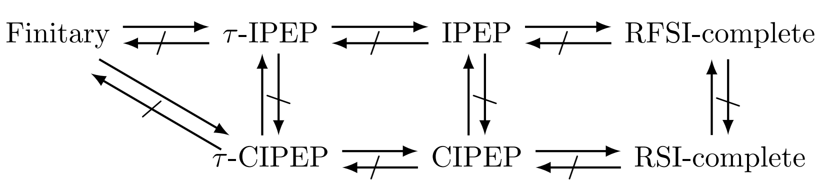
$$\begin{aligned}
[\bar{a}] \in \uparrow \bar{q}^{\boldsymbol{C}} &\iff \{\alpha \in \kappa \mid q \to \bar{a}(\alpha) = 1\} \in \mathcal{F} \\
&\iff \{\alpha \in \kappa \mid \bar{a}(\alpha) \in \uparrow q\} \in \mathcal{F} \\
&\iff [\bar{a}] \in G,
\end{aligned}$$

The claim follows because $\uparrow \bar{q}^{\boldsymbol{B}} = \uparrow \bar{q}^{\boldsymbol{C}} \cap B$.

It is now easy, by virtue of all the claims above, to conclude that $F = \bigcap_{q \notin C_F} \uparrow \bar{q}^{\boldsymbol{A}}$, where for each $q \notin C_F$, $\uparrow \bar{q}^{\boldsymbol{A}}$ is a filter and does not coincide with F –a contradiction with the fact that F is completely intersection-prime. ■

6. Conclusions

In this paper we have considered the transfer problem, left open in [21], for the properties IPEP and CIPEP, which naturally arise in AAL as generalizations of finitarity in the study of propositional logical systems and their completeness theorems. We have shown that these properties do not transfer in general and, hence, we have obtained a richer hierarchy depicted in the following figure:



Moreover, we have used the transferred properties to obtain a characterization of subdirect representation for the matricial semantics of propositional logics (and for prevarieties in universal algebra). We have seen that, outside the scope of finitary logics (or quasivarieties), subdirect representation is not guaranteed (Section 5.3.2); however, there are still positive examples of this phenomenon in Section 5.3.1.

The meaning of the classes in the hierarchy deserves some further remarks. Indeed, determining the position of a logic in the hierarchy essentially amounts to determining how many R(F)SI-models the given logic has. Namely: (1) τ-(C)IPEP logics are those with the biggest amount of such models, enough to build all the remaining models by subdirect products; (2) (C)IPEP logics are complete w.r.t. R(F)SI-models built over the algebra of formulas (i.e. Lindebaum–Tarski models); (3) R(F)SI-complete logics may not have as many Lindebaum–Tarski models, but still have enough R(F)SI models to provide a complete semantics.

Moreover, we have proved that the extension properties are preserved under axiomatic extensions, while, in general, they may be lost when adding rules. We have also found some conditions that ensure preservation under axiomatic *expansions*, hence solving another of the open problems of [21]. All these results are based on two auxiliary technical notions: natural extensions and natural expansions; the former already known before, the latter introduced in this paper. Both notions allow to prove the preservation of properties of Th(L) to logics in expanded languages; other properties for which such technique might turn out useful in future work include distributivity and filtrality (because they are preserved under lattice isomorphisms, cf. Proposition 3.9). Finally, the study we have performed has incidentally brought some new insight on semilinear logics that we would like to stress:

1. IPEP is the smallest class in the hierarchy that contains all semilinear logics. Indeed, Π_∞ does not enjoy the τ-IPEP and the example in Section 5.3.3 does not have the CIPEP.
2. All semilinear logics are RFSI-complete (because they have the IPEP), but they are not all RSI-complete as shown by the example in Section 5.3.3, or another semilinear logic with rational truth-constants introduced in [21, Subsection 3.2].
3. Semilinear logics take their name from the fact their finitely subdirectly irreducible models are linear, that is, $\mathbf{MOD}^\ell(\mathrm{L}) = \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$, and moreover they are complete w.r.t. these models. However, this does not entail that they are always finitely subdirectly representable, as shown again by the infinitary product logic Π_∞.

4. Our preservation result of the τ-CIPEP and the τ-IPEP under axiomatic expansions allows to show that many other interesting infinitary semilinear logics enjoy these properties, such as usual the expansions with the projection connective Δ or other truth hedges, logics with additional truth-constants, logics with additional involutive negation, etc. [12].

Let us end by collecting a few (minor) open problems that arise from the investigation of this paper:

- Are τ-CIPEP and the τ-IPEP preserved under axiomatic expansions for non-protoalgebraic logics?
- Is also R(F)SI-completeness preserved under axiomatic expansions?
- Are the extension properties preserved when adding rules (and keeping the language) to protoalgebraic logics?
- Is there an example of an algebraizable logic with the τ-IPEP but not RSI-complete?
- Since almost all the examples in this paper are semilinear logics, this open problem from [21] still remains: is there a natural RFSI-complete logic without the IPEP, besides the rather complicated system that was built there?

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T. Lávička, C. Noguera
Institute of Information Theory and Automation
Czech Academy of Sciences
Pod Vodárenskou Věží 4
182 08 Prague
Czech Republic
lavickat@utia.cas.cz

C. Noguera
noguera@utia.cas.cz
URL: http://www.utia.cas.cz/people/noguera