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# Continuous SSB representation of preferences

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ABSTRACT

We propose a topological variant of skew-symmetric bilinear (SSB) representation of preferences. First, semi-Fishburn relations are defined by assuming convexity and coherence, a newly considered topological property. We show that lower and upper semi-Fishburn relations admit the existence of a minimal element and a maximal element, respectively. Then axiom of "balance" is stated and we prove that a binary relation has a continuous SSB representation if and only if it is a balanced (lower and upper semi-)Fishburn relation. The relationship between the above definitions and the original axioms of (algebraic) SSB representation is fully discussed. Finally, by applying this theory to probability measures, we show the existence of a maximal preferred measure for an infinite set of pure outcomes, thus generalizing all available existence theorems of (algebraic) SSB representation. Note that by using this framework to, e.g., finitely additive measures, one may develop a non-probabilistic variant of SSB representation as well.


## 1. Introduction

During the second half of the 20th century, many systematic violations of the *expected utility theory* (Von Neumann and Morgenstern, 1944) were observed stimulating the development of numerous generalizations and alternatives (Fishburn, 1982, 1988; Starmer, 2000; Machina, 2004). In particular, the intuitively appealing axiom of *transitivity* of preferences is not supported by empirical evidence, see Tversky (1969) among others. This axiom has recently been questioned also from the normative point of view (Anand et al., 2009; Fishburn, 1984b; Blavatskyy, 2006). Moreover, non-transitive preferences are already well-established in the context of social choice (Fishburn, 1999; Aziz et al., 2015). Thus, we further develop the theory of skew-symmetric bilinear (SSB) representation of preferences (Kreweras, 1961; Fishburn, 1982, 1988), i.e. a concise mathematical model of non-transitive decision making.

In the heart of SSB representation, there are three axioms (of continuity, convexity, and symmetry of preferences) which, once satisfied, are equivalent to the existence of SSB functional representing the preferences, see Fishburn (1982). In more detail, for a non-empty convex set $P$, we denote $\succ$ a binary asymmetric relation of strict preferences on $P$. Further, let $\sim$ and $\succsim$ be indifference and preference-or-indifference relations defined in a standard way using $\succ$. Then, the axioms of (algebraic) SSB representation stated for all $p, q, r \in P$ and all $\lambda \in ]0, 1[$ are as follows:

(C1) Continuity: $p \succ q, q \succ r \Longrightarrow q \sim \alpha p + (1-\alpha)r$ for some $\alpha \in ]0, 1[$,

(C2) Convexity: $p \succ q, p \succsim r \Longrightarrow p \succ \lambda q + (1-\lambda)r$,
$p \sim q, p \sim r \Longrightarrow p \sim \lambda q + (1-\lambda)r$,
$q \succ p, r \succsim p \Longrightarrow \lambda q + (1-\lambda)r \succ p$,

(C3) Symmetry: $p \succ q, q \succ r, p \succ r, q \sim \frac{p+r}{2} \Longrightarrow \left(\lambda p + (1-\lambda)r \sim \frac{p+q}{2} \Longleftrightarrow \lambda r + (1-\lambda)p \sim \frac{r+q}{2}\right)$.

If $P$ is, moreover, a set of probability measures on a Boolean algebra of subsets of a set $X$, axioms (C1), (C2), and (C3) hold if and only if there exists a SSB functional $\phi$ on $P \times P$ such that $p \succ q \Longleftrightarrow \phi(p, q) > 0$ for all $p, q \in P$, see Fishburn (1982, Theorem 1).

In this article, we review the above result mainly from a purely mathematical point of view. We show that the theory of SSB representation can be developed in a more abstract setting, assuming only that $P$ is a non-empty convex subset of a topological vector space $X$. Then, however, the purely algebraic continuity axiom (C1) does not provide the (topological) continuity of preferences if $P$ is infinite, and so the existence of a maximal element may be in jeopardy, c.f. Fishburn (1988, Theorem 6.2). Assumption of stronger (topological) continuity of preferences, see axiom (F1) below, results in several refinements of SSB representation. First, convexity axiom (C2) can be weakened, see axiom (F2). Moreover, axioms (F1) and (F2) may be divided into two symmetric parts

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such that the first one provides the existence of a minimal element, whereas the other provides the existence of a maximal element. We propose to name such relations lower and upper semi-Fishburn relations. Thus, Fishburn relations, i.e. relations that are both lower and upper semi-Fishburn, admit the existence of a minimal element and a maximal element at the same time, see Corollary 3.4. This seems to be a suitable property of preference relations. Finally, we simplify axiom (C3) into the axiom of "balance", which is more amenable to empirical verification, see (F3) below. Note that axioms (F3) and (C3) are equivalent in the context of SSB representation, see Proposition 5.2. To sum up, the newly proposed axioms of continuous SSB representation stated for all $p, q, r \in P$ and all $\lambda \in ]0, 1[$ read as follows:

(F1) Continuity: $\{s \in P : p \succ s\}$, $\{s \in P : s \succ p\}$ are open,

(F2) Convexity: $p \succ q, p \succsim r \Longrightarrow p \succ \lambda q + (1-\lambda)r$,
$$q \succ p, r \succsim p \Longrightarrow \lambda q + (1-\lambda)r \succ p,$$

(F3) Balance: $q \sim \frac{p+r}{2}, \lambda p + (1-\lambda)r \sim \frac{p+q}{2} \Longrightarrow$
$$\lambda r + (1-\lambda)p \sim \frac{r+q}{2}.$$

Note that the topology used in axiom (F1) is the relative topology of $P$. Theorem 5.3 shows that a binary relation is a balanced Fishburn relation, i.e. a relation satisfying axioms (F1), (F2), and (F3), if and only if there exists a continuous SSB representation. As a consequence, existence of maximal probabilistic measure for infinite set of outcomes is shown in Theorem 6.2. Note that our framework may also be used to develop SSB representation beyond the usual probabilistic setting. For instance, $P$ may be a set of finitely additive measures, c.f. Marinacci (1997). This idea is left for future research.

The article is organized as follows. Section 2 presents the basic notation and preliminary results. Then in Section 3 we introduce a notion of (semi)Fishburn relations, and show that these relations attain both minimal and maximal elements. Section 4 develops the tool of continuous partial representation of preferences. The main theorem of continuous SSB representation is presented in Section 5. Finally, an application for the case of infinite set of outcomes is presented in Section 6.

## 2. Notation and preliminary results

For real numbers $x, y \in \mathbb{R}$ we denote by $[x, y]$ and $]x, y[$ closed and open intervals, respectively, and for $\lambda \in ]0, 1[$ we denote $\lambda^\star \equiv \frac{1-\lambda}{\lambda} > 0$. For a real function $f$ satisfying a specified condition, we say that it is *unique up to a similarity transformation* if all functions satisfying the given condition are of the form $\alpha f$ with $\alpha > 0$. For a set $X$ we denote $2^X$ the set of all subsets of $X$.

A *topological space* is a set $X$ equipped with a family of subsets $\tau \subset 2^X$ (called *open* sets) satisfying the following conditions: $\emptyset, X \in \tau$; every union of open subsets of $X$ is open; every finite intersection of open subsets of $X$ is open. A set $Y \subset X$ is a *closed* set if $X \setminus Y \in \tau$. A topological space $X$ is called *compact* if every collection of open sets that covers $X$ has a finite subcover. A topological space is a *Hausdorff* space if any two distinct points are, respectively, contained in disjoint open sets. A Hausdorff topological space $X$ is a *topological vector space* (t.v.s), if it is moreover a vector space such that operations of addition and multiplication are continuous. Then, for $Y \subset X$ we denote by $\overline{Y}$ and $\operatorname{co} Y$ the topological closure and convex hull of $Y$, i.e. the smallest closed and convex set containing $Y$, respectively. For points $p, q \in X$ we use $[p, q] \equiv \{\lambda p + (1-\lambda)q : \lambda \in [0, 1]\}$, and analogously we define $]p, q[$ and $]p, q]$.

For a set $X$ and a binary relation $S$ defined on $X$, $S \subset X \times X$, we write $xSy$ if $(x, y) \in S$. We denote by $X^\star \equiv \{y \in X : xSy, ySz \text{ for some } x, z \in X\}$ the *preference interior* of $X$ with respect to $S$, and define *inverse* relation to $S$ as $\{(x, y) \in X \times X : ySx\}$. We say that $x \in X$ is a *maximal (minimal) element* of $X$ with respect to $S$ if set $\{y \in X : ySx\}$ ($\{y \in X : xSy\}$) is empty. The set of all maximal (minimal) elements of $X$ with respect to $S$ is denoted by $\max_S X$ ($\min_S X$). Relation $S$ is *asymmetric* if for all $x, y \in X$, $xSy$ implies that $ySx$ is not satisfied. Given a (preference) binary relation $\succ$, the *indifference* relation $\sim$ and *preference-or-indifference* relation $\succsim$ are

$$p \sim q \equiv \text{neither } p \succ q \text{ nor } q \succ p,$$
$$p \succsim q \equiv p \succ q \text{ or } p \sim q.$$

Then we denote by $\prec$ and $\precsim$ the inverse relation to $\succ$ and $\succsim$, respectively, and state the basic properties of relations $\sim$ and $\succsim$.

**Lemma 2.1.** *If $\succ$ is an asymmetric binary relation defined on a t.v.s. $X$, then for all $p, q \in X$ one of the following alternatives is satisfied: $p \succ q$, $p \sim q$, or $p \prec q$. Moreover, it holds that $p \sim q$ if and only if $p \succsim q$ and $q \succsim p$.*

## 3. Fishburn's relations

To develop a continuous SSB representation of preferences, we define a notion of coherence of a preference relation with the underlying topology.

**Definition 3.1** (*Coherent Relation*). A binary relation $\succ$ defined on a topological space $P$ is coherent (with topology of $P$) if $\overline{\{q \in P : q \succ p\}} = \{q \in P : q \succsim p\}$ for all $p \in P$ such that $\{q \in P : q \succ p\} \neq \emptyset$.

Note that a coherent relation is, necessarily, asymmetric.

**Definition 3.2** (*Fishburn Relation*). Let a convex subset $P$ of a t.v.s. be equipped with relative topology, and $\succ$ be a coherent relation on $P$. We say that $\succ$ is upper semi-Fishburn if $\{q \in P : q \succ p\}$ is convex for all $p \in P$.

A binary relation is lower semi-Fishburn if its inverse relation is upper semi-Fishburn. Finally, a binary relation is a Fishburn relation if it is both lower and upper semi-Fishburn.

The main motivation for the above definition follows.

**Theorem 3.3.** *Let $P$ be a non-empty compact convex subset of a t.v.s., and $\succ$ be an upper semi-Fishburn relation defined on $P$. Then there exists a maximal element of $P$ with respect to $\succ$, i.e. $\max_\succ P \neq \emptyset$.*

The following proof is inspired by Bergstrom (2012) where various conditions for existence of maximal element are discussed. In particular, for non-transitive relation $\succ$ and finite-dimensional $P$ the analogous result has already been achieved in Sonnenschein (1971).

**Proof.** We assume for a contradiction that $\{q \in P : q \succ p\} \neq \emptyset$ for all $p \in P$. Since relation $\succ$ is coherent, it is also asymmetric, and so $\{q \in P : q \prec p\} = P \setminus \{q \in P : q \succsim p\}$ for all $p \in P$ using Lemma 2.1. Then $\{q \in P : q \succsim p\}$ is closed due to coherency of $\succ$, thus $\{q \in P : p \succ q\}$ is open. Moreover, set $\{q \in P : q \succ p\}$ is convex for all $p \in P$ from the definition. By applying a fixed point theorem (Yannelis and Prabhakar, 1983, Theorem 3.3), we obtain $\bar{p} \in P$ such that $\bar{p} \succ \bar{p}$, hence we reached the contradiction with asymmetry of $\succ$. □

Analogously we may prove that there exists a minimal element for a lower semi-Fishburn relation.

**Corollary 3.4.** *Let $P$ be a non-empty compact convex subset of a t.v.s., and $\succ$ be a Fishburn relation defined on $P$. Then there exist both minimal and maximal elements of $P$ with respect to $\succ$.*

Further, we will show that Fishburn relations are precisely such binary relations that satisfy axioms (F1) and (F2). Note, however, that the respective part of axiom (F1) may be violated by a semi-Fishburn relation.

**Example 3.5.** Consider a binary relation $\succ$ such that for all $p, q \in [0, 1]$, $p \succ q$ if and only if $p > q$ and $q = 0$. Then, $\succ$ is an upper semi-Fishburn relation on $[0, 1]$, and set $\{q \in [0, 1] : p \succ q\}$ is not open for any $p > 0$.

**Theorem 3.6.** *Let $P$ be a non-empty convex subset of a t.v.s., and $\succ$ be a binary relation on $P$. Then $\succ$ is a Fishburn relation if and only if $\succ$ satisfies (F1) and (F2).*

First we show two auxiliary lemmas.

**Lemma 3.7.** *Let $\succ$ be a Fishburn relation defined on a convex subset $P$ of a t.v.s., then sets $\{q \in P : p \succ q\}$ and $\{q \in P : q \succ p\}$ are open for all $p \in P$.*

**Proof.** We provide the proof for sets $\{q \in P : p \succ q\}$, $p \in P$; the latter case is similar. For $p \notin \max_\succ P$ we have $\{q \in P : q \succ p\} \neq \emptyset$ and so $\{q \in P : p \succ q\}$ is open as shown in the proof of Theorem 3.3.

To prove the statement for $p \in \max_\succ P$, we denote $L \equiv \{q \in P : p \succ q\}$ and show that any $r \in L$ has an open neighbourhood contained in $L$. Set $N \equiv \{q \in [r, p] : q \succ r\}$ is convex using upper semi-Fishburn property of $\succ$. Moreover, $p \in N \neq \emptyset$, and so $r \in \overline{N}$ using $r \succsim r$. Since $r \notin N$ we necessarily have $N = ]r, p]$, in particular $\frac{p+r}{2} \succ r$. Thus we have shown that $r \in W \equiv \{q \in P : \frac{p+r}{2} \succ q\}$. Analogously, $p \succ \frac{p+r}{2}$ is implied by lower semi-Fishburn property of $\succ$, and so $W$ is open using the result of the above paragraph. Since $p \in \max_\succ P$, we necessarily have $p \succsim w$ for all $w \in W$, thus $\left\{\frac{w+r}{2} : w \in W\right\} \subset L$ is an open neighbourhood of $r$. $\square$

**Lemma 3.8.** *Let $P$ be a non-empty convex subset of a t.v.s., and $\succ$ be a binary relation on $P$ that satisfies axioms (F1) and (F2), then $\succ$ satisfies (C1).*

**Proof.** For $p, q, r \in P$ such that $p \succ q$ and $q \succ r$ we define $G \equiv \{t \in P : t \succ q\}$, $E \equiv \{t \in P : t \sim q\}$, and $L \equiv \{t \in P : t \prec q\}$. Note that $p \in G$, $q \in E$ and $r \in L$. Moreover, $G$ and $L$ are open and convex using (F1) and (F2), respectively. Thus, applying a separation theorem (Treves, 1967, Proposition 18.1) to $G$ and $L$, there exists a closed hyperplane $H$ strictly separating $G$ and $L$. Now, taking $s \equiv [p, r] \cap H$, we have $s \sim q$ since $H \subset P \setminus (L \cup G)$. To finish the proof, observe that $s = \alpha p + (1 - \alpha)r$ for some $\alpha \in ]0, 1[$. $\square$

**Proof of Theorem 3.6.** The fact that a Fishburn relation satisfies axiom (F1) is due to Lemma 3.7. To verify axiom (F2), note first that for any $s \in P$ sets $\{q \in P : q \succ s\}$ and $\{q \in P : s \succ q\}$ are convex. Then we consider $p, q, r \in P$ such that $p \succ q$ and $p \sim r$. Denoting $K \equiv \{t \in P : p \succ t\}$, $K$ is open and convex using Lemma 3.7, and $q \in K$. Since $K \neq \emptyset$, we also have $\overline{K} = \{t \in P : p \succsim t\}$, thus $r \in \overline{K}$. Then $]q, r[ \subset K$, yielding $p \succ \lambda q + (1 - \lambda)r$ for any $\lambda \in ]0, 1[$. The proof of the last variant of (F2) when $q \succ p$ and $p \sim r$ is analogous.

Further, we have to verify that a binary relation $\succ$ satisfying (F1) and (F2) is a Fishburn relation. We show that $\succ$ is upper semi-Fishburn; the rest of the proof is analogous. For any $p \in P$, set $\{q \in P : p \succ q\}$ is open due to (F1), thus set $\{q \in P : q \succsim p\}$ is closed using Lemma 2.1. Moreover, set $\{q \in P : q \succ p\}$ is convex using (F2), and from the definition we have $\overline{\{q \in P : q \succ p\}} \subset \{q \in P : q \succsim p\}$ as the latter set is closed. To show that relation $\succ$ is coherent, we need to prove that the previous inclusion is equality provided $\{q \in P : q \succ p\} \neq \emptyset$. Take some $r \succsim p$ such that $r \notin \overline{\{q \in P : q \succ p\}}$ for a contradiction. Then $r \sim p$, and for any $q \succ p$ there exists $t \in ]q, r[$ such that $t \not\succ p$, a contradiction to (F2). $\square$

We have shown that Fishburn relations satisfy axioms (C1), (F1), and (F2). Henceforth, we will refer directly to these axioms when needed. To conclude this section, we show to what extent Fishburn relations satisfy axiom (C2).

**Lemma 3.9.** *Let $P$ be a non-empty convex subset of a t.v.s., and $\succ$ be a Fishburn relation on $P$. Then $\succ$ satisfies axiom (C2) on any convex subset of $P^\star$.*

**Proof.** Consider a convex set $K \subset P^\star$, fix some $p \in K$ and define $M \equiv \{q \in K : p \sim q\}$, $G \equiv \{q \in P : q \succ p\}$ and $L \equiv \{q \in P : p \succ q\}$. Sets $G$ and $L$ are non-empty due to $p \in P^\star$. Thus $\overline{G} = \{q \in P : q \succsim p\}$ and $\overline{L} = \{q \in P : p \succsim q\}$ as $\succ$ is a Fishburn relation. Further, $M = K \cap \overline{G} \cap \overline{L}$ using Lemma 2.1, and since $G$ and $L$ are convex, so is $M$. The rest of axiom (C2) is valid due to Theorem 3.6. $\square$

## 4. Continuous partial representation of preferences

Motivated by Corollary 3.4 and Theorem 3.6, we will develop continuous SSB representation on the basis of Fishburn relations. The main workhorse is the so-called continuous partial representation of preferences.

**Definition 4.1.** Let $P$ be a convex subset of a t.v.s., and $\succ$ be a binary relation on $P$. Functional $\phi : P \times P^\star \to \mathbb{R}$ is a continuous partial representation of $\succ$ if $\phi(p, q)$ is continuous and linear in $p$ for all $q \in P^\star$, and for all $(p, q) \in P \times P^\star$ it holds that $p \succ q \iff \phi(p, q) > 0$ and $p \prec q \iff \phi(p, q) < 0$.

Next, we present a result analogous to Fishburn (1988, Theorem 4.2).

**Theorem 4.2.** *Let $P$ be a non-empty convex subset of a t.v.s., and $\succ$ be a Fishburn relation on $P$. Then $\succ$ admits a continuous partial representation $\phi : P \times P^\star \to \mathbb{R}$. Moreover, for any fixed $q \in P^\star$, function $p \to \phi(p, q)$ is unique up to a similarity transformation.*

**Proof.** Fixing any $q \in P^\star$, we will construct an appropriate $\phi(p, q)$ for all $p \in P$. First define $G \equiv \{t \in P : t \succ q\}$, $E \equiv \{t \in P : t \sim q\}$, and $L \equiv \{t \in P : t \prec q\}$. Note that $q \in E$, and $G$ and $L$ are non-empty due to $q \in P^\star$. Moreover, $G$ and $L$ are convex as $\succ$ is a Fishburn relation, and are open due to Lemma 3.7. Thus, by applying a separation theorem (Treves, 1967, Proposition 18.1) there exists a closed hyperplane $H$ strictly separating $G$ and $L$. Further, since $\{G, E, L\}$ is a disjunctive cover of $P$ using Lemma 2.1, we necessarily have $H \subset E$. Moreover, as $\succ$ is a Fishburn relation, it holds that $E = \overline{G} \cap \overline{L}$ due to Lemma 2.1. Then we have $E \subset H$ since $G$ and $L$ are strictly separated, and therefore $H = E$. Using $q \in E$ there exists continuous linear functional $\nu$ on $P$ such that $E = \{p \in P : \nu(p - q) = 0\}$, $G = \{p \in P : \nu(p - q) > 0\}$, and $L = \{p \in P : \nu(p - q) < 0\}$. Then $\phi(p, q) = \nu(p - q)$ satisfies the statement. $\square$

In the rest of this section we assume that a non-empty convex subset $P$ of a t.v.s. is equipped with relative topology, $\succ$ is a Fishburn relation defined on $P$, and $\phi$ is a continuous partial representation of $\succ$ due to Theorem 4.2.

**Corollary 4.3.** *For any $q \in P^\star$, set $\{p \in P : p \sim q\}$ is convex.*

**Proof.** For any $q \in P^\star$, set $\{p \in P : \phi(p, q) = 0\}$ is well-defined. Moreover, it is convex due to linearity of $\phi$ in the first variable. $\square$

Note that in the following lemma we assume only axiom (F2), whereas a stronger axiom (C2) was used in an analogous result in Fishburn (1982, Lemma 4).

**Lemma 4.4.** *Let $p, q, r \in P^\star$ be such that $p \succ q$, $q \succ r$ and $r \succ p$, then*

$$\phi(q,p)\phi(p,r)\phi(r,q) + \phi(q,r)\phi(r,p)\phi(p,q) = 0. \quad (1)$$

**Proof.** There exists $r' \in ]p, q[$ such that $r' \sim r$, and $p' \in ]q, r[$ such that $p' \sim p$ due to axiom (C1). Using Corollary 4.3 for $p \in P^\star$ and $r \in P^\star$ there is $s \in ]r, r'[\bigcap]p, p'[$ such that $s \sim p$ and $s \sim r$. Thus $s = \alpha p + \beta q + \gamma r$ with $\alpha, \beta, \gamma > 0$, and $\phi(s,p) = \phi(s,r) = 0$ imply $\frac{\beta}{\gamma} = -\frac{\phi(r,p)}{\phi(q,p)}$ and $\frac{\alpha}{\beta} = -\frac{\phi(q,r)}{\phi(p,r)}$. Moreover, if we show that $\frac{\gamma}{\alpha} = -\frac{\phi(p,q)}{\phi(r,q)}$, the proof will be finished since the previous three equations imply (1).

To this end it suffices to prove that $\phi(s,q) = 0$ using linearity of $\phi$. Denote $E \equiv \{t \in P : t \sim q\}$ and assume $s \notin E$ for a contradiction. Note that $E$ is convex as $q \in P^\star$, see Corollary 4.3. Considering $\tilde{s} \in ]p, r[$ such that $s \in ]\tilde{s}, q[$, $s \notin E$, and $q \in E$ implies $\tilde{s} \notin E$. Assuming without loss of generality that $s \succ q$, we have $\tilde{s} \succ s$ due to (F2). Thus $s \in P^\star$, and $\phi(p,s) = \phi(r,s) = 0$ implies $\phi(\tilde{s}, s) = 0$, a contradiction to $\tilde{s} \succ s$. The case of $s \prec q$ is similar. □

**Lemma 4.5.** *For $p, q \in P$ and $r \in P^\star$ it holds that $\lambda p + (1-\lambda) q \sim r$ if and only if $\lambda^\star \equiv \frac{1-\lambda}{\lambda} = -\frac{\phi(p,r)}{\phi(q,r)}$.*

**Proof.** We have $0 = \phi(\lambda p + (1-\lambda)q, r) = \lambda\phi(p,r) + (1-\lambda)\phi(q,r)$. □

**Lemma 4.6.** *Let $p, q, r, s \in P^\star$ satisfy $p \succ r$, $p \succ q$, $q \succ r$, $p \succ s$ and $s \succ r$, then*

$$\lim_{\lambda \uparrow 1} \frac{\phi(p, \lambda q + (1-\lambda)s)}{\phi(r, \lambda q + (1-\lambda)s)} = \frac{\phi(p,q)}{\phi(r,q)}. \quad (2)$$

**Proof.** There exists $\mu \in ]0,1[$ such that $q \sim \mu p + (1-\mu) r$ due to (C1). Moreover, such $\mu$ is unique with regard to (F2). Using the same argument, there exists unique $\mu_\lambda \in ]0,1[$ such that $\lambda q + (1-\lambda)s \sim \mu_\lambda p + (1-\mu_\lambda) r$ for any $\lambda \in ]0,1[$. According to Lemma 4.5 we may equivalently write

$$\mu_\lambda^\star = -\frac{\phi(p, \lambda q + (1-\lambda)s)}{\phi(r, \lambda q + (1-\lambda)s)}, \qquad \mu^\star = -\frac{\phi(p,q)}{\phi(r,q)}.$$

Further, for any $z \in \mathrm{co}\{p,q,r,s\}$ either $z \in \{p,r\}$, or $p \succ z$ and $z \succ r$, and so $\mathrm{co}\{p,q,r,s\} \subset P^\star$. Thus axiom (C2) is satisfied on $\mathrm{co}\{p,q,r,s\}$, see Lemma 3.9, and so employing (Fishburn, 1982, Lemma 6) we conclude that $\lim_{\lambda\uparrow 1}\mu_\lambda^\star = \mu^\star$. □

The proof of the following result was inspired by Fishburn (1982, Theorem 1), see the "transitive triple" section of Case 1B in "Skew Symmetry" part of the proof.

**Lemma 4.7.** *Let $p, q, r \in P^\star$ be such that $p \succ q$, $q \succ r$ and $p \succ r$. If $\phi$ satisfies $\phi(u,q) = -\phi(q,u)$ for all $u \in P^\star$ then, for any $w \in \{p,q,r\}$ and $z \in \{p,r\}$, it holds that*

$$\lim_{\lambda\uparrow 1} \phi(w, \lambda z + (1-\lambda)q) = \phi(w,z). \quad (3)$$

**Proof.** We show the statement only for $z = r$, as the other variant with $z = p$ is symmetric. We denote $r' \equiv \lambda r + (1-\lambda) q$ and show (3) first for $w = q$. It holds that

$$\phi(q, r') = -\phi(r', q) = -\lambda\phi(r,q) = \lambda\phi(q,r), \quad (4)$$

thus (3) is valid for $w = q$. Now, for $w = r$, we start with $0 = \phi(r', r') = \lambda\phi(r, r') + (1-\lambda)\phi(q, r')$. Next, substituting from (4) we obtain $\phi(r, r') = (1-\lambda)\phi(r,q)$, thus showing (3) for $w = r$ since $\phi(r,r) = 0$.

To prove the last variant, $w = p$, we first take some $v \in P$ such that $r \succ v$. Then, using openness of $\{x \in P : p \succ x\}$ and $\{x \in P : q \succ x\}$, there exists $t \in ]r, v[$ such that $p \succ t$, $q \succ t$ and $r \succ t$. From the definition it holds that $t \in P^\star$. Since for any $y \in \mathrm{co}\{p,q,r,t\}$ either $y \in \{p,t\}$, or $p \succ y$ and $y \succ t$, we have $\mathrm{co}\{p,q,r,t\} \subset P^\star$. Then by using Lemma 4.6 for $q \succ r$ and $r \succ t$ (with $s = \frac{1}{2}q + \frac{1}{2}r$), we obtain $\lim_{\lambda\uparrow 1}\frac{\phi(q,r')}{\phi(t,r')} = \frac{\phi(q,r)}{\phi(t,r)}$. This equation together with (3) for $w = q$ implies (3) for $w = t$. Similarly, by applying Lemma 4.6 to $p \succ r$ and $r \succ t$ (with $s = q$), we obtain $\lim_{\lambda\uparrow 1}\frac{\phi(p,r')}{\phi(t,r')} = \frac{\phi(p,r)}{\phi(t,r)}$, which, using (3) for $w = t$, implies (3) for $w = p$. □

## 5. Continuous SSB representation of preferences

We use axiom (F3) to define balanced relations, and examine the relationship of axioms (C3) and (F3) inspired by the first paragraph of Fishburn (1982, p. 44).

**Definition 5.1** (*Balanced Relation*)**.** Let $P$ be a non-empty convex set. We say that a binary relation $\succ$ defined on $P$ is balanced if for all $p, q, r \in P$ and all $\lambda \in ]0,1[$ such that $q \sim \frac{1}{2}p + \frac{1}{2}r$ and $\lambda p + (1-\lambda) r \sim \frac{1}{2}p + \frac{1}{2}q$ it holds that $\lambda r + (1-\lambda)p \sim \frac{1}{2}r + \frac{1}{2}q$.

**Proposition 5.2.** *Let $P$ be a non-empty convex set. A balanced relation on $P$ satisfies axiom (C3). Conversely, a binary relation on $P$ that satisfies axioms (C1), (C2) and (C3), is balanced.*

**Proof.** To show that a balanced relation satisfies (C3) it suffices to apply (F3) twice with $p$ and $r$ alternated. More effort is needed to show that given (C1) and (C2), axiom (C3) implies (F3). Consider $p, q, r \in P$, denote $q' \equiv \frac{p+r}{2}$, $r' \equiv \frac{p+q}{2}$ and $p' \equiv \frac{r+q}{2}$, and for fixed $\lambda \in ]0,1[$ take $t \equiv \lambda p + (1-\lambda) r$ and $s \equiv \lambda r + (1-\lambda) p$. Then axiom (F3) reads $q \sim q', t \sim r' \implies s \sim p'$ and (C3) may be stated as $p \succ q, q \succ r, p \succ r, q \sim q' \implies \left(t \sim r' \iff s \sim p'\right)$. We will see that, with an exception of one particular setting of $p$, $q$ and $r$, axiom (F3) is implied already by axioms (C1) and (C2). In what follows we use axiom (C2) intensively; however, once (C1) is used it is explicitly stated. The fact that (C2) implies uniqueness of $\alpha$ in (C1) is used as well.

Note first that we may assume $q \sim q'$ since otherwise (F3) is trivially satisfied. Then we have three alternatives: (*i*) $p \sim q$ and $q \sim r$; (*ii*) $r \succ q$ and $q \succ p$; (*iii*) $p \succ q$ and $q \succ r$.

Starting with variant (*i*), if we moreover assume $r \sim p$ then (F3) is direct. For $r \succ p$ we will show that existence of $t \in ]p, r[$ such that $t \sim r'$ leads to contradiction. First, $q \sim t$ is implied by $q \sim p$ and $q \sim r$. Then, $q \sim t$ together with $r' \sim t$ yields $p \sim t$, a contradiction with $r \succ p$. Thus $t \not\sim r'$ and (F3) is valid. The case of $p \succ r$ may be shown by alternating $p$ and $r$.

Similarly, by alternating $r$ and $p$ we observe that variant (*ii*) is equivalent to (*iii*), thus we further assume $p \succ q$ and $q \succ r$. We will deal with three cases based on the relationship of $p$ and $r$. For $p \succ r$ axiom (C3) directly implies (F3). For $r \sim p$ we first observe that $q' \sim p'$ and $q' \sim r'$. Since $p \succ r'$ and $r \succsim r'$ yields $q' \succ r'$, a contradiction, we have $p \succ r'$ and $r' \succ r$. Then $q'$ is the only point in $[p, r]$ that is indifferent to $r'$ due to (C1). Thus, either $\lambda = 0.5$ and $r' \sim t = q' = s \sim p'$, or $t \not\sim r'$; either way (F3) is satisfied.

To finish the proof, we have to deal with the case $p \succ q$, $q \succ r$ and $r \succ p$. We see that $p' \succ q'$ since $q \sim q'$ and $q' \prec r$, and similarly $q' \succ r'$. Now, denoting $u = [t, q] \cap [p', r']$, we observe that $t \sim r'$ and $r' \succ q$ implies $r' \succ u$, thus also $r' \succ p'$. We have shown $p' \succ q'$, $q' \succ r'$ and $r' \succ p'$, thus using (C1) there exist unique $\alpha, \beta, \gamma \in ]0,1[$ such that $\alpha p' + (1-\alpha) r' \sim q'$, $\beta q' + (1-\beta) p' \sim r'$, and $\gamma r' + (1-\gamma) q' \sim p'$. Observing $\alpha = \frac{1}{2}$, we obtain $\beta^\star\gamma^\star = 1$ due to Fishburn (1982, Lemma 4), thus $\beta = 1 - \gamma$. This implies $t \sim r' \iff s \sim p'$ and so (F3) is satisfied. □

Now we may state the main theorem of this article.

**Theorem 5.3** (*Continuous SSB Representation*). *Suppose $P$ is a non-empty convex subset of a t.v.s. Then, a binary relation $\succ$ on $P$ is a balanced Fishburn relation if and only if there exists a continuous skew-symmetric bilinear functional $\Phi$ on $P \times P$ such that for all $p, q \in P$, $p \succ q \Leftrightarrow \Phi(p, q) > 0$. Moreover, functional $\Phi$ is unique up to a similarity transformation.*

The next lemma will be used to verify skew-symmetry of $\Phi$ in the above theorem.

**Lemma 5.4.** *Let $P$ be a non-empty convex subset of a t.v.s., $\succ$ be a balanced Fishburn relation defined on $P$, and $\phi$ be a continuous partial representation of $\succ$. Then for any $p, p', q, r, r' \in P^\star$ such that $p \succ q$, $q \succ r$, $p \succ r$, $p' \in ]p, q[$ and $r' \in ]r, q[$, it holds that*

$$\phi(q, p')\left[\phi(p, r')\phi(r, q) - \phi(p, q)\phi(r, r')\right] = \phi(q, r')\left[\phi(r, q)\phi(p, p') - \phi(r, p')\phi(p, q)\right]. \tag{5}$$

**Proof.** Using (C1) there exist $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma \in ]0, 1[$ such that

$$\alpha_1 p + (1 - \alpha_1) r \sim p', \qquad \beta p + (1 - \beta) q = p',$$
$$\alpha_2 p + (1 - \alpha_2) r \sim q, \qquad \gamma q + (1 - \gamma) r = r',$$
$$\alpha_3 p + (1 - \alpha_3) r \sim r';$$

axiom (C2) is valid on $K \equiv \mathrm{co}\{p, p', q, r, r'\} = \mathrm{co}\{p, q, r\} \subset P^\star$ using Lemma 3.9. Axiom (C3) is satisfied due to Proposition 5.2, thus we may use (Fishburn, 1982, Lemma 11), the statement of which may be reformulated as $\alpha_1^\star(\alpha_3^\star - \alpha_2^\star) = \beta^\star\gamma^\star(\alpha_2^\star - \alpha_1^\star)$. Expressing all constants in terms of $\phi$, see Lemma 4.5, we obtain (5). □

The following lemma considerably simplifies the proof of Theorem 5.3. Similar idea could be probably used also in the proof of Fishburn (1982, Theorem 1).

**Lemma 5.5.** *Let $P$ be a non-empty convex subset of a t.v.s., $\succ$ be a balanced Fishburn relation defined on $P$, and $\phi$ be a continuous partial representation of $\succ$. Then for $p, q, r \in P^\star$ such that $p \succ q$, $q \succ r$ and $p \succ r$, it holds that*

$$\phi(q, p)\phi(p, r)\phi(r, q) + \phi(q, r)\phi(r, p)\phi(p, q) = 0. \tag{6}$$

**Proof.** For all $(x, y) \in P \times P^\star$ we define function $\psi : P \times P^\star \to \mathbb{R}$ as

$$\psi(x, y) = \begin{cases} \phi(x, y) & \text{if } y \sim q, \\ -\phi(x, y)\dfrac{\phi(y, q)}{\phi(q, y)} & \text{if } y \not\sim q, \end{cases}$$

and observe that $\psi$ is a continuous partial representation of $\succ$ such that $\psi(t, q) = -\psi(q, t)$ for all $t \in P^\star$. Moreover, (6) may be written equivalently in terms of $\psi$ as $\psi(q, p)\psi(p, r)\psi(r, q) + \psi(q, r)\psi(r, p)\psi(p, q) = 0$. This equation is, however, a direct consequence of Lemma 5.4 in the limit of $p' \to p$ and $r' \to r$, where the limits are calculated due to Lemma 4.7. □

Finally we will need two auxiliary lemmas.

**Lemma 5.6.** *Let $P$ be a non-empty convex subset $P$ of a t.v.s., and $\succ$ be a Fishburn relation defined on $P$. Then $P \setminus P^\star = (\min_\succ P) \cup (\max_\succ P)$. Moreover, if $P^\star = \emptyset$ then $P = \min_\succ P = \max_\succ P$, otherwise $\overline{P^\star} = P$.*

**Proof.** Denoting $P_{min} \equiv \min_\succ P$ and $P_{max} \equiv \max_\succ P$, identity $P \setminus P^\star = P_{min} \cup P_{max}$ is direct from the definitions. Thus, for the case of $P^\star = \emptyset$, we have $P = P_{min} \cup P_{max}$. Assume moreover that $P_{min} \neq P_{max}$, and, without loss of generality, that there is $p \in P_{max} \setminus P_{min}$. Then there exists $r \in P$ such that $p \succ r$, and defining $q \equiv \frac{1}{2}p + \frac{1}{2}r$ we observe $p \succ q$ and $q \succ r$ via (F2). Thus $q \in P^\star$, which is a contradiction.

To finish the proof it is to be shown that $\overline{P^\star} = P$ provided $P^\star \neq \emptyset$. Since $P \setminus P^\star = (P_{min} \cap P_{max}) \cup (P_{min} \setminus P_{max}) \cup (P_{max} \setminus P_{min})$ there are three alternatives to be considered. In the case of $z \in P_{min} \cap P_{max}$ all elements of $P$ are indifferent to $z$. Assuming $P^\star \neq \emptyset$ there exist $q \in P^\star$ and $p \in P$ such that $z \sim p \succ q \sim z$. Then for all $r \in ]p, z[$ we have $p \succ r$ and $r \succ q$ using (F2), thus $]p, z[ \subset P^\star$ and $z \in \overline{P^\star}$. For the case of $z \in P_{max} \setminus P_{min}$ there exists $p \in P$ such that $z \succ p$, and similarly to the previous case we obtain $z \in \overline{P^\star}$ due to $]p, z[ \subset P^\star$; the proof for the case of $z \in P_{min} \setminus P_{max}$ is analogous. □

**Lemma 5.7.** *Let $P$ be a non-empty convex subset $P$ of a t.v.s., and $\succ$ be a Fishburn relation defined on $P$. For any $p, q \in P^\star$ such that $q \neq p \sim q$ there exists $r \in P^\star$ such that $r \not\sim p$ and $r \not\sim q$.*

**Proof.** Due to the definition of $P^\star$ there exist $\bar{p}, \bar{q} \in P$ such that $\bar{p} \succ p$ and $\bar{q} \succ q$. We take any $\tilde{p} \in ]\bar{p}, p[$ and $\tilde{q} \in ]\bar{q}, q[$ and observe $\tilde{p} \succ p$, $\tilde{q} \succ q$, and $\tilde{p}, \tilde{q} \in P^\star$. If $\tilde{p} \not\sim q$ then $r \equiv \tilde{p}$ satisfies the statement; if $\tilde{q} \not\sim p$ then $r \equiv \tilde{q}$ satisfies the statement. Otherwise, we set $z \equiv \frac{1}{2}\tilde{p} + \frac{1}{2}\tilde{q}$ and verify that $z \succ p$ and $z \succ q$, using (F2). Finally, taking $r \in ]z, p[$ close enough to $z$, we will have $r \succ p$, $r \in P^\star$, and $r \succ q$, using (F1). □

**Proof of Theorem 5.3.** To show that axioms (F1) and (F2) are necessary, observe that they are direct consequences of continuity and bilinearity of $\Phi$, respectively. A short calculation reveals that any SSB functional satisfies axiom (F3). To show sufficiency, first note that for the case of $P^\star = \emptyset$ we may set $\Phi \equiv 0$, see Lemma 5.6. Thus we further assume $P^\star \neq \emptyset$.

Denoting by $\phi$ a continuous partial representation of $\succ$ due to Theorem 4.2, we construct functional $\Phi$ fulfilling the statement. To maintain the preference structure of $\succ$, there has to be a function $\lambda : P^\star \to ]0, +\infty[$ such that $\Phi(p, q) = \lambda(q)\phi(p, q)$ for or all $p, q \in P^\star$. Note that for any $q \in P^\star$, functional $\Phi(p, q)$ is linear and continuous in $p$ using properties of $\phi$. Thus, if there exists $\lambda$ such that functional $\Phi$ is also skew-symmetric on $P^\star \times P^\star$, then $\Phi$ is automatically bilinear and continuous on $P^\star \times P^\star$. Moreover, such functional $\Phi$ can be continuously extended to $P \times P = \overline{P^\star \times P^\star}$ since $P^\star \neq \emptyset$, see Lemma 5.6.

Thus it suffices to construct a function $\lambda$ such that $\Phi$ will be skew-symmetric on $P^\star \times P^\star$. In other words, we need to satisfy $\lambda(q)\phi(p, q) + \lambda(p)\phi(q, p) = 0$ for all $p, q \in P^\star$. This is valid for $p \sim q$; otherwise one equivalently has

$$\lambda(q) = -\lambda(p)\frac{\phi(q, p)}{\phi(p, q)}. \tag{7}$$

Fixing any $\bar{p} \in P^\star$ and $\lambda(\bar{p}) > 0$, one can thus define $\lambda(q)$ for all $q \in P^\star$ such that $q \not\sim \bar{p}$. Then, for $q \in P^\star$ such that $q \sim \bar{p}$, we define $\lambda(q)$ via (7) using any $r \in P^\star$ satisfying $r \not\sim \bar{p}$ and $r \not\sim q$ due to Lemma 5.7. To verify that function $\lambda$ is thus well defined, we take any $p, q, r \in P^\star$ such that $p \not\sim q \not\sim r \not\sim p$ and apply (7) three times, obtaining

$$\begin{aligned}\lambda(p) &= -\lambda(r)\frac{\phi(p, r)}{\phi(r, p)} = \lambda(q)\frac{\phi(r, q)}{\phi(q, r)}\frac{\phi(p, r)}{\phi(r, p)} \\ &= -\lambda(p)\frac{\phi(q, p)}{\phi(p, q)}\frac{\phi(r, q)}{\phi(q, r)}\frac{\phi(p, r)}{\phi(r, p)}.\end{aligned} \tag{8}$$

Thus $\lambda$ is well defined if and only if, for all $p, q, r \in P^\star$ such that $p \not\sim q \not\sim r \not\sim p$, both sides of (8) are equal, i.e. $\phi(q, p)\phi(r, q)\phi(p, r) + \phi(p, q)\phi(q, r)\phi(r, p) = 0$. Without loss of generality we may verify this equation only for two possibilities, either $p \succ q$, $q \succ r$ and $p \succ r$; or $p \succ q$, $q \succ r$ and $r \succ p$. The first is justified by Lemma 5.5, the latter by Lemma 4.4.

Finally we show that functional $\Phi$ constructed above is unique up to a similarity transformation. For $\Psi \neq \Phi$ maintaining the preference structure of $\succ$, there has to be a function $\mu : P^\star \to ]0,+\infty[$ such that $\Psi(p,q) = \mu(q)\phi(p,q)$ for all $p,q \in P^\star$. Moreover, to guarantee skew-symmetry of $\Psi$, function $\mu$ has to satisfy an equation analogous to (7). Thus for $\bar{p} \in P^\star$ used in the above paragraph and any $q \in P^\star$ we have $\frac{\mu(q)}{\mu(\bar{p})} = \frac{\lambda(q)}{\lambda(\bar{p})}$, and so $\Psi = \frac{\mu(\bar{p})}{\lambda(\bar{p})}\Phi$. □

## 6. Application to infinite set of outcomes

In this section we show that Theorem 5.3 (together with Corollary 3.4) strictly generalizes all the currently available existence results of a maximal element for algebraic SSB theory. Indeed, results for finite dimensional $P$, e.g. Fishburn (1988, Theorem 6.2), are obtained using discrete topology. Then, to our best knowledge, there is only one existence theorem for infinite-dimensional $P$, see Fishburn (1984a, Theorem 5)[1] . At the end of this section we provide a more general theorem, see Theorem 6.2.

First we should give some more terminology, see, e.g., Dunford and Schwartz (1958). For any set $X$ system $\mathcal{F} \subset 2^X$ is a *$\sigma$-field* if $X \in \mathcal{F}$, and $\mathcal{F}$ is closed on set complements and countable unions. Given a topological space $X$, the system of *Borel* sets $\mathscr{B}(X)$ is the smallest $\sigma$-field containing all open sets of $X$. A set function $\mu : \mathscr{B}(X) \to [0,+\infty)$ is a *Borel measure* if $\mu(\emptyset) = 0$, and $\mu(\bigcup_{i=1}^n K_i) = \sum_{i=1}^n \mu(K_i)$ for pairwise disjoint Borel sets $K_i$. A Borel measure $\mu$ is *regular*, if, for each Borel set $E \subset X$ and $\epsilon > 0$, there exists a closed set $F$ and open set $G$ such that $F \subset E \subset G$ and $|\mu(C)| < \epsilon$ for all Borel sets $C$ satisfying $C \subset G \setminus F$. By $\mathscr{P}(X)$ we denote a convex set of all *regular Borel probability* measures on $X$, i.e. regular Borel measures $\mu$ such that $\mu(X) = 1$. For any $x \in X$ we denote $\delta_x \in \mathscr{P}(X)$ a measure such that $\delta_x(\{x\}) = 1$. A sequence $\mu_n \in \mathscr{P}(X)$ converges to $\mu \in \mathscr{P}(X)$ in the so-called weak$^\star$ (w$^\star$) topology if for all continuous functions $f$ on $X$, it holds that $\int_X f\, d\mu_n \to \int_X f\, d\mu$.

Theorem 5 in Fishburn (1984a) is stated for a compact Hausdorff space $X$ and a set of measures $P$ such that $\mathscr{P}(X) \subset P$. Then, one may either define SSB representation $\phi$ on $X$ and extend it to $p,q \in P$ using

$$\Phi(p,q) \equiv \int_{X\times X} \phi(x,y)dp(x)dq(y), \tag{9}$$

or restrict SSB functional $\Phi$ defined on $P$ to SSB functional $\phi$ on $X$ by $\phi(x,y) \equiv \Phi(\delta_x,\delta_y)$. In general, an interplay of these two variants may be demanding, see technical conditions developed in Fishburn (1984a). Nevertheless, the existence of a maximal probabilistic measure is then shown in $\mathscr{P}(X)$, and so we may restrict only to $P = \mathscr{P}(X)$. Then, for continuous SSB functionals the above discussed definitions coincide.

**Lemma 6.1.** *Let set of outcomes $X$ be a compact Hausdorff space. For a balanced Fishburn relation $\succ$ on $X$ there exists a unique balanced Fishburn relation $>$ on $\mathscr{P}(X)$ such that*

$$x \succ y \text{ if and only if } \delta_x > \delta_y \text{ for all } x,y \in X. \tag{10}$$

*On the other hand, for a balanced Fishburn relation $>$ on $\mathscr{P}(X)$ there exists a unique balanced Fishburn relation $\succ$ on $X$ such that (10) holds.*

**Proof.** Denote by $\phi$ a continuous SSB functional representing $\succ$ on $X$ due to Theorem 5.3, $\phi$ is unique up to a similarity transformation. Then, $\Phi$ given by (9) is well-defined due to boundedness of $\phi$ on $X \times X$, and SSB on $\mathscr{P}(X)$. To show that $\Phi$ is, moreover, w$^\star$-continuous, it suffices to consider sequences of $\delta_{x_n} \in \mathscr{P}(X)$ such that $x_n \to x \in X$. Defining real functions $f(y) = \phi(x,y)$ and $f_n(y) = \phi(x_n,y)$ for all $n$, we have $\lim_n f_n(y) = f(y)$. Since $|f_n(y)|$ is dominated on $X$ by a constant, a dominated convergence theorem implies

$$\Phi(\delta_{x_n},q) = \int_X f_n(y)dq(y) \to \int_X f(y)dq(y) = \Phi(\delta_x,q)$$

for any $q \in \mathscr{P}(X)$. Thus $\Phi$ is a continuous SSB functional on $\mathscr{P}(X)$, and from the above construction we see that it is unique up to a similarity transformation. Such $\Phi$ yields uniquely given balanced Fishburn relation $>$ on $\mathscr{P}(X)$ due to Theorem 5.3. One may verify that condition (10) is satisfied by $>$ based on the construction of $\Phi$. The proof of the second part of the statement is analogous. □

**Theorem 6.2.** *Let set of outcomes $X$ be a compact Hausdorff space, and $\succ$ be a balanced Fishburn relation on $X$. Denote by $>$ the extension of $\succ$ to $\mathscr{P}(X)$ using (9). Then, $>$ is a balanced Fishburn relation, and for a closed and convex set $K \subset \mathscr{P}(X)$ there exists $p \in K$ such that $p \geq q$ for all $q \in K$.*

**Proof.** Since $X$ is compact, $\mathscr{P}(X)$ is compact in weak$^\star$ topology, see Goodearl (2010). Then, a closed subset $K$ of $\mathscr{P}(X)$ is also compact. Next, by using Lemma 6.1 we observe that $>$ is a balanced Fishburn relation on $\mathscr{P}(X)$, and the statement is due to Corollary 3.4. □

By stating Theorem 6.2 for $P = \mathscr{P}(X)$, the assumptions about relation $>$ on $P \setminus \mathscr{P}(X)$ may be omitted, c.f. Fishburn (1984a, Theorem 5). Note that this is without loss of generality, since $>$ may be arbitrarily extended to any superset of $\mathscr{P}(X)$, not affecting the above existence result. Moreover, we have shown existence of a maximal measure for any closed and convex subset of $\mathscr{P}(X)$. Thus we have generalized (Fishburn, 1984a, Theorem 5) in these two aspects.

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[1] This theorem has been stated for a locally compact set of outcomes $X$, referring to a minimax theorem from Nikaido (1954). There, however, the compactness of $X$ is required, and so we believe that the word "locally" was most likely used unintentionally in Fishburn (1984a). Indeed, local compactness of $X$ is not sufficient; consider $X = \mathbb{R}$, $P = \mathscr{P}(\mathbb{R})$, $\phi = 1$ as a counterexample. Thus, when referring to Fishburn (1984a, Theorem 5), we consider a modified statement where the word "locally" is removed from its assumptions.

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