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Network-based control of nonlinear large-scale systems composed of identical subsystems

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Abstract

This paper deals with a control of coupled nonlinear identical systems that admit full exact feedback input-output linearization. The subsystems are linearized using this nonlinear transformation. In the next step, an auxiliary low-dimensional system is derived whose stability implies stability of the original large-scale system. The control law is designed so that the control loops are only local, no information exchange between subsystems is required. Unknown time delay in the feedback are allowed. Two cases are studied: equal time delay for all subsystems or different delay in all subsystems. Results are illustrated by two examples.

1. Introduction

1.1. State of the art

In practice, many control systems have a large number of inputs and outputs. It is not desirable to control such systems centrally as such a control would be too complicated, costly to implement and prone to failures. Therefore it is desirable to develop a decentralized control law. This means, the system is decomposed into subsystems; the control is designed for these subsystems with the following feature in mind: to control one particular subsystem,

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only information gained from this subsystem are used; the control must be robust enough to stabilize the system even in presence of interactions caused by other subsystems. See also [1,2] for further details.

A special case of large-scale systems are systems composed of a large number of identical subsystems. In practice, this kind of systems describes e.g. crystal growth furnaces [3], applications in optics [4] or control of paper machines [5]. Control of spatially distributed systems like vehicle platoons [6] or flight formations is a related problem.

A particular kind of these systems are systems where every subsystem is connected to every other subsystem. These systems will be called symmetrically interconnected systems in accordance with Chapter 12 in [1]. An introduction to these systems appeared in [7] where application of this theory to the control of parallel reactors with combined precooling is described. Let us mention control of power systems [8,9] which is another important application of these systems.

The essential tool for analysis of systems composed of identical interconnected subsystems is a state transformation that converts the interconnected system into a block-diagonal one. See [1] for symmetrically coupled systems or [10] or [11] for systems with a general coupling. Therefore, stability of the interconnected system is converted into the problem of robust control of a system with dimension equal to the dimension of one subsystem. The stabilizing solution for this system is obtained using linear matrix inequalities (LMI). These ideas were later elaborated e.g. in [12,13], a thorough analysis together with analysis of fault tolerance can be found in [14]. Control of a switched system composed of identical subsystems is treated in [15]. Let us also mention LQ-optimal control of a large number of identical coupled systems in [16]. Decomposition approach to the control of symmetric system is presented in [17].

Control of large-scale systems is naturally combined with communication networks. Usage of communication networks helps to save costs through reducing complexity of wiring and provides a high degree of flexibility and reliability. On the other hand, several issues arise, e.g. random delays in the communication channel and packet dropouts occur, quantization is inevitable etc., see [2] and references therein. Control of large-scale symmetrically interconnected systems is treated in [18]. Analysis of the maximal allowable time delay in the networked control systems can be found in [19].

The problem of time delay systems control is usually solved using LMI. Recently, the so-called descriptor approach ([20,21] and others) gained attention as it is easy-to-implement and yields results that are not overly conservative. The estimates are usually based on the Jensen's inequality as presented e.g. in [22]. The aforementioned approach is used in this paper as well. The estimates can be further improved e.g. by replacing the Jensen inequality by the Wirtinger inequality [23–26] and others.

In the early days, the control of large systems was focused on linear systems. However, interest in control of large-scale nonlinear systems is rising. A large group is formed by methods that handle the nonlinearity using the Lipschitz inequality and subsequent employing linear robust design methods, mostly based on H_∞ control. In [27], this approach is used while employing Jensen's inequality to prove exponential stability. Other results from this class are described in [28,29]. In other works, the interconnected systems are handled e.g. by backstepping. Two cases of interconnections are distinguished: weakly and strongly coupled large-scale systems. In the case of weakly-coupled systems, the interconnections between subsystems are realized through functions of outputs of subsystems, see e.g. in [30–32] and others. On the other hand, the interconnection terms depend on arbitrary state variables of other subsystems in the case of strongly interconnected systems, see [33]. A control of large-scale nonlinear systems where all subsystems are similar (the system matrices are coupled by the similarity transformation) is derived in [34] using LMI. Furthermore, control of large-scale nonlinear systems via exact linearization is presented in [35]. In this paper, the output-feedback control is designed so that asymptotic tracking is achieved.

1.2. Contribution of the paper

Networked control of nonlinear weakly coupled large-scale systems composed of identical subsystems is studied. The main contribution is to present a novel method based on full exact feedback linearization ([36], Chapter 12.2) of the subsystems in connection with the dimension reduction, as explained in [1]. Moreover, time delays in the control loop are considered and treated using the descriptor approach. The authors believe this combination has never been explored before. The controller design is less conservative while the method remains easy-to-implement and does not pose overly demanding requirements on the computational resources. The most significant features are:

- Exploiting the nonlinear structure of the system rather than merely approximating it by the Lipschitz inequality as is the usual approach for control of identical interconnected systems.
- Formulating the LMI problem with help of the descriptor approach specifically tailored to handle the uncertainties resulting from the exact feedback linearization and demonstrating viability of the descriptor approach to solve the aforementioned problem.
- Formulating the above mentioned LMI problem so that it involves only matrices whose dimension does not depend on the number of the subsystems but solely on the dimension of the subsystems.

The proposed method is an extension of the control design for linear symmetric systems with a single delay in the control loop described in [12] to nonlinear systems and in case of multiple delays in [13]. Effects of other phenomena occurring in the networked control (e.g. quantization) are not considered in this paper.

The results could potentially be extended to the application to complex dynamical networks [37,38] and chaotic systems [39–41].

1.3. Outline

The problem is defined in the second section. All state transformations are presented in the third section. The controller design follows; analysis of an auxiliary system from the Section 4.2 is a crucial part of this section. The fifth section contains examples together with brief discussion about the results. The proofs of theorems concerning the auxiliary system are concentrated in the Appendix. Finally, conclusions and outlooks finish the paper.

2. Problem setting

Notation:

- The symbol $L_f\lambda$ denotes the so-called Lie derivative: $L_f\lambda(x) = \frac{d\lambda}{dx}.f(x)$ where λ: $R^n \to R$, f: $R^n \to R^n$ and the function λ is sufficiently smooth.

- If P is a square matrix, then $P > \mathbf{0}$ means the matrix P is symmetric positive definite.
- For any function f depending on time, the time argument is omitted where no confusion can arise: f means the same as $f(t)$. If the argument is different from t, it is written in full.
- If M, N are matrices, then $\mathrm{diag}(M, N) = \begin{pmatrix} M & \mathbf{0} \\ \mathbf{0} & N \end{pmatrix}$. Here as well as in the subsequent text, $\mathbf{0}$ denotes the zero matrix of appropriate dimension.
- The symbol I denotes the identity matrix. Its dimension can depends on the context where this symbol is used. If a confusion can arise, then the symbol I_k which denotes the k-dimensional identity matrix is used.
- In symmetric matrices, the part under the diagonal is not written in full, rather, the symbol $*$ is used: $\begin{pmatrix} X & Y \\ * & Z \end{pmatrix} = \begin{pmatrix} X & Y \\ Y^T & Z. \end{pmatrix}$
- If f is a function of time, then f_τ denotes this function with a delayed argument: $f_\tau = f(t - \tau)$ for all $\tau > 0$.

The system to be controlled is composed of N identical subsystems. The ith subsystem ($i = 1, \dots, N$) is described by the equation

$$\dot{x}_i = f(x_i) + g(x_i)u_i + \sum_{j=1, j\neq i}^{N} J_{ij} l \lambda(x_j). \tag{1}$$

The symbol $x_i = (x_{1,i}, \dots, x_{n,i})^T$ denotes the state of the ith subsystem, u_i is its control. It is supposed that the functions f: $R^n \to R^n$, λ: $R^n \to R$ are sufficiently smooth, g: $R^n \to R^n$ is continuous, $l = (l_1, \dots, l_n)^T \in R^n$. The term $\sum_{j=1, j\neq i}^{N} J_{ij} l \lambda(x_j)$ represents interconnections, that means, it describes the influence of other subsystems on the ith one. The matrix $J \in R^{N\times N}$ is the adjacency matrix with $J_{ij} \in \{0, 1\}$ such that $J_{ii} = 0$ for all i, $J_{ij} = 1$ if and only if the derivative of the ith state depends on the jth one.

The goal is to find a feedback control $u_i = \mathbf{F}(x_i)$ for all $i = 1, \dots, N$ such that, if the control input u_i is applied to the ith subsystem, the overall system is stabilized. Note also that to control the ith subsystem, only values of state of the ith subsystem are needed as this is the main objective for the decentralized control.

Time delay in the control loop is considered: the control action is computed from delayed state values. This (time varying) delay in the control of the ith subsystem is denoted by $\tau_i(t)$. In accordance with the above notation, the symbol τ_i means the same as $\tau_i(t)$. The functions τ_i are not known.

Assumption 1. There exists a constant $\bar{\tau} > 0$ such that $\tau_i(t) \in [0, \bar{\tau}]$ for each $i = 1, \dots, N$. This constant is assumed to be available for the control design.

Assumption 2. The matrix J is diagonalizable; this means, there exists an invertible matrix U and a diagonal matrix D such that $J = U^{-1}DU$. Moreover, all eigenvalues of the matrix J are real.

Remark 1. The form of the system (1) where the interconnections depend on a certain function λ (which plays the role of the output in the subsequent text) seems to be restrictive. However, several important practical applications, e.g. control of power electric networks or decentralized control of platoons of vehicles with nonlinear dynamics, lead to a problem satisfying this assumption.

3. Transformation of the subsystems

Assumption 3. The auxiliary system $\dot{x} = f(x) + g(x)u, \;\; y = \lambda(x)$ has relative degree n.

Full exact feedback input-output linearization transforms the system (1) by using the following variables:

$$\xi_i = \mathcal{T}(x_i) = (\lambda(x_i), \dots, L_f^{n-1}\lambda(x_i))^T. \tag{2}$$

Moreover, let us define functions Φ: $R^n \to R$, Ψ: $R^n \to R$ as follows:

$$\Phi(\xi_i) = L_f^n \lambda(x_{1,i}), \;\; \Psi(\xi_i) = L_g L_f^{n-1} \lambda(x_{1,i}), \;\; v_i = \Psi(\xi_i)u_i + \Phi(\xi_i).$$

Using the matrices $A \in R^{n \times n}, B \in R^{n \times 1}$ and $L \in R^{n \times n}$ defined by

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}, L = \begin{pmatrix} l_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ l_n & \underbrace{0 \quad \dots \quad 0}_{n-1} \end{pmatrix}, B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

one can write the feedback linearization transforms the ith subsystem into the linear form (here, the symbol $\frac{\partial \mathcal{T}}{\partial x}$ denotes the Jacobi matrix of the mapping $\mathcal{T}$)

$$\begin{aligned} \dot{\xi}_i &= A\xi_i + B\Big(\Psi(\xi_i)u_i + \Phi(\xi_i)\Big) + \frac{\partial \mathcal{T}}{\partial \xi}(\mathcal{T}^{-1}(\xi_i)) \sum_{j=1, j\neq i}^{N} J_{ij} L \xi_j \\ &= A\xi_i + Bv_i + \frac{\partial \mathcal{T}}{\partial \xi}(\mathcal{T}^{-1}(\xi_i)) \sum_{j=1, j\neq i}^{N} J_{ij} L \xi_j. \end{aligned} \tag{3}$$

Assume that the system is stabilized by a linear feedback using the local loops. The ith subsystem is stabilized by using information contained in the vector ξ_i only. In the delay-free case, the ith subsystem is stabilized by applying the classical setting of feedback linearization ([36], Chapter 12.3.). Note also that these assumptions correspond to the assumptions made in [35], namely

$$u_i = \frac{1}{\Psi(\xi_i)}(v_i - \Phi(\xi_i)) \tag{4}$$

with $v_i = K\xi_i$ chosen so that the matrix $A + BK$ is Hurwitz. The nonlinear term in the formula (4) matches exactly the nonlinear terms in the first row of Eq. (3). However, the networked control imposes some time delay caused e.g. by finite time needed to acquire information about the state ξ_i or to pass the control to the plant. Hence, the control u_i^* defined by

$$u_i^* = \frac{1}{\Psi(\xi_{i,\tau_i})}(K\xi_{i,\tau_i} - \Phi(\xi_{i,\tau_i})) \tag{5}$$

is used instead. If we define

$$\Delta u_i = \left(\frac{1}{\Psi(\xi_{i,\tau_i})} - \frac{1}{\Psi(\xi_i)}\right) K\xi_{i,\tau_i} + \frac{\Phi(\xi_i)}{\Psi(\xi_i)} - \frac{\Phi(\xi_{i,\tau_i})}{\Psi(\xi_{i,\tau_i})} \tag{6}$$

then the relation $u_i^* = \frac{1}{\Psi(\xi_i)}(K\xi_{i,\tau_i} - \Phi(\xi_i)) + \Delta u_i$, analogous to Eq. (4), holds. The term Δu_i will be converted into an uncertainty in the subsequent text. In this case, the system (3) can be expressed as

$$\dot{\xi}_i = A\xi_i + \frac{\partial \mathcal{T}}{\partial \xi}(\mathcal{T}^{-1}(\xi_i)) \sum_{j=1, j\neq i}^{N} J_{ij} L\xi_j + BK\xi_{i,\tau_i} + B\Psi(\xi_i)\Delta u_i. \tag{7}$$

To sum up, if we find the matrix K so that the system (7) is stable even in presence of the uncertainty represented by the term $B\Psi(\xi)\Delta u_i$ then the system (3) is stable with the control u_i^*.

Remark 2. Assume $\Psi = 1$ and let there exists an n-dimensional vector $\bar{\Phi} = (\bar{\Phi}_1, \ldots, \bar{\Phi}_n)$ and a smooth function $\tilde{\Phi} : R^n \to R$ vanishing at zero together with its derivatives such that $\Phi(\xi) = \bar{\Phi}\xi + \tilde{\Phi}(\xi)$. Then, one can use $\tilde{A} = A + \binom{\mathbf{0}_{n-1\times n}}{\bar{\Phi}_1, \ldots, \bar{\Phi}_n}$ instead of the matrix A and the function $\tilde{\Phi}$ instead of Φ. This reduces the uncertainties which, in turn, improves efficiency of the algorithm which will be presented later.

In the subsequent text, two problems are studied: first, all delays in the network for all subsystems are equal (albeit not constant in time), that means, $\tau_i = \tau_j$ for all $i, j = 1, \ldots, N$ while this condition is not satisfied in the second problem.

Assumption 4. The differences can be estimated as follows:

- There exist matrices $D_2 \in R^{n\times n}$, $E_2 \in R^{n\times n}$ and measurable functions $F_{2,i}$: $R \to R^{n\times n}$, $i = 1, \ldots, N$ such that for every t holds $B(\Phi(\xi_i) - \Phi(\xi_{i,\tau_i})\frac{\Psi(\xi_i)}{\Psi(\xi_{i,\tau_i})})) = D_2 F_{2,i}(t) E_2(\xi_i - \xi_{i,\tau_i})$ and $\|F_{2,i}(t)\| \le 1$.
- There exist matrices $D_3 \in R^{n\times n}$, $E_3 \in R^{n\times n}$ and measurable functions $F_{3,i}$: $R \to R^{n\times n}$ such that $B(\frac{\Psi(\xi_i)}{\Psi(\xi_{i,\tau_i})} - 1) = D_3 F_{3,i}(t) E_3$ and $\|F_{3,i}(t)\| \le 1$ for every t. Here, the symbol $\|.\|$ denotes the spectral norm (largest singular value).
- There exist matrices $M \in R^{n\times n}$, $D_4 \in R^{n\times n}$, $E_4 \in R^{n\times n}$ and measurable matrix-valued functions $F_{4,i}$: $R \to R^{n\times n}$, $i = 1, \ldots, N$ so that $\frac{\partial \mathcal{T}}{\partial \xi}(\mathcal{T}^{-1}(\xi_i)) = M + D_4 F_{4,i}(t) E_4$ and $\|F_{4,i}(t)\| \le 1$ for every t.

Remark 3. Let us focus on the practical choice of the matrices D_j, E_j for $j = 2, 3, 4$.

- The functions $F_{2,i}$, $F_{3,i}$, $F_{4,i}$ are usually unknown. For the controller design, knowledge of these functions is not necessary, however, the matrices D_j and E_j must be known. They are not unique and can be determined in various ways.
- One method how to find the matrices D_2 and E_2 is described in the sequel. Let I_n be the n-dimensional identity matrix. Assume first that there exists a constant $c > 0$ so that $|\frac{1}{\Psi(\xi)}| > c$ (this holds at least on some neighborhood of the origin). If the function $B\Phi$ has continuous second order derivatives, then, with help of the n-dimensional Taylor formula and the function $\Gamma = \Phi\Psi$, one can write $B(\Gamma(\xi_i) - \Gamma(\xi_{i,\tau_i})) = \mathbf{R}(\xi_i - \xi_{i,\tau_i})$ where the remainder is given by $\mathbf{R} = \sum_{j=1}^n \int_0^1 \mathbf{D}_j B\Gamma(\xi_i + \sigma(\xi_i - \xi_{i,\tau_i}))d\sigma$ (the symbol $\mathbf{D}_j$ denotes the derivative of the function $B\Gamma$ with respect to the jth variable). If the differential of $B\Phi$ is bounded by a constant $\gamma > 0$, one can set $D_2 = \frac{\gamma}{c} I_n$, $E_2 = I_n$ and $F_{2,i}(t) = \frac{1}{\gamma}\mathbf{R}$.

- If the above estimates hold locally on a certain neighborhood of the origin, then it is necessary to ensure that the states of the system remains in this neighborhood for all t. For large initial conditions, this condition might be violated, hence the stabilization is only local.
- The matrices D_3 and E_3 can be obtained e.g. as $D_3 = \frac{2}{c}\|\Psi\|_\infty$, E_3 is the identity matrix. Again, this estimate can be local.
- The matrices D_4, E_4 are determined using a similar reasoning as the matrices D_2, E_2. Assume there exists a constant $\bar{c}$ so that $\bar{c} > \|\frac{\partial\mathcal{T}}{\partial\xi}(\mathcal{T}^{-1}(\xi_i))\|_\infty$. Note that, thanks to continuity of the mapping $\mathcal{T}$, such a constant exists on a neighborhood of the origin. The matrices D_4 and E_4 can then be determined as $D_4 = \bar{c}I$, E_4 is the identity matrix. Again, if this estimate holds only locally, the stabilization is only local.
- From the practical point of view, quality of these estimates has a strong impact on the behavior of the LMI optimization problem which will be presented later.

Note that we assume that the matrices D_j and E_j, $j = 2, 3, 4$ are the same for all subsystems. The closed loop obeys the equation

$$\begin{aligned}\dot{\xi}_i =& A\xi_i + (B + D_3F_{3,i}(t)E_3)K\xi_{i,\tau_i} + (M + D_4F_{4,i}(t)E_4)\sum_{j=1,j\neq i}^{N} LJ_{ij}\xi_j\\ &+ D_2F_{2,i}(t)E_2\int_{t-\tau_i}^{t}\dot{\xi}_i(s)ds. \end{aligned} \tag{8}$$

Moreover, introducing another uncertainty will be useful in the sequel. The new uncertainty is described by n-dimensional matrices D_1, E_1 with measurable matrix-valued functions $F_{1,i}$: $R \to R^{n\times n}$ such that $\|F_{1,i}\| \leq 1$. Using these matrices, the transformed subsystem will be investigated in the form

$$\begin{aligned}\dot{\xi}_i =& (A + D_1F_{1,i}E_1)\xi_i + (B + D_3F_{3,i}(t)E_3)K\xi_{i,\tau_i}\\ &+ (M + D_4F_{4,i}(t)E_4)\sum_{j=1,j\neq i}^{N} LJ_{ij}\xi_j + D_2F_{2,i}(t)E_2\int_{t-\tau_i}^{t}\dot{\xi}_i(s)ds. \end{aligned} \tag{9}$$

Remark 4. Let us explain the motivation for the uncertainty defined by the matrices D_1, E_1 and the functions $F_{1,i}$. In many practical cases, the interconnections are given by the differences $J_{ij}L(\lambda(x_j) - \lambda(x_i))$. Then, the system (3) attains a slightly modified form (compared to Eq. (3)):

$$\dot{\xi}_i = A\xi_i + Bv_i + \frac{\partial\mathcal{T}}{\partial\xi}(\mathcal{T}^{-1}(\xi_i))\sum_{j=1,j\neq i}^{N} J_{ij}L(\xi_j - \xi_i). \tag{10}$$

Then the right-hand side of Eq. (10) can be rewritten as

$$A\xi_i + \frac{\partial\mathcal{T}}{\partial\xi}(\mathcal{T}^{-1}(\xi_i))\sum_{j=1,j\neq i}^{N} J_{ij}L\xi_i + Bv_i + \frac{\partial\mathcal{T}}{\partial\xi}(\mathcal{T}^{-1}(\xi_i))\sum_{j=1,j\neq i}^{N} J_{ij}L\xi_j.$$

The second term is time-dependent and is not equal for all subsystems. Hence, it is treated as uncertainty as follows. First, assume that there exist a matrix $\widetilde{L}$ and a function η: $R^n \to R^n$ so that $\widetilde{L}\xi_i + \eta(\xi_i)\xi_i = \frac{\partial\mathcal{T}}{\partial\xi}(\mathcal{T}^{-1}(\xi_i))\sum_{j=1,j\neq i}^{N} J_{ij}L\xi_i$. The choice of the function η and the matrix $\widetilde{L}$ is not specified here. However, to facilitate the computation of the resulting LMI problem, the linear part $\widetilde{L}\xi_i$ should approximate the term $\frac{\partial\mathcal{T}}{\partial\xi}(\mathcal{T}^{-1}(\xi_i))\sum_{j=1,j\neq i}^{N}J_{ij}L\xi_i$ as closely as possible.

Then, the linear part of the term $A\xi_i+\frac{\partial\mathcal{T}}{\partial\xi}(\mathcal{T}^{-1}(\xi_i))\sum_{j=1,j\neq i}^{N}J_{ij}L\xi_i$ equals to

$$(A+\sum_{j=1,j\neq i}^{N}J_{ij}\widetilde{L})\xi_i.$$

This term is substituted in place of the term $A\xi_i$ in (3). Then, assume the function η is bounded: $\|\eta(\zeta)\|\le s$ for all $\zeta\in R^n$. In the next step, one can define $D_1=sI$, $E_1=I$ and $F_{1,i}(t)=\frac{1}{s}\mathrm{diag}((\eta(\xi_i(t)))_1,\dots,(\eta(\xi_i(t))_n))$. The function $F_{1,i}$ is measurable and satisfies $\|F_{1,i}(t)\|\le 1$ for all t. The system is thus transformed into the form Eq. (9). This way of determining the matrices D_1 and E_1 is not unique; rather it is a very rough estimate. As precise as possible estimate of the function η should be used, however, this estimate depends on the specific problem solved, hence it cannot be given here. If the function η is not bounded, then we assume it is bounded on a neighborhood $\mathcal{U}$ of the origin. Then, the above estimate can be done on the neighborhood $\mathcal{U}$, however, the resulting controller will guarantee stability only if the trajectories do not leave this neighborhood. The functions $F_{1,i}$ are not known for the controller design; the design algorithm is set up so that knowledge of the matrices D_1 and E_1 is sufficient.

Let $\phi_i(t,s)=1$ for $s\in[t-\tau_i,t]$ and $\phi_i(t,s)=0$ elsewhere. Also, let $J_i=\mathrm{diag}(\underbrace{0,\dots,0}_{i-1},I,\underbrace{0,\dots,0}_{N-i})$ with all blocks are $n\times n$-dimensional. Note that $\phi(t,t)=1$. Using this function, one finds that

$\xi(t-\tau_i)=\xi(t)-\int_{t-\bar{\tau}}^{t}\phi_i(t,s)J\dot{\xi}(s)ds$. Hence Eq. (9) can be transformed into

$$\begin{aligned}\dot{\xi_i}=&(A+BK+D_1F_{1,i}(t)E_1+D_3F_{3,i}(t)E_3K)\xi_i\\&+(M+D_4F_{4,i}(t)E_4)\sum_{j=1,j\neq i}^{N}J_{ij}L\xi_j\\&+\Big(D_2F_{2,i}(t)E_2-BK-D_3F_{3,i}(t)E_3\Big)\int_{t-\bar{\tau}}^{t}\phi_i(t,s)J_i\dot{\xi}(s)ds.\end{aligned}\tag{11}$$

$\mathcal{A}=I\otimes A$, $\mathcal{B}=I\otimes B$, $\mathcal{D}_i=I\otimes D$, $\mathcal{E}_i=I\otimes E_i$, $\mathcal{K}=I\otimes K$, $\mathcal{M}=I\otimes M$, $\mathcal{L}=J\otimes L$ and $\mathcal{F}_j(t)=\mathrm{diag}(F_{j,1}(t),\dots,F_{j,N}(t))$, $j=1,2,3,4$, the overall system with the local feedback can be expressed in the compact form

$$\begin{aligned}\dot{\xi}=&(\mathcal{A}+\mathcal{D}_1\mathcal{F}_1(t)\mathcal{E}_1+\mathcal{M}\mathcal{L}+\mathcal{B}\mathcal{K}+\mathcal{D}_3\mathcal{F}_3(t)\mathcal{E}_3\mathcal{K}+\mathcal{D}_4\mathcal{F}_4(t)\mathcal{E}_4\mathcal{L})\xi\\&+\Big(\mathcal{D}_2\mathcal{F}_2(t)\mathcal{E}_2-\mathcal{B}\mathcal{K}-\mathcal{D}_3\mathcal{F}_3(t)\mathcal{E}_3\mathcal{K}\Big)\int_{t-\bar{\tau}}^{t}\sum_{i=1}^{N}\phi_i(t,s)J_i\dot{\xi}(s)ds.\end{aligned}\tag{12}$$

The overall feedback system (12) can be transformed such that the system matrix will be block diagonal. The transformation reads $\tilde{\xi}=T\xi$, the definition of the matrix T is given below.

Remark 5. Two cases are distinguished in the subsequent text. The "General case" means that the matrix T is given in terms of eigenvectors of the matrix J. This approach is applicable to systems with a general kind of interconnections of subsystems. However, analysis of symmetrically interconnected dynamical systems is traditionally carried out using the matrix T as described e.g. in [1,14]. This is the reason why this case is singled out as the "Symmetrically interconnected systems" case in this paper.

Case 1: (General case) Let $I \in R^{N\times N}$ be the identity matrix. The transformation matrix T is defined as (see, e.g. [17])

$$T = U \otimes I, \tag{13}$$

where existence of the matrix U is guaranteed by the Assumption 2. Define also and $\tilde{A} = \mathcal{A} + D \otimes L$. Note that, due to the properties of the Kronecker product (see [42]) one has $T(\mathcal{A} + \mathcal{M}\mathcal{L})T^{-1} = \mathcal{A} + (D \otimes ML) = \tilde{A}$.

Case 2: (Symmetrically interconnected systems) In this case, the matrix T is defined by

$$T = \frac{1}{N}\begin{pmatrix} (N-1)I & -I & \dots & -I & -I \\ -I & (N-1)I & \dots & -I & -I \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -I & -I & \dots & (N-1)I & -I \\ I & I & \dots & I & I \end{pmatrix}. \tag{14}$$

Further, define matrices $A_s = A - ML, A_o = A + NML$ and
$\tilde{A} = \mathrm{diag}(\underbrace{A_s, \dots, A_s}_{N-1 \text{ times}}, A_o)$.

The subsequent analysis is the same for both cases.

In the variable $\tilde{\xi}$ the feedback system (12) attains the form

$$\begin{aligned}\dot{\tilde{\xi}} =& (\tilde{A} + \mathcal{D}_1\widetilde{\mathcal{F}}_1(t)\mathcal{E}_1 + \mathcal{B}\mathcal{K} + \mathcal{D}_3\widetilde{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K} + \mathcal{D}_4\widetilde{\mathcal{F}}_4(t)\mathcal{E}_4\tilde{\mathcal{L}})\tilde{\xi} \\ &+ \left(\mathcal{D}_2\widetilde{\mathcal{F}}_2(t)\mathcal{E}_2 - \mathcal{B}\mathcal{K} - \mathcal{D}_3\widetilde{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}\right)\int_{t-\bar{\tau}}^{t}\sum_{i=1}^{N}\phi_i(t,s)\tilde{J}_i\dot{\tilde{\xi}}(s)ds,\end{aligned} \tag{15}$$

where $\widetilde{\mathcal{F}}_i(t) = T\mathcal{F}_i(t)T^{-1}$, $i = 1, 2, 3$, $\tilde{J}_i = TJ_iT^{-1}$ and $\tilde{\mathcal{L}} = D \otimes L$. Note also $\mathcal{D}_i = T\mathcal{D}_iT^{-1}$ and $\mathcal{E}_i = T\mathcal{E}_iT^{-1}$, $i = 1, 2$, $\mathcal{E}_3\mathcal{K} = T\mathcal{E}_3\mathcal{K}T^{-1}$ as this state transformation does not change block-diagonal matrices with the same block on the diagonal. Contrary to this, the matrices $\widetilde{\mathcal{F}}$ and $\tilde{J}_i$ are not diagonal. With $\mu > 0$ defined as $\mu^2 = \|T^{-1}T^{-1^T}\|\|TT^T\|$ and properties of the uncertainties $F_{j,i}(t)$, $j = 1, 2, 3, 4$, one obtains

$$\widetilde{\mathcal{F}}_i\widetilde{\mathcal{F}}_i^T \le \mu^2 I, \quad i = 1, 2, 3, 4. \tag{16}$$

4. Controller design

In this section, the decentralized controller is designed. In the first subsection, an auxiliary problem is defined. After that, it is shown how the results of this auxiliary problem allow to define a stabilizing solution of the original problem.

4.1. Preliminaries

In the general case, define $\bar{d}$ as the largest eigenvalue of the matrix J and $\underline{d}$ as the smallest eigenvalue of the matrix J. Let the matrices D_A and E_A be chosen as

$$D_AE_A = \frac{\bar{d} - \underline{d}}{2}ML \tag{17}$$

in the general case or

$$D_A E_A = \frac{N+1}{2} ML \tag{18}$$

for the symmetrically interconnected systems. Assume $D_A \in R^{n\times n}$, $E_A \in R^{n\times n}$.

Let us denote the n-dimensional identity matrix by I, then, define the matrix A_m and the constant matrix function $F_A(t) \in R^{Nn\times Nn}$ in the general case:

$$\tilde{d} = \frac{\bar{d} - \underline{d}}{2},\ A_m = A + \frac{1}{2}(\bar{d} + \underline{d})ML,\ F_A(t) = \mathrm{diag}\left(\frac{d_i - \frac{\bar{d}+\underline{d}}{2}}{\tilde{d}} I\right) \tag{19}$$

or

$$A_m = A + \frac{N-1}{2} ML,\ F_A(t) = \mathrm{diag}\left(\underbrace{I, \ldots, I}_{N-1\ \text{times}}, -I\right) \tag{20}$$

for the symmetrically interconnected systems if the transformation matrix T is given by Eq. (14).

$\mathcal{A}_m = I \otimes A_m$, $\mathcal{D}_A = I \otimes D_A$, $\mathcal{E}_A = I \otimes E_A$. In both cases,

$\tilde{A} = \mathcal{A}_m + \mathcal{D}_A F_A(t)\mathcal{E}_A$, $\|F_A(t)\| \le 1$.

4.2. Auxiliary problem

The stabilizing control for the system (15) is closely related to the stabilization of the auxiliary system defined in this subsection.

Assume the functions $F_{m,A}$ and $F_{m,j}$, $j = 1, 2, 3, 4$ are measurable functions satisfying $\|F_{m,A}(t)\| \le 1$, $\|F_{m,j}(t)\| \le 1$, $0 \le \tau \le \bar{\tau}$ being (possibly time varying) unknown time delay. Let also $\widehat{d} = \max(|\mathrm{Eig}\ D|)$ where the matrix D was defined in Assumption 2.

$$\begin{aligned}\dot{z} =& (A_m + \mu D_1 F_{m,1}(t)E_1 + \mu\widehat{d} D_4 F_{m,4}(t)E_4 L + D_A F_{m,A}(t)E_A)z \\ &+ (B + \mu D_3 F_{m,3}(t)E_3)Kz(t-\tau) + \mu D_2 F_{m,2}(t)E_2 \int_{t-\tau}^{t} \dot{z}(s)ds,\end{aligned} \tag{21}$$

which can be using $z(t-\tau) = z(t) - \int_0^\tau \dot{z}(s)ds$ rewritten as

$$\begin{aligned}\dot{z} =& (A_m + \mu D_1 F_{m,1}(t)E_1 + \mu\widehat{d} D_4 F_{m,4}(t)E_4 L + D_A F_A(t)E_A + BK \\ &+ \mu D_3 F_{m,3}(t)E_3 K)z \\ &+ (\mu D_2 F_{m,2}(t)E_2 - BK - \mu D_3 F_{m,3}(t)E_3 K)\int_{t-\tau}^{t} \dot{z}(s)ds\end{aligned} \tag{22}$$

is stable for all uncertainties $F_{m,A}(t)$ and $F_{m,j}(t)$. In the following subsection, we will show that robust stabilization of Eq. (22) for all uncertainties $F_A, F_{m,j}$, $j = 1, 2, 3, 4$ guarantees stability of Eq. (15). This is the reason to introduce the system (22).

The controller for the system (22) is designed using the descriptor approach to the Lyapunov–Krasovskii functional as explained in e.g. [20,21] in detail. First, define the matrix Ψ by

$$\Psi = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} \\ * & \psi_{2,2} & \varepsilon\psi_{1,3} \\ * & * & -\bar{\tau}R \end{pmatrix}, \tag{23}$$

$$
\begin{aligned}
\psi_{1,1} &= P_2^T\big(A_m + \mu D_1 F_{m,1}(t)E_1 + \mu\widehat{d}D_4 F_{m,4}(t)E_4 L + D_A F_{m,A}(t)E_A + BK \\
&\quad + \mu D_3 F_{m,3}(t)E_3 K\big) \\
&\quad + (A_m + \mu D_1 F_{m,1}(t)E_1 + \mu\widehat{d}D_4 F_{m,4}(t)E_4 L + D_A F_{m,A}(t)E_A + BK \\
&\quad + \mu D_3 F_{m,3}(t)E_3 K)^T P_2, \\
\psi_{1,2} &= P_1 - P_2^T + \big(A_m + \mu D_1 F_{m,1}(t)E_1 + \mu\widehat{d}D_4 F_{m,4}(t)E_4 L + D_A F_{m,A}(t)E_A \\
&\quad + BK + \mu D_3 F_{m,3}(t)E_3 K\big)\big)^T \varepsilon P_2, \\
\psi_{1,3} &= \bar{\tau} P_2^T\big(\mu D_2 F_m(t)E_2 - BK - \mu D_3 F_{m,3}(t)E_3 K\big), \\
\psi_{2,2} &= -\varepsilon(P_2 + P_2^T) + \bar{\tau}R.
\end{aligned}
$$

All proofs of theorems formulated in this section are concentrated in the Appendix.

Theorem 1. *Let there exist the parameter $\varepsilon > 0$ and matrices $P_1 > \mathbf{0}$, P_2, $R > \mathbf{0}$ such that $\Psi \leq \mathbf{0}$ for every value of $F_{m,A}(t)$ and $F_{m,j}(t)$, $j = 1, 2, 3, 4$. Then the system (22) is stable.*

The proof can be found in the Appendix A.

Assumption 5. The matrix P_2 is invertible.

Remark 6. This assumption is crucial for finding of the controller gain K. In the descriptor approach, regularity of the matrix P_2 is often supposed, see e.g. [21].

Theorem 2. *Let there exist $n \times n$-dimensional matrices $\bar{P}_1 > \mathbf{0}$, Q_2, $\bar{R} > \mathbf{0}$, $Z > \mathbf{0}$, $Z_B > \mathbf{0}$, positive constants ν_A, ν_1, ν_3, ε, ρ_A, ρ_1, ρ_3, ρ_4, $\bar{\rho}$, ρ_{2B} and a matrix $Y \in R^{1\times n}$ such that the matrix Q_2 is regular and*

$$\mathbf{0} > \Sigma, \tag{24}$$

$$\mathbf{0} > \begin{pmatrix} -Z & Q_2^T E_2^T \\ * & -\frac{1}{\bar{\rho}} I \end{pmatrix}, \tag{25}$$

$$\mathbf{0} > \begin{pmatrix} -Z_B & Y^T E_3^T \\ * & -\frac{1}{\rho_{2B}} I \end{pmatrix}, \tag{26}$$

$$\mathbf{0} > -\bar{\tau}\bar{R} + Z + Z_B, \tag{27}$$

$$
\begin{aligned}
\mathbf{0} > &-\varepsilon(Q_2 + Q_2^T) + \bar{\tau}\bar{R} + \rho_A D_A D_A^T + \rho_1 D_1 D_1^T \\
&+ \rho_3 D_3 D_3^T + \rho_4 D_4 D_4^T
\end{aligned} \tag{28}
$$

where the matrix Σ is defined as (the symbol I denotes the identity matrix with an appropriate dimension)

$$\Sigma = \begin{pmatrix}
\sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & \sigma_{1,4} & \sigma_{1,5} & \sigma_{1,6} & \sigma_{1,7} & \sigma_{1,8} & \sigma_{1,9} & \sigma_{1,10} & \sigma_{1,11} & \sigma_{1,12} & \sigma_{1,13} \\
* & \sigma_{2,2} & \varepsilon\sigma_{1,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \varepsilon\sigma_{1,10} & \varepsilon\sigma_{1,11} & \mathbf{0} & \mathbf{0} \\
* & * & \sigma_{3,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & -\nu_A I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & -\nu_1 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & -\nu_3 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & -\rho_A I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & -\rho_1 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & -\rho_3 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & -\bar{\rho} I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & -\rho_{2B} I & \mathbf{0} & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & -\nu_4 I & \mathbf{0} \\
* & * & * & * & * & * & * & * & * & * & * & * & -\rho_4 I
\end{pmatrix}$$

$$\sigma_{1,1} = (A_m Q_2 + BY)^T + A_m Q_2 + BY + \nu_A D_A D_A^T + \nu_1 D_1 D_1^T + \nu_3 D_3 D_3^T + \nu_4 D_4 D_4^T,$$

$$\sigma_{1,2} = \bar{P}_1 - Q_2 + \varepsilon(Q_2^T A_m^T + Y^T B^T),$$

$$\sigma_{1,3} = -\bar{\tau} BY,$$

$$\sigma_{1,4} = \mu Q_2^T E_A^T,$$

$$\sigma_{1,5} = \mu Q_2^T E_1^T,$$

$$\sigma_{1,6} = \mu Y^T E_3^T$$

$$\sigma_{1,7} = \varepsilon Q_2^T E_A^T,$$

$$\sigma_{1,8} = \varepsilon\mu Q_2^T E_1^T,$$

$$\sigma_{1,9} = \varepsilon\mu Y^T E_3^T,$$

$$\sigma_{1,10} = -\bar{\tau}\mu D_2,$$

$$\sigma_{1,11} = -\bar{\tau}\mu D_3,$$

$$\sigma_{1,12} = \mu\widehat{d} Q_2^T L^T E_4^T,$$

$$\sigma_{1,13} = \varepsilon\mu\widehat{d} Q_2^T L^T E_4^T,$$

$$\sigma_{2,2} = -\varepsilon(Q_2 + Q_2^T) + \bar{\tau}\bar{R} + \rho_A D_A D_A^T + \rho_1 D_1 D_1^T + \rho_3 D_3 D_3^T + \rho_4 D_4 D_4^T,$$

$$\sigma_{3,3} = -\bar{\tau}\bar{R} + Z + Z_B.$$

Then the system (22) is stabilized by the feedback law $u(t) = Kz(t-\tau)$ with $K = YQ_2^{-1}$.

The proof is in the Appendix A.

Remark 7. The constants ν_A, ν_j, $j = 1, 2, 3, 4$ and ρ_A, ρ_1, ρ_3, ρ_4 can be obtained as a part of the solution of the LMI optimization problem. On the other hand the constants $\bar{\rho}$, $\rho_{2,B}$, ε must be defined a-priori. Including them into the optimization problem leads to the loss of convexity.

4.3. Identical delays in the network

In this subsection, the case when $\tau_i = \tau_j$ for all $t > 0$ and all $i, j = 1, \ldots, N$ is considered. Networks with this property have been studied e.g. in [43], however, with the value of the delay being constant in time which is not needed here. If the time delays are equal throughout the network, one has $\phi_i = \phi_j$ for all $i, j = 1, \dots, N$. Consequently, the formula (12) attains the form

$$\begin{aligned}\dot{\xi} =&(\mathcal{A} + \mathcal{D}_1\mathcal{F}_1(t)\mathcal{E}_1 + \mathcal{ML} + \mathcal{BK} + \mathcal{D}_3\mathcal{F}_3(t)\mathcal{E}_3\mathcal{K} + \mathcal{D}_4\mathcal{F}_4(t)\mathcal{E}_4\mathcal{L})\xi \\ &+ \left(\mathcal{D}_2\mathcal{F}_2(t)\mathcal{E}_2 - \mathcal{BK} - \mathcal{D}_3\mathcal{F}_3(t)\mathcal{E}_3\mathcal{K}\right)\int_{t-\tau}^{t}\dot{\xi}(s)ds \end{aligned}\tag{29}$$

and the formula (15) is converted into

$$\begin{aligned}\dot{\tilde{\xi}} =&(\tilde{A} + \mathcal{D}_1\widetilde{\mathcal{F}}_1(t)\mathcal{E}_1 + \mathcal{BK} + \mathcal{D}_3\widetilde{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}+\mathcal{D}_4\widetilde{\mathcal{F}}_4(t)\mathcal{E}_4\tilde{\mathcal{L}})\tilde{\xi} \\ &+ \left(\mathcal{D}_2\widetilde{\mathcal{F}}_2(t)\mathcal{E}_2 - \mathcal{BK} - \mathcal{D}_3\widetilde{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}\right)\int_{t-\tau}^{t}\dot{\tilde{\xi}}(s)ds,\end{aligned}$$

which, with functions $\widehat{\mathcal{F}}_i(t),\ i = 1, 2, 3, 4$ defined as $\widehat{\mathcal{F}}_i(t) = \frac{1}{\mu}\widetilde{\mathcal{F}}_i(t)$ may be rewritten as

$$\begin{aligned}\dot{\tilde{\xi}} =&(\tilde{A} + \mu\mathcal{D}_1\widehat{\mathcal{F}}_1(t)\mathcal{E}_1 + \mathcal{BK} + \mu\mathcal{D}_3\widehat{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}+\mu\mathcal{D}_4\widehat{\mathcal{F}}_4(t)\mathcal{E}_4\tilde{\mathcal{L}})\tilde{\xi} \\ &+ \left(\mu\mathcal{D}_2\widehat{\mathcal{F}}_2(t)\mathcal{E}_2 - \mathcal{BK} - \mu\mathcal{D}_3\widehat{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}\right)\int_{t-\tau}^{t}\dot{\tilde{\xi}}(s)ds.\end{aligned}\tag{30}$$

Note that $\|\widehat{\mathcal{F}}_i(t)\| \le 1$ for $i = 1, 2, 3, 4$ due to Eq. (16).

Moreover, the matrix $\tilde{A}$ is block-diagonal. In the general case, the blocks are equal to $A + d_i ML$ where d_i are eigenvalues of the matrix J. In the case of symmetrically interconnected systems, the blocks equal either to $A + NML$ or $A - ML$; see Eq. (20). As mentioned therein, one can replace the matrix $\tilde{A}$ by the matrix $\mathcal{A}_m + \mathcal{D}_A\widetilde{\mathcal{F}}_A(t)\mathcal{E}_A$. Define $\widehat{\mathcal{F}}_i = \mu\widetilde{\mathcal{F}}_i$, $i = 1, 2, 3, 4$. Then, one finally arrives at

$$\begin{aligned}\dot{\tilde{\xi}} =&\Big(\mathcal{A}_m + \mu\mathcal{D}_A\widehat{\mathcal{F}}_A(t)\mathcal{E}_A + \mu\mathcal{D}_1\widehat{\mathcal{F}}_1(t)\mathcal{E}_1 + \mathcal{BK} + \mu\mathcal{D}_3\widehat{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}\tilde{\xi} \\ &+\mu\mathcal{D}_4\widehat{\mathcal{F}}_4(t)\mathcal{E}_4\tilde{\mathcal{L}}\Big) + \left(\mu\mathcal{D}_2\widehat{\mathcal{F}}_2(t)\mathcal{E}_2 - \mathcal{BK} - \mu\mathcal{D}_3\widehat{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}\right)\int_{t-\tau}^{t}\dot{\tilde{\xi}}(s)ds\end{aligned}\tag{31}$$

Note also that

$$\mathcal{D}_4\widehat{\mathcal{F}}_4(t)\mathcal{E}_4\tilde{\mathcal{L}} = \mathcal{D}_4\widehat{\mathcal{F}}_4(t)\Big(\mathrm{diag}(\frac{d_1}{\widehat{d}}, \dots, \frac{d_N}{\widehat{d}}) \otimes I\Big)\widehat{d}\mathcal{E}_4(I_N \otimes L)$$

and define $\overline{\mathcal{F}}_4(t) = \widehat{\mathcal{F}}_4(t)\mathrm{diag}(\frac{d_1}{\widehat{d}}, \dots, \frac{d_N}{\widehat{d}})$. One can see that $\|\overline{\mathcal{F}}_4(t)\| \le \|\widehat{\mathcal{F}}_4(t)\|\|\mathrm{diag}(\frac{d_1}{\widehat{d}}, \dots, \frac{d_N}{\widehat{d}})\| \le 1$. Hence one can reformulate the Eq. (31) into

$$\begin{aligned}\dot{\tilde{\xi}} =&\Big(\mathcal{A}_m + \mu\mathcal{D}_A\widehat{\mathcal{F}}_A(t)\mathcal{E}_A + \mu\mathcal{D}_1\widehat{\mathcal{F}}_1(t)\mathcal{E}_1 + \mathcal{BK} + \mu\mathcal{D}_3\widehat{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}\tilde{\xi} \\ &+\mu\widehat{d}\mathcal{D}_4\overline{\mathcal{F}}_4(t)\mathcal{E}_4(I \otimes L)\Big) \\ &+ \left(\mu\mathcal{D}_2\widehat{\mathcal{F}}_2(t)\mathcal{E}_2 - \mathcal{BK} - \mu\mathcal{D}_3\widehat{\mathcal{F}}_3(t)\mathcal{E}_3\mathcal{K}\right)\int_{t-\tau}^{t}\dot{\tilde{\xi}}(s)ds\end{aligned}\tag{32}$$

Before stating the proof of the main theorem, let us introduce the following notation: define matrices $\mathcal{P}_i, \bar{\mathcal{P}}_i,\ i = 1, 2,\ i = 1, 2,\ \mathcal{Q}_2$ and $\mathcal{R}$ by $\mathcal{P}_i = \mathrm{diag}(P_i, \dots, P_i)$, $\bar{\mathcal{P}}_i = \mathrm{diag}(\bar{P}_i, \dots, \bar{P}_i)$, $\mathcal{Q}_2 = \mathrm{diag}(Q_2, \dots, Q_2)$ and $\mathcal{R} = \mathrm{diag}(R, \dots, R)$ so that the matrices $\mathcal{P}_i$, $\mathcal{Q}_2$ and $\mathcal{R}$ have N blocks on the diagonal. Analogously, let the matrices $\mathcal{A}_m, \mathcal{D}_A, \mathcal{E}_A$ are defined as diagonal matrices with N diagonal blocks A_m, D_A, E_A, respectively.

Corollary 1. *Under assumptions of Theorem 2, the overall system (15) is stable with* $\mathcal{K} = \operatorname{diag}(K, \dots, K)$ *with the matrix* $K = YQ_2^{-1}$.

Proof: Let $\bar{\xi}_i = (\tilde{\xi}_{(i-1)n+1}, \dots, \tilde{\xi}_{in})$ for $i = 1, \dots, N$. Then we define the vector $\tilde{\eta}$ by $\tilde{\eta} = (\bar{\xi}_1^T, \dot{\bar{\xi}}_1^T, \frac{1}{\bar{\tau}}\int_{t-\bar{\tau}}^t \dot{\bar{\xi}}_1^T(s)ds, \dots, \bar{\xi}_N^T, \dot{\bar{\xi}}_N^T, \frac{1}{\bar{\tau}}\int_{t-\bar{\tau}}^t \dot{\bar{\xi}}_N^T(s)ds)^T$.

In Eq. (32), the only matrices without the required block-diagonal structure are the matrices $\widehat{\mathcal{F}}_i$ and $\overline{\mathcal{F}}_4$. Let us define the symmetric $Nn \times Nn$ matrix $\widetilde{\Psi}$ as follows: in the definition Eq. (23), replace the matrices A_m, B, D_A, $F_{m,A}(t)$, E_A, D_j, E_j, $F_{m,j}(t)$, K, P_1, P_2, R by matrices $\mathcal{A}_m, \mathcal{B}, \mathcal{D}_A, F_A(t)\mathcal{E}_A, \mathcal{D}_j, \mathcal{E}_j, \widehat{\mathcal{F}}_j(t), \mathcal{K}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{R}$, respectively. Define also the Lyapunov function $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ where $\mathcal{V}_1 = \tilde{\xi}^T\mathcal{P}_1\tilde{\xi}$, $\mathcal{V}_2 = \int_{-\bar{\tau}}^0\int_{t+\theta}^t \dot{\tilde{\xi}}^T(s)\mathcal{R}\dot{\tilde{\xi}}(s)dsd\theta$. Then

$$\dot{\mathcal{V}}_2 \le \bar{\tau}\dot{\tilde{\xi}}^T\mathcal{R}\dot{\tilde{\xi}} - \int_{t-\bar{\tau}}^t \dot{\tilde{\xi}}^T(s)\mathcal{R}\dot{\tilde{\xi}}(s)ds$$

and finally, proceeding as in the proof of the Theorem 1 yields $\dot{\mathcal{V}} \le \tilde{\eta}^T\widetilde{\Psi}\tilde{\eta}$.

Proceeding further analogously as in the proof of Theorem 2, using the Proposition 1 and its corollaries, one can introduce the matrix $\widetilde{\Sigma}$ defined by replacing the matrices $A_m, B, D_A, E_A, D_j, E_j, Y, \bar{P}_1, Q_2, \bar{R}, Z, Z_B$ by matrices $\mathcal{A}_m, \mathcal{B}, \mathcal{D}_A$, $\mathcal{E}_A, \mu\mathcal{D}_j, \mathcal{E}_j, \mathcal{Y}, \bar{\mathcal{P}}_1, \mathcal{Q}_2, \mathcal{R}$, $\mathcal{Z} = \operatorname{diag}(Z, \dots, Z)$, $\mathcal{Z}_B = \operatorname{diag}(Z_B, \dots, Z_B)$, respectively. (The matrices $\mathcal{Z}$ and $\mathcal{Z}_B$ have N diagonal blocks again.) Also use $\mathcal{Y} = \mathcal{K}\mathcal{Q}_2$. The same substitution is done in the inequalities (25)–(28). A permutation of rows and columns can rearrange the matrix $\widetilde{\Sigma}$ into a block-diagonal matrix composed of N blocks where each of the blocks contains one copy of the matrix Σ. This means, there exists a permutation matrix V such that $V^T\widetilde{\Sigma}V = I \otimes \Sigma$ ($I \in R^{N\times N}$ again). Then, properties of the Kronecker product imply $\widetilde{\Sigma} < \mathbf{0}$.

Moreover, inequalities corresponding to LMI Eqs. (25)–(28) obtained by the above substitution are also valid, hence Corollaries 2 and 3 from the Appendix are applicable. Note that in their formulation, the matrices $\mathcal{F}_i$, $\mathcal{F}_A$ do not need to possess the block-diagonal structure. Using the same reasoning as in the proof of Theorem 2, one infers that the inequality $\widetilde{\Sigma} < \mathbf{0}$ implies $\widetilde{\Psi} < \mathbf{0}$ for every $t \ge 0$. Hence $\dot{\mathcal{V}} < 0$ if $\tilde{\xi} \ne 0$.

Remark 8. Application of the Proposition 1 brings some conservatism since the condition in Proposition 1 can be sharp inequality.

Remark 9. Several parameters can be computed in the process of the LMI solution. On the other hand, other parameters cannot be obtained in this way as the resulting problem would be nonconvex. They must be determined before the LMI optimization, finding their values so that the set of LMI constitutes a feasible problem is a matter of trial and error.

Theorem 3. *Let the assumptions of Theorem 2 hold. The overall system is stabilized if each subsystem (1) is controlled by the local feedback*

$$u_i = \frac{1}{\Psi(\mathcal{T}(x_{i,\tau_i}))}\Big(K\mathcal{T}(x_{i,\tau}) - \Phi(\mathcal{T}(x_{i,\tau_i}))\Big)$$

where $\mathcal{T}$ *is defined by Eq. (2).*

Proof: Since the transformation $\mathcal{T}$ is a diffeomorphism, Theorem 3 is a direct consequence of Corollary 1.

4.4. Non-equal network delays

In this subsection, the requirement of equal time delay throughout the whole network is removed. This leads to more conservative results. The notation from the preceding subsection is retained. Moreover, define also the vector v by

$$v = \frac{1}{\bar{\tau}} \sum_{i=1}^{N} \int_{t-\tau_i}^{t} \tilde{J}_i \dot{\tilde{\xi}}(s) ds = \frac{1}{\bar{\tau}} \sum_{i=1}^{N} \int_{t-\bar{\tau}}^{t} \phi_i(t, s) \tilde{J}_i \dot{\tilde{\xi}}(s) ds.$$

Then Eq. (15) can be reformulated as

$$\begin{aligned}\dot{\tilde{\xi}} &= (\tilde{A} + \mathcal{D}_1 \widetilde{\mathcal{F}}_1(t) \mathcal{E}_1 + \mathcal{D}_4 \widetilde{\mathcal{F}}_4(t) \mathcal{E}_4 \tilde{\mathcal{L}} + \mathcal{B}\mathcal{K} + \mathcal{D}_3 \widetilde{\mathcal{F}}_3(t) \mathcal{E}_3 \mathcal{K}) \tilde{\xi} \\ &\quad + \left(\mathcal{D}_2 \widetilde{\mathcal{F}}_2(t) \mathcal{E}_2 - \mathcal{B}\mathcal{K} - \mathcal{D}_3 \widetilde{\mathcal{F}}_3(t) \mathcal{E}_3 \mathcal{K} \right) v. \end{aligned} \tag{33}$$

Note that the Young inequality applied on each term $J_i^T \phi_i(t, s) \mathcal{R} J_j \phi_j(t, s)$ with $i \neq j$ and subsequent use of the Jensen's inequality N times together with the fact that $\phi_i \in \{0, 1\}$ for all i yields the following:

$$\begin{aligned}\bar{\tau}^2 v^T \mathcal{R} v &= \left(\sum_{i,j=1}^{N} \int_{t-\bar{\tau}}^{t} \phi_i(t, s) \dot{\tilde{\xi}}^T(s) \tilde{J}_i^T ds \right) \mathcal{R} \left(\int_{t-\bar{\tau}}^{t} \dot{\tilde{\xi}}^T(s) \tilde{J}_j^T \phi_j(t, s) ds \right) \\ &\leq N \sum_{i=1}^{N} \left(\int_{t-\tau_i}^{t} \dot{\tilde{\xi}}^T(s) \tilde{J}_i^T ds \right) \mathcal{R} \left(\int_{t-\tau_i}^{t} \tilde{J}_i \dot{\tilde{\xi}}(s) ds \right) \\ &\leq \bar{\tau} N \sum_{i=1}^{N} \int_{t-\bar{\tau}}^{t} \phi_i(t, s) \dot{\tilde{\xi}}^T(s) \tilde{J}_i^T \mathcal{R} \tilde{J}_i \dot{\tilde{\xi}}(s) ds \\ &\leq \bar{\tau} N \int_{t-\bar{\tau}}^{t} \dot{\tilde{\xi}}^T(s) \left(\sum_{i=1}^{N} \tilde{J}_i^T \mathcal{R} \tilde{J}_i \right) \dot{\tilde{\xi}}(s) ds. \end{aligned}$$

Assume there exists a constant $\gamma > 0$ such that

$$N \sum_{i=1}^{N} \tilde{J}_i^T \mathcal{R} \tilde{J}_i \leq \gamma \mathcal{R}.$$

This definition in combination with the above estimate of the term $v^T \mathcal{R} v$ yields

$$\bar{\tau} v^T \mathcal{R} v \leq \gamma \int_{t-\bar{\tau}}^{t} \dot{\tilde{\xi}}^T(s) \mathcal{R} \dot{\tilde{\xi}}(s) ds. \tag{34}$$

Finally, choose the constant k such that $k > \gamma$.

Based on the definition of the matrix Ψ in Eq. (23), define now the matrix $\bar{\Psi}$ by

$$\bar{\Psi} = \begin{pmatrix} \psi_{1,1} & \psi_{1,2} & \psi_{1,3} \\ * & \bar{\psi}_{2,2} & \varepsilon \psi_{1,3} \\ * & * & -\bar{\tau} R \end{pmatrix} \tag{35}$$

where $\psi_{i,j}$ were used in Eq. (23). The element $\bar{\psi}_{2,2}$ is defined by

$$\bar{\psi}_{2,2} = -\varepsilon (P_2 + P_2^T) + k \bar{\tau} R.$$

Theorem 4. *Assume the matrices $P_1 > \mathbf{0}$, P_2 and $R > \mathbf{0}$ solve the problem $\bar{\Psi} < \mathbf{0}$ for all t. Then the system (33) is stable.*

Proof. Define now the $Nn \times Nn$-dimensional matrix $\bar{\tilde{\Psi}}$ by formal replacement of matrices A, B etc. in the definition of the matrix $\bar{\Psi}$ by matrices $\hat{A}$, $\mathcal{B}$, etc. respectively, in the same manner as in the previous section. With the Lyapunov function $\mathcal{V}$ defined as $\bar{\mathcal{V}} = \mathcal{V}_1 + k\mathcal{V}_2$, one has

$$\dot{\mathcal{V}}_1 \leq (\tilde{\xi}^T, \dot{\tilde{\xi}}^T, v^T)\bar{\tilde{\Psi}}\begin{pmatrix}\tilde{\xi}\\ \dot{\tilde{\xi}}\\ v\end{pmatrix} - k\bar{\tau}\dot{\tilde{\xi}}^T\mathcal{R}\dot{\tilde{\xi}} + \bar{\tau}v^T\mathcal{R}v,$$

$$\dot{\mathcal{V}}_2 \leq \bar{\tau}\dot{\tilde{\xi}}^T\mathcal{R}\dot{\tilde{\xi}} - \int_{t-\bar{\tau}}^{t}\dot{\tilde{\xi}}^T(s)\mathcal{R}\dot{\tilde{\xi}}(s)ds.$$

Hence, using Eq. (34), one arrives at

$$\begin{aligned}\dot{\bar{\mathcal{V}}} &\leq (\tilde{\xi}^T, \dot{\tilde{\xi}}^T, v^T)\bar{\tilde{\Psi}}\begin{pmatrix}\tilde{\xi}\\ \dot{\tilde{\xi}}\\ v\end{pmatrix} + \bar{\tau}v^T\mathcal{R}v - k\int_{t-\bar{\tau}}^{t}\dot{\tilde{\xi}}^T(s)\mathcal{R}\dot{\tilde{\xi}}(s)ds\\ &\leq (\tilde{\xi}^T, \dot{\tilde{\xi}}^T, v^T)\bar{\tilde{\Psi}}\begin{pmatrix}\tilde{\xi}\\ \dot{\tilde{\xi}}\\ v\end{pmatrix} - (k-\gamma)\int_{t-\bar{\tau}}^{t}\dot{\tilde{\xi}}^T(s)\mathcal{R}\dot{\tilde{\xi}}(s)ds \end{aligned} \tag{36}$$

Since $\bar{\tilde{\Psi}} < 0$ and $k > \gamma$, one gets $\frac{d}{dt}\bar{\mathcal{V}} < 0$. □

Theorem 5. *Let the matrix $\bar{\Sigma}$ be defined as*

$$\bar{\Sigma} = \begin{pmatrix}
\sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & \sigma_{1,4} & \sigma_{1,5} & \sigma_{1,6} & \sigma_{1,7} & \sigma_{1,8} & \sigma_{1,9} & \sigma_{1,10} & \sigma_{1,11} & \sigma_{1,12} & \sigma_{1,13}\\
* & \bar{\sigma}_{2,2} & \varepsilon\sigma_{1,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \varepsilon\sigma_{1,10} & \varepsilon\sigma_{1,11} & \mathbf{0} & \mathbf{0}\\
* & * & \sigma_{3,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & -\nu_A I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & -\nu_1 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & -\nu_3 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & -\rho_A I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & -\rho_1 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & -\rho_3 I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & -\bar{\rho} I & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & -\rho_{2B} I & \mathbf{0} & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & * & -\nu_4 I & \mathbf{0}\\
* & * & * & * & * & * & * & * & * & * & * & * & -\rho_4 I
\end{pmatrix}$$

with $\sigma_{i,j}$ defined in the Theorem 2 and $\bar{\sigma}_{2,2} = -\varepsilon(Q_2 + Q_2^T) + k\bar{\tau}\bar{R} + \rho_1 D_A D_A^T + \rho_3 D_3 D_3^T + \rho_4 D_4 D_4^T$. Let $Q_1 > \mathbf{0}$, Q_2 and $R > \mathbf{0}$, Q_2 regular, be the solution of the problem $\bar{\Sigma} < \mathbf{0}$, $-\varepsilon(Q_2 + Q_2^T) + k\bar{\tau}\bar{R} + \rho_1 D_A D_A^T + \rho_3 D_3 D_3^T + \rho_4 D_4 D_4^T \leq \mathbf{0}$ combined with inequalities (25) and (26), $K = YQ_2^{-1}$. Then the system (33) is stable.

Proof. Using the function $\bar{\mathcal{V}}$, the same reasoning as in the proof of Corollary 1 together with Eq. (36) yields the result. □

Remark 10. Solvability of the LMI introduced in Theorems 2 and 5 is a complex issue. There are many factors having influence on existence of a solution. First, the longer maximal allowable delay $\bar{\tau}$ the more difficult it is to find a solution. A trickier part is tuning the constants ε and ρ_{2B}. Even though many problems lead to LMIs depending on such parameters, no tuning method could be found in the available literature. Remarks about influence of these parameters on the computational example is given in Remark 12.

5. Examples

5.1. Example 1

The system is composed of N identical symmetrically connected subsystems (for simulations, $N = 8$ was chosen) with the ith subsystem given by

$$\dot{x}_{1,i} = x_{1,i}^2 + x_{2,i}$$
$$\dot{x}_{2,i} = -2x_{1,i}(x_{1,i}^2 + x_{2,i}) + 0.6x_{1,i}^2 + 0.3(x_{1,i}^2 + x_{2,i}) + \frac{25}{25 + x_{1,i}^2}u_i + \sum_{j=1, j\neq i}^{N} 0.1x_{1,j}$$

The exact linearization (see e.g. [36], denoted by $\mathcal{T}$ above) which transforms the ith subsystem into the form Eq. (3) is defined by

$$\xi_{1,i} = x_{1,i}, \;\; \xi_{2,i} = x_{1,i}^2 + x_{2,i}.$$

By setting $\Phi(\xi_i) = 0.6\xi_{1,i}^2 + 0.3\xi_{2,i}^2$ and $\Psi(\xi_i) = (1 + \frac{\xi_{1,i}^2}{25})$ the dimension reduction can now be carried out as described in the above sections.

The matrices D_i and E_i, $i = 1, \ldots, 4$, are determined using the method described in Remark 2. In this example,we have $D_1 = 0,\; E_1 = 0$ and $D_4 = 0,\; E_4 = 0$. If we assume $|\xi_{1,i}(t)| \leq 1$ and $|\xi_{2,i}(t)| \leq 1$ for all $t > 0$ then one has $\|B(\frac{\Psi(\xi_i)}{\Psi(\xi_{i,\tau_i})} - 1)\| \leq \frac{1}{25}$. We choose $F_{3,i}(t) = 25(\frac{\Psi(\xi_i)}{\Psi(\xi_{i,\tau_i})} - 1)$, then clearly $\|F_{3,i}(t)\| \leq 1$ for all $t > 0$. Defining $D_3 = \binom{0}{0.2}$ and $E_3 = 0.2I$, one has $B(\frac{\Psi(\xi_i)}{\Psi(\xi_{i,\tau_i})} - 1) = D_3F_{3,i}E_3$. Note that the function $F_{3,i}$ is not available for the controller design or implementation. However, knowledge of this function is not needed. Similarly, one can choose $D_2 = 1.1I,\; E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0.5 \end{pmatrix}$.

Note that the matrix L is not changed by this transformation, $L = \begin{pmatrix} 0 & 0 \\ 0.1 & 0 \end{pmatrix}$. The next step is to define the transformation matrix T as in Eq. (14). Using Eqs. (18) and (20) define also the matrices

$$A_m = \begin{pmatrix} 0 & 1 \\ \frac{N-1}{20} & 0 \end{pmatrix}, D_A = \begin{pmatrix} 0 & 0 \\ \frac{N+1}{20} & 0 \end{pmatrix}, E_A = I.$$

This yields the problem of finding a feedback matrix K such that the system (21) is stable. A sufficient condition is to find a solution of the LMI Eqs. (24)–(28).

The design parameters were chosen as $\varepsilon = 0.5,\; \rho_2 = 1,\; \bar{\rho} = 1$ and the delay in the feedback was $\bar{\tau} = 0.1s$. The algorithm described in the previous section yields the control gain $K = (-2.04, -1.99)$ for $N = 8$. Simulations are shown on the Fig. 1 for initial conditions $x(0) = (r\cos\frac{\pi i}{8})$ with $r = 0.6\, i = 1, \ldots, N$. For the sake of simplicity, only the state of the first subsystem is depicted: state $x_{1,\,1}$ (solid line) and $x_{1,\,2}$ (dashed line). For comparison, the

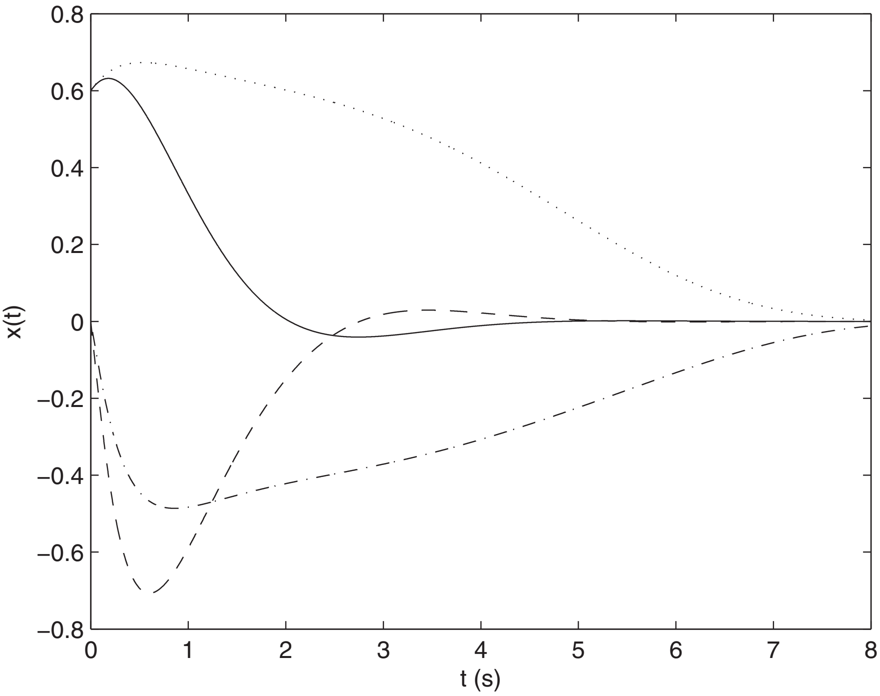

Fig. 1. State $x_{1,1}$ and $x_{1,2}$ in Example 1.

Table 1
Values of the feedback gain.

N	2	4	6	8
K^T	$\begin{pmatrix}-1.67\\-2.19\end{pmatrix}$	$\begin{pmatrix}-1.38\\-2.28\end{pmatrix}$	$\begin{pmatrix}-1.32\\-2.38\end{pmatrix}$	$\begin{pmatrix}-2.04\\-1.99\end{pmatrix}$

state of the same system controlled by linear controllers with equal gain K is depicted as well: $x_{1,\,1}$ (dotted line) and $x_{1,\,2}$ (dash-dot line).

The values of the feedback gain matrix K for different number of subsystems are summarized in Table 1.

Remark 11. The advantage of the presented method is twofold: not only is the convergence better, but also a larger range of initial conditions can be handled. While $r = 0.6$ seems to be the largest value for which the linear controller performs acceptably (higher values lead to instability of the control loop), the nonlinear controller yields good results for larger values of this parameter.

Remark 12. Values of the parameters ε, ρ_2 and ρ_{2B} were found using error and trial. While the algorithm seems to tolerate a fairly wide range of values of the parameter ε, the region of acceptable values of the parameters ρ_1 and $\bar{\rho}$ is probably rather small, roughly between 1 and 10 in the example. Values out of this range lead to an infeasible optimization problem, similarly as too large uncertainties. No further investigation of this dependence was carried out.

5.2. Example 2

Consider a set of N interconnected identical oscillators given by the equation

$$\ddot{y}_i + \frac{y_i^2 - 1}{5(1 + y_i^2 + \dot{y}_i^2)}\dot{y}_i + y_i + I_i = u_i,\ \ y_i(0) = y_{0,i}, \dot{y}_i(0) = \hat{y}_{0,i}.$$

The term I_i represents the interconnections; let k be a constant and

$$I_1 = k(y_1 - y_2),$$
$$I_i = k(y_i - y_{i+1}) + k(y_i - y_{i-1}) \text{ for } i = 2, \dots, N-1,$$
$$I_N = k(y_N - y_{N-1}).$$

This oscillator is based on the example presented in [44], however, unlike that article, no disturbances are considered here; on the other hand interconnection terms are present.

Define $\xi_{1,0} = \xi_{1,N+1} = 0$. Using the exact linearization of the auxiliary system: $y = x_1$, $\xi_1 = x_1$, $\xi_2 = x_2$ and $v = u - \frac{2x_1^2+\xi_2^2}{5(1+x_1^2+x_2^2)}x_2$ one can obtain the equation describing the ith subsystem in the form

$$\dot{\xi}_{1,i} = \xi_{2,i},$$
$$\dot{\xi}_{2,i} = -\frac{\xi_{1,i}^2 - 1}{5(1 + \xi_{1,i}^2 + \xi_{2,i}^2)}\xi_{2,i} - (1 + \alpha_i)\xi_{1,i} + u_i - \xi_{1,i-1} - \xi_{1,i+1}$$

where $\alpha_1 = \alpha_N = k$, $\alpha_i = 2k$ if $i = 2, \dots, N-1$. The slightly different form of the interconnections of the first and last subsystems can be treated using the uncertainty expressed by the terms $D_1F_1(t)E_1$ as introduced in Eq. (9). Since $-\frac{\xi_{1,i}^2-1}{5(1+\xi_{1,i}^2+\xi_{2,i}^2)}\xi_{2,i} = \frac{1}{5}\xi_{2,i} - \frac{2\xi_{1,i}^2+\xi_{2,i}^2}{5(1+\xi_{1,i}^2+\xi_{2,i}^2)}$, one can with help of the Remark 1 express the subsystem as

$$\dot{\xi}_i = (A\xi_i + D_1F_{1,i}(t)E_1)\xi_i + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(v_i - \xi_{1,i-1} - \xi_{1,i+1})$$

with matrices A, D_1, E_1 defined as

$$A = \begin{pmatrix} 0 & 1 \\ -(1 + \frac{3}{2})k & \frac{1}{5} \end{pmatrix},\ D_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix},\ E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$F_{1,1}(t) = F_{1,N}(t) = -1$ and $F_{1,i}(t) = 1$ for $i \in \{2, \dots, N-1\}$. Also, one can see that $D_3 = 0, E_3 = 0,\ D_4 = 0,\ E_4 = 0$. Moreover,

$$u_i = v_i + \frac{2\xi_{1,i}^2 + \xi_{2,i}^2}{5(1 + \xi_{1,i}^2 + \xi_{2,i}^2)}\xi_{2,i}.$$

The overall system can be written as

$$\frac{d}{dt}\begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \\ \vdots \\ \xi_{2,N} \end{pmatrix} = \Big(\mathcal{A} + \mathcal{D}_1\mathcal{F}_1(t)\mathcal{E}_1 + k(J \otimes L)\Big)\begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \\ \vdots \\ \xi_{2,N} \end{pmatrix} + \mathcal{B}\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

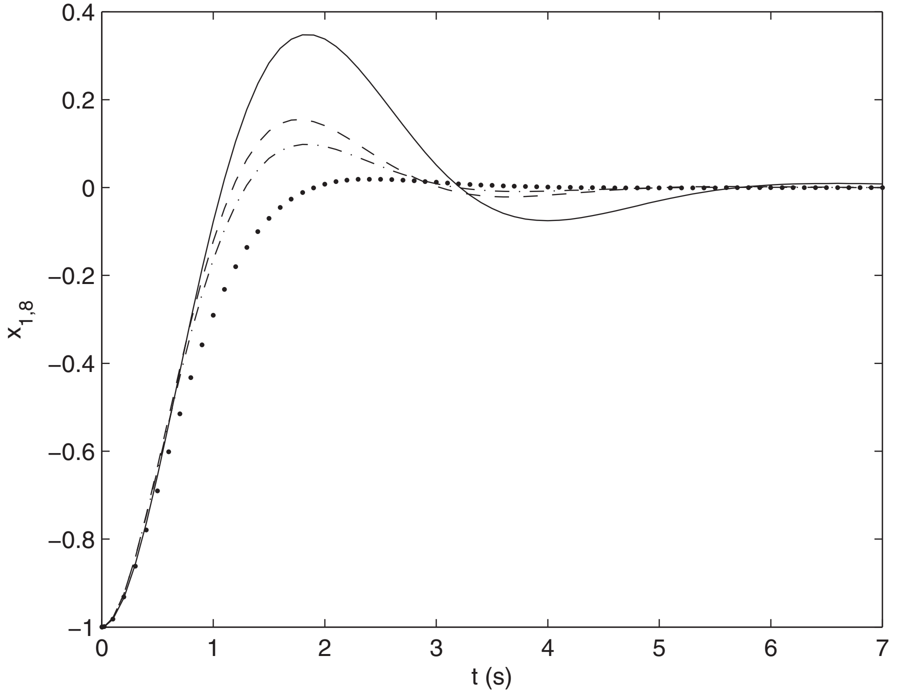

Fig. 2. State $x_{1,8}$ from Example 2 for different values of k.

with matrices J and L are given by

$$J = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \ L = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The matrix T is composed of eigenvectors of the matrix J. Due to symmetry of the matrix J, there exists a full set of orthogonal eigenvectors. Hence $TT^T = I$ which implies $\mu = 1$. We can define $\bar{d} = 2, \underline{d} = -2$ as these values bound the eigenvalues of the matrix J. In the sequel we set $D_A = \binom{0}{2k}$, $E_A = (1, 0)$. This choice then satisfies Eq. (17).

Since the control is delayed, one has to define the uncertainties described by the matrices D_2, E_2 and the functions $F_{2,i}$. To do this, define $\Phi(\xi_1, \xi_2) = \frac{2\xi_1^2\xi_2+\xi_2^3}{5(1+\xi_1^2+\xi_1^2)}$. Hence,

$$\nabla\Phi(\xi_1, \xi_2) = \begin{pmatrix} 0 & 0 \\ \frac{2\xi_1\xi_2^2(2+\xi_2^2)}{5(1+\xi_1^2+\xi_1^2)} & \frac{2\xi_2(\xi_1^4-1)}{5(1+\xi_1^2+\xi_1^2)^2} \end{pmatrix}.$$

If no variables exceed the value 2, one can choose the uncertainties as $D_2 = 5I$, $E_2 = I$.

Moreover, it is assumed that the delay in the feedback is $\tau \in [0,\ 0.1]s$.

For the evaluation of the solution of the LMI problem, the values $\varepsilon = \frac{1}{2}$ and $\rho_2 = 1$ were used. With these settings, the LMI optimization problem is feasible. The resulting feedback gain in $K = (-3.50, -1.65)$ for $k = 0.1$.

In the example, 11 subsystems were interconnected as described above. The initial conditions were chosen as $(x_{1,1}(0), x_{1,2}(0), \dots, x_{1,11}(0)) = (2, -2, 1.5, -1, -2, 2, -2, -1, 0, 1, -2)$. The states $x_{2,i}$ were zero in all subsystems.

There is a varying time delay in the feedback loop which models transmission of the measured values and/or the control signals only in some discrete time values. The time delay in the control loop satisfied $\tau \in [0, 0.1]s$. Hence the feedback of the ith subsystem introduced in Eq. (5) can be expressed as

$$u_i^*(t) = -3.50x_{1,i}(t-\tau) - 1.65x_{2,i}(t-\tau) + \frac{2x_{1,i}^2(t-\tau) + x_{2,i}^2(t-\tau)}{5(1 + x_{1,i}^2(t-\tau) + x_{2,i}^2(t-\tau))} x_{2,i}(t-\tau).$$

The response of the 8th subsystem to the above choice of initial conditions is shown in Fig. 2 for various values of the constant k: $k = \frac{1}{2}$ (solid line), $k = \frac{1}{3}$ (dashed line), $k = \frac{1}{4}$ (dash-dot line) and $k = \frac{1}{10}$ (dotted line). The condition that no variables exceed the value 2, used in the design process, has been verified for these initial conditions.

The LMI problem is infeasible for k larger than $\frac{1}{2}$.

6. Conclusions and outlooks

An algorithm for the control of large-scale nonlinear symmetric system has been introduced. The method consists of two steps. The subsystem is linearized using exact input-output linearization first. Then, a system with a reduced dimension (equal to the double of the dimension of one subsystem) is found. Robust stabilization of this low-dimensional system implies stability of the overall large system. The results are illustrated by examples. Further research will be devoted to the application of the proposed algorithm to chaotic systems and dynamical networks with complex structure. Application of this method to the consensus problem will also be investigated in future.

Appendix A. Proofs of Theorems

Proof of Theorem 1

For the sake of the proof of the Theorem 1, define the vector η by $\eta^T = (z^T, \dot{z}^T, \frac{1}{\bar{\tau}} \int_{t-\tau}^{t} \dot{z}(s)ds)$.

The Lyapunov function candidate for the system (22) is chosen in accordance with [20,21] as $V = \frac{1}{2} z^T P_1 z + \int_{-\bar{\tau}}^{0} \int_{t+\alpha}^{t} \dot{z}^T(s) R \dot{z}(s) ds d\alpha$. The derivative of the function V is expressed using Eq. (22) as

$$\begin{aligned}\dot{V} = & z^T P_1 \dot{z} + (z^T P_2^T + \varepsilon \dot{z}^T P_2^T)(A_m + \mu D_1 F_{m,1}(t) E_1 + \mu \tilde{d} D_4 F_{m,4}(t) E_4 L \\ & + D_A F_A(t) E_A + BK + \mu D_3 F_{m,3}(t) E_3 K) z \\ & + (\mu D_2 F_{m,2}(t) E_2 - BK - \mu D_3 F_{m,3}(t) E_3 K) \int_{t-\tau}^{t} \dot{z}(s) ds) + \bar{\tau} \dot{z}^T R \dot{z} \\ & - \int_{t-\bar{\tau}}^{t} \dot{z}^T(s) R \dot{z}^T(s) ds. \end{aligned} \quad \text{(A.1)}$$

Proceeding as in [21], the last term is estimated using the Jensen's inequality as

$$-\int_{t-\bar{\tau}}^{t} \dot{z}^T(s)R\dot{z}^T(s)ds \leq -\int_{t-\tau}^{t} \dot{z}^T(s)R\dot{z}^T(s)ds$$
$$\leq -\tau^{-1}\int_{t-\tau}^{t} \dot{z}^T(s)dsR\int_{t-\tau}^{t} \dot{z}^T(s)ds$$
$$\leq -\bar{\tau}^{-1}\int_{t-\tau}^{t} \dot{z}^T(s)dsR\int_{t-\tau}^{t} \dot{z}^T(s)ds.$$

Standard manipulations on the equation for $\dot{V}$ turn this inequality into the form $\dot{V} = \eta^T \Psi \eta$ which is negative for $\eta \neq 0$.

Remark 13. For the sake of stability analysis, a more general form of the Lyapunov function may be used. Namely, one can look for matrices P_1, P_2, P_3, R as proposed in [21]. However, the controller design problem requires setting $P_3 = \varepsilon P_2$ with $\varepsilon > 0$.

Thanks to the Assumption 5, one can define $Q_2 = P_2^{-1}$, $\bar{P}_1 = Q_2^T P_1 Q_2$, $\bar{R} = Q_2^T R Q_2$, $Y = KQ_2$.

The following propositions will be useful (see [45] for the proof, also [46], Lemma 2 or [14]):

Proposition 1. *Let the matrices $\mathcal{M}, \mathcal{N}, \mathcal{Q}, \mathcal{F}(t), \mathcal{P}$ have compatible dimensions, $\mathcal{P}$ be symmetric positive definite and $\|\mathcal{F}(t)\| \leq 1$ for all t. Let $\rho > 0$ satisfy $\mathcal{P} - \rho\mathcal{N}\mathcal{N}^T > \mathbf{0}$. Then*

$$(\mathcal{M} + \mathcal{N}\mathcal{F}\mathcal{Q})^T \mathcal{P}^{-1}(\mathcal{M} + \mathcal{N}\mathcal{F}\mathcal{Q}) < \mathcal{M}^T(\mathcal{P} - \rho\mathcal{N}\mathcal{N}^T)^{-1}\mathcal{M} + \frac{1}{\rho}\mathcal{Q}^T\mathcal{Q}.$$

Corollary 2. *If $\mathcal{X}$, $\mathcal{Y}$ are symmetric, $\mathcal{Y} < \mathbf{0}$ and let the following LMI hold:*

$$\mathbf{0} > \begin{pmatrix} \mathcal{X} & \mathcal{M}^T & \mathcal{Q}^T \\ * & \mathcal{Y} + \rho\mathcal{N}\mathcal{N}^T & \mathbf{0} \\ * & * & -\rho I \end{pmatrix} \tag{A.2}$$

$$\mathbf{0} > \mathcal{Y} + \rho\mathcal{N}\mathcal{N}^T \tag{A.3}$$

for a some fixed value of the parameter $\rho > 0$. Then for every $\mathcal{F}$ such that $\|\mathcal{F}\| \leq 1$ holds:

$$\begin{pmatrix} \mathcal{X} & (\mathcal{M} + \mathcal{N}\mathcal{F}\mathcal{Q})^T \\ * & \mathcal{Y} \end{pmatrix} < \mathbf{0}. \tag{A.4}$$

Proof. Application of the Schur complement twice on the inequality (A.2) yields $\mathcal{X} + \frac{1}{\rho}\mathcal{Q}^T\mathcal{Q} + \mathcal{M}^T(-\mathcal{Y} - \rho\mathcal{N}\mathcal{N}^T)^{-1}\mathcal{M} < \mathbf{0}$. Then, Proposition 1 implies

$\mathcal{X} + (\mathcal{M} + \mathcal{N}\mathcal{F}\mathcal{Q})^T(-\mathcal{Y})^{-1}(\mathcal{M} + \mathcal{N}\mathcal{F}\mathcal{Q})^T < \mathbf{0}$. The Schur complement then yields Eq. (A.4). □

Corollary 3. *Let $\mathcal{X}$ be symmetric, $\mathcal{Y} < \mathbf{0}$, $\mathcal{Z} > \mathbf{0}$. Assume the following LMI hold for a parameter $\rho > 0$:*

$$\mathbf{0} > \begin{pmatrix} \mathcal{X} & \mathcal{M} & \mathcal{N} \\ * & \mathcal{Y}+\mathcal{Z} & \mathbf{0} \\ * & * & -\frac{1}{\rho}I \end{pmatrix} \tag{A.5}$$

$$\mathbf{0} > \begin{pmatrix} -\mathcal{Z} & \mathcal{Q}^T \\ * & -\rho I \end{pmatrix} \tag{A.6}$$

$$\mathbf{0} > \mathcal{Y}+\mathcal{Z}. \tag{A.7}$$

Then the following holds for every $\mathcal{F}$, $\|\mathcal{F}\| \le 1$:

$$\begin{pmatrix} \mathcal{X} & (\mathcal{M}+\mathcal{N}\mathcal{F}\mathcal{Q}) \\ * & \mathcal{Y} \end{pmatrix} < \mathbf{0}. \tag{A.8}$$

Proof. Using the Schur complement on Eq. (A.5) yields

$$\mathbf{0} > \begin{pmatrix} \mathcal{X}+\rho\mathcal{N}\mathcal{N}^T & \mathcal{M} \\ * & \mathcal{Y}+\mathcal{Z} \end{pmatrix}.$$

Moreover, the LMI Eq. (A.6) implies $\mathcal{Z} > \frac{1}{\rho}\mathcal{Q}^T\mathcal{Q}$, hence Eq. (A.7) yields $\mathcal{Y}+\frac{1}{\rho}\mathcal{Q}^T\mathcal{Q} < \mathbf{0}$. From this inequality and Eq. (A.5) follows

$$\mathbf{0} > \begin{pmatrix} \mathcal{X}+\rho\mathcal{N}\mathcal{N}^T & \mathcal{M} \\ * & \mathcal{Y}+\frac{1}{\rho}\mathcal{Q}^T\mathcal{Q} \end{pmatrix}.$$

Let $\Xi = \mathcal{X}+\rho\mathcal{N}\mathcal{N}^T+\mathcal{M}(\mathcal{Y}+\frac{1}{\rho}\mathcal{Q}^T\mathcal{Q})^{-1}\mathcal{M}^T$. The last LMI is equivalent to $\mathbf{0} > \Xi$. In turn, the Proposition 1 guarantees that $\Xi > \mathcal{X}+(\mathcal{M}+\mathcal{N}\mathcal{F}\mathcal{Q})\mathcal{Y}^{-1}(\mathcal{M}+\mathcal{N}\mathcal{F}\mathcal{Q})^T$. Taking the Schur complement, one sees that the latter inequality is equivalent to Eq. (A.8). □

Proof of Theorem 2

We proceed with following manipulations:

1. Multiply the matrix Ψ by $\mathrm{diag}(Q_2, Q_2, Q_2)$ from the right and by $\mathrm{diag}(Q_2^T, Q_2^T, Q_2^T)$ from the left. Substitute the matrices $\bar{P}_1, \bar{R}, Y$ where possible.
2. In the block at the position (1,1), use

$$(D_A F_{m,A}(t)E_A Q_2)^T + D_A F_{m,A}(t)E_A Q_2 < \nu_A D_A^T D_A + \frac{1}{\nu_A} Q_2^T E_A^T E_A Q_2$$

$$(D_1 F_{m,1}(t)E_1 Q_2)^T + D_1 F_{m,1}(t)E_1 Q_2 < \nu_1 D_1^T D_1 + \frac{1}{\nu_1} Q_2^T E_1^T E_1 Q_2$$

$$(D_4 F_{m,4}(t)E_4 L Q_2)^T + D_4 F_{m,4}(t)E_4 L Q_2 < \nu_4 D_4^T D_4 + \frac{1}{\nu_4} Q_2^T L^T E_4^T E_4 L Q_2$$

with $\nu_A > 0$, $\nu_1 > 0$, $\nu_4 > 0$. The first inequality introduces both $\sigma_{1,\,4}$ and $\sigma_{4,\,4}$, the second one $\sigma_{1,\,5}$ and $\sigma_{5,\,5}$ and the fourth one $\sigma_{1,\,12}$ and $\sigma_{12,\,12}$.
3. Again, in the block at the position (1,1), use

$$(D_3 F_{m,3}(t)E_3 K Q_2)^T + D_3 F_{m,3}(t)E_3 K Q_2 < \nu_3 D_3^T D_3 + \frac{1}{\nu_3} Y^T E_3^T E_3 Y$$

with $\nu_3 > 0$. Then, $\sigma_{1,\,6}$ and $\sigma_{6,\,6}$ appear.

4. Using the Corollary 2 and (28), remove the terms containing uncertainties in the blocks on the positions (1,2) and (2,1). This changes also the block on the position (2,2), also $\sigma_{1,7}$, $\sigma_{7,7}$, $\sigma_{1,8}$, $\sigma_{8,8}$, $\sigma_{1,13}$, $\sigma_{13,13}$ appear.
5. Apply the Corollary 3 together with (27) on the term $-\bar{\tau}\mu\binom{D_2F_{m,2}(t)E_2Q_2}{\varepsilon D_2F_{m,2}(t)E_2Q_2}$ on the positions $(1,\ 3),(2,\ 3)$ and $(3,\ 1),(3,\ 2)$ with $\mathcal{N}=\bar{\tau}\mu\binom{D_2}{\varepsilon D_2}$, $\mathcal{F}=F_{m,2}$, $\mathcal{Q}=E_2Q_2$.
6. In the same positions, apply the Corollary 3 on the term $-\mu\bar{\tau}\binom{D_3F_{m,3}(t)E_3Y}{\varepsilon D_3F_{m,3}(t)E_3Y}$. The terms on the positions (1,10), (2,10), (1,11), (2,11), (10,10) and (11,11) arise.

Finally, use Theorem 1.

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