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Short communication

A sufficient condition of equivalence of the Choquet and the pan-integral

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Abstract

In this note, we continue to investigate the relationship between the Choquet integral and the pan-integral on infinite space. We will show that the *(M)-property* of monotone measures is a sufficient condition that the Choquet integral coincides with the pan-integral. In this discussion, the spaces are not restricted to be finite, thus the previous results obtained in finite space are further generalized and developed. A characteristic of the (M)-property of monotone measures is also presented.

Keywords: Monotone measure; Choquet integral; Pan-integral; (M)-property

1. Introduction

The Choquet integral ([1]), the pan-integral ([14]) and the concave integral ([4]) are three kinds of well-known nonlinear integrals. Recently, the relationship among these integrals was investigated and many interesting results were obtained. Lehrer and Teper [4] showed that the Choquet integral coincides with the concave integral if and only if the monotone measure μ is convex (or supermodular [2]). In [9] we introduced the concept of *minimal atom* of a monotone measure and by means of the characteristics of minimal atoms we presented a necessary and sufficient condition that the concave integral coincides with the $(+, \cdot)$-based pan-integral on a finite space X. The equivalence of these two integrals on infinite space was also discussed in [11].

For the Choquet integral and the pan-integral, in [7] we introduced the concept of *(M)-property* of a monotone measure and discussed the relation between these two integrals on finite space. Following these ideas, in [10] we further proved that the *(M)-property* of a monotone measure is a necessary and sufficient condition that the Choquet integral is equivalent to the $(+, \cdot)$-based pan-integral on finite space. In there we did not know whether our result

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is still true for infinite spaces. In this paper, we will prove that the sufficient condition that the Choquet integral coincides with the $(+, \cdot)$-based pan-integral on finite space remains valid for infinite spaces, that is, the *(M)-property* of monotone measure is a sufficient condition that the Choquet integral is equivalent to the $(+, \cdot)$-based pan-integral on an arbitrary space (not necessarily finite). We will also show a characteristic of the (M)-property, as we will see, the structure of a monotone measure is very "close to" additivity, that is, if a monotone measure μ possessing the (M)-property is null-additive, then it becomes to be additive.

2. Preliminaries

Let X be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of X. A set function $\mu : \mathcal{A} \to [0, +\infty]$ is called a *monotone measure* on a measurable space $(X, \mathcal{A})$ if it satisfies the following conditions:

(1) $\mu(\emptyset) = 0$ and $\mu(X) > 0$;

(2) $\mu(A) \le \mu(B)$ whenever $A \subset B$ and $A, B \in \mathcal{A}$.

When μ is a monotone measure, the triple $(X, \mathcal{A}, \mu)$ is called a monotone measure space ([12,13]).

In this paper we always assume that μ is a monotone measure on $(X, \mathcal{A})$.

The monotone measure μ is said to be (i) *superadditive*, if $\mu(A \cup B) \ge \mu(A) + \mu(B)$ for any $A, B \in \mathcal{A}$, $A \cap B = \emptyset$; (ii) *null-additive* [12,13], if for any $A, B \in \mathcal{A}$, $\mu(B) = 0$ implies $\mu(A \cup B) = \mu(A)$; (iii) *converse null-additive* [13], if for any $A, B \in \mathcal{A}$, $A \subset B$ and $\mu(A) = \mu(B) < \infty$ imply $\mu(B - A) = 0$.

The superadditivity implies converse null-additivity.

Let $(X, \mathcal{A}, \mu)$ be a monotone measure space and f a nonnegative measurable function. The $(+, \cdot)$-based *pan-integral* of f on X with respect to μ is given by

$$\int^{pan} f \, d\mu = \sup\left\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \le f, \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a partition of } X, \lambda_i \ge 0, n \in \mathbb{N} \right\}.$$

Notice that the concept of a pan-integral was introduced in [14] and it involves two binary operations, the pan-addition $\oplus$ and pan-multiplication $\otimes$ of real numbers (see also [12,13]). As we did in [7,9,10], in this note we only consider the pan-integrals based on the usual addition $+$ and multiplication $\cdot$.

The *Choquet integral* [1] of f on X with respect to μ, is defined by

$$\int^{Cho} f \, d\mu = \int_0^{\infty} \mu(\{x : f(x) \geqslant t\}) \, dt,$$

where the right-hand side integral is the improper Riemann integral. The *Choquet integral* can also be defined equivalently by

$$\int^{Cho} f \, d\mu = \sup\left\{ \sum_{i=1}^{n} \lambda_i \mu(A_i) : \sum_{i=1}^{n} \lambda_i \chi_{A_i} \le f, \{A_i\}_{i=1}^{n} \subset \mathcal{A} \text{ is a chain}, \lambda_i \ge 0, n \in \mathbb{N} \right\}.$$

In [7] we have proved that if the Choquet integral coincides with the $(+, \cdot)$-based pan-integral, then the monotone measure μ is superadditive, and hence it is converse null-additive.

Proposition 2.1. ([7]) *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. Then μ is superadditive if and only if $\int^{pan} f d\mu \le \int^{Cho} f d\mu$ holds for each nonnegative measurable function f.*

3. The (M) property of monotone measures

The concept of (M)-property of a monotone measure played an important role in the discussions of equivalence of the Choquet integral and the pan-integral on finite spaces (see [7,10]). We recall the concept which was proposed by Mesiar, see [7].

Definition 3.1. *([7])* Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If for any $A, B \in \mathcal{A}$, $A \subset B$, there exists $C \in A \cap \mathcal{A}$ such that

$$\mu(C) = \mu(A) \text{ and } \mu(B) = \mu(C) + \mu(B \setminus C),$$

then μ is called to have (M)-property.

Obviously, if μ is additive, then μ has (M)-property.

From the definition of (M)-property, we can easily see the following result:

Proposition 3.2. ([7]) *If the monotone measure μ has (M)-property, then μ is superadditive.*

The following example shows that the (M)-property is indeed stronger than superadditivity (we can also refer to [10] for a similar example).

Example 3.3. Let $X = [0, 1]$ and $\mathcal{A}$ be the Borel algebra. The monotone measure μ is defined by

$$\mu(A) = \begin{cases} 2 & \text{if } A \supsetneqq \{0, 1\} \\ 1 & \text{if } A = \{0, 1\} \\ 0 & \text{else.} \end{cases}$$

Then μ is superadditive. In fact, for any $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, either $\mu(A) = 0$ or $\mu(B) = 0$. Then the superadditivity follows from the monotonicity of μ, i.e., $\mu(A \cup B) \geq \max(\mu(A), \mu(B)) = \mu(A) + \mu(B)$. But μ has not (M)-property. For $A = \{0, 1\}$ and $B = X$, to ensure $\mu(C) = \mu(A)$, it is the only case that $C = A = \{0, 1\}$, but then $\mu(B) = 2 > 1 = \mu(C) + \mu(B \setminus C)$.

Noting that the superadditivity implies converse null-additivity (in fact, for any $A, B \in \mathcal{A}$, $A \subset B$ and $\mu(A) = \mu(B) < \infty$, by the superadditivity of μ, then $\mu(B) = \mu(A \cup (B - A)) \geq \mu(A) + \mu(B - A)$, therefore $\mu(B - A) = 0$), from Proposition 3.2, we get the following result:

Proposition 3.4. *If μ has (M)-property, then μ is converse null-additive.*

From definition of the (M)-property and by using Proposition 3.4, we can easily obtain the following result.

Proposition 3.5. *Let μ be null-additive. If μ has (M)-property, then μ is additive.*

4. Main result

In [10] (see also [7]) we proved that the *(M)-property* is a necessary and sufficient condition that the Choquet integral coincides with the $(+, \cdot)$-based pan-integral on finite spaces (Theorem 4.6 in [10]). Now we show that the sufficiency in this result remains valid for infinite spaces. The following theorem is our main result.

Theorem 4.1. *Let $(X, \mathcal{A}, \mu)$ be a monotone measure space. If μ has (M)-property, then for any nonnegative measurable function f, we have*

$$\int^{Cho} f d\mu = \int^{pan} f d\mu. \tag{4.1}$$

Proof. Let f be an arbitrary nonnegative measurable function. Since μ has (M)-property, by Proposition 3.2, μ is superadditive. For a superadditive measure, by Theorem 10.7 in [13] (see also [7]), it holds that

$$\int^{Cho} f d\mu \geq \int^{pan} f d\mu.$$

Thus, to reach our result, it suffices to prove that for any f,

$$\int^{Cho} f d\mu \le \int^{pan} f d\mu. \tag{4.2}$$

For an arbitrarily given expression $\sum_{i=1}^{n} \lambda_i \chi_{B_i} \le f$ with $\lambda_i \ge 0$ and $B_1 \subset B_2 \subset \cdots \subset B_n$, we need only to find a sequence of mutual disjoint sets $\{A_i\}_{i=1}^{n}$ and a sequence of nonnegative numbers $\{l_i\}_{i=1}^{n}$ such that

$$\sum_{i=1}^{n} l_i \chi_{A_i} \le \sum_{i=1}^{n} \lambda_i \chi_{B_i} \le f, \tag{4.3}$$

and

$$\sum_{i=1}^{n} \lambda_i \mu(B_i) = \sum_{i=1}^{n} l_i \mu(A_i) \le \int^{pan} f d\mu. \tag{4.4}$$

Then, by the arbitrariness of $\sum_{i=1}^{n} \lambda_i \chi_{B_i}$, the formula (4.2) (and thus the equality (4.1)) follows.

Let such an expression be given. By the (M)-property of μ, for $B_1 \subset B_2$ there is $B_1^{(1)} \subset B_1$ such that $\mu(B_1^{(1)}) = \mu(B_1)$ and

$$\mu(B_2) = \mu(B_1^{(1)}) + \mu(B_2 \setminus B_1^{(1)}).$$

Similarly, for $B_1^{(i-1)} \subset B_{i+1}$ there is $B_1^{(i)} \subset B_1^{(i-1)}$ such that $\mu(B_1^{(i)}) = \mu(B_1^{(i-1)}) = \mu(B_1)$ and

$$\mu(B_{i+1}) = \mu(B_1^{(i)}) + \mu(B_{i+1} \setminus B_1^{(i)}), i = 2, 3, \cdots, n-1.$$

Denote $A_1 = B_1^{(n-1)}$, $B_{i+1}^{(1)} = B_{i+1} \setminus B_1^{(i)}$, $i = 1, 2, \cdots, n-1$ and $l_1 = \sum_{i=1}^{n} \lambda_i$. Then

$$A_1 = B_1^{(n-1)} \subset B_1^{(n-2)} \subset \cdots \subset B_1^{(1)} \subset B_1.$$

Thus,

$$\begin{aligned} l_1 \chi_{A_1} + \sum_{i=2}^{n} \lambda_i \chi_{B_i^{(1)}} &= \left(\sum_{i=1}^{n} \lambda_i\right) \chi_{A_1} + \sum_{i=2}^{n} \lambda_i \chi_{B_i^{(1)}} \\ &\le \lambda_1 \chi_{B_1} + \sum_{i=2}^{n} \lambda_i \left(\chi_{B_i^{(1)}} + \chi_{B_1^{(i-1)}}\right) \\ &= \sum_{i=1}^{n} \lambda_i \chi_{B_i} \le f. \end{aligned}$$

Moreover, due to the fact that

$$\mu(A_1) = \mu(B_1^{(n-1)}) = \mu(B_1^{(n-2)}) = \cdots = \mu(B_1^{(1)}) = \mu(B_1),$$

we have

$$\begin{aligned} l_1 \mu(A_1) + \sum_{i=2}^{n} \lambda_i \mu(B_i^{(1)}) &= \left(\sum_{i=1}^{n} \lambda_i\right) \mu(A_1) + \sum_{i=2}^{n} \lambda_i \mu(B_i^{(1)}) \\ &= \lambda_1 \mu(B_1) + \sum_{i=2}^{n} \lambda_i \left(\mu(B_i^{(1)}) + \mu(B_1^{(i-1)})\right) \\ &= \sum_{i=1}^{n} \lambda_i \mu(B_i). \end{aligned}$$

Similarly, for $B_2^{(1)} \subset B_3^{(1)} \subset \cdots \subset B_n^{(1)}$, we can find $\{B_2^{(i)}\}_{i=2}^{n-1}$ with $B_2^{(n-1)} \subset B_2^{(n-2)} \subset \cdots \subset B_2^{(2)} \subset B_2^{(1)}$ such that $\mu(B_2^{(i)}) = \mu(B_2^{(1)})$ and

$$\mu(B_{i+1}^{(1)}) = \mu(B_2^{(i)}) + \mu(B_{i+1}^{(1)} \setminus B_2^{(i)}).$$

Denote $A_2 = B_2^{(n-1)}$, $B_{i+1}^{(2)} = B_{i+1}^{(1)} \setminus B_2^{(i)}$, $i = 2, 3, \cdots, n-1$ and $l_2 = \sum_{i=2}^{n} \lambda_i$. Then we have,

$$\sum_{i=1}^{2} l_i \chi_{A_i} + \sum_{i=3}^{n} \lambda_i \chi_{B_i^{(2)}} \le l_1 \chi_{A_1} + \sum_{i=2}^{n} \lambda_i \chi_{B_i^{(1)}} \le f,$$

and

$$\sum_{i=1}^{2} l_i \mu(A_i) + \sum_{i=3}^{n} \lambda_i \mu(B_i^{(2)}) = l_1 \mu(A_1) + \sum_{i=2}^{n} \lambda_i \mu(B_i^{(1)}) = \sum_{i=1}^{n} \lambda_i \mu(B_i).$$

Generally, for $B_i^{(i-1)} \subset B_{i+1}^{(i-1)} \subset \cdots \subset B_n^{(i-1)}$, $i = 2, 3, \cdots, n-1$, we can find $\{B_i^{(j)}\}_{j=i}^{n-1}$ with $B_i^{(n-1)} \subset \cdots \subset B_i^{(i)} \subset B_i^{(i-1)}$ such that $\mu(B_i^{(j)}) = \mu(B_i^{(i-1)})$ and

$$\mu(B_{j+1}^{(i-1)}) = \mu(B_i^{(j)}) + \mu(B_{j+1}^{(i-1)} \setminus B_i^{(j)}).$$

Denote $A_i = B_i^{(n-1)}$, $B_{j+1}^{(i)} = B_{j+1}^{(i-1)} \setminus B_i^{(j)}$, $j = i, \cdots, n-1$ and $l_i = \sum_{j=i}^{n} \lambda_j$. Then

$$\sum_{j=1}^{i} l_j \chi_{A_j} + \sum_{j=i+1}^{n} \lambda_j \chi_{B_j^{(i)}} \le f,$$

and

$$\sum_{j=1}^{i} l_j \mu(A_j) + \sum_{j=i+1}^{n} \lambda_j \mu(B_j^{(i)}) = \sum_{i=1}^{n} \lambda_i \mu(B_i).$$

Finally, we denote $A_n = B_n^{(n-1)}$ and $l_n = \lambda_n$.

In this way, we obtain a sequence of mutual disjoint sets $\{A_i\}_{i=1}^{n}$. In fact, by our construction, for any $1 \le i < j \le n$, we have $A_i \cap B_k^{(i)} = B_i^{(n-1)} \cap B_k^{(i)} = \emptyset, k = i+1, \cdots, n$ and $A_j = B_j^{(n-1)} \subset B_j^{(i)}$, thus $A_i \cap A_j = \emptyset$. Moreover, together with $\{A_i\}_{i=1}^{n}$, the sequence of nonnegative numbers $\{l_i\}_{i=1}^{n}$ satisfies both the formulas (4.3) and (4.4).

The proof is complete. □

The above theorem provides a sufficient condition to ensure the equivalence of the Choquet integral and the $(+, \cdot)$-based pan-integral. We do not know whether this condition is also necessary (by Proposition 2.1, the superadditivity of μ is necessary). But we know that to ensure the equivalence of these two integrals, the (M)-property of μ can not be weakened to the superadditivity.

Example 4.2. Let the monotone measure μ be considered as in Example 3.3. Then μ is superadditive. Let the measurable function f be defined as

$$f(x) = \begin{cases} 2 & \text{if } x = 0, 1 \\ 1 & \text{else.} \end{cases}$$

For any $\sum_{i=1}^{n} \lambda_i \chi_{A_i} \le f$ with $A_i \cap A_j = \emptyset$, without loss of generality, we suppose $A_1 \cap \{0, 1\} \ne \emptyset$. Then $\mu(A_i) = 0$ for $i \ge 2$. If $\mu(A_1) = 2$ then $\lambda_1 \le 1$ which implies that $\sum_{i=1}^{n} \lambda_i \mu(A_i) \le 2$. If $\mu(A_1) = 1$ then $\lambda_1 \le 2$ which also implies that $\sum_{i=1}^{n} \lambda_i \mu(A_i) \le 2$. Thus $\int^{pan} f d\mu = 2$, but

$$\int^{Cho} f d\mu = \mu(X) + \mu(\{0, 1\}) = 3 > \int^{pan} f d\mu.$$

5. Concluding remarks

In this note, we have shown that the (M)-property of a monotone measure is sufficient for the equivalence of the Choquet integral and the $(+, \cdot)$-based pan-integral on an arbitrary monotone measure space X (i.e., X is not necessarily finite) (Theorem 4.1). But we do not know whether it is also necessary.

As we have mentioned in Section 1, in the case of finite space, the *(M)-property* is not only sufficient, but also necessary for which the Choquet integral coincides with the $(+, \cdot)$-based pan-integral. So, the first open question is to prove (or disprove) the necessity of (M)-property on an infinite space.

Notice that the concave integral introduced by Lehrer (see [4]), which is based on $+$ and $\cdot$, was generalized to the pseudo-concave integral [6] based on pseudo-operations $\oplus$ and $\otimes$. Also, the Choquet integral was generalized to the Choquet-like integral [5]. So the second open question is to investigate the relationships among the Choquet-like, pseudo-concave and pan-integrals (related to $\oplus$ and $\otimes$ operations) on arbitrary spaces.

Finally, as a dual counterpart of decomposition integral ([3]), the superdecomposition integrals were introduced in [8]. It is of interest to investigate the relationships among various superdecomposition integrals.

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