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Monte Carlo integration for Choquet integral

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Abstract

In this paper, a numerical Monte Carlo integration for Choquet integrals is proposed by using a generalized version of mean value theorem based on Choquet integral. In special cases, this generalization can help us to have the classical Monte Carlo integration and the mean value theorem over some unbounded regions.

KEYWORDS

Choquet integral, mean value theorem, Monte Carlo integration, simulation

1 | INTRODUCTION

One of the most powerful and flexible methods of statistical simulation of physical systems and mathematical problems is the Monte Carlo method. The essential idea of this method is using randomness to solve problems that might be deterministic in principle.[1–3] There are many statistical applications of the Monte Carlo method in different branches of physics, including quantum systems,[4–6] nuclear physics,[7] particle physics,[8] biological science,[9,10] and financial science.[11] This method is one of the most important statistical simulations, which can be applied to approximate an integral.[12–14] For a function of one variable, the steps are

(i) Pick n randomly distributed points $y_1, y_2,\ldots,y_n$ in the interval $[a, b]$, where they are selected from a uniform distribution.

(ii) Obtain the function

$$\widehat{g} = \frac{1}{n}\sum_{i=1}^{n} g(y_i).$$

(iii) Compute

$$\int_a^b g(x)\,dx \approx (b-a)\widehat{g}.$$

(iv) The estimate for the error is

$$\text{Error} \approx (b-a)\sqrt{\frac{\widehat{g^2} - \widehat{g}^2}{n}},$$

where $\widehat{g^2} = (1/n)\sum_{i=1}^{n} g^2(y_i)$.

The main idea of the approximation of integrals is related to the following theorem.

Theorem 1 (Flett,[15] mean value theorem for integrals). *If f: $[a, b] \to \mathbb{R}$ is a continuous function, then there exists a point c in (a, b) such that*

$$\int_a^b f(x)\,dx = (b-a)f(c).$$

The following example shows that the Monte Carlo integration and the classical mean value theorem cannot work for the unbounded interval.

Example 2. Let $f_1(x) = e^{-x}, f_2(x) = e^{-x^2}, f_3(x) = 1/(1+x^2), x \in [0, +\infty)$. Then, clearly, we cannot use the Monte Carlo integration and the classical mean value theorems in Theorem 1 for f_i, $i = 1, 2, 3$. In the Monte Carlo integration, the random choice of points is from a uniform distribution on the domain. Then the domain $[0, \infty)$ cannot be directly covered by the Monte Carlo method.

Choquet integral is one of the most important concepts in the theory of fuzzy measures, which was presented by Choquet[16] in 1954 and was considered by many researchers.[16–24]

In this paper, we introduce a mean value theorem and Monte Carlo method for Choquet integral, which can be applied in different fields of applied science.

The paper is organized as follows. In Section 2, some preliminaries of the Choquet integral are given. In Section 3, we introduce the mean value theorem for Choquet integral and discuss the Monte Carlo integration for Choquet integral with some numerical examples. Finally, we present some conclusions.

2 | PRELIMINARIES AND NOTATIONS

We first recall the definition of Choquet integral and monotone measures.[16–18] For more details, see the following references.[19,20,22–24]

Definition 3. A monotone measure μ on a measurable space $(\Omega, \mathcal{F})$ is a set function $\mu: \mathcal{F} \to [0, \infty]$ satisfying

(i) $\mu(\varnothing) = 0$,
(ii) $\mu(E) \leq \mu(F)$ whenever $E \subseteq F$.

Definition 4 (Torra et al[25]). A monotone measure μ is called *a fuzzy measure* (or *monotone probability* or *capacity*) if $\mu(\Omega) = 1$.

Definition 5. Let $m: \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function such that $m(0) = 0$. Then, a monotone measure μ_m is called a distorted Lebesgue measure, if

$$\mu_m(\cdot) = m(\delta(\cdot)),$$

where δ is the Lebesgue measure.

Definition 6 (Choquet,[16] Denneberg,[26] and Pap[27]). The Choquet integral of a nonnegative real-valued measurable function X with respect to a monotone measure μ on $A \in \mathcal{F}$ is defined by

$$(C)\int_A X\, d\mu = \int_0^{+\infty} \mu(A \cap \{X \geq t\})\, dt,$$

where the right-hand side integral is the (improper) Riemann integral.

Definition 7. We say that two real-valued measurable functions f and g are comonotonic on A for any $\omega, \omega' \in A$, if

$$f(\omega) < f(\omega') \Rightarrow g(\omega) \leq g(\omega').$$

Some basic properties of the Choquet integral were summarized by Denneberg,[26] we list some of them:

(a) $(C)\int_A d\mu = \mu(A)$;
(b) $(C)\int_A X\, d\mu \leq (C)\int_A Y\, d\mu$ whenever $X \leq Y$ (monotonicity);
(c) $(C)\int_A \beta X\, d\mu = \beta (C)\int_A X\, d\mu$ for any real $\beta \geq 0$ (positive homogeneity);
(d) $(C)\int_A (X + \beta)\, d\mu = (C)\int_A X\, d\mu + \beta\mu(A)$ whenever $X + \beta$ is nonnegative on A (translatability);
(e) $(C)\int_A (X + Y)\, d\mu = (C)\int_A X\, d\mu + (C)\int_A Y\, d\mu$ whenever X, Y are comonotone (comonotone additivity).

Definition 8 (Torra et al[23]). Let $(\Omega, \mathcal{F})$ be a measurable space and $\mu, \upsilon: \mathcal{F} \to \mathbb{R}^+$ be monotone measures. Then υ is called a Choquet integral of μ if there exists a measurable function $g: \Omega \to \mathbb{R}^+$ with

$$\upsilon(A) = (C)\int_A g\, d\mu \tag{1}$$

for all $A \in \mathcal{F}$.

Definition 9 (Torra et al[23]). Let μ and υ be two monotone measures. If υ is a Choquet integral of μ, and g satisfies (1). Then, we use notation

$$\frac{d\upsilon}{d\mu} = g,$$

and g is called a Radon-Nikodym derivative of υ with respect to μ.

Henceforth, $\mathcal{A}^+$ always denotes the class of measurable, continuous, nonnegative, and increasing bounded functions. Also, $\mathcal{A}^-$ be the class of measurable, continuous, nonnegative, and decreasing bounded functions.

3 | MAIN RESULTS

Fix $(\Omega, \mathcal{F})$ to be $([-\infty, +\infty], \mathcal{B}([-\infty, +\infty]))$, where $\mathcal{B}([-\infty, +\infty])$ is the σ-algebra of all Borel subsets of $[-\infty, +\infty]$. Throughout this section, we assume that $\mu: \mathcal{B}([-\infty, +\infty]) \to [0, +\infty]$ is a monotone measure and $f: [a, b] \to \mathbb{R}$ is a continuous bounded function.

Theorem 10. *Let $f \in \mathcal{A}^-$ and $h: [f(b), f(a)] \to [0, \infty]$ be a continuous function such that $h(x) = \mu([a, f^{-1}(x)])$ for $x \in [f(b), f(a)]$. Then there exists a point $\xi \in (f(b), f(a))$ such that*

$$(C)\int_a^b f\, d\mu = (f(a) - f(b))h(\xi) + f(b)\mu([a, b]).$$

Proof. For any $f \in \mathcal{A}^-$, we have

$$\begin{aligned}(C)\int_a^b f\, d\mu &= \int_{f(b)}^{f(a)} \mu([a, f^{-1}(r)])\, dr + f(b)\mu([a, b]) \\ &= \int_{f(b)}^{f(a)} h(r)\, dr + f(b)\mu([a, b]),\end{aligned} \tag{2}$$

where $h(x) = \mu([a, f^{-1}(x)])$ for all $x \in [f(b), f(a)]$. Since h is a continuous function on $[f(a), f(b)]$, then by (2) and the classical mean value theorem, there exists a point $\xi \in (f(b), f(a))$ such that

$$(C)\int_a^b f\,d\mu = (f(a) - f(b))h(\xi) + f(b)\mu([a, b]).$$

This completes the proof. □

Example 11. Let μ be defined as a distorted Lebesgue measure with distortion $m(x) = x^{\alpha}$, $\alpha > 0$. Using Example 2, then

$$\begin{aligned}(C)\int_0^{+\infty} f_i\,d\mu &= (f_i(0) - \lim_{\gamma\to+\infty} f_i(\gamma))\mu([0, f_i^{-1}(\xi)]) + \lim_{\gamma\to+\infty} f_i(\gamma)\gamma^{\alpha}\\ &= (f_i(0) - \lim_{\gamma\to+\infty} f_i(\gamma))\mu([0, f_i^{-1}(\xi)]).\end{aligned}$$

- For $i = 1$, we have $f_1(x) = e^{-x}$ and

$$\begin{aligned}(C)\int_0^{+\infty} f_1\,d\mu &= \Gamma[1 + \alpha] = \mu([0, f_1^{-1}(\xi)])\\ &= (-\ln \xi)^{\alpha}.\end{aligned}$$

  So, $\xi = \exp\{-(\Gamma(1 + \alpha))^{\frac{1}{\alpha}}\} \in (f(b), f(a)) = (0, 1)$.
- For $i = 2$, we have $f_2(x) = e^{-x^2}$ and

$$\begin{aligned}(C)\int_0^{+\infty} f_2\,d\mu &= \Gamma[1 + \tfrac{\alpha}{2}] = \mu([0, f_2^{-1}(\xi)])\\ &= (-\ln \xi)^{\alpha/2}.\end{aligned}$$

  So, $\xi = \exp\{-(\Gamma(1 + \frac{\alpha}{2}))^{\frac{2}{\alpha}}\} \in (f(b), f(a)) = (0, 1)$.
- For $i = 3$, we have $f_3(x) = 1/(1 + x^2)$. Then for $0 < \alpha < 2$, we have

$$\begin{aligned}(C)\int_0^{+\infty} f_3\,d\mu &= \tfrac{\alpha\pi}{2}\csc\left(\tfrac{\alpha\pi}{2}\right) = \mu([0, f_3^{-1}(\xi)])\\ &= \left(\tfrac{1-\xi}{\xi}\right)^{\alpha/2}\end{aligned}$$

  So, $\xi = 1/(1 + ((\pi\alpha/2)\csc(\pi\alpha/2))^{2/\alpha}) \in (f(b), f(a)) = (0, 1)$.

We see that for $\alpha = 1$, the Choquet integral becomes the classical integral and the classical mean value theorem can work for the unbounded interval $[0, +\infty)$.

Example 12. Let $f(x) = 1/x$, $x \in [1, 2]$ and μ be defined as a distorted Lebesgue measure with distortion $m(x) = x^2$. Then, the mean value theorems in Theorem 10 are valid,

$$\begin{aligned}(C)\int_1^2 f\,d\mu &= 2 - \ln 4 = 0.613706 = (f(1) - f(2))\mu([1, f^{-1}(\xi)]) + f(2)\mu[1, 2]\\ &= \frac{1}{2}\left\{1 + \left(\frac{1}{\xi} - 1\right)^2\right\}.\end{aligned}$$

So, $\xi = 1/(1 + \sqrt{3 - 2\ln 4}) = 0.677105 \in (f(b), f(a)) = (0.5, 1)$.

Theorem 13. *Let $f \in \mathcal{A}^+$ and g: $[f(a), f(b)] \to [0, \infty]$ be a continuous function such that $g(x) := \mu([f^{-1}(x), b])$ for $x \in [f(a), f(b)]$. Then there exists a point $\zeta \in (f(a), f(b))$ such that*

$$(C)\int_a^b f\,d\mu = (f(b) - f(a))\mu([f^{-1}(\zeta), b]) + f(a)\mu([a, b]).$$

Proof. For any $f \in \mathcal{A}^+$, we have

$$\begin{aligned}(C)\int_a^b f\,d\mu &= \int_{f(a)}^{f(b)} \mu([f^{-1}(r), b])\,dr + f(a)\mu([a, b]) \\ &= \int_{f(a)}^{f(b)} g(r)\,dr + f(a)\mu([a, b]),\end{aligned} \tag{3}$$

where $g(x) = \mu([f^{-1}(x), b])$ for all $x \in [f(a), f(b)]$. Since g is a continuous function on $[f(a), f(b)]$, then by (3) and the classical mean value theorem, there exists a point $\zeta \in (f(a), f(b))$ such that

$$(C)\int_a^b f\,d\mu = (f(b) - f(a))g(\zeta) + f(a)\mu([a, b]).$$

This completes the proof. □

Example 14. Let $f(x) = x^3$, $x \in [0, 2]$ and μ be defined as a distorted Lebesgue measure with distortion $m(x) = x^2$. Then, the mean value theorems in Theorem 13 are valid,

$$\begin{aligned}(C)\int_0^2 f\,d\mu &= \tfrac{16}{5} = (f(2) - f(0))\mu([f^{-1}(\zeta), 2]) + f(0)\mu[0, 2] \\ &= 8(2 - \sqrt[3]{\zeta})^2.\end{aligned}$$

So, $\sqrt[3]{\zeta} = 2 - \sqrt{2/5}$ and $\zeta = 2.558 \in (f(a), f(b)) = (0, 8)$.

3.1 | Monte Carlo integration for Choquet integral

3.1.1 | Case 1

For a bounded function $f \in \mathcal{A}^+$, the Monte Carlo integration for Choquet integral has the following steps.

(i) Pick n randomly distributed points $y_1, y_2, \ldots, y_n$ in the interval $[f(a), f(b)]$, where they are selected from a uniform distribution.

(ii) Obtain

$$\hat{g} = \frac{1}{n}\sum_{i=1}^{n} g(y_i),$$

where $g(x) = \mu([f^{-1}(x), b])$.

(iii) Compute

$$(C)\int_a^b f\,d\mu \approx (f(b) - f(a))\hat{g} + f(a)\mu([a, b]).$$

(iv) The estimate for the error is

$$\text{Error} \approx (f(b) - f(a))\sqrt{\frac{\widehat{g^2} - \hat{g}^2}{n}},$$

where $\widehat{g^2} = (1/n)\sum_{i=1}^{n} g^2(y_i)$.

Example 15 (Torra et al,[23] Example 2). Let $f(x) = \sqrt{\sin((\pi/2)x)}$ and $\mu = m\circ\mathbf{P}$ be a distorted probability with

$$m(x) = \frac{s(x) - s(0)}{s(1) - s(0)},$$

where $s(x) = 1/(1 + e^{-10(x-0.5)})$ and $\mathbf{P}$ is a truncated normal distribution $N(0.5, 0.1)$ in [0,1] with

$$\mathbf{P}[a, b] = \frac{F(b, 0.5, 0.1) - F(a, 0.5, 0.1)}{F(1, 0.5, 0.1) - F(0, 0.5, 0.1)},$$

where

$$F(x, 0.5, 0.1) = \frac{1}{\sqrt{2\pi} \times 0.1}\int_{-\infty}^{x} e^{-(t-0.5)^2/(2\times(0.1)^2)}\,dt$$

is the cumulative distribution function of $N(0.5, 0.1)$. The code of the Monte Carlo integration in Mathematica has evaluated $(C)\int_0^1 f\,d\mu$ for $n = 10$, 100, 1000, 10 000, 100 000, and 1 000 000. The results are presented in Table 1. Errors and estimated errors are presented in the last two columns of this table.

Example 16. Let $f(x) = x^3$, $x \in [0, 2]$ and μ be defined as a distorted Lebesgue measure with distortion $m(x) = x^2$. Then, the code of the Monte Carlo integration in Mathematica has evaluated $(C)\int_0^2 f\,d\mu$ for $n = 10$, 100, 1000, 10 000, and 100 000. The results are presented in Table 2. Errors and estimated errors are presented in the last two columns of this table.

3.1.2 | Case 2

For a bounded function $f \in \mathcal{A}^-$, the Monte Carlo integration for Choquet integral has the following steps:

TABLE 1 Monte Carlo integration for $(C)\int_0^2 f\,d\mu$ for $f(x)=\sqrt{\sin((\pi/2)x)}$ and the distorted probability $\mu = m\circ\mathbf{P}$ presented in Example 15

n	Exact value (E)[23]	Monte Carlo (MC)	|E − MC|	$(f(b)-f(a))\sqrt{(\widehat{g^2}-\hat{g}^2)/n}$
10	0.83897	0.809194	0.0297761	0.118032
100	0.83897	0.857496	0.0185262	0.0323732
1000	0.83897	0.845578	0.00660789	0.0106104
10 000	0.83897	0.835902	0.00306769	0.00344733
100 000	0.83897	0.839939	0.000968507	0.00108109
1 000 000	0.83897	0.838762	0.000207615	0.000342904

(i) Pick n randomly distributed points $y_1, y_2, \ldots, y_n$ in the interval $[f(b), f(a)]$, where they are selected from a uniform distribution.

(ii) Obtain

$$\hat{h} = \frac{1}{n}\sum_{i=1}^{n} h(y_i),$$

where $h(x) = \mu([a, f^{-1}(x)])$.

(iii) Compute

$$(C)\int_a^b f d\mu \approx (f(a)-f(b))\hat{h} + f(b)\mu([a,b]).$$

(iv) The estimate for the error is

$$\text{Error} \approx (f(a)-f(b))\sqrt{\frac{\widehat{h^2}-(\hat{h})^2}{n}},$$

where $\widehat{h^2} = (1/n)\sum_{i=1}^{n} h^2(y_i)$.

TABLE 2 Monte Carlo integration for $(C)\int_0^2 f\,d\mu$ for $f(x)=x^3$ and the distorted Lebesgue measure with distortion $m(x)=x^2$ in Example 16

n	Exact value (E)	Monte Carlo (MC)	|E − MC|	$(f(b)-f(a))\sqrt{(\widehat{g^2}-\hat{g}^2)/n}$
10	3.2	4.06188	0.861875	1.3259
100	3.2	3.28381	0.0838089	0.470845
1000	3.2	3.24331	0.0433094	0.139418
10 000	3.2	3.22217	0.0221691	0.0437121
100 000	3.2	3.19378	0.00621813	0.0137401

TABLE 3 Monte Carlo integration for $(C)\int_0^2 f\,d\mu$ for $f(x) = 1/x$ and the distorted Lebesgue measure μ with distortion $m(x) = x^2$ presented in Example 17

n	Exact value (E)	Monte Carlo (MC)	|E − MC|	$(f(a)-f(b))\sqrt{(\widehat{f^2}-(\widehat{f})^2)/n}$
10	0.613706	0.596105	0.0176009	0.0353673
100	0.613706	0.618482	0.00477639	0.0149729
1000	0.613706	0.612331	0.00137449	0.00412911
10 000	0.613706	0.612825	0.000880464	0.00132314
100 000	0.613706	0.613909	0.000203327	0.000417607

TABLE 4 Monte Carlo integration for $(C)\int_0^\infty f_1\,d\mu$ for $f_1(x) = e^{-x}$ with distortion $m(x) = x^{1/2}$ presented in Example 18

n	Exact value (E)	Monte Carlo (MC)	|E − MC|	$(f(a)-f(b))\sqrt{(\widehat{f^2}-(\widehat{f})^2)/n}$
10	0.886227	1.0251	0.138877	0.149227
100	0.8862276	0.902649	0.0164225	0.0507259
1000	0.8862276	0.894439	0.00821172	0.0147983
10 000	0.886227	0.888633	0.00240627	0.00468254
100 000	0.886227	0.886555	0.000327842	0.00147469

Example 17. Let $f(x) = 1/x$, $x \in [1, 2]$ and μ be defined as a distorted Lebesgue measure with distortion $m(x) = x^2$. Then, the code of the Monte Carlo integration in Mathematica has evaluated $(C)\int_0^2 f\,d\mu$ for $n = 10$, 100, 1000, 10 000, and 100 000. The results are presented in Table 3. Errors and estimated errors are presented in the last two columns of this table.

Example 18. Let μ be defined as a distorted Lebesgue measure with distortion $m(x) = x^{1/2}$, and $f_i(x)$, $x \in [0, +\infty)$, $i = 1, 2, 3$ are the same as Example 2. Then, the code of the Monte Carlo integration in Mathematica has evaluated $(C)\int_0^\infty f_i\,d\mu$ for $n = 10$, 100, 1000, 10 000, and 100 000. For $f_1(x) = e^{-x}$, $f_2(x) = e^{-x^2}$, and $f_3(x) = 1/(1 + x^2)$, the results are, respectively, presented in Tables 4, 5, and 6. Errors and estimated errors are presented in the last two columns of these tables.

TABLE 5 Monte Carlo integration for $(C)\int_0^\infty f_2\,d\mu$ for $f_2(x) = e^{-x^2}$ with distortion $m(x) = x^{1/2}$ presented in Example 18

n	Exact value (E)	Monte Carlo (MC)	|E − MC|	$(f(a)-f(b))\sqrt{(\widehat{f^2}-(\widehat{f})^2)/n}$
10	0.906402	0.985397	0.0789949	0.0581737
100	0.906402	0.888126	0.0182766	0.0237233
1000	0.906402	0.895629	0.0107732	0.00817426
10 000	0.906402	0.911433	0.0050302	0.00257403
100 000	0.906402	0.906858	0.000455929	0.0008031

TABLE 6 Monte Carlo integration for $(C)\int_0^\infty f_3\, d\mu$ for $f_3(x) = 1/(1 + x^2)$ with distortion $m(x) = x^{1/2}$ presented in Example 18

n	Exact value (E)	Monte Carlo (MC)	|E − MC|	$(f(a) - f(b))\sqrt{(\widehat{f^2} - (\widehat{f})^2)/n}$
10	1.11072	0.969293	0.141428	0.132081
100	1.11072	1.12595	0.0152341	0.0475496
1000	1.11072	1.10088	0.00984405	0.0176675
10 000	1.11072	1.10707	0.00364735	0.0055651
100 000	1.11072	1.11099	0.000266366	0.00212135

4 | CONCLUSIONS

We have introduced a generalized version of the mean value theorem for Choquet integral. This generalization can help us to have the mean value theorem for some bounded functions on unbounded regions, thus to overcome impossibility of applying the standard approaches in these cases. Using our mean value theorem, we have proposed the method of the Monte Carlo integration for Choquet integrals with some numerical examples.

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