How many market makers does a market need?

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Abstract

We consider a simple model for the evolution of a limit order book in which limit orders of unit size arrive according to independent Poisson processes. The frequency of buy limit orders below a given price level, respectively sell limit orders above a given level are described by fixed demand and supply functions. Buy (resp. sell) limit orders that arrive above (resp. below) the current ask (resp. bid) price are converted into market orders. There is no cancellation of limit orders. This model has independently been reinvented by several authors, including Stigler in 1964 and Luckock in 2003, who was able to calculate the equilibrium distribution of the bid and ask prices. We extend the model by introducing market makers that simultaneously place both a buy and sell limit order at the current bid and ask price. We show how the introduction of market makers reduces the spread, which in the original model is unrealistically large. In particular, we are able to calculate the exact rate of market makers needed to close the spread completely.

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1 Introduction

1.1 Description of the model

We will be interested in a simple mathematical model for the evolution of a limit order book, as used on a stock market or commodity market. The basic model we are interested in has been independently (re-)invented at least four times, by [Sti64, Luc03, Pla11, Yud12b]. We first describe the model, including our extension of the model, and then discuss its history, motivation, and related models.

Let $I = (I_-, I_+) \subset \mathbb{R}$ be a nonempty open interval, modelling the possible prices of limit orders, and let $\overline{I} = [I_-, I_+] \subset [-\infty, \infty]$ denote its closure. Recall that a *counting measure* on I is a measure μ such that $\mu(A) \in \mathbb{N}$ for all measurable $A \subset I$. Each finite counting measure can be written as a finite sum of delta measures. At any given time, we represent the state of the order book by a pair $(\mathcal{X}^-, \mathcal{X}^+)$ of counting measures on I, where we interpret the delta measures that $\mathcal{X}^-$ (resp. $\mathcal{X}^+$) is composed of as buy (resp. sell) limit orders of unit size at a given price. We assume that:

(i) there are no $x, y \in I$ such that $x \leq y$ and $\mathcal{X}^+(\{x\}) > 0$, $\mathcal{X}^-(\{y\}) > 0$,

(ii) $\mathcal{X}^-\big([x, I_+)\big) < \infty$ and $\mathcal{X}^+\big(I_-, x]\big) < \infty$ for all $x \in I$.

Here, the first condition says that the order book cannot simultaneously contain a buy and sell limit order when the ask price of the seller is lower than or equal to the bid price of the buyer. The second condition guarantees that

$$\begin{aligned} M^- &:= \max\big(\{I_-\} \cup \{x \in I : \mathcal{X}^-(\{x\}) > 0\}\big), \\ M^+ &:= \min\big(\{I_+\} \cup \{x \in I : \mathcal{X}^+(\{x\}) > 0\}\big), \end{aligned} \tag{1.1}$$

are well-defined, which can be interpreted as the current bid and ask prices. Note that $M^\pm := I_\pm$ if the order book contains no limit orders of the given type. It is often convenient to represent the order book by the signed counting measure $\mathcal{X} := \mathcal{X}^+ - \mathcal{X}^-$. We let $\mathcal{S}_{\text{ord}}$ denote the space of all signed measures of this form, with $\mathcal{X}^-$ and $\mathcal{X}^+$ satisfying the conditions (i) and (ii) above.

The dynamics of the model are described by two functions $\lambda_\pm : \overline{I} \to \mathbb{R}$, which we call the *demand function* λ_- and *supply function* λ_+, and a nonnegative constant $\rho \geq 0$, which will represent the *rate of market makers*. We assume that:

(A1) λ_- is nonincreasing, λ_+ is nondecreasing,

(A2) $\lambda_\pm$ are continuous functions,

(A3) $\lambda_+ - \lambda_-$ is strictly increasing,

(A4) $\lambda_\pm > 0$ on I.

We let $\mathrm{d}\lambda_\pm$ denote the measures on I defined by $\mathrm{d}\lambda_\pm\big([x, y]\big) := \lambda_\pm(y) - \lambda_\pm(x)$ $(x, y \in I,\ x \leq y)$. In particular, $\mathrm{d}\lambda_-$ is a negative measure and $\mathrm{d}\lambda_+$ is a positive measure. We consider a continuous-time Markov process $(\mathcal{X}_t)_{t\geq 0}$ that takes values in the space $\mathcal{S}_{\text{ord}}$ and whose dynamics have the following description.

Buy market orders With Poisson rate $\lambda_-(I_+)$, a trader comes and takes the best available sell limit order, if there is one, i.e., $\mathcal{X} \mapsto \mathcal{X} - \delta_{M^+}$ if $M^+ < I_+$ and nothing happens otherwise.

Sell market orders With Poisson rate $\lambda_+(I_-)$, a trader comes and takes the best available buy limit order, if there is one, i.e., $\mathcal{X} \mapsto \mathcal{X} + \delta_{M^-}$ if $M^- > I_-$ and nothing happens otherwise.

Buy limit orders With Poisson local rate $-\mathrm{d}\lambda_-$, a trader comes and places a buy limit order at a price x, or takes the best available sell limit order at a price $\leq x$, if there is one, i.e., $\mathcal{X} \mapsto \mathcal{X} - \delta_{x\wedge M^+}$.

Sell limit orders With Poisson local rate $\mathrm{d}\lambda_+$, a trader comes and places a sell limit order at a price x, or takes the best available buy limit order at a price $\geq x$, if there is one, i.e., $\mathcal{X} \mapsto \mathcal{X} + \delta_{x\vee M^-}$.

Market makers With Poisson rate ρ, a market maker arrives who places both a buy and sell limit order at the current ask and bid prices, provided these lie inside I, i.e., $\mathcal{X} \mapsto \mathcal{X} - 1_{\{M^- > I_-\}}\delta_{M^-} + 1_{\{M^+ < I_+\}}\delta_{M^+}$.

Here, the phrase "with Poisson local rate $\mathrm{d}\lambda_+$" means that sell limit orders with prices inside some measurable set $A \subset I$ arrive with Poisson rate $\mathrm{d}\lambda_+(A)$, which is independent for disjoint sets A. We assume that all Poisson processes governing different mechanisms (buy/sell market/limit orders, and market makers) are independent. After [Sti64, Luc03], we call the Markov process $(\mathcal{X}_t)_{t\geq 0}$ the *Stigler-Luckock model* with demand and supply functions $\lambda_\pm$ and rate of market makers ρ.

We make the assumptions (A2)–(A4) for technical simplicity. As explained in Appendix A.1 of [Swa16], these assumptions can basically be made without loss of generality. In particular, models for which (A2) and (A3) fail can be obtained as functions of models for which (A2) and (A3) hold. In particular, this applies to discrete models in which limit orders can only be placed at integer prices. To explain this on a concrete example, consider a model with a price interval of the form $I = (0, 2n)$ where $n \geq 1$ is some integer, and demand and supply functions that satisfy

$$\mathrm{d}\lambda_- = -1_{\{\lceil x\rceil \text{ is even}\}}\mathrm{d}x \quad \text{and} \quad \mathrm{d}\lambda_+ = -1_{\{\lceil x\rceil \text{ is odd}\}}\mathrm{d}x, \tag{1.2}$$

i.e., the measure $\mathrm{d}\lambda_-$ has a density with respect to the Lebesgue measure which is -1 on the intervals $(1,2]$, $(3,4]$,... and zero elsewhere, and likewise, the density of $\mathrm{d}\lambda_+$ is $+1$ on the intervals $(0,1]$, $(2,3]$,... and zero elsewhere. Let $(\mathcal{X}_t)_{t\geq 0}$ denote a model with such demand and supply functions (which satisfy (A1)–(A4)) and let $\mathcal{X}'_t := \mathcal{X}_t \circ \psi^{-1}$ denote the image of the measure $\mathcal{X}_t$ under the map

$$\psi(x) := \lceil x/2 \rceil \qquad (x \in I). \tag{1.3}$$

Then $(\mathcal{X}'_t)_{t\geq 0}$ is a model in which limit orders can ony be placed at discrete prices in $\{1, \ldots, n\}$. In particular, buy and sell limit orders at prices that in the original model lie in an interval of the form $(2(k-1), 2k)$ are placed in such a way that they always match, with buy orders on the right of sell orders. After applying the map ψ all these orders are mapped to the price k, i.e., they still match.

1.2 History of the model

The first reference for a model of the type we have just described is Stigler [Sti64], who simulated a model without market orders or market makers where $-\mathrm{d}\lambda_-$ and $\mathrm{d}\lambda_+$ are the uniform distributions on a set of 10 prices. Luckock [Luc03] (who was apparently unaware of Stigler's work) considered the model with general demand and supply functions, but without market orders or market makers. Assuming a special sort of stationarity, Luckock was able to find explicit expressions for the equilibrium distribution of the bid and ask prices of his model. In [Pla11], the model was once again independently reinvented, this time with $-\mathrm{d}\lambda_-$ and $\mathrm{d}\lambda_+$ the uniform distributions on a set of 100 prices. Building on this and Luckock's work, models with market orders were considered in [Swa16], who was able to give a precise criterion for the positive recurrence of such models. In the meantime, Yudovina [Yud12a, Yud12b], who was unaware of the previous references, in her Ph.D. thesis considered the model for a general class of demand and supply functions (though less general than those of Luckock) and also introduced models with market orders (although her formulation is somewhat different). Together with Kelly [KY16], under certain technical conditions, they were able to prove that the limit inferior and limit superior as time tends to infinity of the bid and ask prices have certain deterministic values, that they were able to calculate explicitly.

A characteristic feature of the Stigler-Luckock model is that buy and sell orders arrive at a rate that is independent of the current price. By contrast, a number of authors have considered models where limit orders are placed at rates that are relative to the price of the last transaction [Mas00] or the opposite best quote [CST10, SRR16]. A very general but rather complicated model is formulated in [Smi12]. See also [CTPA11] and chapter 4 of [Sla13] for a (partial) overview of the literature up to that point. Several authors also allow cancellation of orders.

In real markets, much of the trade seems to come from traders who speculate on the price going up or down. In view of this, a model where orders are placed relative to the current price may appear more realistic than the model we are interested in. Nevertheless, for an asset to be interesting for traders, there must always be some real demand and supply in the background, no matter how much this may be obscured by other effects. An unrealistic aspect of our model is that even traders who have a genuine interest in the asset and are nor merely speculating will usually not place limit orders very far from the current price, but rather wait until the price reaches a level that is acceptable to them. Thus, limit orders that are visibly written into the order book in our model may in reality not be visible, although they are in a sense still there in the form of traders silently waiting for the price to go up or down.

The impossibility to cancel a limit order is surely an unrealistic aspect of the Stigler-Luckcock model, that moreover greatly affects its long-time behvior. Nevertheless, on intermediate time scales (in the order of minutes), neglecting cancellation of orders may be realistic. Thus, the limit behavior of the Stigler-Luckcock model as time tends to infinity should in reality be viewed as an approximate description of the order book at intermediate time scales, when the number of orders is already large but cancellation is not yet an important aspect of the market.

In the days before electronic trading, market makers on the floor of the exchange would match buy and sell orders. Even though nowadays, market makers are not formally separated from other traders, they in effect still exist. They are distinguished from other traders by having a different motivation for trading. Rather than being interested in buying or selling an asset or speculating on the future development of its price, market makers place both buy and sell orders, at a high volume, with the aim of profiting from the small difference between the bid and ask prices. The strategy we have chosen for market makers is extremely simple. Depending on the current state of the order book and the expected behavior of the other traders, more intelligent choices may be possible. We will see, however, that the presence of market makers in itself has a huge effect on the shape of the order book. After this is taken into account, their present strategy may prove not to be too unrealistic.

From a purely mathematical prespective, the Stigler-Luckock model is similar to a number of other models that are motivated by other applications. We mention in particular the Bak Sneppen model [BS93] and its modification by Meester and Sarkar [MS12], a model for canyon formation [Swa15], as well as the queueing models for email communication of Barabási [Bar05] and Gabrielli and Caldarelli [CG09]. All these models are "rank-based" in the sense that the dynamics are based on the relative order of the particles and all models contain some rule of the form "kill the lowest (or highest) particle". For the model of [CG09], the shape of the stationary process near the critical point has been studied in [FS15] and these authors conjecture that their results also hold for the Stigler-Luckock model.

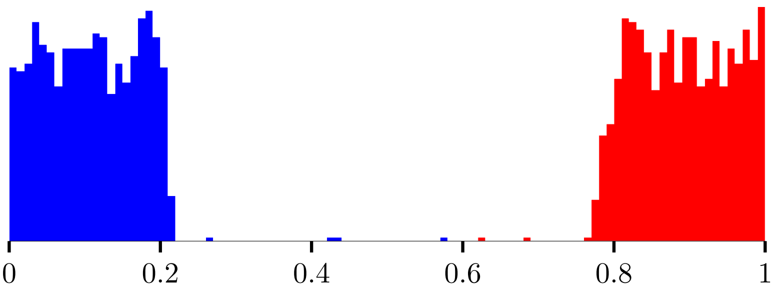

Figure 1: Simulation of the "uniform" Stigler-Luckock model with $\overline{I} = [0, 1]$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$. Shown is the state of the order book after the arrival of $10,000$ traders (starting from an empty order book).

2 Behavior of the model without market makers

2.1 The competitive window

Consider a Stigler-Luckock model without market orders (i.e., $\lambda_-(I_+) = 0$ and $\lambda_+(I_-) = 0$) and without market makers (i.e., $\rho = 0$). Assumptions (A1)–(A4) imply that there exists a unique price $x_{\rm W} \in I$ and constant $V_{\rm W} > 0$ such that

$$\lambda_-(x_{\rm W}) = \lambda_+(x_{\rm W}) =: V_{\rm W}. \tag{2.1}$$

Classical economic theory going back to Walras [Wal74] says that in a perfectly liquid market in equilibrium, a commodity with demand and supply functions $\lambda_\pm$ is traded at the price $x_{\rm W}$ and the volume of trade is given by $V_{\rm W}$. We call $x_{\rm W}$ the *Walrasian price* and $V_{\rm W}$ the *Walrasian volume of trade.*

Perhaps not surprisingly, in the absence of market makers, Stigler-Luckock models turn out to be highly non-liquid. Indeed, buyers willing to pay a price above the Walrasian price $x_{\rm W}$ and sellers willing to sell for a price below $x_{\rm W}$ may have to wait a considerable time before they get their trade, since the bid and ask prices do not settle at $x_{\rm W}$ but instead keep fluctuating in a *competitive window* (x_-, x_+) which satisfies $\lambda_-(x_-) = \lambda_+(x_+)$. As a result, *Luckock's volume of trade* $V_{\rm L} := \lambda_-(x_-) = \lambda_+(x_+)$ is larger than the Walrasian volume of trade $V_{\rm W}$ and in fact larger than it could be at any fixed price level.

Figure 1 shows the result of a numerical simulation of the uniform model with $\overline{I} = [0, 1]$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = 1$. Depicted is the state of the order book, started from the empty initial state, after the arrival of 10,000 traders. This and more precise simulations suggest that the boundaries of the competitive window are given by $x_- \approx 0.218$ and $x_+ \approx 0.782$. In the long run, buy limit orders at prices below x_- and sell limit orders at prices above x_+ stay in the order book forever, while all other orders are eventually matched. As a result, Luckcock's volume of trade $V_{\rm L} \approx 0.782$ is considerably higher than the Walrasian volume of trade $V_{\rm W} = 0.5$. Luckock [Luc03] described a method how to calculate x_-, x_+, and $V_{\rm L}$. In particular, for the uniform model, his method predicts that $V_{\rm L} = 1/z$ with z the unique solution of the equation $e^{-z} - z + 1 = 0$. To explain Luckock's formula for $V_{\rm L}$, we need to look at restricted models.

2.2 Restricted models

Let $(\mathcal{X}_t)_{t \geq 0}$ be a Stigler-Luckock model defined by demand and supply functions $\lambda_\pm : \overline{I} \to \mathbb{R}$ and rate of market makers $\rho \geq 0$. Let $(J_-, J_+) = J \subset I$ be an open subinterval of I and let $\lambda'_\pm : \overline{J} \to \mathbb{R}$ be the restrictions of the functions $\lambda_\pm$ to J. Let $(\mathcal{X}'_t)_{t \geq 0}$ be the Stigler-Luckock model on J defined by the by the demand and supply functions $\lambda'_\pm$ and the rate of market makers ρ. We call $(\mathcal{X}'_t)_{t\geq 0}$ the *restricted model on* J. Its dynamics are the same as for the original model $(\mathcal{X}_t)_{t\geq 0}$, except that limit orders arriving outside J cannot be written into the order book. Instead, buy limit orders arriving at prices in $[J_+, I_+)$ are converted into buy market orders while buy limit orders arriving at prices in $(I_-, J_-]$ have no effect. Similar rules apply to sell limit orders. Note that as long as the bid and ask prices $M_t^\pm$ stay inside J, the evolution of both models inside J is the same, i.e., restricting the measure $\mathcal{X}_t$ to J yields $\mathcal{X}'_t$.

Consider, in particular, a Stigler-Luckock model without market orders (i.e., $\lambda_-(I_+) = 0$ and $\lambda_+(I_-) = 0$) and without market makers (i.e., $\rho = 0$). Let $\lambda_-^{-1} : [0, \lambda_-(I_-)] \to \overline{I}$ and $\lambda_+^{-1} : [0, \lambda_+(I_+)] \to \overline{I}$ denote the left-continuous inverses of the functions λ_- and λ_+, respectively, i.e.,

$$\lambda_-^{-1}(V) := \sup\{x \in \overline{I} : \lambda_-(x) \geq V\} \quad \text{and} \quad \lambda_+^{-1}(V) := \inf\{x \in \overline{I} : \lambda_+(x) \geq V\}. \tag{2.2}$$

Let $V_{\max} := \lambda_-(I_-) \wedge \lambda_+(I_+)$ denote the maximal possible volume of trade. To avoid trivialities, let us assume that

(A5) $V_\mathrm{W} < V_{\max}$.

By the continuity of the demand and supply functions, for each $V \in (V_\mathrm{W}, V_{\max}]$, setting $J(V) := (\lambda_-^{-1}(V), \lambda_+^{-1}(V))$ defines a subinterval $J(V) \subset I$ such that $\lambda_-(J_-(V)) = V = \lambda_+(J_+(V))$. For later use, we define a continuous, strictly increasing function $\Phi : [V_\mathrm{L}, V_{\max}] \to \mathbb{R}$ with $\Phi(0) = 0$ by

$$\Phi(V) := \int_{V_\mathrm{W}}^{V} \Big\{ \frac{1}{\lambda_+\big(\lambda_-^{-1}(W)\big)} + \frac{1}{\lambda_-\big(\lambda_+^{-1}(W)\big)} \Big\} \frac{1}{W^2} \mathrm{d}W. \tag{2.3}$$

By definition, a Stigler-Luckock model is *positive recurrent* if started from an empty order book, it returns to the empty state in finite expected time. The following facts have been proved in [Swa16].

Proposition 1 (Luckock's volume of trade) *Assume (A1)–(A5), $\lambda_\pm(I_\mp) = 0$ (no market orders) and $\rho = 0$. Then, for each $V \in (V_\mathrm{W}, V_{\max})$, the restricted Stigler-Luckock model on $J(V)$ is positive recurrent if and only if $\Phi(V) < 1/V_\mathrm{W}^2$.*

Proof This follows from Proposition 2, Theorem 3, and formula (1.22) in [Swa16]. ∎

Proposition 1 suggests that Luckcock's volume of trade should be given by

$$V_\mathrm{L} = \sup\big\{V \in [V_\mathrm{W}, V_{\max}) : \Phi(V) < 1/V_\mathrm{W}^2\big\}, \tag{2.4}$$

and that the competitive window is given by $(x_-, x_+) = J(V_\mathrm{L}) = \big(\lambda_-^{-1}(V_\mathrm{L}), \lambda_+^{-1}(V_\mathrm{L})\big)$. These formulas agree well with numerical simulations and also agree with the (somewhat more complicated) method for calculating V_L described in [Luc03]. For the uniform model, one can check that one obtains for V_L the constant described at the end of the previous subsection. Under certain additional technical assumptions on $\lambda_\pm$, which include the uniform model, it has been proved in [KY16, Thms 2.1 and 2.2] that the limit inferior and limit superior of the bid and ask prices are a.s. given by the boundaries of the competitive window, as we have just calculated it.

We note that $V_\mathrm{L} > V_\mathrm{W}$ always but it is possible that $V_\mathrm{L} = V_{\max}$. In the latter case, the competitive window is the whole interval I. For example, this happens for the model with $\overline{I} = [0, 1]$, $\lambda_-(x) = (1 - x)^\alpha$, and $\lambda_+(x) = x^\alpha$ if $0 < \alpha \leq 1/2$. In the next subsection, we will see that if $V_\mathrm{L} < V_{\max}$ and one assumes that the restricted model on the competitive window has an invariant law, then the equilibrium distributions of the bid and ask prices are given by the unique solutions of a certain differential equation.

2.3 Stationary models

By definition, an invariant law for a Stigler-Luckock model is a probability law on $\mathcal{S}_{\rm ord}$ so that the process started in this initial law is stationary. We let

$$\mathcal{S}_{\rm ord}^{\rm fin} := \big\{\mathcal{X} \in \mathcal{S}_{\rm ord} : \mathcal{X}^- \text{ and } \mathcal{X}^+ \text{ are finite measures}\big\} \tag{2.5}$$

denote the subspace of $\mathcal{S}_{\rm ord}$ consisting of all states in which the order book contains only finitely many orders. If a Stigler-Luckock model is positive recurrent, then it has a unique invariant law that is moreover concentrated on $\mathcal{S}_{\rm ord}^{\rm fin}$ (see [Swa16, Thm 3]). In particular, this applies to the restricted model on $J(V)$ for any $V < V_{\rm L}$. If $V_{\rm L} < V_{\max}$, then it is believed that the restricted model on the competitive window $J(V_{\rm L})$ also has a unique invariant law, but this invariant law is not concentrated on $\mathcal{S}_{\rm ord}^{\rm fin}$. Instead, in equilibrium, the competitive window contains a.s. infinitely many limit orders of each type. In [FS15], a precise conjecture is made about the asymptotics of $\mathcal{X}^-$ near $J_-(V_{\rm L})$ and $\mathcal{X}^+$ near $J_+(V_{\rm L})$ in equilibrium.

On a rigorous level, even just proving existence of an invariant law for the restricted model on $J(V_{\rm L})$ is so far an open problem. However, postulating the existence of such an invariant law, Luckock was able to calculate the equilibrium distribution of the bid and as prices. We cite the following result from [Swa16, Thm 1]. Essentially, this goes back to [Luc03, formulas (20) and (21)], although he does not consider market orders.

Theorem 2 (Luckock's differential equation) *Assume that a Stigler-Luckock model with demand and supply functions satisfying (A1)–(A4) and $\rho = 0$ has an invariant law. Let $(\mathcal{X}_t)_{t\geq 0}$ denote the process started in this invariant law, and let $M_t^\pm = M^\pm(\mathcal{X}_t)$ denote the bid and ask price at time $t \geq 0$. Define functions $f_\pm : \overline{I} \to \mathbb{R}$ by*

$$f_-(x) := \mathbb{P}\big[M_t^- \leq x\big] \quad \text{and} \quad f_+(x) := \mathbb{P}\big[M_t^+ \geq x\big] \qquad (x \in \overline{I}), \tag{2.6}$$

which by stationarity do not depend on $t \geq 0$. Then $f_\pm$ are continuous and solve the equations

$$\begin{aligned} &\text{(i)} \quad f_-\mathrm{d}\lambda_+ + \lambda_-\mathrm{d}f_+ = 0,\\ &\text{(ii)} \quad f_+\mathrm{d}\lambda_- + \lambda_+\mathrm{d}f_- = 0,\\ &\text{(iii)} \quad f_-(I_+) = 1 = f_+(I_-), \end{aligned} \tag{2.7}$$

where $f_-\mathrm{d}\lambda_+$ denotes the measure $\mathrm{d}\lambda_+$ weighted with the density f_-, and the other terms have a similar interpretation.

Consider a Stigler-Luckock model satisfying (A1)–(A5), $\lambda_\pm(I_\mp) = 0$ (no market orders) and $\rho = 0$. Let J be a subinterval such that $\overline{J} \subset I$. Then it has been shown in [Swa16, Prop. 2] that Luckock's equation (2.7) for the restricted model $(\mathcal{X}'_t)_{t\geq 0}$ on J has a unique solution (f_-, f_+). By Theorem 2, if the restricted model on J has an invariant law, then

$$f_-(J_-) = \mathbb{P}\big[\mathcal{X}_t'^{\,-} = 0\big] \quad \text{and} \quad f_+(J_+) = \mathbb{P}\big[\mathcal{X}_t'^{\,+} = 0\big] \tag{2.8}$$

are the equilibrium probabilities that the restricted model $(\mathcal{X}'_t)_{t\geq 0}$ contains no buy or sell limit orders, respectively. In particular, if the restricted model on J has an invariant law, then these quantities must be ≥ 0, and if the restricted model is positive recurrent they must be > 0. In [Swa16, Thm 3] it is shown that conversely, if $f_-(J_-) \wedge f_+(J_+) > 0$, then the restricted model on J is positive recurrent. For intervals of the form $J(V) = \big(\lambda_-^{-1}(V), \lambda_+^{-1}(V)\big)$ as in (2.2), it is shown in [Swa16, Prop 2 and formula (1.22)] that

- If $\Phi(V) < 1/V_{\rm W}^2$, then $f_-(\lambda_-^{-1}(V)) > 0$ and $f_+(\lambda_+^{-1}(V)) > 0$.
- If $\Phi(V) = 1/V_{\rm W}^2$, then $f_-(\lambda_-^{-1}(V)) = 0 = f_+(\lambda_+^{-1}(V))$.

(Here Φ is the function defined in (2.3).) In particular, if $V_{\rm L} < V_{\max}$, then Luckock's equation has a unique solution (f_-, f_+) on the competitive window $J(V_{\rm L})$, and this solution satisfies $f_-(J_-(V_{\rm L})) = 0 = f_+(J_+(V_{\rm L}))$, which indicates that the bid and ask prices never leave the competitive window.

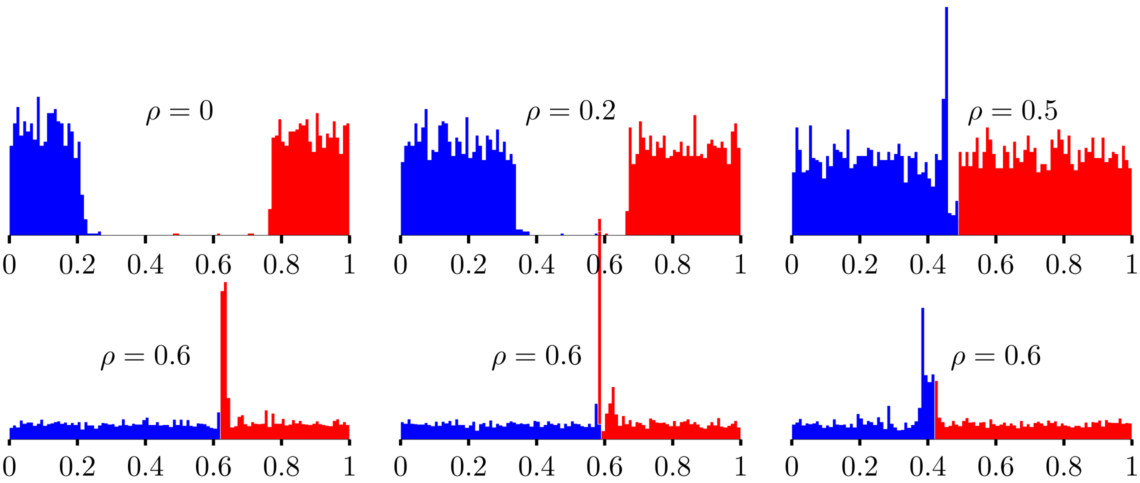

Figure 2: Simulation of the uniform Stigler-Luckock model of Figure 1 for different values of the rate ρ of market makers. Shown is the state of the order book after the arrival of $10,000$ traders. The histograms for $\rho = 0.6$ have a different vertical scale.

3 Behavior of the model with market makers

3.1 Numerical simulation

In Figure 2, we show the results of numerical simulations of the "uniform" Stigler-Luckock model with $\overline{I} = [0, 1]$, $\lambda_-(x) = 1 - x$, and $\lambda_+(x) = x$, for different rates ρ of market makers. We observe that as ρ is increased, the size of the competitive window decreases, until for $\rho = \rho_c = 0.5$, it closes completely and the bid and ask prices settle at the Walrasian price x_W. If the rate ρ of market orders is increased even more beyond this point, we observe an interesting phenomenon. In this regime, the bid and ask prices converge to a random limit which is different each time we run the simulation, and which in general also differs from the Walrasian price x_W. The reason for this is a huge surplus of limit buy and sell orders placed by market makers on the current bid and ask prices, which is capable of "freezing" the price at a random position.

In the coming subsections, we will demonstrate that the critical rate ρ_c of market makers for which the competitive window closes completely is for continuous models given by $\rho_c = V_W$, the Walrasian volume of trade. We will argue that for $\rho < V_W$, the equilibrium distributions of the bid and ask prices are still given by the unique solutions of a differential equation, similar to the one for the model without market makers. For $\rho \geq V_W$, we will prove that the bid and ask price converge to a common limit and determine the subinterval of possible prices where this limit can take values.

3.2 Stationary models

In the present subsection, we show how for $0 < \rho < V_W$, one can calculate the competitive window and the equilibrium distributions of the bid and ask prices by methods similar to those for $\rho = 0$. We first investigate how Luckock's differential equation changes in the presence of market makers.

Theorem 3 (Luckock's differential equation) *Theorem 2 generalizes to $\rho \geq 0$ provided* *we modify Luckock's equation (2.7) to*

$$\begin{aligned} &\text{(i)} \quad f_- \mathrm{d}\lambda_+ + (\lambda_- - \rho)\mathrm{d}f_+ = 0, \\ &\text{(ii)} \quad f_+ \mathrm{d}\lambda_- + (\lambda_+ - \rho)\mathrm{d}f_- = 0, \\ &\text{(iii)} \quad\quad f_-(I_+) = 1 = f_+(I_-). \end{aligned} \tag{3.1}$$

Proof We first show that $f_\pm$ are continuous. By symmetry, it suffices to do this for f_-. Right continuity is immediate from the continuity of the probability measure $\mathbb{P}$. To prove continuity, it suffices to prove that $\mathbb{P}[M_t^- = x] = 0$ for all $x \in (I_-, I_+]$. This is clear for $x = I_+$. Imagine that $\mathbb{P}[M_0^- = x] > 0$ for some $x \in (I_-, I_+)$. Since $\mathcal{X}_0 \in \mathcal{S}_{\mathrm{ord}}$, there are initially finitely many buy limit orders in $[x, I_+)$. By assumption (A4), there is a positive probability that these buy limit orders are all removed at some time before time one, while by assumption (A2), the probability of a new buy limit order arriving at x after such a time is zero. This proves that $\mathbb{P}[M_1^- = x] < \mathbb{P}[M_0^- = x]$, contradicting stationarity.

To prove (3.1), we observe that by stationarity, for each measurable $A \subset I$ that is bounded away from I_-, sell limit orders are added in A at the same rate as they are removed. This yields the equation

$$\int_A \mathbb{P}[M_t^- < x]\, \mathrm{d}\lambda_+(\mathrm{d}x) + \rho \int_A \mathbb{P}[M_t^+ \in \mathrm{d}x] = \int_A \lambda_-(x)\, \mathbb{P}[M_t^+ \in \mathrm{d}x]. \tag{3.2}$$

Here, the first term on the left-hand side is the frequency at which sell limit orders are added at a price $x \in A$ while the current bid price is lower than x, the second term on the left-hand side is the frequency at which market makers add sell limit orders at the current ask price, and the right-hand side is the frequency at which sell limit orders at the current ask price are removed because of the arrival of a buy limit order at a lower price or the arrival of a buy market order. Using also continuity of f_-, (3.2) proves (3.1) (i). The proof of (ii) is similar while the boundary conditions (iii) follow from the fact that $M_t^- < I_+$ and $M_t^+ > I_-$ a.s. ∎

Assume (A1)–(A5), fix ρ and define $\tilde{\lambda}_\pm := \lambda_\pm - \rho$. Then $\mathrm{d}\tilde{\lambda}_\pm = \mathrm{d}\lambda_\pm$ and hence (3.1) is just Luckock's original equation (2.7) with $\lambda_\pm$ replaced by $\tilde{\lambda}_\pm$. In particular, if $\rho < V_{\mathrm{W}}$, then

$$\tilde{V}_{\max} := \sup\big\{V \geq V_{\mathrm{W}} : \tilde{\lambda}_-(\lambda_+^{-1}(V)) \wedge \tilde{\lambda}_+(\lambda_-^{-1}(V)) > 0\big\} \tag{3.3}$$

satisfies $V_{\mathrm{W}} < \tilde{V}_{\max}$, and for each $V \in (V_{\mathrm{W}}, \tilde{V}_{\max})$, the functions $\tilde{\lambda}_\pm$ are positive on the subinterval $J(V) = (\lambda_-^{-1}(V), \lambda_+^{-1}(V))$. This suggests that for the model with market makers, Luckock's volume of trade should be given by (2.4) but with $V_{\max}$ replaced by $\tilde{V}_{\max}$ and with the functions $\lambda_\pm$ in the definition of Φ in (2.3) replaced by $\tilde{\lambda}_\pm$.

Defining V_{L} by this formula, if $V_{\mathrm{L}} < \tilde{V}_{\max}$, then [Swa16, Prop. 2] tells us that (3.1) has a unique solution (f_-, f_+) on the competitive window $J(V_{\mathrm{L}}) = (\lambda_-^{-1}(V_{\mathrm{L}}), \lambda_+^{-1}(V_{\mathrm{L}}))$, which should give the equilibrium distribution of the bid and ask prices. Moreover, since $\tilde{V}_{\max}$ (which depends on ρ) decreases to V_{W} as $\rho \uparrow V_{\mathrm{W}}$, we see that $V_{\mathrm{L}} \downarrow V_{\mathrm{W}}$ and the size of the competitive window decreases to zero as $\rho \uparrow V_{\mathrm{W}}$.

3.3 The regime with many market makers

In the previous subsection, we have argued that the competitive window has a positive length for each $\rho < V_{\mathrm{W}}$ but its length decreases to zero as $\rho \uparrow V_{\mathrm{W}}$. In the present subsection, we look at the regime $\rho \geq V_{\mathrm{W}}$. It will be necessary to strengthen assumptions (A1) and (A3) on the demand and supply functions $\lambda_\pm$, to:

(A6) λ_- is strictly decreasing on I and λ_+ is strictly increasing on I.

We have argued in Subsection 1.1 that the assumptions (A1)–(A3) can basically be made without loss of generality. Moreover, (A4) and (A5) only exclude trivial cases. Assumption (A6) is restrictive, however. As explained at the end of Subsection 1.1, we can include models where prices assume only discrete values in our analysis by constructing such models as functions of other models which satisfy (A1)–(A3). However, as is clear from (1.2), these models will not satisfy (A6), so our result Theorem 4 below does not apply to discrete models.

For models with market orders, we generalize our previous definition of the Walrasian volume of trade $V_{\rm W}$ by setting

$$V_W := \sup_{x \in \overline{I}} \big(\lambda_-(x) \wedge \lambda_+(x)\big). \tag{3.4}$$

Under the assumptions (A2) and (A6), the function $\lambda_- \wedge \lambda_+$ assumes its maximum over $\overline{I}$ in a unique point $x_{\rm W}$, which we call the Walrasian price. For models without market orders, these definitions agree with our earlier definitions. The following theorem describes the behavior of Stigler-Luckcock models with $\rho \geq V_{\rm W}$.

Theorem 4 (Fixation of the price) *Let $(\mathcal{X}_t)_{t\geq 0}$ be a Stigler-Luckcock model with demand and supply functions $\lambda_\pm$ satisfying (A2), (A4), and (A6), and rate of market makers ρ satisfying $\rho \geq V_{\rm W}$, started in an initial state in $\mathcal{S}_{\rm ord}$. Let $M^\pm_t = M^\pm(\mathcal{X}_t)$ denote the bid and ask price at time $t \geq 0$. Then there exists a random variable M_∞ such that*

$$\lim_{t\to\infty} M^-_t = \lim_{t\to\infty} M^+_t = M_\infty \quad \text{a.s.} \tag{3.5}$$

Moreover, the support of the law of M_∞ is given by $\{x \in \overline{I} : \lambda_-(x) \vee \lambda_+(x) \leq \rho\}$. In particular, if $\rho = V_{\rm W}$, then $M_\infty = x_{\rm W}$ a.s.

We prepare for the proof of Theorem 4 with a number of lemmas, some of which are of independent interest.

Lemma 5 (Lower bound on freezing probability) *Let $(\mathcal{X}_t)_{t\geq 0}$ be a Stigler-Luckcock model on an interval I with demand and supply functions $\lambda_\pm$ satisfying (A1)–(A4) and rate of market makers $\rho \geq 0$. Assume that initially $M^-_0 = y$ where $y \in I$ satisfies $\lambda_+(y) < \rho$. Then*

$$\mathbb{P}\big[M^-_t \geq y \ \forall t \geq 0\big] \geq 1 - \frac{\lambda_+(y)}{\rho}. \tag{3.6}$$

Proof Consider the number $\mathcal{X}^-_t(\{y\})$ of buy limit orders that are placed exactly at the price y. At times when $M^-_t = y$, this quantity goes up by one with rate ρ and down by one with rate $\lambda_+(y)$, while at times when $M^-_t > y$, this quantity does not change at all. Thus, up to the first time that $\mathcal{X}^-_t(\{y\}) = 0$, this process is a random time change of the random walk on $\mathbb{Z}$ that jumps up one step with rate ρ and down one step with rate $\lambda_+(y)$. If $\lambda_+(y) < \rho$, then by the well-known gambler's ruin, this random walk, started in 1, stays positive with probability $1 - \lambda_+(y)/\rho$. ∎

Lemma 6 (Bound on the competitive window) *Let $(\mathcal{X}_t)_{t\geq 0}$ be a Stigler-Luckcock model on an interval I with demand and supply functions $\lambda_\pm$ satisfying (A1)–(A4) and rate of market makers $\rho \geq 0$. Assume that $x, y \in I$ satisfy $\lambda_-(x) > \lambda_-(y)$ and $\lambda_+(y) < \rho$. Then*

$$\mathbb{P}\big[\liminf_{t\to\infty} M^-_t < x \text{ and } \limsup_{t\to\infty} M^+_t > y\big] = 0. \tag{3.7}$$

By symmetry, the same conclusion can be drawn if $\lambda_+(x) < \lambda_+(y)$ and $\lambda_-(x) < \rho$.

Proof If we start the process in an intial state such that $M_0^+ \geq y$, then there is a probability

$$p := \frac{\lambda_-(x) - \lambda_-(y)}{\lambda_-(I_-) + \lambda_+(I_+) + \rho} > 0 \tag{3.8}$$

that the first trader arriving at the market places a buy limit order somewhere in the interval (x, y). By Lemma 5, there is then a probability of at least $q := 1 - \lambda_+(y)/\rho > 0$ that after this event, the best buy price M_t^- never drops to values $\leq x$ anymore. Thus, letting σ denote the first time that a trader arrives at the market, we have that

$$\mathbb{P}\big[M_t^- > x \ \forall t \geq \sigma \,|\, M_0^+ \geq y\big] \geq pq > 0. \tag{3.9}$$

We claim that this implies (3.7). To see this, set $\tau_0 := 0$ and define inductively

$$\begin{aligned} \sigma_k &:= \inf\{t \geq \tau_k : M_t^+ \geq y\} && (k \geq 0),\\ \sigma'_k &:= \inf\{t > \sigma_k : \text{ a trader arrives}\} && (k \geq 0),\\ \tau_k &:= \inf\{t \geq \sigma'_{k-1} : M_t^- \leq x\} && (k \geq 1), \end{aligned} \tag{3.10}$$

where the infimum over the empty set is $:= \infty$. By the strong Markov property, $P[\tau_k < \infty] \leq (1 - pq)^k$ and hence $\mathbb{P}[\tau_k < \infty \ \forall k \geq 0] = 0$, which implies (3.7). ∎

Lemma 7 (Freezing) *Let $(\mathcal{X}_t)_{t\geq 0}$ be a Stigler-Luckock model with demand and supply functions $\lambda_\pm$ satisfying (A2), (A4), and (A6), and rate of market makers ρ satisfying $\rho \geq V_{\mathrm{W}}$. Then there exists a random variable M_∞ such that*

$$\lim_{t\to\infty} M_t^- = \lim_{t\to\infty} M_t^+ = M_\infty \quad \text{a.s.} \tag{3.11}$$

Proof If (3.11) does not hold, then there must exist $x, y \in I$ with $x < y$ such that

$$\mathbb{P}\big[\liminf_{t\to\infty} M_t^- < x \text{ and } \limsup_{t\to\infty} M_t^+ > y\big] > 0. \tag{3.12}$$

By (A6), making the interval (x, y) smaller if necessary we can assume without loss of generality that we are in one of the following two cases: I. $\lambda_+(y) < \rho$, and II $\lambda_-(x) < \rho$. Using again (A6), we see that (3.12) contradicts Lemma 6. ∎

Lemma 8 (Bound on possible limit values) *Under the assumptions of Lemma 7, the random variable M_∞ from (3.11) satisfies*

$$\lambda_-(M_\infty) \vee \lambda_+(M_\infty) \leq \rho \quad \text{a.s.} \tag{3.13}$$

Proof By symmetry, it suffices to prove that $\lambda_+(M_\infty) \leq \rho$ a.s. Assume the converse. Then there exists some $z \in I$ with $\lambda_+(z) > \rho$ such that $\mathbb{P}\big[M_\infty \in (z, I_+]\big] > 0$. By the continuity of λ_-, for each $\varepsilon > 0$, we can cover the compact interval $[z, I_+]$ with finitely many intervals of the form (x, y) (if $y < I_+$) or $(x, y]$ (if $y = I_+$) such that $\lambda_-(x) - \lambda_-(y) \leq \varepsilon$. In view of this, we can find $x < y$ and $u > 0$ such that $\lambda_+(x) > \rho + \big(\lambda_-(x) - \lambda_-(y)\big)$ and $\mathbb{P}[x \leq M_t^- \leq M_t^+ \leq y \ \forall t \geq u] > 0$.

During the time interval $[u, \infty)$, the number of buy limit orders in $[x, y)$ can only increase when a market maker arrives or a buyer places a buy limit order in $[x, y)$. On the other hand, the number of buy limit orders in $[x, y)$ decreases each time a trader places a sell market order or a sell limit order at some price in $(I_-, x]$, which happens at times according to a Poisson process with rate $\lambda_+(x)$. Since $\lambda_+(x) > \rho + \big(\lambda_-(x) - \lambda_-(y)\big)$, by the strong law of large numbers applied to the Poisson processes governing the arrival of different sorts of traders, we see that a.s. on the event that $x \leq M_t^- \leq M_t^+ \leq y \ \forall t \geq u$, there must come a time when there are no buy limit orders left in $[x, y)$, which is a contradiction. ∎

Proof of Theorem 4 Lemmas 7 and 8 show that $M_t^\pm$ converge a.s. to a common limit M_∞ which takes values in the compact interval $C := \{x \in \overline{I} : \lambda_-(x) \vee \lambda_+(x) \leq \rho\}$. If $\rho = V_{\rm W}$, then by (A6), C consists of the single point $C = \{x_{\rm W}\}$. On the other hand, if $\rho > V_{\rm W}$, then by (A6), $C = [C_-, C_+]$ is an interval of positive length. To complete the proof, we must show that in the latter case, for each $C_- < x < y < C_+$, the event $M_\infty \in (x, y)$ has positive probability. It is not hard to see that for each $\mathcal{X}_0 \in \mathcal{S}_{\rm ord}$ and $t > 0$, there is a positive probability that $x < M_t^- < M_t^+ < y$. Thus, it suffices to prove that if $x < M_0^- < M_0^+ < y$, then $\mathbb{P}[M_\infty \in (x, y)] > 0$. This is similar to Lemma 5, but we use a slightly different argument.

Note that by (A6), $\lambda_-(x) < \rho$ and $\lambda_+(y) < \rho$. As long as $x \leq M_t^- \leq M_t^+ \leq y$, the number $\mathcal{X}_t^-\big([x, y]\big)$ of buy limit orders in $[x, y]$ goes up by one with rate at least ρ and decreases by one with rate at most $\lambda_+(y)$. A similar statement holds for the number of sell limit orders in (x, y). Let $(N_t^-, N_t^+)_{t\geq 0}$ be a Markov process in $\mathbb{Z}^2$ that jumps with rates

$$\begin{aligned} &(n_-, n_+) \mapsto (n_- + 1, n_+) \quad \text{at rate} \quad \rho, &&(n_-, n_+) \mapsto (n_- - 1, n_+) \quad \text{at rate} \quad \lambda_+(y), \\ &(n_-, n_+) \mapsto (n_-, n_+ + 1) \quad \text{at rate} \quad \rho, &&(n_-, n_+) \mapsto (n_-, n_+ - 1) \quad \text{at rate} \quad \lambda_-(x). \end{aligned} \tag{3.14}$$

Then $(N_t^-)_{t\geq 0}$ and $(N_t^+)_{t\geq 0}$ are independent random walks with positive drift, and hence by the strong law of large numbers, if $N_0^- > 0$ and $N_0^+ > 0$, then

$$\mathbb{P}[N_t^- > 0 \text{ and } N_t^+ > 0 \ \forall t \geq 0] > 0. \tag{3.15}$$

The claim now follows from a simple coupling argument, comparing $\mathcal{X}_t^\pm\big([x, y]\big)$ with $N_t^\pm$. ∎

3.4 Conclusion

The Stigler-Luckock model is one of the most basic and natural models for traders interacting through a limit order book, so natural, in fact, that it has been at least four times independently (re-)invented [Sti64, Luc03, Pla11, Yud12b]. Although it is based on natural assumptions, its behavior is unrealistic since the bid and ask prices do not settle at the Walrasian equilibrium price but rather keep fluctuating inside an interval of positive length called the competitive window. This provides an opportunity for market makers who make money from buying at a low price and selling at a higher price.

In this paper, we have added such market makers to the model who trade using a very simple strategy, namely, by placing one buy and sell limit order at the current bid and ask prices. We have seen that the addition of market makers makes the model more realistic in the sense that the size of the competitive window decreases. In particular, for continuous models, if the rate at which market makers arrive equals the Walrasian volume of trade, then the size of the competitive window decreases to zero and the bid and ask prices converge to the Walrasian equilibrium price. If the rate of market makers is even higher, then the bid and ask prices also converge to a common limit, but now the limit price is random and in general differs from the Walrasian equilibrium price. Moreover, in this regime, some of the limit orders placed by market makers are never matched by market orders but stay in the order book forever (on the time scale we are interested in).

In reality, market makers make profit only if their limit orders are matched, and this profit is proportional to the size of the competitive window. Therefore, in real markets, there is no motivation for market makers to trade once the size of the competitive window has shrunk to zero. In view of this, in reality, we can expect a self-regulating mechanism that makes sure that in the long run, the rate at which market makers place orders is approximately equal to the Walrasian volume of trade. The effect of this is that in the limit, all trade involves market makers, i.e., the buyers and sellers of the original Stigler-Luckock model do not directly interact with each other but make all their trade with the market makers.

We conclude from this that adding market makers to the Stiger-Luckock model has produced a more realistic model, especially if the rate of market makers is chosen equal to the Walrasian volume of trade. Future, better models should include a self-regulating mechanism that links the rate at which market makers place orders to the present state of the order book by weighing their expected profit against the costs and risks. Realistic models should also consider prices that can take only discrete values since in reality the size of the competitive window and hence the potential for profit for market makers are bounded from below by the tick size.

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