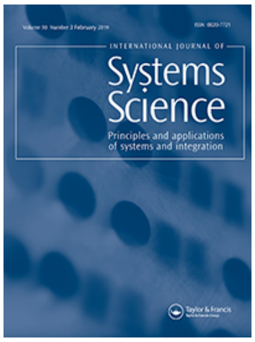

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Optimal fuzzy controller based on non-monotonic Lyapunov function with a case study on laboratory helicopter

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ABSTRACT
This paper presents a new approach to design an observer-based optimal fuzzy state feedback controller for discrete-time Takagi–Sugeno fuzzy systems via LQR based on the non-monotonic Lyapunov function. Non-monotonic Lyapunov stability theorem proposed less conservative conditions rather than common quadratic method. To compare with optimal fuzzy feedback controller design based on common quadratic Lyapunov function, this paper proceeds reformulation of the observer-based optimal fuzzy state feedback controller based on common quadratic Lyapunov function. Also in both methodologies, the dependence of optimisation problem on initial conditions is omitted. As a practical case study, the controllers are implemented on a laboratory twin-rotor helicopter to compare the controllers' performance.

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Takagi–Sugeno fuzzy systems; common quadratic Lyapunov function; non-monotonic Lyapunov function; optimal fuzzy control; laboratory twin-rotor helicopter; linear matrix inequalities

1. Introduction

Fuzzy control systems have had a vast growth of interest in engineering applications during recent years. In fact, fuzzy control, as one of the applications of fuzzy sets and fuzzy logic theory, has proven to be one of the most successful controllers for many complex nonlinear systems. However, their stability proof has remained a challenging issue.

The Takagi and Sugeno (T-S) model is provided by Takagi and Sugeno (1985) to represent or approximate the nonlinear systems by fuzzy IF-THEN rules. The main feature of this fuzzy model is to express the nonlinear system by some local linear dynamic models of each rule. This fuzzy model provides an appropriate basis for stability analysis and control design of fuzzy control systems.

Stability analysis is an important issue to design control systems. The Lyapunov stability theorem is the main applicable theorem for stability analysis or stabilisation of control systems. One of the prominent properties of T-S fuzzy systems is the ability to use state feedback structure in order to analyse the stability and design stabilised controller by the Lyapunov theorem.

In the Lyapunov direct method, a single quadratic function should be found to prove the global stability of the T-S fuzzy systems (Tanaka & Sugeno, 1992; Wang, Tanaka, & Griffin, 1996). Based on a common quadratic Lyapunov function (Tanaka & Sugeno, 1992), an approach to design a state feedback controller is developed in Tanaka, Ikeda, and Wang (1998). Finding a common positive definite matrix, which can satisfy the conditions of the Lyapunov theorem, results in more conservatism in this method, especially when the number of fuzzy rules increases. In this regard, other Lyapunov functions have been proposed like piecewise quadratic Lyapunov function (Feng, 2003; Wang & Feng, 2004) and fuzzy Lyapunov function (Guerra & Vermeiren, 2004; Tanaka, Hori, & Wang, 2003). Based on these two methods, parallel distributed compensation (PDC) structures are applied to control discrete-time T-S fuzzy systems (Guerra & Perruquetti, 2001; Guerra & Vermeiren, 2004).

In all of the methods as mentioned above, the Lyapunov function monotonically decreases. To relax the stability conditions, the non-monotonic Lyapunov function has been introduced (Butz, 1969; Derakhshan & Fatehi, 2014), which allows the Lyapunov function to decrease every few steps, but might be increased in between. This means that if the Lyapunov function decreases, for instance, every two steps, it does not necessarily decrease in each step; it might increase for one step provided that it decreases in the next step so that it is decreasing every two steps. As a result, the feasible space of the stability condition in two-step Lyapunov function is larger than the one of the one-step Lyapunov function. In fact, the feasible space of the

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one-step Lyapunov function is a sub-space of the feasible space of the two-step Lyapunov function. In Derakhshan and Fatehi (2014) the quadratic non-monotonic Lyapunov function is studied, while in Kruszewski, Wang, and Guerra (2008) and Nasiri, Nguang, Swain, and Almakhles (2016, 2018) the non-quadratic Lyapunov function (NQLF) is considered as the k-sample variations of the Lyapunov function. However, the non-quadratic approach has a larger feasible space it may be more difficult to find the larger space practically. Based on the non-monotonic Lyapunov function, an approach to design a fuzzy state feedback controller in Derakhshan, Fatehi, and Sharabiany (2014) and also an observer-based fuzzy controller design in Derakhshan, Fatehi, and Sharabiany (2012) is obtained for discrete-time T-S fuzzy systems. An observer-based H_∞ controller design is introduced by using a fuzzy Lyapunov function for discrete-time T-S fuzzy systems (El Haiek, Hmamed, El Hajjaji, & Tissir, 2017) and also for a nonlinear system with uncertainty which is described as a T-S fuzzy model (Derakhshan & Fatehi, 2015) drives a robust H_2 fuzzy observer-based controller which is considered as a sufficient condition. The main purpose is to minimise the upper bound of the cost function based on the non-monotonic Lyapunov function similar to what proposed in Derakhshan et al. (2012). The cost function is selected as a vector of the states and error of the estimation of the states.

In all methods, the controller design conditions are expressed in the form of linear matrix inequalities (LMIs). As an analytical solution, the design of an optimal controller can be described by an algebraic Riccati equation for linear systems. For a general nonlinear system, this problem reduces to the Hamilton–Jacobi equations which are usually hard to be solved; therefore, few approaches have been provided to design an optimal fuzzy controller as an effective method so far (Tanaka & Wang, 2004). Based on the stability conditions of the Lyapunov function, Li, Wang, Bushnell, and Hong proposed a new scheme of optimal control design for discrete-time T-S fuzzy systems considering PDC structure (2000). The Lyapunov function is considered a common quadratic function. In this way, the problem of controller design is presented as an optimisation problem subject to satisfy some LMIs. The purpose of this problem is minimising a cost function upper bound, which provides the parameters of the controller. It should be noted that the assumed cost function is in the same structure with LQR control problems. In Zhao, Xie, and Zhu (2007) an optimal controller for a harmonic drive system is applied which is modelled as a fuzzy T-S system, and LMI constraints from the optimisation are solved to obtain the Lyapunov matrix respect to guarantee the stability of the closed-loop system. Besides, recently an NQLF and a non-PDC controller are proposed to formulate a robust quadratic-optimal control problem as an optimisation problem for uncertain continuous-time T-S fuzzy systems (Horng, Fang, & Chou, 2017) which assumed disturbance attenuation. With the same idea, Chen et al. (2014) applied the k-sample variations of the Lyapunov function as a non-monotonic approach to design a fuzzy optimal controller based on an NQLF. Also, another latest research on fuzzy control systems tried to improve optimality and robustness. A fuzzy controller design is presented for discrete-time nonlinear systems by using a quadratic Lyapunov function which considered a mixed performance criterion which consists of a nonlinear quadratic regulator (NLQR) and a dissipativity-type performance index by considering the disturbance reduction (Wang & Yaz, 2016).

The laboratory scale twin-rotor helicopter is an appropriate plant for verification of control designs. Several control strategies have been applied to this laboratory scale helicopter. Azimian, Fatehi, and Araabi acquired the linear models of CE 150 in the presence of nonlinear distortions (2012). Also, a high order classical model has been utilised for modelling of CE 150 laboratory helicopter (John & Mija, 2014; Tao, Taur, Chang, & Chang, 2010; Wen & Lu, 2008). In Wen and Lu (2008) a robust deadbeat control scheme is applied for a decoupled system into two SISO systems. Furthermore, a fuzzy-sliding and a fuzzy-integral-sliding controller (FSFISC) are designed to control the yaw and pitch angles of twin-rotor helicopter (Tao et al., 2010), and an H_∞ control strategy of robust control for a twin-rotor MIMO system is introduced in John and Mija (2014). In addition, the T-S fuzzy model of CE 150 laboratory helicopter platform is obtained, and a controller based on the non-monotonic Lyapunov function by considering an attenuation ratio for stability condition is implemented for the elevation movement (Nategh, 2017). Finally, in Azarmi, Tavakoli-Kakhki, Sedigh, and Fatehi (2015), a fractional order robust PID controller is applied to that.

In this paper, in order to reach relaxed stability conditions and with less conservatism, a new approach is presented to design a fuzzy optimal control for discrete-time T-S fuzzy systems based on the non-monotonic quadratic Lyapunov function. In a word, this study extends the optimisation problem of Li et al. (2000) by the assumption of the less conservatism Lyapunov stability conditions of Derakhshan et al. (2012) and Derakhshan and Fatehi (2015). Furthermore, a fuzzy observer is designed simultaneously to estimate the states of the plant. Finally, the proposed method is compared with the fuzzy optimal controller designed by the common Lyapunov function. Designed controllers are experimentally evaluated on a laboratory scale twin-rotor helicopter.

The organisation of the rest of the paper is as follows. In Section 2, the discrete-time T-S fuzzy model and PDC structure together are introduced together with some preliminary definitions and lemmas. In Section 3, the main results of the observer-based fuzzy optimal controller design conditions with concerning for the non-monotonic Lyapunov function are presented. Section 4 presents the experimental results of the controllers' implementation on the laboratory scale helicopter. Then, in Section 5, some concluding remarks are presented.

2. Preliminary

2.1. Discrete-time T-S fuzzy control system

A discrete-time T-S fuzzy system is described by fuzzy IF-THEN rules which represent a local linear state-space model of a nonlinear system. The lth rule of these fuzzy models are of the following form:

$$\begin{aligned} &R^l: \ \mathit{If}\, z_1 \text{ is } F_1^l, \quad z_2 \text{ is } F_2^l, \dots, z_v \text{ is } F_v^l, \\ &\mathit{Then} \quad x(t+1) = A_l x(t) + B_l u(t), \\ &\qquad\quad y(t) = C_l x(t) + D_l u(t), \\ &\qquad\quad l \in S = \{1, 2, \dots, r\} \end{aligned} \tag{1}$$

Here, F_j^l are fuzzy sets, $z = [z_1, z_2, \dots, z_v]^{\mathrm{T}}$ is the premise variable vector that its elements are states or measurable external variables, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^p$ is the input vector, $y(t) \in \Re^q$ is the output vector, the number of inference rules is shown by r, and (A_l, B_l, C_l, D_l) are the matrices of the lth local model.

By the singleton fuzzifier, the product inference engine and centre average defuzzification, the final output of (1) is inferred as

$$\begin{aligned} x(t+1) &= A(\mu)x(t) + B(\mu)u(t) \\ y(t) &= C(\mu)x(t) + D(\mu)u(t) \end{aligned} \tag{2}$$

where

$$\begin{aligned} A(\mu) &= \sum_{l=1}^{r} \mu_l A_l, \quad B(\mu) = \sum_{l=1}^{r} \mu_l B_l \\ C(\mu) &= \sum_{l=1}^{r} \mu_l C_l, \quad D(\mu) = \sum_{l=1}^{r} \mu_l D_l \end{aligned} \tag{3}$$

μ_l is the normalised membership function defined as

$$\epsilon_l = \Pi_{i=1}^{r} F_i^l(z_i), \quad \mu_l(z) = \frac{\epsilon_l}{\sum_{i=1}^{r} \epsilon_i} \tag{4}$$

The membership grade of premise variables (z_i) is defined as $F_i^l(z_i)$ in the fuzzy set F_i^l. Then it is simple to show that $\sum_{l=1}^{r} \mu_l = 1$. Moreover, the following fuzzy observer is considered to estimate the states of the fuzzy system:

$$\begin{aligned} \hat{x}(t+1) &= A(\mu)\hat{x}(t) + B(\mu)u(t) - L(\mu)(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= C(\mu)\hat{x}(t) + D(\mu)u(t), \quad \hat{x}(0) = 0 \end{aligned} \tag{5}$$

where $\hat{x}(t) \in \Re^n$ is the estimated state vector, $\hat{y}(t) \in \Re^q$ is the estimated output vector and $L(\mu) = \sum_{i=1}^{r} \mu_l L_i$ where L_i is the observer gain matrix of the ith sub-space.

For the stabilisation of the closed-loop discrete-time fuzzy system, by using fuzzy observer (5), the following PDC fuzzy controller is used:

$$u(t) = \sum_{i=1}^{r} \mu_i F_i \hat{x}(t) = F(\mu)\hat{x}(t) \tag{6}$$

The fuzzy system and fuzzy observer error system should be augmented to obtain the closed-loop observer-based feedback fuzzy control system, which can be determined as

$$\begin{aligned} \tilde{x}_{cl}(t+1) &= \tilde{A}_{cl}(\mu)\tilde{x}_{cl}(t) \\ y(t) &= \tilde{C}_{cl}(\mu)\tilde{x}_{cl}(t) \\ u(t) &= \tilde{F}_{cl}(\mu)\tilde{x}_{cl}(t) \end{aligned} \tag{7}$$

where

$$\begin{aligned} \tilde{x}_{cl}(t) &= \begin{bmatrix} x(t) \\ e_s(t) \end{bmatrix}, \\ \tilde{A}_{cl}(\mu) &= \begin{bmatrix} A(\mu) + B(\mu)F(\mu) & -B(\mu)F(\mu) \\ 0 & A(\mu) + L(\mu)C(\mu) \end{bmatrix} \\ \tilde{C}_{cl}(\mu) &= [C(\mu) \quad 0] \\ \tilde{F}_{cl}(\mu) &= [F(\mu) \quad -F(\mu)] \end{aligned} \tag{8}$$

In which $e_s = x(t) - \hat{x}(t)$ denotes the estimation error. To design an observer-based feedback fuzzy optimal controller, the following quadratic cost function is considered:

$$J = \sum_{t=0}^{\infty} \{y^{\mathrm{T}}(t)Wy(t) + u^{\mathrm{T}}(t)Ru(t)\} \tag{9}$$

where $R = R^{\mathrm{T}} > 0$, $W = W^{\mathrm{T}} > 0$. The control objective is to minimise the upper bound of this cost function.

2.2. Optimal fuzzy controller design based on common quadratic Lyapunov function

The optimal fuzzy controller has been designed for closed-loop system (8) using the common quadratic Lyapunov function (Li et al., 2000). This type of optimal fuzzy controller is, in essence, a sub-optimal controller, since instead of **J**, its upper bound is minimised. The control design procedure is given in the following theorem.

Theorem 2.1 (Li et al., 2000): *The fuzzy T-S model* (2) *is stabilisable with PDC control* (6) *if there exist a* $Q > 0$ *and* $Y_i, i = 1, 2, \ldots, r$ *such that LMI conditions* (10)–(12) *are satisfied. Consequently, the performance measure J will be less than* γ*, and the parameters of the fuzzy optimal controller are given by* $F_i = Y_i Q^{-1}$*, where the Lyapunov function is* $V(x(t)) = x(t)^{\mathrm{T}} Q^{-1} x(t)$.

$$\begin{aligned} &\text{Minimise} \quad \gamma \\ &\text{subject to} \quad \begin{bmatrix} 1 & \mathbf{x}^{\mathrm{T}}(0) \\ \mathbf{x}(0) & Q \end{bmatrix} > 0 \end{aligned} \tag{10}$$

$$\begin{bmatrix} Q & (A_iQ + B_iY_i)^{\mathrm{T}} & QC_i^{\mathrm{T}}W^{1/2} & Y_i^{\mathrm{T}}R^{1/2} \\ A_iQ + B_iY_i & Q & 0 & 0 \\ W^{1/2}C_iQ & 0 & \gamma I & 0 \\ R^{1/2}Y_i & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{11}$$

$$\begin{bmatrix} 4Q & S^{\mathrm{T}} & \sqrt{2}QC_i^{\mathrm{T}}W^{1/2} & \sqrt{2}QC_j^{\mathrm{T}}W^{1/2} & \sqrt{2}Y_i^{\mathrm{T}}R^{1/2} & \sqrt{2}Y_j^{\mathrm{T}}R^{1/2} \\ S & Q & 0 & 0 & 0 & 0 \\ \sqrt{2}W^{1/2}C_iQ & 0 & \gamma I & 0 & 0 & 0 \\ \sqrt{2}W^{1/2}C_jQ & 0 & 0 & \gamma I & 0 & 0 \\ \sqrt{2}R^{1/2}Y_i & 0 & 0 & 0 & \gamma I & 0 \\ \sqrt{2}R^{1/2}Y_j & 0 & 0 & 0 & 0 & \gamma I \end{bmatrix} > 0 \tag{12}$$

where $S = A_iQ + B_iY_j + A_jQ + B_jY_i$.

To prove this theorem and driving the constraints, Li et al. (2000) expand the condition of $\Delta V(x(k)) < 0$ by substituting the definition of the state space and quadratic form of the Lyapunov function corresponding to $V(x(t)) = x(t)^{\mathrm{T}} Q^{-1} x(t)$. Afterward, the sufficient conditions were provided as the constraints of the controller design problem. More details on the proof are given in Li et al. (2000). A drawback of this proof is the dependency of the upper bound of the cost function on the initial condition x(0). This dependency will be removed later in Section 3.

2.3. Stability conditions based on non-monotonic Lyapunov function

To obtain less conservative stability conditions, the non-monotonic stability conditions for discrete-time systems is proposed in Aeyels and Peuteman (1998) and Derakhshan and Fatehi (2014) according to the following theorem. Based on this theorem, the Lyapunov function declines every two steps; however, it can grow for one step.

Theorem 2.2 (Derakhshan & Fatehi, 2014): *Consider a discrete system described by* (1)*, where* $\mathbf{x}(k) \in \Re^n$, $f : \Re^n \to \Re^n$ *and satisfies* $f(0) = 0$*. If there exists a continuous scalar function* $V(\mathbf{x}(k))$ *satisfying*

(1) $V(\mathbf{x}(k))$ *is a positive definite function (pdf),*
(2) $V(\mathbf{x}(k)) \to \infty$ *as* $\mathbf{x}(k) \to \infty$,
(3) $V(\mathbf{x}(k+2)) - V(\mathbf{x}(k)) < 0$ *for* $\mathbf{x}(k) \neq 0$,

then the equilibrium state $\mathbf{x}(k) = 0$ *is globally asymptotically stable, and* $V(\mathbf{x}(k))$ *is a Lyapunov function.*

Condition (3) guarantees that the Lyapunov function decreases every two steps. This means it is possible that V function increases for one step without missing the stability. This ends to non-monotonic Lyapunov function. Since the T-S fuzzy system includes some local linear models and every membership function is bounded and satisfies the Lipschitz condition, it means that the T-S fuzzy system satisfies the Lipschitz condition (Derakhshan & Fatehi, 2014). Then the stability analysis and stabilisation conditions can be described based on Theorem 2.2.

Before presenting the main result, let's notice the two following lemmas.

Lemma 2.3 (Mozelli & Palhares, 2011): *If* $P > 0$*, then* $GP^{-1}G^{\mathrm{T}} \geq G + G^{\mathrm{T}} - P$.

Lemma 2.4 (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994, Schur Complement): *The LMI*

$$\begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} > 0$$

where $Q = Q^{\mathrm{T}}$, $R = R^{\mathrm{T}}$ *, is equivalent to* $R > 0$, $Q - SR^{-1}S^{\mathrm{T}} > 0$.

3. Main result

In this section, an optimal observer-based controller design is presented for fuzzy system (7) which is developed based on non-monotonic Lyapunov function. Similar to Theorem 2.1 for the monotonic Lyapunov function, it is described as an optimisation problem to minimise a provided upper bound for cost function (9). On the contrary to Theorem 2.1, the proposed method is independent of the initial value of the state vector. So afterward, Theorem 2.1 is modified to remove its dependency on the initial state vector.

As mentioned in Theorem 2.2, closed-loop fuzzy system (7) is globally asymptotically stable if the following inequality holds:

$$V(\tilde{x}_{cl}(t+2)) - V(\tilde{x}_{cl}(t)) < 0 \tag{13}$$

It is obvious that if the following inequality holds

$$\begin{aligned} &V(\tilde{x}_{cl}(t+2)) - V(\tilde{x}_{cl}(t)) + y^{\mathrm{T}}(t)Wy(t) + u^{\mathrm{T}}(t)Ru(t) \\ &\quad + y^{\mathrm{T}}(t+1)Wy(t+1) + u^{\mathrm{T}}(t+1)Ru(t+1) < 0 \end{aligned} \tag{14}$$

then inequality (13) is confirmed. Using (14), the upper bound of cost function (9) can be obtained as follows:

$$\begin{aligned} &\sum_{t=2n,n=0}^{\infty} V(\tilde{x}_{cl}(t+2)) - V(\tilde{x}_{cl}(t)) + y^{\mathrm{T}}(t)Wy(t) \\ &\quad + u^{\mathrm{T}}(t)Ru(t) + y^{\mathrm{T}}(t+1)Wy(t+1) \\ &\quad + u^{\mathrm{T}}(t+1)Ru(t+1) < 0, \\ &V(\tilde{x}_{cl}(\infty)) - V(\tilde{x}_{cl}(0)) \\ &\quad + \sum_{t=0}^{\infty}\{u^{\mathrm{T}}(t)Ru(t) + y^{\mathrm{T}}(t)Wy(t)\} < 0. \end{aligned}$$

Provided the stability of the closed-loop system $V(\tilde{x}_{cl}(\infty)) \Rightarrow 0$; therefore,

$$J < V(\tilde{x}_{cl}(0)). \tag{15}$$

As shown in (15) the provided upper bound for the cost function depends on the initial condition. In the following main theorem, this dependency will be replaced with another assumption.

Theorem 3.1: *Consider augmented closed-loop fuzzy system (7). If there exist the positive definite matrices P^1, Q^3, P^1_{ijkl} and Q^3_{ijkl} and matrices G_1, G_2, N_i, M_i, Q^2 and Q^2_{ijkl} for every $i, j, k, l \in L$ such that*

$$\begin{aligned} &\Omega_{ij}^{kl} + \Omega_{ji}^{kl} > 0 \\ &\Psi_{kl}^{ij} + \Psi_{lk}^{ij} > 0 \end{aligned} \tag{16}$$

where

$$\Omega_{ij}^{kl} = \begin{bmatrix} P^1_{ijkl} & * & * & * & G_1^{\mathrm{T}}C_i^{\mathrm{T}}W^{1/2} & N_j^{\mathrm{T}}R^{1/2} \\ Q^2_{ijkl} & Q^3_{ijkl} & * & & C_i^{\mathrm{T}}W^{1/2} & * \\ A_iG_1 + B_iN_j & A_i & G_1 + G_1^{\mathrm{T}} - P^1 & & * & * \\ 0 & G_2^{\mathrm{T}}A_i + M_jC_i & I - Q^2 & G_2^{\mathrm{T}} + G_2 - Q^3 & * & * \\ W^{1/2}C_iG_1 & W^{1/2}C_i & 0 & 0 & I & * \\ R^{1/2}N_j & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

$$\Psi_{kl}^{ij} = \begin{bmatrix} P^1 & * & * & * & G_1^{\mathrm{T}}C_k^{\mathrm{T}}W^{1/2} & N_l^{\mathrm{T}}R^{1/2} \\ Q^2 & Q^3 & * & * & C_k^{\mathrm{T}}W^{1/2} & * \\ A_kG_1 + B_kN_l & A_k & G_1 + G_1^{\mathrm{T}} - P^1_{ijkl} & * & * & * \\ 0 & G_2^{\mathrm{T}}A_k + M_lC_k & I - Q^2_{ijkl} & G_2 + G_2^{T} - Q^3_{ijkl} & * & * \\ W^{1/2}C_kG_1 & W^{1/2}C_k & 0 & 0 & I & * \\ R^{1/2}N_l & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{17}$$

where $W = W^{\mathrm{T}} > 0$ and $R = R^{\mathrm{T}} > 0$ then fuzzy system (7) is globally asymptotically stable and an upper bound is provided for cost function (9).

Proof: If conditions (16) hold, then matrices G_1 and G_2 can be found that satisfy $G_1 + G_1^{\mathrm{T}} - P^1 > 0$ and $G_2^{\mathrm{T}} + G_2 - Q^3 > 0$ which means $G_1 + G_1^{\mathrm{T}} > 0$ and $G_2^{\mathrm{T}} + G_2 > 0$ and ensures that G_1 and G_2 are non-singular. For these two non-singular matrices G_1 and G_2 and a positive symmetric matrix P, we can define the candidate Lyapunov function as

$$V(\tilde{x}(t)) = \tilde{x}_{cl}^{\mathrm{T}}(t)G^{-T}PG^{-1}\tilde{x}_{cl}(t) \tag{18}$$

where

$$P = \begin{bmatrix} P^1 & * \\ P^2 & P^3 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & G_2^{-1} \\ 0 & G_2^{-1} \end{bmatrix} \tag{19}$$

By substituting (18) and (19) in (14), the following inequality can be obtained

$$\begin{aligned} &\left[A_{cl}(\mu(t+1))A_{cl}(\mu(t))\tilde{x}_{cl}(t)\right]^{\mathrm{T}}G^{-T}PG^{-1} \\ &\quad \times \left[A_{cl}(\mu(t+1))A_{cl}(\mu(t))\tilde{x}_{cl}(t)\right] \end{aligned}$$

$$
\begin{aligned}
&- \tilde{x}_{cl}^{\mathrm{T}}(t)G^{-T}PG^{-1}\tilde{x}_{cl}(t) \\
&+ \left[C_{cl}(\mu(t))\tilde{x}_{cl}(\mu(t))\right]^{\mathrm{T}} W\left[C_{cl}(\mu(t))\tilde{x}_{cl}(t)\right] \\
&+ \left[F_{cl}(\mu(t))\tilde{x}_{cl}(t)\right]^{\mathrm{T}} R\left[F_{cl}(\mu(t))\tilde{x}_{cl}(t)\right] \\
&+ \left[C_{cl}(\mu(t+1))A_{cl}(\mu(t))\tilde{x}_{cl}(t)\right]^{\mathrm{T}} W \\
&\times \left[C_{cl}(\mu(t+1))A_{cl}(\mu(t))\tilde{x}_{cl}(t)\right] \\
&+ \left[F_{cl}(\mu(t+1))A_{cl}(t)\tilde{x}_{cl}(\mu(t))\right]^{\mathrm{T}} R \\
&\times \left[F_{cl}(\mu(t+1))A_{cl}(\mu(t))\tilde{x}_{cl}(t)\right] < 0
\end{aligned} \tag{20}
$$

Inequality (20) can be rewritten as follows:

$$
\begin{aligned}
&\tilde{x}_{cl}^{\mathrm{T}}(t)A_{cl}^{\mathrm{T}}(\mu(t))A_{cl}^{\mathrm{T}}(\mu(t+1))G^{-T}PG^{-1}A_{cl}(\mu(t+1)) \\
&\times A_{cl}(\mu(t))\tilde{x}_{cl}(t) \\
&- \tilde{x}_{cl}^{\mathrm{T}}(t)G^{-T}PG^{-1}\tilde{x}_{cl}(t) \\
&+ \tilde{x}_{cl}^{\mathrm{T}}(t)C_{cl}^{\mathrm{T}}(\mu(t))WC_{cl}(\mu(t))\tilde{x}_{cl}(t) \\
&+ \tilde{x}_{cl}^{\mathrm{T}}(t)F_{cl}^{\mathrm{T}}(\mu(t))RF_{cl}(\mu(t))\tilde{x}_{cl}(t) \\
&+ \tilde{x}_{cl}^{\mathrm{T}}(t)A_{cl}^{\mathrm{T}}(\mu(t))C_{cl}^{\mathrm{T}}(\mu(t+1))WC_{cl}(\mu(t+1)) \\
&\times A_{cl}(\mu(t))\tilde{x}_{cl}(t) \\
&+ \tilde{x}_{cl}^{\mathrm{T}}(t)A_{cl}^{\mathrm{T}}(\mu(t))F_{cl}^{\mathrm{T}}(\mu(t+1))RF_{cl}(\mu(t+1)) \\
&\times A_{cl}(\mu(t))\tilde{x}_{cl}(\mu(t)) \\
&+ \tilde{x}_{cl}^{\mathrm{T}}(t)A_{cl}^{\mathrm{T}}(\mu(t))G^{-T}\hat{P}(\mu(t,t+1)) \\
&\times G^{-1}A_{cl}(\mu(t))\tilde{x}_{cl}(t) \\
&- \tilde{x}_{cl}^{\mathrm{T}}(t)A_{cl}^{\mathrm{T}}(\mu(t))G^{-T}\hat{P}(\mu(t,t+1)) \\
&\times G^{-1}A_{cl}(\mu(t))\tilde{x}_{cl}(t) < 0
\end{aligned} \tag{21}
$$

where $\hat{P}(\mu(t,t+1))$ is defined as follows:

$$
\begin{aligned}
\hat{P}(\mu(t,t+1)) &= \begin{bmatrix} \hat{P}_{ijkl}^{1}(\mu(t,t+1)) & * \\ \hat{P}_{ijkl}^{2}(\mu(t,t+1)) & \hat{P}_{ijkl}^{3}(\mu(t,t+1)) \end{bmatrix} \\
&= \sum \mu_{ijkl}(t,t+1) \begin{bmatrix} P_{ijkl}^{1} & * \\ P_{ijkl}^{2} & P_{ijkl}^{3} \end{bmatrix}
\end{aligned} \tag{22}
$$

which $i, j, k, l \in L$. By defining the left-hand side of inequality (21) as Δ:

$$
\begin{aligned}
\Delta = {} & \tilde{x}_{cl}^{\mathrm{T}}(t)A_{cl}^{\mathrm{T}}(\mu(t))\big[A_{cl}^{\mathrm{T}}(\mu(t+1))G^{-T}PG^{-1}A_{cl}(\mu(t+1)) \\
&- G^{-T}\hat{P}(\mu(t,t+1))G^{-1} \\
&+ C_{cl}^{\mathrm{T}}(\mu(t+1))WC_{cl}(\mu(t+1)) \\
&+ F_{cl}^{\mathrm{T}}(\mu(t+1))RF_{cl}(\mu(t+1))\big]A_{cl}(\mu(t))\tilde{x}_{cl}(t) \\
&+ \tilde{x}_{cl}^{\mathrm{T}}(t)\big[A_{cl}^{\mathrm{T}}(\mu(t))G^{-T}\hat{P}(\mu(t,t+1))G^{-1}A_{cl}(\mu(t)) \\
&- G^{-T}PG^{-1} + C_{cl}^{\mathrm{T}}(\mu(t))WC_{cl}(\mu(t)) \\
&+ F_{cl}^{\mathrm{T}}(\mu(t))RF_{cl}(\mu(t))\big]\tilde{x}_{cl}(t)
\end{aligned} \tag{23}
$$

It is necessary to show that $\Delta < 0$ holds in order to show that the stability condition based on the non-monotonic Lyapunov function for a closed-loop fuzzy system is satisfied and an upper bound for cost function (9) is provided. Thus, if $\forall\ \tilde{x}_{cl} \in \mathbb{R}^n, \tilde{x}_{cl} \neq 0$, the following inequalities, which can be simply extracted from (23), hold then (23) is satisfied.

$$
\begin{aligned}
\Big[&G^{\mathrm{T}}A_{cl}^{\mathrm{T}}(\mu(t+1))G^{-T}PG^{-1}A_{cl}(\mu(t+1))G \\
&- \hat{P}(\mu(t,t+1)) \\
&+ G^{\mathrm{T}}C_{cl}^{\mathrm{T}}(\mu(t+1))WC_{cl}(\mu(t+1))G \\
&+ G^{\mathrm{T}}F_{cl}^{\mathrm{T}}(\mu(t+1))RF_{cl}(\mu(t+1))G\Big] < 0
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\Big[&G^{\mathrm{T}}A_{cl}^{\mathrm{T}}(\mu(t))G^{-T}\hat{P}(\mu(t,t+1))G^{-1}A_{cl}(\mu(t))G \\
&- P + G^{\mathrm{T}}C_{cl}^{\mathrm{T}}(\mu(t))WC_{cl}(\mu(t))G \\
&+ G^{\mathrm{T}}F_{cl}^{\mathrm{T}}(\mu(t))RF_{cl}(\mu(t))G\Big] < 0
\end{aligned} \tag{25}
$$

According to lemma 2.4, (24) and (25) can be written as follows:

$$
\begin{bmatrix}
\hat{P}(\mu(t,t+1)) & * & G^{\mathrm{T}}C_{cl}^{\mathrm{T}}(\mu(t+1))W^{1/2} & G^{\mathrm{T}}F_{cl}^{\mathrm{T}}(\mu(t+1))R^{1/2} \\
A_{cl}(\mu(t+1))G & GP^{-1}G^{\mathrm{T}} & * & * \\
W^{1/2}C_{cl}(\mu(t+1))G & 0 & I & * \\
R^{1/2}F_{cl}(\mu(t+1))G & 0 & 0 & I
\end{bmatrix} > 0 \tag{26}
$$

$$
\begin{bmatrix}
P & * & G^{\mathrm{T}}C_{cl}^{\mathrm{T}}(\mu(t))W^{1/2} & G^{\mathrm{T}}F_{cl}^{\mathrm{T}}(\mu(t))R^{1/2} \\
A_{cl}(\mu(t))G & G\hat{P}(\mu(t,t+1))^{-1}G^{\mathrm{T}} & * & * \\
W^{1/2}C_{cl}(\mu(t))G & 0 & I & * \\
R^{1/2}F_{cl}(\mu(t))G & 0 & 0 & I
\end{bmatrix} > 0 \tag{27}
$$

By substituting (8), (22) and (19) in (26) and (27) and according to Lemma 2.3, we have

$$\begin{bmatrix}
\hat{P}^1_{ijkl}(\mu(t,t+1)) & * & * & * & G_1^{\mathrm{T}}C^{\mathrm{T}}(\mu(t+1))W^{1/2} & G_1^{\mathrm{T}}F^{\mathrm{T}}(\mu(t+1))R^{\frac{1}{2}} \\
\hat{P}^2_{ijkl}(\mu(t,t+1)) & \hat{P}^3_{ijkl}(\mu(t,t+1)) & * & * & G_2^{-T}C^{\mathrm{T}}(\mu(t+1))W^{1/2} & * \\
A(\mu(t+1))G_1 + B(\mu(t+1))F(\mu(t+1))G_1 & A(\mu(t+1))G_2^{-1} & G_1 + G_1^{\mathrm{T}} - P^1 & * & * & * \\
0 & A(\mu(t+1))G_2^{-1} + L(\mu(t+1))C(\mu(t+1))G_2^{-1} & G_2^{-T} - P^2 & G_2^{-1} + G_2^{-T} - P^3 & * & * \\
W^{1/2}C(\mu(t+1))G_1 & W^{1/2}C(\mu(t+1))G_2^{-1} & 0 & 0 & I & * \\
R^{1/2}F(\mu(t+1))G_1 & 0 & 0 & 0 & 0 & I
\end{bmatrix} > 0 \tag{28}$$

$$\begin{bmatrix}
P^1 & * & * & * & G_1^{\mathrm{T}}C^{\mathrm{T}}(\mu(t))W^{1/2} & G_1^{\mathrm{T}}F^{\mathrm{T}}(\mu(t))R^{1/2} \\
P^2 & P^3 & * & * & G_2^{-T}C^{\mathrm{T}}(\mu(t))W^{1/2} & * \\
A(\mu(t))G_1 + B(\mu(t))F(\mu(t))G_1 & A(\mu(t))G_2^{-1} & G_1 + G_1^{\mathrm{T}} - \hat{P}^1_{ijkl}(\mu(t,t+1)) & * & * & * \\
0 & A(\mu(t))G_2^{-1} + L(\mu(t))C(\mu(t))G_2^{-1} & G_2^{-T} - \hat{P}^2_{ijkl}(\mu(t,t+1)) & G_2^{-1} + G_2^{-T} - \hat{P}^3_{ijkl}(\mu(t,t+1)) & * & * \\
W^{1/2}C(\mu(t))G_1 & W^{1/2}C(\mu(t))G_2^{-1} & 0 & 0 & I & * \\
R^{1/2}F(\mu(t))G_1 & 0 & 0 & 0 & 0 & I
\end{bmatrix} > 0 \tag{29}$$

By defining $T_1 = diag\{I, G_2, I, G_2, I, I\}$ and $Q^2 = G_2^{\mathrm{T}}P^2, Q^3 = G_2^{\mathrm{T}}P^3G_2$, we pre- and post-multiply (28) and (29) by T_1^{T} and T_1. Then, we have

$$\begin{bmatrix}
\hat{P}^1_{ijkl}(\mu(t,t+1)) & * & * & * & G_1^{\mathrm{T}}C^{\mathrm{T}}(\mu(t+1))W^{\frac{1}{2}} & G_1^{\mathrm{T}}F^{\mathrm{T}}(\mu(t+1))R^{1/2} \\
G_2^{\mathrm{T}}\hat{P}^2_{ijkl}(\mu(t,t+1)) & G_2^{\mathrm{T}}\hat{P}^3_{ijkl}(\mu(t,t+1))G_2 & * & * & C^{\mathrm{T}}(\mu(t+1))W^{1/2} & * \\
A(\mu(t+1))G_1 + B(\mu(t+1))F(\mu(t+1))G_1 & A(\mu(t+1)) & G_1 + G_1^{\mathrm{T}} - P^1 & * & * & * \\
0 & G_2^{\mathrm{T}}A(\mu(t+1)) + G_2^{\mathrm{T}}L(\mu(t+1))C(\mu(t+1)) & I - Q^2 & G_2^{\mathrm{T}} + G_2 - Q^3 & * & * \\
W^{1/2}C(\mu(t+1))G_1 & W^{1/2}C(\mu(t+1)) & 0 & 0 & I & * \\
R^{1/2}F(\mu(t+1))G_1 & 0 & 0 & 0 & 0 & I
\end{bmatrix} > 0 \tag{30}$$

$$
\begin{bmatrix}
P^1 & * & * & * & G_1^{\mathrm{T}}C^{\mathrm{T}}(\mu(t))W^{\frac{1}{2}} & G_1^{\mathrm{T}}F^{\mathrm{T}}(\mu(t))R^{1/2} \\
Q^2 & Q^3 & * & * & C^{\mathrm{T}}(\mu(t))W^{1/2} & * \\
A(\mu(t))G_1 + B(\mu(t))F(\mu(t))G_1 & A(\mu(t)) & G_1 + G_1^{\mathrm{T}} - \hat{P}^1_{ijkl}(\mu(t,t+1)) & * & * & * \\
0 & G_2^{\mathrm{T}}A(\mu(t)) + G_2^{\mathrm{T}}L(\mu(t))C(\mu(t)) & I - G_2^{\mathrm{T}}\hat{P}^2_{ijkl}(\mu(t,t+1)) & G_2 + G_2^T - G_2^{\mathrm{T}}\hat{P}^3_{ijkl}(\mu(t,t+1))G_2 & * & * \\
W^{1/2}C(\mu(t))G_1 & W^{1/2}C(\mu(t)) & 0 & 0 & I & * \\
R^{1/2}F(\mu(t))G_1 & 0 & 0 & 0 & 0 & I
\end{bmatrix} > 0 \tag{31}
$$

By defining the following matrices

$$
N_i = F_iG_1, \quad M_i = G_2^{\mathrm{T}}L_i,
$$
$$
Q^2_{ijkl} = G_2^{\mathrm{T}}P^2_{ijkl}, \quad Q^3_{ijkl} = G_2^{\mathrm{T}}P^3_{ijkl}G_2 \tag{32}
$$

and by substituting (22) and (3) in (30) and (31), Ω^{kl}_{ij} and Ψ^{ij}_{kl} can be defined as (17). Considering the following properties

$$
\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\mu_i(t+1)\mu_j(t+1)\mu_k(t)\mu_l(t)\Omega^{kl}_{ij}
$$
$$
= \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\mu_{ijkl}(\Omega^{kl}_{ij} + \Omega^{kl}_{ji}) > 0 \tag{33}
$$

$$
\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\mu_i(t+1)\mu_j(t+1)\mu_k(t)\mu_l(t)\Psi^{ij}_{kl}
$$
$$
= \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r}\mu_{ijkl}(\Psi^{ij}_{kl} + \Psi^{ij}_{lk}) > 0 \tag{34}
$$

it is sufficient that the following inequalities hold in order to satisfy (33) and (34) :

$$
\Omega^{kl}_{ij} + \Omega^{kl}_{ji} > 0 \tag{35}
$$
$$
\Psi^{ij}_{kl} + \Psi^{ij}_{lk} > 0 \tag{36}
$$

It is clear from (15) that the upper bound of cost function (9) is depended on the initial condition; in order to remove this dependency, we define matrix Λ such that the following inequality holds:

$$
\begin{bmatrix} \Lambda & * \\ I & G + G^{\mathrm{T}} - P \end{bmatrix} > 0 \tag{37}
$$

where $\Lambda = \begin{smallmatrix} \Lambda^1 & * \\ \Lambda^2 & \Lambda^3 \end{smallmatrix}$. By substituting (19) in (37)

$$
\begin{bmatrix}
\Lambda^1 & * & * & * \\
\Lambda^2 & \Lambda^3 & * & * \\
I & 0 & G_1 + G_1^{\mathrm{T}} - P^1 & * \\
0 & I & G^{-T} - P^2 & G_2^{-1} + G_2^{-T} - P^3
\end{bmatrix} > 0 \tag{38}
$$

By defining $T_2 = diag\{I\ I\ I\ G_2\}$, then multiplying before and after (38) by T_2^{T} and T_2 results in

$$
\begin{bmatrix}
\Lambda^1 & * & * & * \\
\Lambda^2 & \Lambda^3 & * & * \\
I & 0 & G_1 + G_1^{\mathrm{T}} - P^1 & * \\
0 & G_2^{\mathrm{T}} & I - Q^2 & G_2 + G_2^{\mathrm{T}} - Q^3
\end{bmatrix} > 0 \tag{39}
$$

By using Lemma 2.3, it can be written as

$$
\begin{bmatrix} \Lambda & * \\ I & GP^{-1}G^{\mathrm{T}} \end{bmatrix} > \begin{bmatrix} \Lambda & * \\ I & G + G^{\mathrm{T}} - P \end{bmatrix} > 0 \tag{40}
$$

By Schur complement based on Lemma 2.4, we have

$$
\Lambda - G^{-T}PG^{-1} > 0 \rightarrow \Lambda > G^{-T}PG^{-1} \tag{41}
$$

Thus, the upper bound of the cost function can be written as follows:

$$
J < V(\tilde{x}_{cl}(0)) = \tilde{x}^{\mathrm{T}}_{cl}(0)\big(G^{-T}PG^{-1}\big)\tilde{x}_{cl}(0)
$$
$$
< \tilde{x}^{\mathrm{T}}_{cl}(0)\Lambda\tilde{x}_{cl}(0) \tag{42}
$$

Without loss of generality, assume $x(0)$ as a random variable that satisfies $E[x(0)x^{\mathrm{T}}(0)] = I$ (Derakhshan & Fatehi, 2015; Ma et al., 2018). Hence the upper bound is achieved as follows:

$$
\rightarrow \bar{J} < E[J] < E\big[\tilde{x}^{\mathrm{T}}_{cl}(0)\Lambda\tilde{x}_{cl}(0)\big]
$$

$$
\begin{aligned}
&< trace\left[\Lambda E\left[\tilde{x}_{cl}(0)\tilde{x}_{cl}^{\mathrm{T}}(0)\right]\right] \\
&= trace\left\{\begin{bmatrix}\Lambda^1 & * \\ \Lambda^2 & \Lambda^3\end{bmatrix}\right. \\
&\quad \times \left.\begin{bmatrix}E\{x(0)x^{\mathrm{T}}(0)\} & * \\ E\{x(0)x^{\mathrm{T}}(0)\} & E\{x(0)x^{\mathrm{T}}(0)\}\end{bmatrix}\right\} \\
&= trace\{\Lambda^1 + 2\Lambda^2 + \Lambda^3\} \qquad (43)
\end{aligned}
$$

Thus, an upper bound for the expected value of the cost function is provided. The purpose of the optimisation problem is to minimise this upper bound, and the proof is completed. ■

Remark 3.1: The above controller has a state observer-based structure. Up to authors knowledge, this has not been considered in available monotonic Lyapunov function. In order to make the same formulation of both approaches based on common Lyapunov and non-monotonic Lyapunov functions, the following lemma is presented.

Lemma 3.2: *Consider augmented closed-loop fuzzy system* (7). *If there exists the positive definite matrices* P^1 *and* Q^3 *and matrices* G_1, G_2, N_i, M_i *and* Q^2 *for every* $i, j \in L$ *such that*

$$\Phi_{ij} + \Phi_{ji} > 0 \qquad (44)$$

where

$$
\Phi_{ij} = \begin{bmatrix}
P^1 & * & * & * & G_1^{\mathrm{T}}C_i^{\mathrm{T}}W^{1/2} & N_j^{\mathrm{T}}R^{1/2} \\
Q^2 & Q^3 & * & * & C_i^{\mathrm{T}}W^{1/2} & * \\
A_iG_1 + B_iN_j & A_i & G_1 + G_1^{\mathrm{T}} - P^1 & * & * & * \\
0 & G_2^{\mathrm{T}}A_i + M_jC_i & I - Q^2 & G_2^{\mathrm{T}} + G_2 - Q^3 & * & * \\
W^{1/2}C_iG_1 & W^{1/2}C_i & 0 & 0 & I & * \\
R^{1/2}N_j & 0 & 0 & 0 & 0 & I
\end{bmatrix} \qquad (45)
$$

then fuzzy system (7) *is globally asymtotically stable and upper bound* (43) *can be minimised for cost function* (9).

The proof is presented in Appendix.

Table 1. T-S fuzzy model of CE 150 helicopter in three operating regions.

Performance region	State space model
60°	$A_1 = \begin{bmatrix}0 & 0 & 0.9527 \\ 1 & 0 & -2.9055 \\ 0 & 1 & 2.9529\end{bmatrix}$, $B_2 = \begin{bmatrix}1.1754\times10^{-4} \\ -6.4319\times10^{-5} \\ 2.9606\times10^{-16}\end{bmatrix}$, $C_1 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$
90°	$A_2 = \begin{bmatrix}0 & 0 & 0.9491 \\ 1 & 0 & -2.8982 \\ 0 & 1 & 2.9491\end{bmatrix}$, $B_2 = \begin{bmatrix}1.1754\times10^{-4} \\ -6.4319\times10^{-5} \\ 2.9606\times10^{-16}\end{bmatrix}$, $C_2 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$
120°	$A_3 = \begin{bmatrix}0 & 0 & 0.9539 \\ 1 & 0 & -2.9081 \\ 0 & 1 & 2.9543\end{bmatrix}$, $B_3 = \begin{bmatrix}9.4478\times10^{-5} \\ -4.4461\times10^{-5} \\ -4.4409\times10^{-16}\end{bmatrix}$, $C_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$

Figure 1. CE 150 laboratory helicopter.

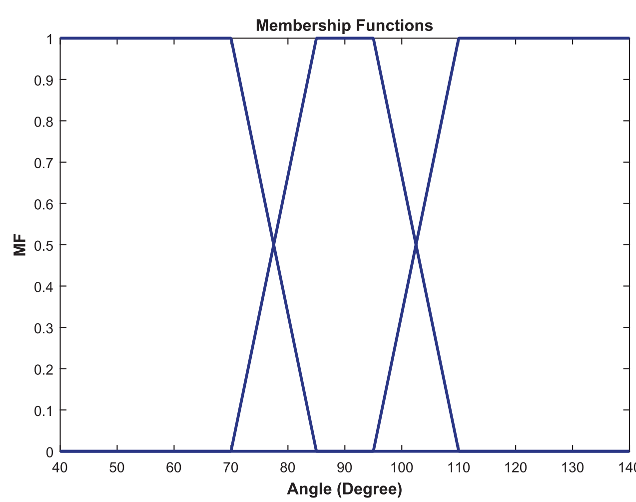

Figure 2. Membership functions of each region.

4. Experimental results

The Humusoft CE150 twin-rotor helicopter, shown in Figure 1, is a laboratory scale helicopter which is selected to experimentally apply the controllers. The apparatus consists of two DC motors that drive the propellers. The main motor provides an ability for elevation angle movement in the vertical plane, and the side motor provides an ability for azimuth angle movement in the horizontal plane. Thus, the voltages to these two motors are the plant inputs, and the measured azimuth and elevation angles are the outputs of this multivariable dynamic plant. The plant is essentially nonlinear and unstable, and all inputs and outputs are coupled. In this paper, the

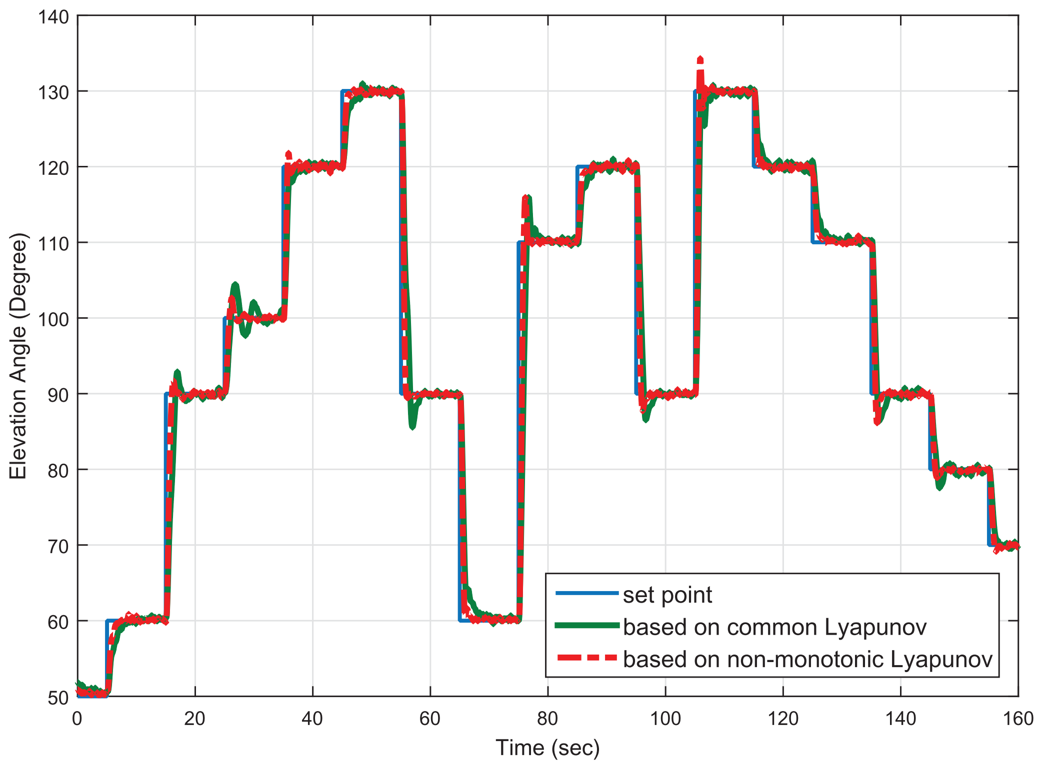

Figure 3. Closed-loop response of T-S fuzzy system.

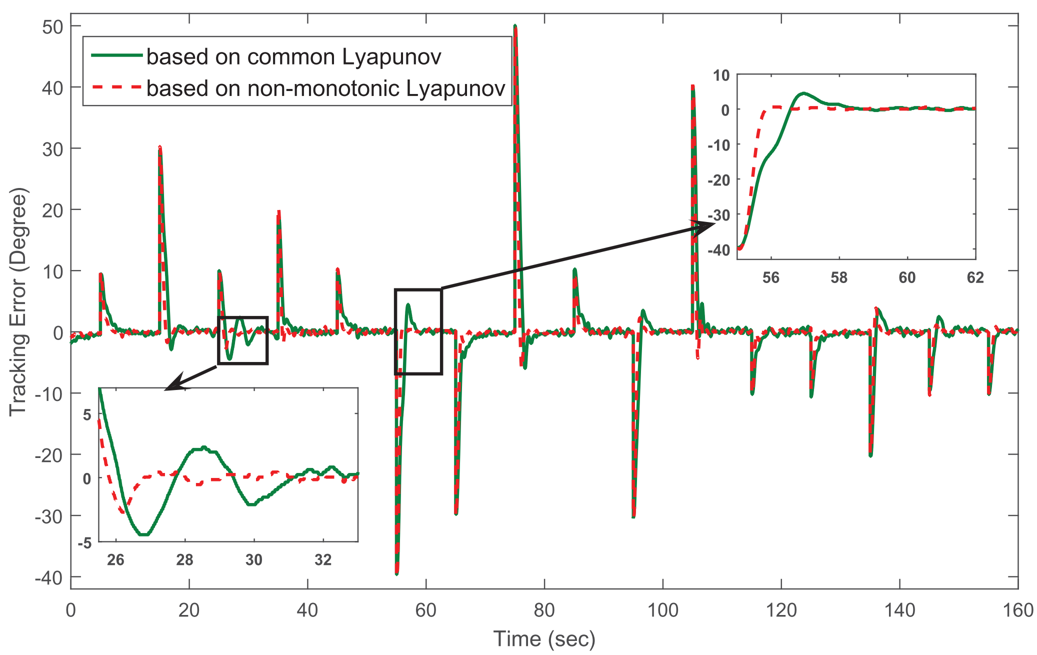

Figure 4. Tracking error of the closed-loop response of a T-S fuzzy system.

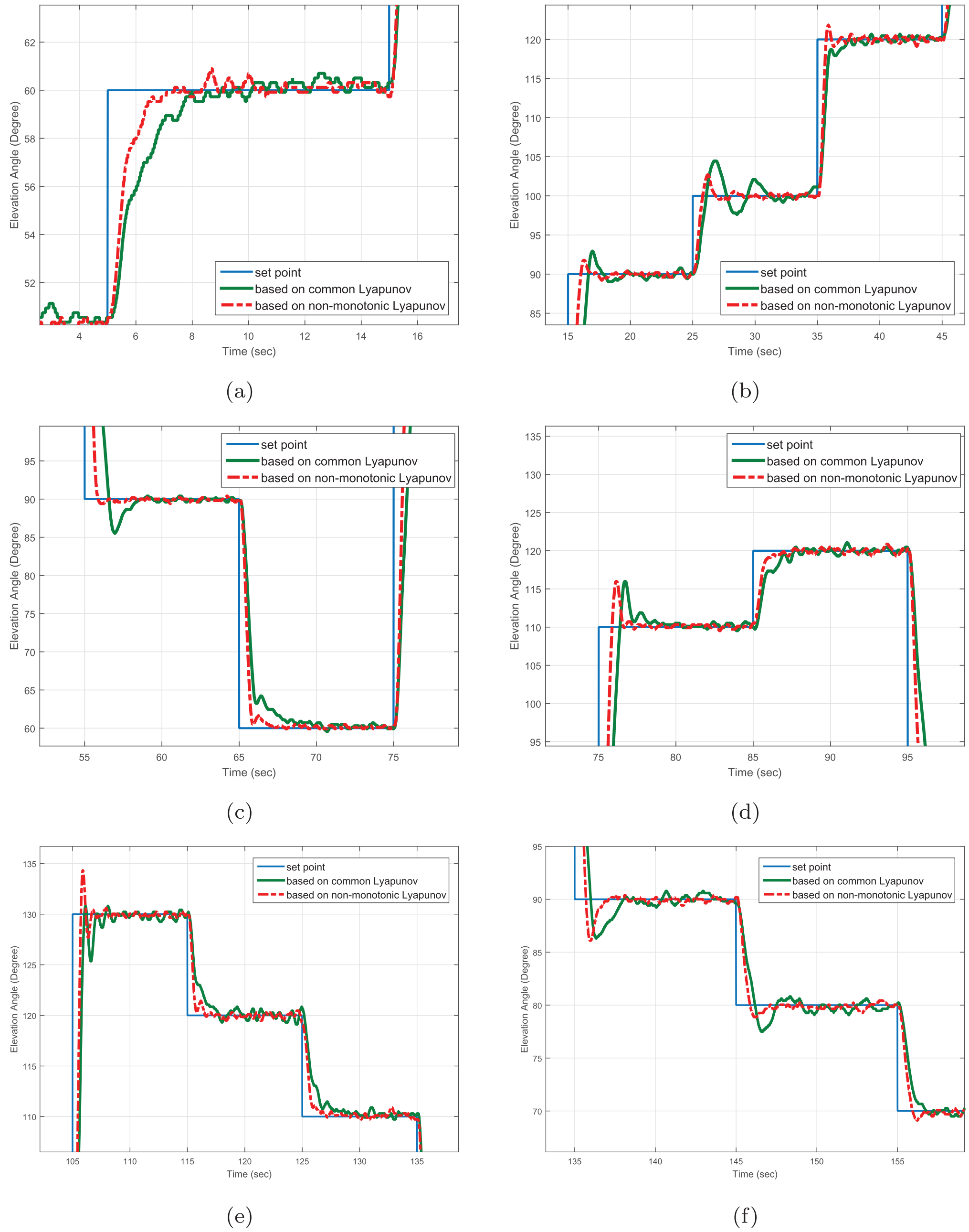

Figure 5. Closed-loop response of T-S fuzzy system.

azimuth channel is mechanically locked, that means the model is reduced to a 1-DOF plant and the aim is control of the elevation angle in the vertical plane. The T-S fuzzy model used in this paper is provided based on the identification of the system in three performance regions of elevation channel that are 60°, 90° and 120° (Nategh, 2017). Table 1 shows state-space models for each region.

The associated membership functions for each region are shown in Figure 2. In order to design and implement the fuzzy controllers and observers, we assume the servo mechanism, which, because of its integrator, removes the steady-state error. This structure adds a new state to state-space equations of the system, and it demands augmented equations for the system. The state feedback matrix and state observer matrix can be obtained by solving (16) and (44) in YALMIP toolbox (Lofberg, 2004), which is a useful toolbox for solving LMIs. The controller and observer gain matrices based on common Lyapunov function (44) are as follows:

$$\begin{aligned} F_1 &= [8107.4 \quad 8600.7 \quad 9112.5 \quad -0.5594] \\ F_2 &= [8134.7 \quad 8626.8 \quad 9134.9 \quad -0.5635] \\ F_3 &= [8336.0 \quad 8842.4 \quad 9368.3 \quad -0.5732] \end{aligned} \tag{46}$$

Table 2. Transient response specification.

Moment	Criteria	Based on common Lyapunov function	Based on non-monotonic Lyapunov function
100° at sec 25	Over/Undershoot (per cent)	4.8	2.7
	Settling Time (s)	9	3.8
90° at sec 55	Over/Undershoot (per cent)	4.98	0.6
	Settling Time (s)	4	3.2
60° at sec 65	Over/Undershoot (per cent)	0	0
	Settling Time (s)	5.9	2.6
110° at sec 75	Over/Undershoot (per cent)	5.4	5.5
	Settling Time (s)	5.5	3.6
130° at sec 105	Over/Undershoot (per cent)	0.4	3.3
	Settling Time (s)	4.6	3.6
Performance Index		417.47	327.85

$$\begin{aligned} L_1 &= [0.3811 \quad -0.8209 \quad 0.4434]^{\mathrm{T}} \\ L_2 &= [0.3847 \quad -0.8282 \quad 0.4472]^{\mathrm{T}} \\ L_3 &= [0.3799 \quad -0.8183 \quad 0.4420]^{\mathrm{T}} \end{aligned} \tag{47}$$

Similarly, the controller and observer gain matrices based on non-monotonic Lyapunov function (16) are as follows

$$\begin{aligned} F_1 &= [7070.3 \quad 7594.5 \quad 8140.9 \quad -0.9493] \\ F_2 &= [6964.9 \quad 7480.2 \quad 8015.1 \quad -0.9367] \\ F_3 &= [7226.4 \quad 7761.4 \quad 8319.4 \quad -0.9686] \end{aligned} \tag{48}$$

$$\begin{aligned} L_1 &= [0.2115 \quad -0.4978 \quad 0.2928]^{\mathrm{T}} \\ L_2 &= [0.2178 \quad -0.5130 \quad 0.3020]^{\mathrm{T}} \\ L_3 &= [0.2119 \quad -0.4986 \quad 0.2932]^{\mathrm{T}} \end{aligned} \tag{49}$$

Matrices R and W are assumed to be equal to I in all computations. The above controllers are applied to the twin-rotor helicopter. The closed-loop responses are shown in Figure 3. To observe the responses with more details, some periods are enlarged in Figure 5. Figure 3 shows that set-point tracking of the controller based on non-monotonic Lyapunov function is better than the controller based on the common Lyapunov function. Also, Figure 4 indicates the errors of the response of each controller to the set-point. The large amplitude sharp errors have happened in those moments of time when set-point changes. In order to further compare these two controllers, the transient response of them are numerically compared in Table 2. The results in Table 2 demonstrate the effectiveness of the proposed fuzzy optimal controller based on the non-monotonic Lyapunov function towards achieving better settling time and performance index compared with the common Lyapunov function-based fuzzy controller. Generally, overshoot/undershoot percentage of the non-monotonic Lyapunov function-based fuzzy controller for most of the steps is less than the common Lyapunov function-based fuzzy controller. The control signal is obtained as shown in Figure 6. Table 3 presents the maximum and minimum amplitude of the control signal. Standard deviation (SD) of the control signal is also given in Table 3. As shown in Figure 6 and from Table 3, the control signal caused by the implementation of the non-monotonic Lyapunov function-based fuzzy controller has wider amplitude, yet its SD is only 2% larger than that of the common Lyapunov based controller. This result means that the non-monotonic Lyapunov function-based fuzzy controller has smoother tracking performance at the expense of slightly more energy to control the helicopter.

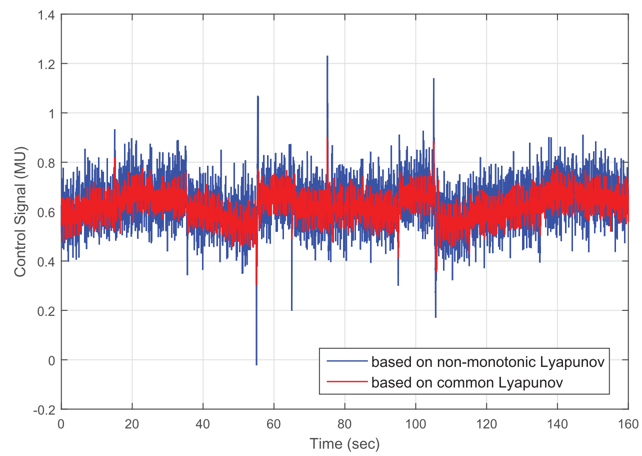

Figure 6. Control signal of closed-loop response.

Table 3. Control signals properties.

	Based on common Lyapunov function	Based on non-monotonic Lyapunov function
Maximum control signal amplitude	0.9022	1.2316
Minimum control signal amplitude	0.3036	−0.0222
Standard deviation	0.6240	0.6387

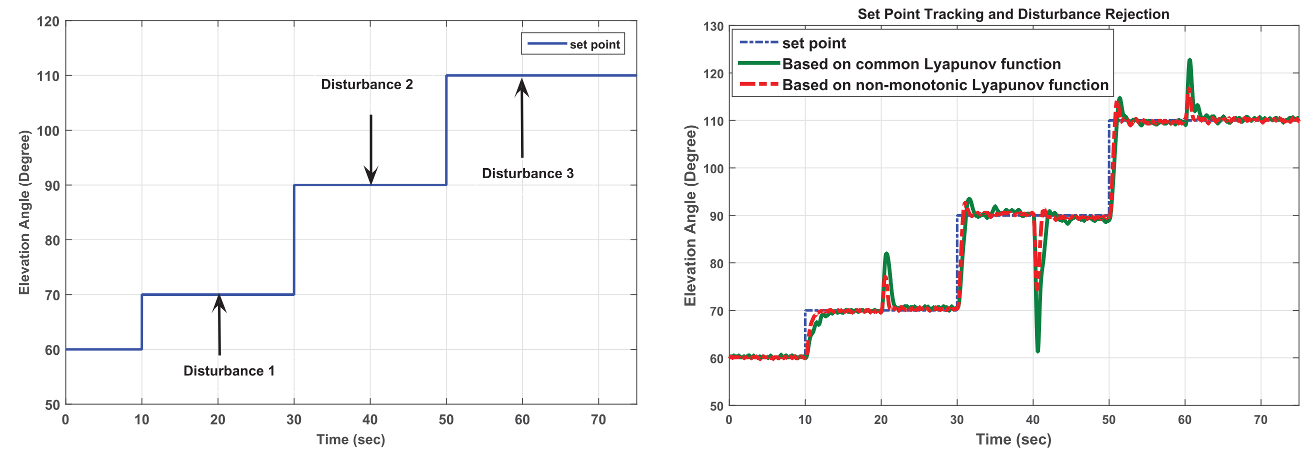

Figure 7. Disturbance effect on fuzzy controllers performance. (a) The set-point used for disturbance effect. (b) Disturbance effect on fuzzy controller performance to set-point tracking.

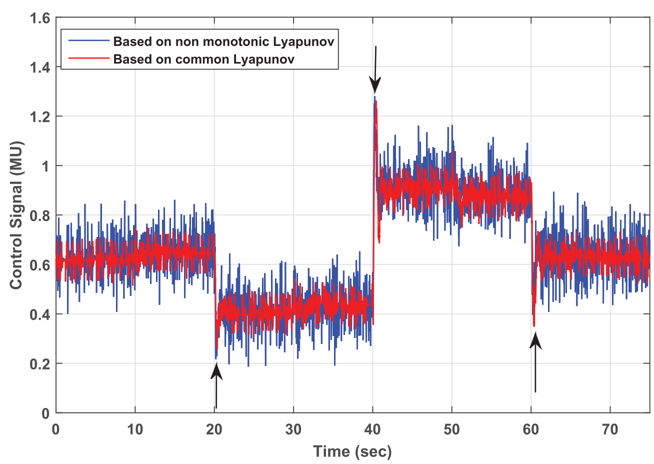

Figure 8. The control signals in the presence of disturbance.

4.1. *Disturbance rejection*

In order to study the effect of disturbance on the performance of both controllers, another experiment was run with input disturbances at three different moments as shown in Figure 7(a). To explain more, this kind of disturbance simulates an external force added to the input control signal causing a sudden change in the elevation angle. Consequently, because of increasing the errors, the controllers should adjust the new control signals to compensate for the errors. The result is shown in Figures 7(b) and 8. The black arrows point to the moments the disturbance applied.

Table 5. The control signal properties in the presence of disturbance.

	Based on common Lyapunov function	Based on non-monotonic Lyapunov function
Maximum control signal amplitude	1.2631	1.2814
Minimum control signal amplitude	0.2573	0.1864
Standard deviation	0.6690	0.6803

The overshoot/undershoot and settling time of the helicopter vertical angle are presented in Tables 4 and 5 as the transient response specifications in addition to the control signal properties. The results show that the non-monotonic Lyapunov function-based fuzzy controller has more successful performance compared with the other controller while the control signals are nearly the same. While the controllers are designed to improve the quadratic performance measure of (9), Table 6 presents the SAE ,[1], MAE[2] and MSE[3] as some other criteria for error in the presence of input disturbance. It shows that the performance errors related to the common Lyapunov function-based fuzzy controller are significantly larger.

Table 4. The transient response specifications in the presence of disturbance.

Position	Criteria	Based on common Lyapunov function	Based on non-monotonic Lyapunov function
70° at sec 20	Over/Undershoot (per cent)	17.3	10.16
	Settling Time (s)	2.2	1.8
90° at sec 40	Over/Undershoot (per cent)	31.7	17.36
	Settling Time (s)	3.25	1.8
110° at sec 60	Over/Undershoot (per cent)	20.7	15.09
	Settling Time (s)	4.6	2
Performance Index		335.56	205.69

Table 6. Error criteria in the presence of disturbance.

Error criteria	Based on common Lyapunov function	Based on non-monotonic Lyapunov function
MSE	43.824	16.5695
SAE	4.9960×10^3	1.9461×10^3
MAE	3.5609	2.1551

5. Conclusion

The Takagi–Sugeno fuzzy model is introduced as an appropriate structure to describe nonlinear systems. Although various methods have been introduced in the literature for stability analysis of T-S systems, reducing the conservatism of them is an open problem. In this paper, less conservative stability condition is provided to minimise the upper bound of the optimal cost function using the non-monotonic Lyapunov function. Furthermore, the independence of the controller design to the initial value of the states has been removed using an optimisation problem on the expected value of its effect. CE 150 laboratory helicopter is selected in order to evaluate both observer-based control systems' performance. The results show that the non-monotonic Lyapunov function-based control system is more successful to track the set-point and disturbance rejection rather than the common Lyapunov function; however, the control signal of it is slightly larger. While both controllers tried to reduce the upper bound of the quadratic performance measure, the non-monotonic Lyapunov function-based controller results much better actual performance index.

The proposed work can be enhanced even further. The same conditions can be obtained considering a multiple-step non-monotonic Lyapunov function required to be only decreasing every three or more steps. Although, it is theoretically expected that the conservatism decreases even further using multiple-step NMLF, increasing the number of steps results in more complexity in the analysis and synthesis formulations. This will limit the expected improvement of the feasible space. Obtaining the optimal steps requires more study. Also, the effect of the weighted matrices R and W on the performance and conservatism of the controller is important from both theoretical and practical point of view.

Notes

1. Sum absolute error.
2. Mean absolute error
3. Mean square error.

Disclosure statement

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Appendix

Proof: Based on Lyapunov stability condition theory, the closed-loop fuzzy system is globally asymptotically stable if for each $x(t) \neq 0$, $V(\tilde{x}_{cl}(t+1)) - V(\tilde{x}_{cl}(t)) < 0$ is satisfied; thus the following equation assures the stability:

$$V(\tilde{x}_{cl}(t+1)) - V(\tilde{x}_{cl}(t)) + y^{\mathrm{T}}(t)Wy(t) + u^{\mathrm{T}}(t)Ru(t) < 0 \tag{A1}$$

By summation of both sides of inequality from step zero to infinity the upper bound for cost function with respect to (9) can be obtained as follows:

$$\begin{aligned} &\sum_{t=0}^{\infty} V(\tilde{x}_{cl}(t+1)) - V(\tilde{x}_{cl}(t)) + y^{\mathrm{T}}(t)Wy(t) + u^{\mathrm{T}}(t)Ru(t) \\ &\quad = -V(\tilde{x}_{cl}(0)) + J < 0 \\ &\quad J < V(\tilde{x}_{cl}(0)). \end{aligned} \tag{A2}$$

By substituting (18) with respect to (7) in (A1)

$$\begin{aligned} &\left[A_{cl}(\mu(t))\tilde{x}_{cl}(t)\right]^{\mathrm{T}} G^{-T} P G^{-1} \left[A_{cl}(\mu(t))\tilde{x}_{cl}(t)\right] \\ &\quad - \tilde{x}_{cl}^{\mathrm{T}}(\mu(t)) G^{-T} P G^{-1} \tilde{x}_{cl}(\mu(t)) \\ &\quad + \left[C_{cl}(t)\tilde{x}_{cl}(\mu(t))\right]^{\mathrm{T}} W \left[C_{cl}(t)\tilde{x}_{cl}(\mu(t))\right] \\ &\quad + \left[F_{cl}(\mu(t))\tilde{x}_{cl}(\mu(t))\right]^{\mathrm{T}} R \left[F_{cl}(\mu(t))\tilde{x}_{cl}(t)\right] < 0 \end{aligned} \tag{A3}$$

And by rewriting (A3)

$$\begin{aligned} \Delta^* = \tilde{x}_{cl}^{\mathrm{T}}(t) G^{-T} \Big[& G^{\mathrm{T}} A_{cl}^{\mathrm{T}}(\mu(t)) G^{-T} P G^{-1} A_{cl}(\mu(t)) G \\ & - P + G^{\mathrm{T}} C_{cl}^{\mathrm{T}}(\mu(t)) W C_{cl}(\mu(t)) G \\ & + G^{\mathrm{T}} F_{cl}^{\mathrm{T}}(\mu(t)) R F_{cl}(\mu(t)) G \Big] G^{-1} \tilde{x}_{cl}(t) < 0 \end{aligned} \tag{A4}$$

In other words for all $x \in \mathcal{R}^n, x(t) \neq 0$, $\Delta^* < 0$ should hold, which means

$$\begin{aligned} & G^{\mathrm{T}} A_{cl}^{\mathrm{T}}(\mu(t)) G^{-T} P G^{-1} A_{cl}(\mu(t)) G \\ & \quad - P + G^{\mathrm{T}} C_{cl}^{\mathrm{T}}(\mu(t)) W C_{cl}(\mu(t)) G \\ & \quad + G^{\mathrm{T}} F_{cl}^{\mathrm{T}}(\mu(t)) R F_{cl}(\mu(t)) G < 0 \end{aligned} \tag{A5}$$

(A5) can be converted to (A6) by Schur complement

$$\begin{bmatrix} P & * & G^{\mathrm{T}} C_{cl}^{\mathrm{T}}(\mu(t)) W^{1/2} & G^{\mathrm{T}} F_{cl}(\mu(t)) R^{1/2} \\ A_{cl}(\mu(t)) G & G P^{-1} G^{\mathrm{T}} & * & * \\ W^{1/2} C_{cl}(\mu(t)) G & 0 & I & * \\ R^{1/2} F_{cl}(\mu(t)) G & 0 & 0 & I \end{bmatrix} > 0 \tag{A6}$$

Substitution of (19) and (8) in (A6) results in

$$\begin{bmatrix} P^1 & * & * & * & G_1^{\mathrm{T}}C^{\mathrm{T}}(\mu(t))W^{1/2} & G_1^{\mathrm{T}}F^{\mathrm{T}}(\mu(t))R^{1/2} \\ P^2 & P^3 & * & * & G_2^{-T}C^{\mathrm{T}}(\mu(t))W^{1/2} & * \\ A(\mu(t))G_1 + B(\mu(t))F(\mu(t))G_1 & A(\mu(t))G_2^{-1} & G_1 + G_1^{\mathrm{T}} - P^1 & * & * & * \\ 0 & A(\mu(t))G_2^{-1} + L(\mu(t))C(\mu(t))G_2^{-1} & G_2^{-T} - P^2 & G_2^{-1} + G_2^{-T} - P^3 & * & * \\ W^{1/2}C(\mu(t))G_1 & W^{1/2}C(\mu(t))G_2^{-1} & 0 & 0 & I & * \\ R^{1/2}F(\mu(t))G_1 & 0 & 0 & 0 & 0 & I \end{bmatrix} > 0 \tag{A7}$$

By defining $T_1 = diag\{I, G_2, I, G_2, I, I\}$ and $Q^2 = G_2^{\mathrm{T}}P^2, Q^3 = G_2^{\mathrm{T}}P^3G_2$, multiplying before and after (A7) by T_1^{T} and T_1. Then we have

$$\begin{bmatrix} P^1 & * & * & * & G_1^{\mathrm{T}}C^{\mathrm{T}}(\mu(t))W^{1/2} & G_1^{\mathrm{T}}F^{\mathrm{T}}(\mu(t))R^{1/2} \\ Q^2 & Q^3 & * & * & C^{\mathrm{T}}(\mu(t))W^{1/2} & * \\ A(\mu(t))G_1 + B(\mu(t))F(\mu(t))G_1 & A(\mu(t)) & G_1 + G_1^{\mathrm{T}} - P^1 & * & * & * \\ 0 & G_2^{\mathrm{T}}A(\mu(t)) + G_2^{\mathrm{T}}L(\mu(t))C(\mu(t)) & I - Q^2 & G_2^{\mathrm{T}} + G_2 - Q^3 & * & * \\ W^{1/2}C(\mu(t))G_1 & W^{1/2}C(\mu(t)) & 0 & 0 & I & * \\ R^{1/2}F(\mu(t))G_1 & 0 & 0 & 0 & 0 & I \end{bmatrix} > 0 \tag{A8}$$

We define $N_i = F_iG_1$ and $M_i = G_2^{\mathrm{T}}L_i$ then substitute in (A8) with respect to (4)

$$\Phi_{ij} = \begin{bmatrix} P^1 & * & * & * & G_1^{\mathrm{T}}C_iW^{1/2} & N_jR^{1/2} \\ Q^2 & Q^3 & * & * & C_iW^{1/2} & * \\ A_iG_1 + B_iN_j & A_i & G_1 + G_1^{\mathrm{T}} - P^1 & * & * & * \\ 0 & G_2^{\mathrm{T}}A_i + M_jC_i & I - Q^2 & G_2^{\mathrm{T}} + G_2 - Q^3 & * & * \\ W^{1/2}C_iG_1 & W^{1/2}C_i & 0 & 0 & I & * \\ R^{1/2}N_j & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{A9}$$

Finally we can rewrite (A9) as follows:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(t)\mu_j(t)\Phi_{ij} = \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(t)\mu_j(t)(\Phi_{ij} + \Phi_{ji}) > 0 \tag{A10}$$

So Lyapunov stability condition $V(\tilde{x}_{cl}(t+1)) - V(\tilde{x}_{cl}(t)) < 0$ holds if the following inequality holds

$$\Phi_{ij} + \Phi_{ji} > 0 \tag{A11}$$

The upper bound of the cost function can be obtained same as the proof of the section main result for the non-monotonic Lyapunov function-based controller design which is given in the form of (43). ■