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METHODOLOGIES AND APPLICATION

Pseudo-exponential distribution and its statistical applications in econophysics

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Abstract

In generalized measure theory, σ-$\oplus$-measure is a generalization of the classical measure defined on a pseudo-addition. In this paper, the class of pseudo-exponential distributions based on a class of σ-$\oplus$-measure is introduced. Some examples of this class are investigated. Then by two real data sets obtained from the last three decades of oil, and the last two decades of the daily natural gas spot prices, we show that the pseudo-exponential distribution is better fitted than exponential distribution using the AIC and BIC information criteria.

Keywords Pseudo-operations · Pseudo-exponential distribution · Moment-generating function · Numerical computation

1 Introduction

Generalized measure theory is very important in numerous applications in engineering, economics, and statistics involving uncertainty. In generalized measure theory, σ-$\oplus$-measure is a generalization of the classical measure defined on a pseudo-addition (Pap 1993). The theory of g-calculus as an important case of pseudo-analysis, which is a mathematical base for fuzzy system and soft computing, was initiated by Pap (1993, 1997) in 1993. There are many applications of pseudo-analysis in applied mathematical sciences, fuzzy sets, fuzzy numbers, optimization, system theory, and nonlinear analysis (Pap and Štajner 1999; Mesiar and Pap 1999; Pap 1993, 2005, 2008; Pap and Ralević 1998; Pap et al. 2014; Bede and O' Regan 2013). For example, pseudo-analysis has been applied in solving uncertain partial differential equations (Pap 2005) and game theory (Litvinov and Maslov 1996).

The exponential distribution is one the simplest and perhaps the most widely applied statistical distribution. This distribution is famous in statistics and statistical physics as the Boltzmann–Gibbs or thermal distributions. The exponential distribution is a commonly used model in lifetime data analysis. In statistics, the random variable X has exponential distribution if its probability density function is given by:

$$f_X(x) = \lambda \exp\{-\lambda x\}, \quad x > 0, \lambda > 0. \tag{1}$$

Econophysics (Mantegna and Stanley 1999) is a new interdisciplinary research field which applies the statistical physics methods to problems in economics and finance. Exponential distribution has many applications in econophysics (Banerjee et al. 2006). For example, in an equilibrium temperature T, the probability of finding a physical system or subsystem in energy E is given by $P(E) = ce^{-E/T}$, where c is the normalizing constant (Wannier 2010). The exponential distributions have many applications in condensed matter physics (Bernasconi 1979; Kakalios et al. 1987; Macdonald 1985), theoretical physics (Budiyono 2013; Drăgulescu and Yakovenko 2001), astronomy and astrophysics (Mao et al.

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2013; Collier 2004), neuroscience (Zeman et al. 2015; Trappenberg 2009; Schwartz 1993), mechanical systems (Granato and Lücke 1956), climate science (Field et al. 2005), and many other physical systems.

In this paper, we expand the applicability of exponential distribution by combining the properties of pseudo-analysis with exponential distribution and introduce the concept of pseudo-exponential distribution. By the real data set obtained from WTI (West Texas Intermediate) crude oil, Oklahoma, dollars per barrel, daily price from 1986/01/02 to 2017/07/03 and the real data set obtained from the daily Henry Hub Natural (HHN) gas spot price per million British Thermal Units (MBTU) in period of 1997/06/30 to 2017/06/30, we show that pseudo-exponential distribution is better fitted than the exponential distribution using the Akaike (AIC) and Bayesian (BIC) information criteria.

The paper is organized as follows. In Sect. 2, some preliminaries of the classical measure and the σ-$\oplus$-measure theory are given. In Sect. 3, we introduce the concept of pseudo-exponential distribution and discuss some of its examples. In Sect. 4, two numerical examples based on two above-mentioned real data sets are provided and the maximum likelihood estimators of the parameters are derived. Finally, we present some conclusions.

2 Preliminaries and notations

In this section, we review some basic concepts that are used in this paper. For more details, see Pap (1995, 2002); Kolokoltsov and Maslov (1997).

2.1 Measures

Definition 1 Let Ω be a non-empty set. A collection of subsets $\mathcal{F}$ of Ω is a σ-algebra if :

(i) $\Omega \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$;
(ii) If $A \in \mathcal{F}$ then its complement $A^c \in \mathcal{F}$;
(iii) If $A_1, A_2, \ldots \in \mathcal{F}$, then their union $\bigcup A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 2 Let $(\Omega, \mathcal{F})$ be a measurable space. A measure is a function $\mu : \mathcal{F} \to [0, \infty]$ which satisfies the following conditions:

(i) $\mu(\emptyset) = 0$;
(ii) (σ-additivity) If $\{A_i\}$ is a sequence of disjoint sets from $\mathcal{F}$, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.
In that case, the triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Definition 3 A probability measure is a measure P with the additional property $P(\Omega) = 1$. In that case, the triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Define $\mathbb{R}$ as the real line and $\mathcal{B}(\mathbb{R})$ as σ-algebra of subsets of $\mathbb{R}$.

Definition 4 A function $X : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable if $X^{-1}(B) \in \mathcal{F}$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$.

Definition 5 A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{F}$-measurable.

Definition 6 The probability density function of a continuous random variable X with support S is an integrable function $f : \mathbb{R} \to [0, \infty)$ such that $f_X(x) \geqslant 0$ if and only if $x \in S$ and $\int_S f_X(x)\,dx = 1$.

Note that if $f_X(x)$ is the probability density function of a continuous random variable X, then the probability that X belongs to A, where A is some interval, is given by

$$P(X \in A) = \int_A f_X(x)\mathrm{d}x.$$

2.2 σ-$\oplus$-measure

Here, we recall some notions and definitions of pseudo-operations and σ-$\oplus$-measure (Pap 1993). Let $[a, b]$ be a closed subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 7 A binary operation $\oplus$ on $[a, b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to $\preceq$), continuous, associative, and with a zero (neutral) element different from b and denoted by $\mathbf{0}$.

Let $[a, b]_+ = \{x \mid x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 8 A binary operation $\odot$ on $[a, b]$ is pseudo-multiplication fitting to the pseudo-addition $\oplus$ if it commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and with a unit element $\mathbf{1} \in [a, b]_+$, i.e., for each $x \in [a, b]$, $\mathbf{1} \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a, b], \oplus, \odot)$ is a *semiring* (see Kuich 1986).

Now, we recall the definitions of a σ-$\oplus$-measure. Let Ω be a non-empty set. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω. For more details, see Pap (1995, 2002); Kolokoltsov and Maslov (1997).

Definition 9 A σ-$\oplus$-measure is a set function $m : \mathcal{F} \to [a, b]_+$ if the following conditions are fulfilled:

(i) $m(\phi) = \mathbf{0}$ (for not idempotent $\oplus$);
(ii) for any sequence $\{A_i\}_{i\in N}$ of pairwise disjoint sets from $\mathcal{A}$, we have

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i).$$

An important real semiring on the interval with continuous operations is when the pseudo-operations are defined by a monotone bijection $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \text{ and } x \odot y = g^{-1}(g(x)\, g(y)).$$

In this case, the structure $([a, b], \oplus, \odot)$ is called a *g-semiring* (see Pap 1993; Kuich 1986). In the case of a g-semiring, a set function m is a σ-$\oplus$-measure if and only if $g \circ m$ is the classical measure (Pap 1993). So, we can introduce the concept of pseudo-density function.

Definition 10 Let $([0, \infty], \oplus, \odot)$ be a g-semiring, where the generator $g : [0, \infty] \to [0, \infty]$ is a strictly increasing function. Let $f_X(x)$ be the probability density function of a nonnegative continuous random variable X. The pseudo-density function of X is a function $f_{X,g}(x)$ satisfying

$$f_{X,g}(x) = Cg^{-1}(f_X(g(x))), \quad x \in (0, +\infty),$$

where C is the normalizing constant, i.e., $C\int_0^{+\infty} g^{-1}(f_X(g(x))\, dx = 1$. If such a normalizing constant C exists, then the pseudo-density function is well defined.

3 Main results: Pseudo-exponential distribution

Now, we introduce the concept of pseudo-exponential distribution.

Definition 11 Let $([0, \infty], \oplus, \odot)$ be a g-semiring, where the generator $g : [0, \infty] \to [0, \infty]$ is a strictly increasing function. A random variable X is said to have a pseudo-exponential distribution, if its pseudo-density function is

$$f_{X,g}(x) = Cg^{-1}(\exp\{-g(x)\}), \quad x \in (0, +\infty),$$

where C is the normalizing constant.

Some examples of the pseudo-exponential distribution are as follows:

- for $g(x) = \lambda x, \lambda > 0$, we have $x \oplus y = x + y$ and the pseudo-exponential distribution coincides with the exponential distribution (1).

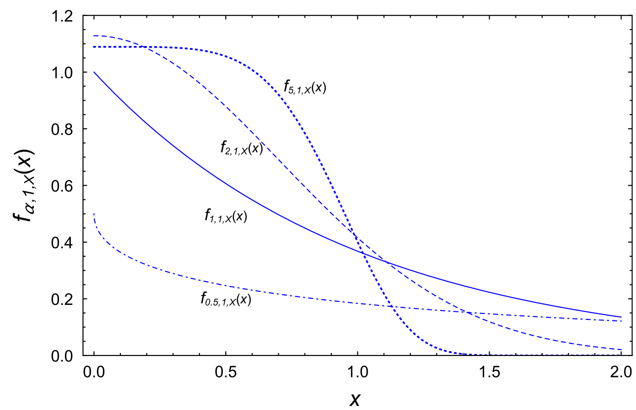

Fig. 1 $f_{X,\alpha,1}(x)$ for $\alpha = 0.5, 1, 2, 5$. $f_{X,1,1}(x) = e^{-x}$, presents the usual exponential distribution

- for $g(x) = x^2,\ x > 0$, we have $x \oplus y = \sqrt{x^2 + y^2}$ and the pseudo-exponential distribution coincides with the half-normal distribution with the probability density function as,

$$h_X(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2}, \quad x > 0. \tag{2}$$

Let $t \in \mathbb{R}$. One can generalize the above distribution to normal distribution as follows:

$$\varphi_X(t) = \begin{cases} \frac{1}{2}h_X(t), & t > 0, \\ \frac{1}{2}h_X(-t), & t < 0. \end{cases} \tag{3}$$

- for $g(x) = \alpha\beta x^\alpha,\ \alpha > 0,\ \beta > 0$, then $x \oplus y = (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}$ and we have a generalized exponential distribution, denoted by $X \sim GE(\alpha, \beta)$, with the probability density function as follows

$$f_{X,\alpha,\beta}(x) = \beta^{\frac{1}{\alpha}} \frac{e^{-\beta x^\alpha}}{\Gamma\left[1 + \frac{1}{\alpha}\right]}, \quad x > 0,\ \alpha > 0, \quad \beta > 0. \tag{4}$$

Figure 1 shows $f_{X,\alpha,1}(x)$ for different α parameters. Also, Fig. 2 presents $f_{X,1,\beta}(x)$ for different β parameters. In both figures, $f_{X,1,1}(x) = e^{-x}$, the exponential distribution, is presented with full curves.

Now we find the moment-generating function of $X \sim GE(\alpha, \beta)$.

Theorem 12 *The moment-generating function of $X \sim GE(\alpha, \beta)$ is given by*

$$M_X(t) = \sum_{n=0}^{\infty} \frac{\beta^{-\frac{n}{\alpha}} t^n \Gamma\left[\frac{1+n}{\alpha}\right]}{\alpha n! \Gamma\left[1 + \frac{1}{\alpha}\right]}.$$

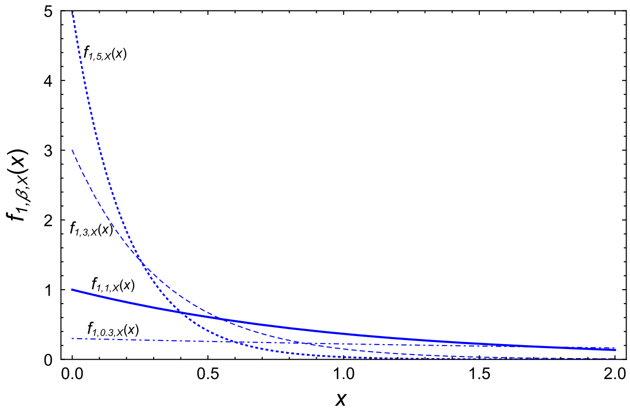

Fig. 2 $f_{X,1,\beta}(x)$ for $\beta = 0.3, 1, 3, 5$. $f_{X,1,1}(x) = e^{-x}$, presents the exponential distribution

Proof It is easy to see that

$$\begin{aligned} M_X(t) = E\left(e^{tX}\right) &= \int_0^\infty e^{tx}\beta^{\frac{1}{\alpha}} \frac{e^{-\beta x^\alpha}}{\Gamma\left[1+\frac{1}{\alpha}\right]}\mathrm{d}x \\ &= \frac{\beta^{\frac{1}{\alpha}}}{\Gamma\left[1+\frac{1}{\alpha}\right]}\int_0^\infty e^{-\beta x^\alpha + tx}\mathrm{d}x \\ &= \frac{\beta^{\frac{1}{\alpha}}}{\Gamma\left[1+\frac{1}{\alpha}\right]}\int_0^\infty \sum_{n=0}^\infty \frac{(tx)^n}{n!} e^{-\beta x^\alpha}\mathrm{d}x \\ &= \sum_{n=0}^\infty \frac{\beta^{-\frac{n}{\alpha}} t^n \Gamma\left[\frac{1+n}{\alpha}\right]}{\alpha n! \Gamma\left[1+\frac{1}{\alpha}\right]}; \end{aligned}$$

this completes the proof. □

If $X \sim GE(\alpha, \beta)$, then we have

$$E\left(X^r\right) = \frac{\beta^{-\frac{r}{\alpha}}\Gamma\left[\frac{1+r}{\alpha}\right]}{\alpha\Gamma\left[1+\frac{1}{\alpha}\right]}, \quad \alpha > 0, \\ \beta > 0, \quad r = 1, 2, 3, \ldots, n. \tag{5}$$

Also, we can also obtain the variance of $X \sim GE(\alpha, \beta)$ as follows:

$$\begin{aligned} \mathrm{Var}(X) &= E\left(X^2\right) - E^2(X) \\ &= \frac{\beta^{\frac{-2}{\alpha}}\Gamma\left[\frac{3}{\alpha}\right]}{\alpha\Gamma\left[1+\frac{1}{\alpha}\right]} - \left(\frac{\beta^{\frac{-1}{\alpha}}\Gamma\left[\frac{2}{\alpha}\right]}{\alpha\Gamma\left[1+\frac{1}{\alpha}\right]}\right)^2 \\ &= \frac{\alpha\Gamma\left[\frac{3}{\alpha}\right]\Gamma\left[\frac{1}{\alpha}(\alpha+1)\right] - \left(\Gamma\left[\frac{2}{\alpha}\right]\right)^2}{\alpha^2\beta^{\frac{2}{\alpha}}\left(\Gamma\left[\frac{1}{\alpha}(\alpha+1)\right]\right)^2}. \end{aligned}$$

4 Statistical applications to oil and gas daily prices

In this section, we would like to compare our proposed distribution with some other distributions. We firstly evaluate the Akaike (AIC) and Bayesian (BIC) information criteria for our distribution. Then we will try to apply our proposed distribution to two real data sets of oil and gas daily prices.

We compare the pseudo-exponential distribution with other distributions using AIC and BIC such that

$$\begin{aligned} &\mathrm{AIC} := 2k - 2\ln\widehat{L}, \\ &\mathrm{BIC} := -2\ln\widehat{L} + k\ln n, \end{aligned}$$

where n is the number of data, k is the number of estimated parameters and L is the maximum value of the likelihood function.

The maximum likelihood estimates (MLEs) of the parameters α and β based on an independent and identically distributed sample $x_1, \ldots, x_n \sim GE(\alpha, \beta)$ can be obtained from

$$\ln L(\alpha, \beta) = \sum\nolimits_{i=1}^n \left[\frac{1}{\alpha}\ln\beta - \beta x^\alpha - \ln\left(\Gamma\left[1+\frac{1}{\alpha}\right]\right)\right]. \tag{6}$$

By taking partial derivatives from (6) with respect to parameters, we obtain the following equations

$$\left.\begin{aligned} \frac{\partial \ln L}{\partial\alpha} &= \sum\nolimits_{i=1}^n \left[-\frac{1}{\alpha^2}\ln\beta - x_i^\alpha\beta\ln x_i + \frac{1}{\alpha^2}\Psi\left(1+\frac{1}{\alpha}\right)\right] = 0, \\ \frac{\partial \ln L}{\partial\beta} &= \sum\nolimits_{i=1}^n \left(\frac{1}{\alpha\beta} - x_i^\alpha\right) = 0, \end{aligned}\right\} \tag{7}$$

where $\Psi(x) = \frac{\partial}{\partial x}\ln\Gamma(x)$.
Equation (7) yields a closed-form expression for the MLE of β as

$$\widehat{\beta} = \frac{n}{\alpha\sum_{i=1}^n x_i^\alpha},$$

and also a likelihood equation for α as

$$-\frac{n}{\alpha^2}\ln n + \frac{n}{\alpha^2}\left(\ln\alpha\sum_{i=1}^n x_i^\alpha\right) + \frac{n}{\alpha^2}\Psi\left(1+\frac{1}{\alpha}\right) \\ -\frac{n}{\alpha\sum_{i=1}^n x_i^\alpha}\sum\nolimits_{i=1}^n x_i^\alpha\ln x_i = 0,$$

which is solved numerically with the package "nlminb" from the statistical software R.

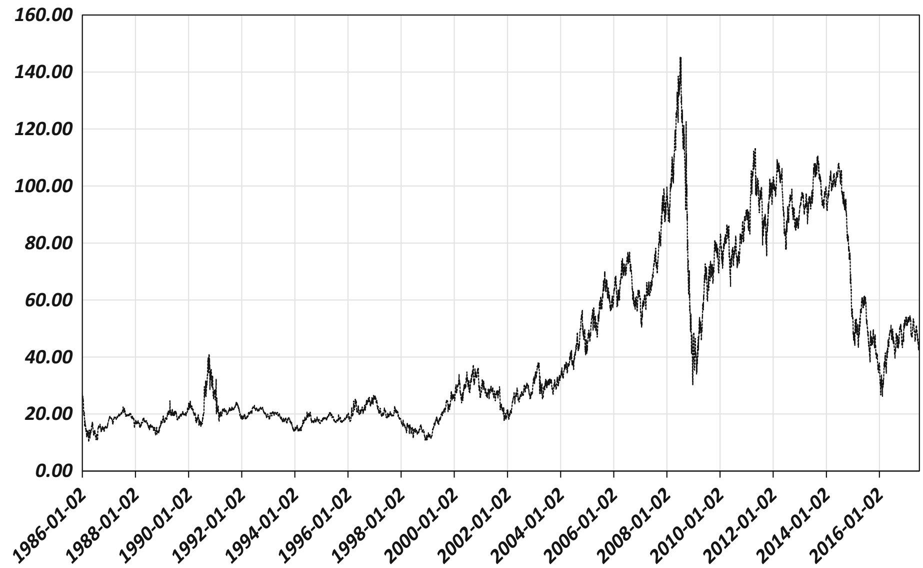

Fig. 3 WTI crude oil, Oklahoma, dollars per barrel, daily price from 1986/01/02 to 2017/07/03

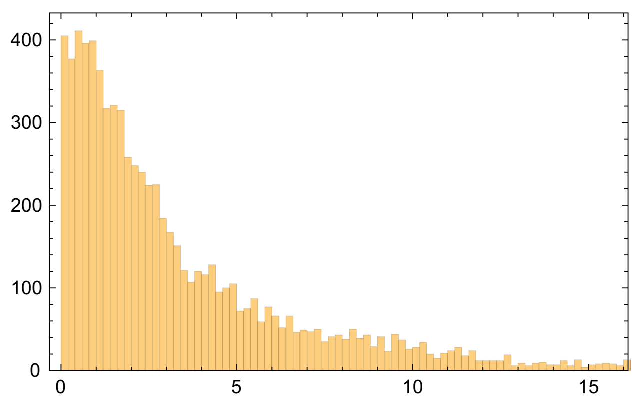

Fig. 4 Distribution of absolute difference of WTI crude oil, from its last 100 days moving average. The histogram obtained from WTI crude oil daily price in period of 1986/01/02 to 2017/07/03

4.1 Applications to WTI crude oil prices-cushing, Oklahoma, dollars per barrel, daily

To illustrate the applicability of the proposed model in Section 2, a real data set is analyzed. Recall the real data set obtained from WTI crude oil, Oklahoma, dollars per barrel, daily price from 1986/01/02 to 2017/07/03, described in Fig. 3. We work on the data with $n = 7849$ of the absolute difference of WTI crude oil price from its last 100 days moving average in period of 1986 to 2017, described in Fig. 4. Throughout this section, we also used the statistical software R version 3.4.1 with the package nlminb for estimating the parameters.

According to AIC and BIC, Table 1 shows that the $GE(\alpha, \beta)$ (G-Exponential distribution) better fitted than exponential distribution. We have considered the next classes of distribution functions: exponential, G-exponential, Weibull, normal, and Laplace distributions.

4.2 Applications to Henry hub natural gas spot price, dollars per million BTU, daily

Here, the real data set of daily Henry Hub Natural (HHN) gas spot price per million British Thermal Units (MBTU) is analyzed. The real data set of the daily HHN gas spot price in period of 1997/06/30 to 2017/06/30 is described in Fig. 5. We work on the data with $n = 4874$ of the absolute difference of HHN gas spot price from its last 100 days moving average in period of 1997–2017, described in Fig. 6.

In Table 2, the values of AIC and BIC indicate that the $GE(\alpha, \beta)$ (G-Exponential distribution) better fitted than other distributions such as exponential, Weibull, normal and Laplace distributions.

5 Conclusion

In this paper, we have introduced the pseudo-exponential distribution. We have also obtained the moments and the moment-generating function of pseudo-exponential distribution $GE(\alpha, \beta)$. Furthermore, for applicability of this distribution, the real data set obtained from WTI crude oil, Oklahoma, dollars per barrel, daily price from 1986/01/02 to 2017/07/03 and the real data set of daily HHN gas spot price per dollars per MBTU in period of 1997/06/30 to 2017/06/30 have been analyzed. For two above-mentioned real data sets, it has been obtained that the $GE(\alpha, \beta)$ distribution is better fitted than other candidate distributions using the AIC and BIC information criteria.

Table 1 Results of fitting models for WTI crude oil

Distribution	α	β	$-\log \widehat{L}$	AIC	BIC
$E(\beta)$ Exponential	–	0.240807	19024.07	38050.15	38057.12
$GE(\alpha, \beta)$ G-Exponential	**0.6845109**	**0.6308010**	**18771.6**	**37547.2**	**37561.14**
$WE(\alpha, \beta)$ Weibull	0.863852	3.82183	18856.49	37716.97	37730.91
$N(\alpha, \beta^2)$ Normal	4.1527	5.7462	24861.53	49727.06	49740.99
$L(\alpha, \beta)$ Laplace	2.30213	3.52587	22256.2	44516.4	44530.34

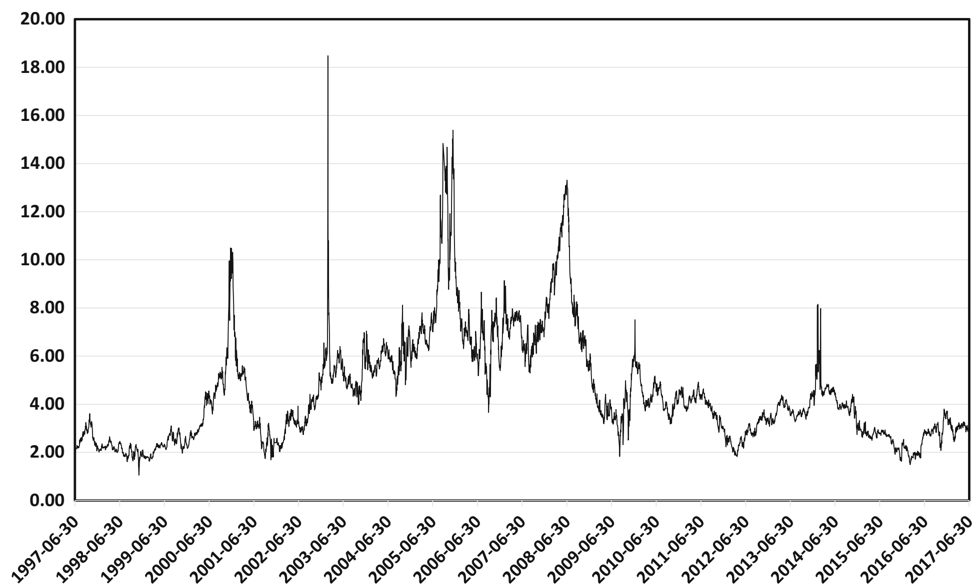

Fig. 5 HHN gas spot, dollars per MBTU, daily price from 1997/06/30 to 2017/06/30

Table 2 Results of fitting models for HHN gas spot daily price

Distribution	α	β	$-\log \widehat{L}$	AIC	BIC
$E(\beta)$ Exponential	–	1.42426	3150.32	6302.63	6309.12
$GE(\alpha, \beta)$ G-Exponential	0.8518199	1.6889330	**3127.439**	**6258.88**	**6271.86**
$WE(\alpha, \beta)$ Weibull	0.952473	0.686087	3139.98	6283.97	6296.95
$N(\alpha, \beta^2)$ Normal	0.702121	0.801484	5837.33	11678.7	11691.7
$L(\alpha, \beta)$ Laplace	0.45915	0.539328	4845.82	9695.64	9708.62

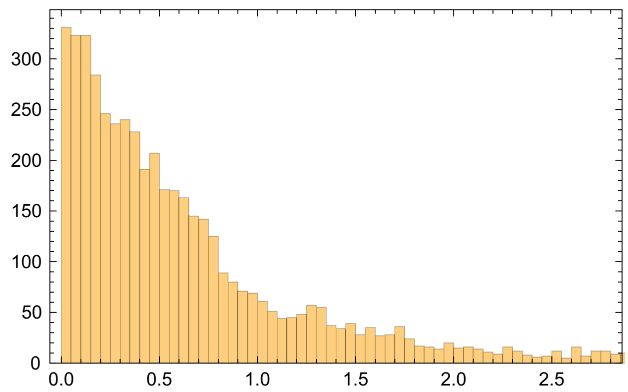

Fig. 6 Distribution of absolute difference of HHN gas spot daily price, from its last 100 days moving average, in period of 1997/06/30 to 2017/06/30

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Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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