Mean-Risk Optimization Problem via Scalarization, Stochastic Dominance, Empirical Estimates

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Abstract. Many economic and financial situations depend simultaneously on a random element and on a decision parameter. Mostly it is possible to influence the above mentioned situation by an optimization model depending on a probability measure.We focus on a special case of one–stage two–objective stochastic "Mean–Risk problem". Of course to determine optimal solution simultaneously with respect to the both criteria is mostly impossible. Consequently, it is necessary to employ some approaches. A few of them are known (from the literature), however two of them are very important; the first of them is based on a scalarizing technique and the second one is based on the stochastic dominance. The first approach has been suggested (in a special case) by Markowitz, the second approach is based on the second order stochastic dominance. The last approach corresponds (under some assumptions) to partial order in the set of the utility functions.

The aim of the contribution is to deal with the both main approaches mentioned above. First, we repeat their properties and further we try to suggest possibility to improve the both values simultaneously with respect to the both criteria. However, we focus mainly on the case when probability characteristics has to be estimated on the data base.

Keywords: Two–objective stochastic optimization problems, scalarization, stochastic dominance, empirical estimates

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1 Introduction

Let $(\Omega, \mathcal{S}, P)$ be a probability space, $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ an s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$, $F := F_\xi$ the distribution function of ξ, P_F, and Z_F the probability measure and the support corresponding to F, respectively. Let, moreover, $g_0 : \mathcal{R}^n \times \mathcal{R}^s \to \mathcal{R}^1$ be real-valued function, $X \subset \mathcal{R}^n$ a nonempty "deterministic" set; E_F, $\rho := \rho_F$ denote the operator of mathematical expectation and the operator of risk measure corresponding to the distribution function F. To introduce mean-risk model let for $x \in X$ there exist finite $\mathsf{E}_F g_0(x, \xi)$, $\rho_F g_0(x, \xi)$. An objective optimization mean–risk problem can be defined as two–objective problem in the following form:

$$\text{Find} \quad \max \mathsf{E}_F g_0(x, \xi), \quad \min \rho_F(g_0(x, \xi)) \quad \text{s.t.} \quad x \in X. \tag{1}$$

Evidently, to optimize simultaneously both objectives is mostly impossible. Different approaches are known from the literature. We recall, at first, three of them; consequently we define new problems:

a.

$$\text{Find} \quad \max \mathsf{E}_F g_0(x, \xi) \qquad \text{s.t.} \qquad \rho_F(g_0(x, \xi)) \le \nu_1, \, x \in X, \tag{2}$$

b.

$$\text{Find} \quad \min \rho_F(g_0(x, \xi)) \qquad \text{s.t.} \qquad \mathsf{E}_F g_0(x, \xi) \ge \nu_2, \; x \in X, \tag{3}$$

c. Markowitz approach [10]

$$\text{Find} \quad \min\{(1-\lambda)\mathsf{E}_F[-g_0(x, \xi)] + \lambda \rho_F(g_0(x, \xi))\} \quad \text{s.t.} \quad x \in X; \quad \lambda \in \langle 0, 1 \rangle. \tag{4}$$

(ν_1, ν_2 are suitable constants.)

Evidently, the properties of the problems (2), (3), (4) depend on the probability measure P_F, on the properties of the function $g_0(x, \xi)$ and on the risk measure $\rho_F(\cdot)$. We recall a few well–known risk measures ρ_F. To this end we set $U = g_0(x, \xi)$; they are:

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1. variance – $\rho_F(U)(:= \mathrm{var}(U)) = \mathsf{E}_F[U - \mathsf{E}_F U]^2$,
2. absolute semi–deviation – $\rho_F(U)\,(:= \bar{\delta}(U)) = \mathsf{E}_F[\max(\mathsf{E}_\mathsf{F}[U] - U),\, 0]$,
3. the standard semi–deviation – $\rho_F(U)\,(:= (\delta(U)) = (\mathsf{E}_F[(\max(\mathsf{E}_F[U] - U,\, 0))^2])^{1/2}$,
4. $\rho_F(U)$ (:= Average Value-at-Risk) = $AV@R_\alpha(U)$ for some fixed $\alpha \in [0,\, 1]$.
   (For the definition of $AV@R_\alpha(U)$ see, e.g., [4].)

Moreover, there exists a relationship between the Mean-Risk model (1), Markowitz approach and the stochastic dominance approach. We employ it for the second order stochastic dominance [4]. However, to introduce the corresponding definition we have to recall, first, a definition of the second order stochastic dominance. To this end let $Y(:= Y(\xi(\omega)))$, $V(:= V(\xi(\omega)))$ be random variables defined on $(\Omega, \mathcal{S}, P)$. If there exist finite $\mathsf{E}_F V(\xi)$, $\mathsf{E}_F Y(\xi)$ and if

$$F^2_{Y(\xi)}(u) = \int_{-\infty}^{u} F_{Y(\xi)}(z)dz, \quad F^2_{V(\xi)}(u) = \int_{-\infty}^{u} F_{V(\xi)}(z)dz, \quad u \in \mathcal{R}^1,$$

then $Y(\xi)$ dominates in second order $V(\xi)$ $(Y(\xi) \succeq_2 V(\xi))$ if

$$F^2_{Y(\xi)}(u) \le F^2_{V(\xi)}(u) \quad \text{for every} \quad u \in \mathcal{R}^1.$$

Definition 1. [4] The mean–risk model (1) is called consistent with the second order stochastic dominance ($\succeq_2$) if for every $x \in X$ and $y \in X$,

$$g_0(x,\, \xi) \succeq_2 g_0(y,\, \xi) \quad \Longrightarrow \quad \mathsf{E}_F g_0(x,\, \xi) \ge \mathsf{E}_F g_0(y,\, \xi) \quad \text{and} \quad \rho_F(g_0(x,\, \xi)) \le \rho_F(g_0(y,\, \xi)). \tag{5}$$

According to the definition of the second order stochastic dominance $g_0(x,\, \xi) \succeq_2 g_0(y,\, \xi)$ means that

$$F^2_{g_0(x,\, \xi)}(u) = \int_{-\infty}^{u} F_{g_0(x,\, \xi)}(z)dz \le F^2_{g_0(y,\, \xi)}(u) = \int_{-\infty}^{u} F_{g_0(y,\, \xi)}(z)dz \quad \text{for every} \quad u \in \mathcal{R}^1. \tag{6}$$

Employing the results [11], the stochastic second order dominance (6) can be rewritten in a more friendly form:

$$\mathsf{E}_F(u - g_0(x, \xi))^+ \le \mathsf{E}_F(u - g_0(y, \xi))^+ \quad \text{for every} \quad u \in \mathcal{R}^1. \tag{7}$$

Evidently to analyze the above mentioned approaches we can employ already known results for deterministic and stochastic optimization. In the next part we repeat some of them. However, before it we try to introduce an organization of the paper. A brief survey of the corresponding definitions and auxiliary assertions is given in Section 2. Section 3 is devoted to Markowitz approach, results determined on the base of the second order stochastic dominance can be found in Section 4. The contribution is closed by Conclusion (Section 5).

2 Some Definitions and Auxiliary Assertions

Setting $f_1(x) = -\mathsf{E}_F g_0(x,\, \xi)$, $\quad f_2(x) = \rho_F(g_0(x,\, \xi))$ we can see that problem (1) is a problem of two-objective deterministic optimization. Consequently it is possible for their investigation to employ the results achieved for multi-objective deterministic problems.

2.1 Deterministic Multi-Objective Problems

To recall suitable results obtained for deterministic problems, let $f_i(x)$, $i = 1,\, \ldots,\, l$ be real–valued functions defined on $\mathcal{R}^n$; $\mathcal{K} \subset \mathcal{R}^n$ be a nonempty set. The multi–objective deterministic optimization problem can be defined by:

$$\text{Find} \quad \min f_i(x),\, i = 1,\, \ldots,\, l \quad \text{subject to} \quad x \in \mathcal{K}. \tag{8}$$

Definition 2. The vector x^* is an efficient solution of the problem (8) if and only if there exists no $x \in \mathcal{K}$ such that $f_i(x) \le f_i(x^*)$ for $i = 1,\, \ldots,\, l$ and such that for at least one i_0 one has $f_{i_0}(x) < f_{i_0}(x^*)$.

Definition 3. The vector x^* is a properly efficient solution of the multi–objective optimization problem (8) if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $x \in \mathcal{K}$ satisfying $f_i(x) < f_i(x^*)$ there exists at least one j such that $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \le M. \tag{9}$$

Proposition 1. [3] *Let $\mathcal{K} \subset \mathcal{R}^n$ be a nonempty convex set and let $f_i(x),\ i = 1, \ldots, l$ be convex functions on $\mathcal{K}$. Then $x^0 \in \mathcal{K}$ is a properly efficient solution of the problem (8) if and only if x^0 is optimal in*

$$\text{Find} \quad \min_{x \in \mathcal{K}} \sum_{i=1}^{l} \lambda_i f_i(x) \quad \text{for some} \quad \lambda_1, \ldots, \lambda_l > 0, \quad \sum_{i=1}^{l} \lambda_i = 1.$$

A relationship between efficient and properly efficient points is introduced, e.g., in [2] or [3]. We summarize it in the following Remark.

Remark 1. Let $f(x) = (f_1(x), \ldots, f_l(x)),\ x \in \mathcal{K}$; $\mathcal{K}^{eff}$, $\mathcal{K}^{peff}$ be sets of efficient and properly efficient points of the problem (8). If $\mathcal{K}$ is a convex set, $f_i(x),\ i = 1, \ldots, l$ are convex functions on $\mathcal{K}$, then

$$f(\mathcal{K}^{peff}) \subset f(\mathcal{K}^{eff}) \subset \bar{f}(\mathcal{K}^{peff}), \quad (\bar{f}(\mathcal{K}^{peff} \quad \text{denotes the closure set of} \quad f(\mathcal{K}^{peff})).$$

Remark 2. Evidently setting $l = 2$, $\mathcal{K} = X$ and $f_1(x) = -\mathsf{E}_F g_0(x, \xi), \quad f_2(x) = \rho_F(g_0(x, \xi))$ we can see that problem (1) corresponds to the deterministic problem (8).

Further, we recall the definition of strongly convex function.

Definition 4. Let $h(x)$ be a real-valued function defined on a nonempty convex set $\mathcal{K} \subset \mathcal{R}^n$. $h(x)$ is strongly convex function with a parameter $\rho' > 0$ if

$$h(\lambda x^1 + (1-\lambda)x^2) \le \lambda h(x^1) + (1-\lambda)h(x^2) - \lambda(1-\lambda)\rho'\|x^1 - x^2\|_2^2$$
$$\text{for very} \quad x^1, x^2 \in \mathcal{K},\ \lambda \in \langle 0, 1\rangle, \quad (\|\cdot\|_2 := \|\cdot\|_2^n \quad \text{denotes the Euclidean norm in} \quad \mathcal{R}^n).$$

2.2 Stochastic Optimization Problems and Empirical Estimates

Let $\mathcal{P}(\mathcal{R}^s)$ denote the set of all (Borel) probability measures on $\mathcal{R}^s$, $\mathcal{M}_1^1(\mathcal{R}^s)$ be defined by the relation:

$$\mathcal{M}_1^1(\mathcal{R}^s) := \left\{\nu \in \mathcal{P}(\mathcal{R}^s) : \int_{\mathcal{R}^s} \|z\|_1 d\nu(z) < \infty\right\}, \quad \|\cdot\|_1^s := \|\cdot\|_1 \text{ denotes } \mathcal{L}_1 \text{ norm in } \mathcal{R}^s. \tag{10}$$

Let, further, $g : \mathcal{R}^n \times \mathcal{R}^s \to \mathcal{R}^1$ be real-valued function such that there exists a finite $\mathsf{E}_F g(x, \xi)$ for every $x \in X$. We introduce a system of assumptions:

A.0 $g(x, z)$ is for $x \in X$ a Lipschitz function of $z \in \mathcal{R}^s$ with the Lipschitz constant (corresponding to the $\mathcal{L}_1$ norm) not depending on x,

A.1 $g(x, z)$ is either a uniformly continuous function on $X \times \mathcal{R}^s$ or there exists $\varepsilon > 0$ such that $g(x, z)$ is a convex function on $X(\varepsilon)$ and bounded on $X(\varepsilon) \times \mathcal{R}^s$ ($X(\varepsilon)$ denotes ε–neighborhood of X),

A.2 • $\{\xi^i\}_{i=1}^{\infty}$ is an independent random sequence corresponding to F,
- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$, $N = 1, 2, \ldots$,
- $F_i, i = 1, \ldots, s$ denote one-dimensional marginal distribution functions corresponding to F.

Let, further, for the random value $Y := Y(\xi)$ and $\varepsilon \in \mathcal{R}^1$ the sets $X_F^{\varepsilon} := X_F^{\varepsilon}(Y(\xi))$ be defined by

$$X_F^{\varepsilon} = \{x \in X : \mathsf{E}_F(u - g_0(x, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \le \varepsilon \quad \text{for every} \quad u \in \mathcal{R}^1\}, \quad \varepsilon \in \mathcal{R}^1. \tag{11}$$

If we set $X_F = X_F^0$, then a rather general optimization problem with second order stochastic dominance constraints can be introduce in the following form:

$$\text{Find} \quad \varphi(F, X_F) = \inf\{\mathsf{E}_F g(x, \xi) : x \in X_F\}. \tag{12}$$

Replacing the distribution function F by the empirical one F^N, then we obtain an empirical optimization problem. The following assertion follows from stability results presented in [9].

Proposition 2. [9] *Let X_F be a nonempty compact set, $P_F \in \mathcal{M}_1^1(\mathcal{R}^s)$. Let, moreover, $g_0(x, z)$, $Y(z)$ be for every $x \in X$ Lipschitz functions of $z \in Z_F$ with the Lipschitz constant not depending on $x \in X$. If*

1. *Assumptions A.0, A.1, A.2 are fulfilled,*
2. *$g(x, z)$ is a Lipschitz function on X with the Lipschitz constant not depending on $z \in Z_F$,*
3. *there exists $\varepsilon_0 > 0$ such that X_F^{ε} are nonempty compact sets for every $\varepsilon \in \langle -\varepsilon_0, \varepsilon_0\rangle$ and, moreover, there exists a constant $\hat{C} > 0$ such that*

$$\Delta_n[X_F^{\varepsilon},\ X_F^{\varepsilon'}] \le \hat{C}|\varepsilon - \varepsilon'| \quad \text{for} \quad \varepsilon, \varepsilon' \in \langle -\varepsilon_0, \varepsilon_0\rangle,$$

4. *there exists a finite first moment of the random vector* ξ,

then

$$P\{\omega : |\varphi(F, X_F) - \varphi(F^N, X_{F^N}| \longrightarrow_{N \longrightarrow \infty} 0\} = 1 \tag{13}$$

A crucial assumptions (in Proposition 2) is the existence of a finite first moment of ξ. Consequently, the relation (13) holds also for stable distributions with the parameter stability greater or equal to 1 (for the definition of stable distribution see, e.g., [8]).

3 Mean-Risk Model via Markowitz Approach

Evidently, the following assertion follows from Proposition 1.

Proposition 3. *Let* $X \subset \mathcal{R}^n$ *be a nonempty convex set,* $\mathsf{E}_F[-g_0(x, \xi)]$, $\rho_F(g_0(x, \xi))$ *be finite convex functions on* X. *Then* $x^\lambda \in X$ *is a properly efficient solution of the problem (1) if and only if* x^λ *is optimal in the problem*

$$\text{Find} \quad \min_{x \in X}\{(1-\lambda)\mathsf{E}_F[-g_0(x, \xi)] + \lambda\rho_F(g_0(x, \xi))\} \quad \text{for} \quad \lambda \in (0, 1). \tag{14}$$

If we denote by $\bar{X}$ the set of all solutions of the problem (14) for some $\lambda \in (0, 1)$, then according to Remark 1 the closure of the set $\bar{X}$ is equal to set of all efficient points of two-objective problem (1). If, moreover, functions $\mathsf{E}_F[-g_0(x, \xi)]$, $\rho_F(g_0(x, \xi))$ are strongly convex on X with the same parameter ρ' and if $\bar{\mathcal{X}}(F, X)$, $\bar{\mathcal{G}}(F, X)$ are defined by the relation:

$$\begin{aligned} \bar{\mathcal{X}}(F, X) &= \{x \in X : x \quad \text{is a properly efficient point of the problem (1)}\}, \\ \bar{\mathcal{G}}(F, X) &= \{t_1, t_2 : t_1 = \mathsf{E}_F[-g_0(x, \xi)],\ t_2 = \rho_F(g_0(x, \xi)) \quad \text{for some} \quad x \in \bar{\mathcal{X}}(X, F)\}, \end{aligned}$$

then the following assertion follows from [6] and [7].

Theorem 1. *Let* $P_F, P_G \in \mathcal{M}_1^1(\mathcal{R}^s)$, X *be a nonempty convex compact set, A.2 be fulfilled. If*

- $g_0(x, z)$ *is for every* ($x \in X$) *a Lipschitz function of* $z \in \mathcal{R}^s$ *with the Lipschitz constant (corresponding to* $\mathcal{L}_1$ *norm) not depending on* $x \in X$,
- *there exists a constant* $C^1 > 0$ *such that*

$$|\rho_F(g_0(x, \xi) - \rho_G(g_0(x, \xi))| \le C^1 \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)| dz_i,$$

- $\mathsf{E}_F g_0(x, \xi)$, $\rho_F(g_0(x, \xi))$ *are strongly convex on* X *with the same parameter* $\rho' > 0$,

then

$$P\{\omega : \Delta_n[\bar{\mathcal{X}}(F, X) - \bar{\mathcal{X}}(F^N, X)] \to_{N \to \infty} 0\} = 1.$$

(Symbol $\Delta[\cdot, \cdot] := \Delta_n[\cdot, \cdot]$ denotes the Hausdorff distance of two nonempty sets in $\mathcal{R}^n$, for the definition Hausdorff distance see, e.g., [14].)

Remark 3. Conditions under which variance and Avarage-value-at-Risk fulfil the assumptions of Theorem 1 can be found in [7].

4 Mean-Risk Model via Second Order Stochastic Dominance

Let us start with model (1), problems (2), (3) and Definition 1. If we can for suitable ν_1, ν_2 find $x_0 \in X$ such that

$$\mathsf{E}_F g_0(x_0, \xi) \ge \nu_2, \quad \rho_F g_0(x_0, \xi)) \le \nu_1,$$

then setting $Y(\xi) = g_0(x_0, \xi)$ we can define the set $X(x_0)$ by

$$X(x_0) = \{x \in X : \mathsf{E}_F(u - g_0(x,\xi))^+ \le \mathsf{E}_F(u - g_0(x_0, \xi))^+ \quad \text{for every} \quad u \in \mathcal{R}^1\}. \tag{15}$$

In the case when the model (1) is consistent with the second order stochastic dominance and $X(x_0)$ is a nonempty set, then

$$x \in X(x_o) \quad \Longrightarrow \quad \mathsf{E}_F g_0(x, \xi) \ge \mathsf{E}_F g_0(x_0, \xi) \quad \text{and simultaneously} \quad \rho_F(g_0(x, \xi)) \le \rho_F(g_0(x_0, \xi)).$$

Further, evidently, if we can determine $x_1 \in X(x_0)$ such that $\mathsf{E}_F g_0(x_1, \xi) > \mathsf{E}_F g_0(x_0, \xi)$, then setting $Y(\xi) := g_0(x_1, \xi)$, we can define the set $X(x_1)$ by

$$X(x_1) = \{x \in X(x_1) : \mathsf{E}_F(u - g_0(x,\xi))^+ \le \mathsf{E}_F(u - g_0(x_1, \xi))^+ \quad \text{for every} \quad u \in \mathcal{R}^1\}. \tag{16}$$

Employing Definition 1 we obtain the following relations

$$x \in X(x_1) \quad \Longrightarrow \quad \mathsf{E}_F g_0(x, \xi) \ge \mathsf{E}_F g_1(x_1, \xi) \quad \text{and simultaneously} \quad \rho_F(g_0(x, \xi)) \le \rho_F(g_0(x_1, \xi)).$$

Further, evidently, if we can determine $x_2 \in X(x_1)$ such that that $\mathsf{E}_F g_0(x_2, \xi) > \mathsf{E}_F g_0(x_1, \xi)$, then we can setting $Y(\xi) := g_0(x_2, \xi)$, employing Definition 1 define the set $X(x_2)$ by

$$X(x_2) = \{x \in X(x_2) : \mathsf{E}_F(u - g_0(x,\xi))^+ \le \mathsf{E}_F(u - g_0(x_2, \xi))^+ \quad \text{for every} \quad u \in \mathcal{R}^1\} \tag{17}$$

such that

$$x \in X(x_2) \quad \Longrightarrow \quad \mathsf{E}_F g_0(x, \xi) \ge \mathsf{E}_F g_1(x_2, \xi) \quad \text{and simultaneously} \quad \rho_F(g_0(x, \xi)) \le \rho_F(g_0(x_2, \xi)).$$

Of course, we can continue looking for $x_3, x_4, \ldots$.

Evidently

$$X(x_2) \subset X(x_1) \subset X(x_0),$$

$$x \in X(x_2) \quad \Longrightarrow \quad \mathsf{E}_F g_0(x, \xi) \ge \mathsf{E}_F g_0(x_2, \xi) > \mathsf{E}_F g_0(x_1, \xi) > \mathsf{E}_F g_0(x_0, \xi),$$

$$\rho_F g_0(x, \xi) \le \rho_F g_0(x_2, \xi) \le \rho_F g_0(x_1, \xi) \le \rho_F g_0(x_0, \xi).$$

Of course, theoretically, it is possible to determine already $x_1 \in X(x_0)$ by an optimization problem:

$$\text{Find} \quad \max \mathsf{E}_F g_0(x, \xi) \quad \text{s.t.} \quad x \in X(x_0). \tag{18}$$

However, problem (18) is generally semi-infinite optimization problem for which Slater's condition is not very often fulfilled. Many authors have dealt with this problems, we can recall e.g., [1], [5], [9], [5].

Till now (in this section) we have assumed that the underlying probability measure is known. This assumption is fulfilled in real situations very seldom and the problem has to be analyzed on the data base. We can recall works dealing with this situation see, e.g., [4], [9] and [15].

Remark 4.

- It has been proven in [13] (see also [4])) that the mean-risk problem using Average Value-at Risk at some level α (as risk measure) is consistent with the second order stochastic dominance.
- Of course, it is possible to interchange the positions of $\mathsf{E}_F g_0(x, \xi)$ and $\rho_F(g_0(x, \xi))$ and consequently to try to improve the value of risk measure.

5 Conclusion

We have dealt with a special case of multi-objective stochastic optimization problems. Especially we consider Mean-Risk problem with Markowitz approach and with the approach based on the stochastic second order dominance. It could seem that our results do not cover many real situations. However, to deal with these others cases is beyond of the scope of this contribution.

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