CHOQUET-LIKE INTEGRALS
WITH RESPECT TO LEVEL-DEPENDENT CAPACITIES
AND $\varphi$-ORDINAL SUMS OF AGGREGATION FUNCTION

Radko Mesiar and Peter Smrek

Dedicated to the memory of Ivan Kramosil

In this study we merge the concepts of Choquet-like integrals and the Choquet integral with respect to level dependent capacities. For finite spaces and piece-wise constant level-dependent capacities our approach can be represented as a $\varphi$-ordinal sum of Choquet-like integrals acting on subdomains of the considered scale, and thus it can be regarded as extension method. The approach is illustrated by several examples.

Keywords: Choquet integral, Choquet-like integral, level-dependent capacity, $\varphi$-ordinal sum of aggregation functions

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1. INTRODUCTION

Based on the problem of integration with respect to inner and outer measures, Vitali \[14\] proposed to merge the information hidden in a monotone measure $m$ (not necessarily $\sigma$-additive) and in a non-negative measurable function $f$ into one source, namely a real function $h_{m,f} : [0, \infty] \rightarrow [0, \infty]$ given by

$$h_{m,f}(t) = m(\{f \geq t\}),$$

where $\{f \geq t\}$ stands for the set of all arguments where the function $f$ attains a value which is at least $t$, i.e., $\{f \geq t\} = \{\omega \in \Omega | f(\omega) \geq t\}$. Note that this is an idea related to the probability theory approach, when survival functions, i.e., complementary functions to distribution functions, are considered. Note that a survival function $S_{P,X}$ is given by $S_{P,X}(t) = P(\{X \geq t\})$, where $(\Omega, \mathcal{A}, P)$ is a given probability space and $X$ a non-negative random variable on $(\Omega, \mathcal{A}, P)$. Recall that then the expected value of $X$ can be computed by means of the (improper) Riemann integral

$$E_P(X) = \int_0^\infty S_{P,X}(t) \, dt, \quad (1)$$

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independently of the type of random variable \( X \) (discrete, with density, etc.). Considering capacities, Choquet \[1\] introduced an integral, which is now called the Choquet integral

\[
Ch_m(f) = \int_0^\infty h_{m,f}(t) \, dt.
\]

(2)

A deep study and discussion concerning the Choquet integral can be found in Denneberg’s monograph \[2\], Pap’s handbook \[7\], see also \[15\], and also in many scientific papers. From among several generalizations of the Choquet integral, we will consider the concept of Choquet-like integrals \[5\] and the concept of the Choquet integral with respect to level-dependent capacities \[4\]. The main aim of this paper is the introduction of Choquet-like integrals with respect to level-dependent capacities and the study of representation of these integrals by means of special ordinal sums introduced in \[6\], see also \[3\].

The paper is organized as follows: In Section 2, we recall the concept of Choquet-like integrals. Section 3 explains the concept of level-dependent capacities and the related Choquet integral. Then, in Section 4, these two concepts are merged into Choquet-like integrals with respect to level-dependent capacities. In Section 5, \( \varphi \)-ordinal sums are recalled, and Section 6 is devoted to finite spaces and piece-wise constant level-dependent capacities. In this section, Choquet-like integrals with respect to level-dependent capacities are represented as \( \varphi \)-ordinal sums of Choquet-like integrals. Finally, some concluding remarks are provided.

2. CHOQUET-LIKE INTEGRALS

Let \((\Omega, \mathcal{A})\) be a fixed measurable space. A set function \( v: \mathcal{A} \to [0, 1] \) is called a capacity if it is monotone (i.e., \( v(A) \leq v(B) \) whenever \( A \subseteq B \)), and \( v(\emptyset) = 0, \ v(\Omega) = 1 \). The set of all \( \mathcal{A} \)-measurable functions \( f: \Omega \to [0, 1] \) will be denoted by \( \mathcal{F}_\mathcal{A} \).

**Definition 2.1.** (Choquet, Denneberg \[1, 2\]) Let \( v \) be a capacity on \((\Omega, \mathcal{A})\). Then the functional \( Ch_v: \mathcal{F}_\mathcal{A} \to [0, 1] \) given by

\[
Ch_v(f) = \int_0^1 h_{v,f}(t) \, dt = \int_0^1 v(\{f \geq t\}) \, dt
\]

(3)

is called the Choquet integral.

**Remark 2.2.**

(i) Having in mind aggregation functions on the interval \([0, 1]\) \[3\], we have constrained the range of considered functions to be contained in \([0, 1]\) and the boundary value of the set function \( v \) to be \( v(\Omega) = 1 \). However, all the next results also stay valid without these constraints, if we suppose the range of functions to be a subset of \([0, \infty]\) and ask the positivity of \( v(\Omega) \) only.
(ii) Due to Schmeidler [10, 11], we also have an axiomatic characterization of the Choquet integral. Recall that two functions \( f, g \in \mathcal{F}_A \) are comonotone whenever

\[
(f(\omega_1) - f(\omega_2))(g(\omega_1) - g(\omega_2)) \geq 0
\]

for any \( \omega_1, \omega_2 \in \Omega \) (i.e., the orderings on \( \Omega \) induced by \( f \) and \( g \), respectively, are not contradictory). Then every comonotone additive functional \( I: \mathcal{F}_A \to [0, 1] \), \( I(1_\Omega) = 1 \) and \( I(f + g) = I(f) + I(g) \) whenever \( f, g, f + g \in \mathcal{F}_A \) and \( f \) and \( g \) are comonotone, is just the Choquet integral, \( I = Ch_v \), where \( v(A) = I(1_A) \) for every \( A \in \mathcal{A} \).

The standard arithmetical operations + and \( \cdot \) acting on the interval \([0, \infty]\) are the background of several integrals, including the Lebesgue and Choquet integrals. Many attempts to generalize these classical integrals were based on a generalization of these operations into a pseudo-addition \( \oplus \) and pseudo-multiplication \( \odot \) [8, 9, 13]. In this paper, we will consider a distinguished kind of couples \((\oplus, \odot)\) generated by an automorphism \( \varphi: [0, \infty] \to [0, \infty] \).

**Definition 2.3.** Let \( \varphi: [0, \infty] \to [0, \infty] \) be an increasing bijection. Then the couple \((\oplus, \odot, \varphi)\) of binary operations on \([0, \infty]\) given by

\[
\begin{align*}
    x \oplus \varphi y &= \varphi^{-1}(\varphi(x) + \varphi(y)), \\
    x \odot \varphi y &= \varphi^{-1}(\varphi(x)\varphi(y))
\end{align*}
\]

is called a \( \varphi \)-generated couple of pseudo-arithmetical operations.

A typical example of a \( \varphi \)-generated couple \((\oplus, \odot, \varphi)\) is generated by a power function \( \varphi: \varphi(x) = x^p, p \in [0, \infty] \), and denoted by \((\oplus_p, \odot_p)\). Clearly, for each \( p \), \( \odot_p \) is the standard multiplication, while \( x \oplus_p y = (x^p + y^p)^{1/p} \) is well known from \( L_p \)-spaces theory (when \( p \geq 1 \)). Note that \( \odot_p \) has a neutral element \( e = \varphi^{-1}(1) \), and thus we will often require \( \varphi(1) = 1 \). Note also that for each automorphism \( \varphi \) on \([0, \infty]\), \( \varphi(1) = \varphi^* \) is an automorphism that satisfies \( \varphi^*(1) = 1, \oplus = \oplus_{\varphi^*} \) and \( 1 \odot_{\varphi^*} x = x \odot_{\varphi^*} 1 = x \) for every \( x \in [0, 1] \). This automorphism is called a normed automorphism.

**Theorem 2.4.** Let \( \varphi: [0, \infty] \to [0, \infty] \) be a normed automorphism and let \( I: \mathcal{F}_A \to [0, 1] \) be a comonotone \( \oplus, \varphi \)-additive functional (i.e., for any \( f, g \in \mathcal{F}_A \) such that \( f \oplus_{\varphi} g \in \mathcal{F}_A \), and \( f, g \) are comonotone, it holds that \( I(f \oplus_{\varphi} g) = I(f) \oplus \varphi I(g) \)), which satisfies \( I(1_\Omega) = 1 \). Then

\[
I(f) = \varphi^{-1}(Ch_{\varphi_{\odot_v}}(\varphi \circ f)) = \varphi^{-1}\left(\int_0^1 \varphi\left(\{f \geq \varphi^{-1}(t)\}\right) \, dt\right),
\]

where \( v: A \to [0, 1] \) is a capacity given by \( v(A) = I(1_A) \).

**Proof.** We first note that \( f, g \in \mathcal{F}_A \) are comonotone if and only if \( \varphi \circ f, \varphi \circ g \in \mathcal{F}_A \) are comonotone. For comonotone functions \( f, g \in \mathcal{F}_A \) such that \( f \oplus_{\varphi} g \) is also in \( \mathcal{F}_A \), the comonotone \( \oplus, \varphi \)-additivity of \( I \) can be written as

\[
I(f \oplus_{\varphi} g) = I \circ \varphi^{-1}(\varphi \circ f + \varphi \circ g) = I(f) \oplus_{\varphi} I(g) = \varphi^{-1}(\varphi(I(f)) + \varphi(I(g))),
\]
i.e., with the notation $\varphi \circ I \circ \varphi^{-1} = J$, we have

$$J(\varphi \circ f + \varphi \circ g) = J(\varphi \circ f) + J(\varphi \circ g).$$

Hence, $J: \mathcal{F}_A \to [0, 1]$ is a comonotone additive functional and by Remark 2.2 (ii),

$$J(h) = Ch_m(h),$$

where the capacity $m: \mathcal{A} \to [0, 1]$ is given by

$$m(A) = J(1_A) = \varphi \left( I \left( \varphi^{-1}(1_A) \right) \right) = \varphi \left( I(1_A) \right) = \varphi \left( v(A) \right),$$

with $v(A) = I(1_A)$. Now, the representation (4) of $I$ follows.

Recall that for a general monotone measure $m: \mathcal{A} \to [0, 1]$ (i.e., with the properties $m(\emptyset) = 0$, $m(\Omega) > 0$, and $m(A) \leq m(B)$ whenever $A \subseteq B$), a measurable function $f: \Omega \to [0, \infty]$ and an automorphism $\varphi: [0, \infty] \to [0, \infty]$, the formula

$$\varphi^{-1} \left( Ch_{\varphi m}(\varphi \circ f) \right) = \varphi^{-1} \left( \int_0^\infty \varphi \left( m(\{ f \geq \varphi^{-1}(t) \}) \right) dt \right)$$

defines a $\varphi$-generated Choquet-like integral introduced in [5]. If we denote this integral by $Ch_{\varphi m}$, then the functional $I$ given by (4) satisfies $I = Ch_{\varphi m}$. Thus, Theorem 2.4 provides an axiomatic characterization of Choquet-like integrals related to a normed automorphism, i.e., such an automorphism $\varphi$, for which it holds that

$$Ch_{\varphi m}(1_A) = v(A), \text{ for any capacity } v: \mathcal{A} \to [0, 1].$$

3. LEVEL-DEPENDENT CAPACITIES AND THE CHOQUET INTEGRAL

Let $X = \{ c_1, \ldots, c_n \}$ be a set of criteria. A capacity $v: \mathcal{A} \to [0, 1]$ can be regarded as a weighting function assigning a weight to a group of criteria $A \in \mathcal{A}$. The idea of such weight being dependent on the level of criteria satisfaction degrees to be aggregated led Greco et al. [4] to the introduction of level-dependent capacities.

**Definition 3.1.** A mapping $M: [0, 1] \times \mathcal{A}$ is called a level-dependent capacity whenever $M(t, \cdot): \mathcal{A} \to [0, 1]$ is a capacity for each $t \in [0, 1]$.

It is obvious that a level-dependent capacity $M$ can be written in the form $M = (m_t)_{t \in [0, 1]}$, i.e., as a system of capacities $m_t$, $t \in [0, 1]$. Inspired by Vitali’s idea to assign to any capacity $v$ on $(\Omega, \mathcal{A})$ and any function $f \in \mathcal{F}_A$ the function $h_{v,f}$, one can introduce the function $h_{M,f}: [0, 1] \to [0, 1]$ as follows:

$$h_{M,f}(t) = m_t(\{ f \geq t \}).$$

Note that the mentioned function $h_{v,f}$ is decreasing and thus Riemann integrable for any capacity $v$ and any $f \in \mathcal{F}_A$, but this need not be true in the case of $h_{M,f}$ (neither monotonicity nor Riemann integrability is guaranteed). A function $f \in \mathcal{F}_A$, such that for a given level-dependent capacity $M$ the function $h_{M,f}$ is Lebesgue integrable, is called an $M$-integrable function.
Definition 3.2. (Greco et al. [4]) Let $M$ be a fixed level-dependent capacity and let $f \in \mathcal{F}_A$ be $M$-integrable. Then the Choquet integral of $f$ with respect to $M$ (with the notation $Ch_M(f)$) is defined by

$$Ch_M(f) = \int_0^1 h_{M,f}(t) \, dt = \int_0^1 m_t(\{ f \geq t \}) \, dt,$$

where the Lebesgue integral with respect to the standard Lebesgue measure on $[0,1]$ is applied.

To ensure the $M$-integrability of every $f \in \mathcal{F}_A$, we introduce the notion of piece-wise constant level-dependent capacities.

Definition 3.3. For a fixed $k \in \mathbb{N}$, let $0 = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1$ and let, for $i = 1, \ldots, k, v_1, \ldots, v_k : A \to [0,1]$ be capacities. Put $M = (m_t)_{t \in [0,1]}$, where

$$m_t = v_i \text{ if } a_{i-1} \leq t < a_i, \text{ and } m_1 = v_k.$$

Then $M$ is called a piece-wise constant level-dependent capacity.

Proposition 3.4. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $A = 2^\Omega$. Let $M$ be a piece-wise constant level-dependent capacity as is described in Definition 3.3. Then each $f \in \mathcal{F}_A$ is $M$-integrable.

Proof. The result follows from the fact that the function $h_{M,f}$ is piece-wise constant. \qed

Remark 3.5. Note that in general, the finiteness of $\Omega$ does not guarantee the $M$-integrability of each $f \in \mathcal{F}_A$. Consider $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $f(\omega_i) = (i - 1)/2$ and let, for a Borel non-measurable set $E \subset [0,1], 0 \notin E$,

$$m_t(A) = \begin{cases} 0 & \text{for each } t \in E, A \neq \Omega, \\ 1 & \text{for each } t \notin E, A \neq \emptyset. \end{cases}$$

Then

$$h_{M,f}(t) = \begin{cases} 0 & \text{if } t \in E, \\ 1 & \text{if } t \notin E, \end{cases}$$

is not Borel measurable and thus not Lebesgue integrable.

4. CHOQUET-LIKE INTEGRALS WITH RESPECT TO LEVEL-DEPENDENT CAPACITIES

In what follows, we merge the concepts discussed in Sections 2 and 3.

Definition 4.1. Let $M$ be a level-dependent capacity, $f \in \mathcal{F}_A$, and let $\varphi$ be a normed automorphism. Let the function $h_{M,f}^\varphi : [0,1] \to [0,1]$ given by $h_{M,f}^\varphi(t) = \varphi(m_t(\{ f \geq \varphi^{-1}(t) \}))$ be Lebesgue integrable. Then $f$ is called $\varphi - M$-integrable and the value

$$Ch_M^\varphi(f) = \varphi^{-1}(Ch_{\varphi \circ M}(\varphi \circ f))$$

is called a $\varphi$-based Choquet-like integral of $f$ with respect to $M$. 
The next result is obvious.

**Corollary 4.2.** Let Ω be a finite space, \(A = 2^\Omega\), and let \(M\) be a piece-wise constant level-dependent capacity. Then, for any normed automorphism \(\varphi\), any function \(f \in F_A\) is \(\varphi - M\)-integrable.

**Example 4.3.** Let \(\Omega = \{\omega_1, \omega_2\}\). Then each \(f \in F_A\) can be identified with a couple \((x, y) \in [0, 1]^2\), where \(x = f(\omega_1)\) and \(y = f(\omega_2)\). Define two capacities \(v_1, v_2 : 2^\Omega \to [0, 1]\), by

\[
\begin{align*}
  v_1(\{\omega_1\}) &= a, & v_2(\{\omega_1\}) &= c, \\
  v_1(\{\omega_2\}) &= b, & v_2(\{\omega_2\}) &= d,
\end{align*}
\]

with \(a, b, c, d \in [0, 1]\) (and obviously, \(v_i(\emptyset) = 0\) and \(v_i(\Omega) = 1\)), and also define a piece-wise constant level-dependent capacity \(M = (m_t)_{t \in [0, 1]}\), where, for \(\alpha \in ]0, 1[\),

\[
m_t = \begin{cases} 
  v_1 & \text{if } t \leq \alpha, \\
  v_2 & \text{if } t > \alpha.
\end{cases}
\]

For an arbitrary normed automorphism \(\varphi\) (i.e., \(\varphi|_{[0, 1]}\) is an automorphism of \([0, 1]\)), consider \(\varphi(x) \leq \alpha < \varphi(y)\). Then

\[
h_{M,(x,y)}^\varphi(t) = \begin{cases} 
  1 & \text{if } t \leq \varphi(x), \\
  \varphi(b) & \text{if } \varphi(x) < t \leq \alpha, \\
  \varphi(d) & \text{if } \alpha < t \leq \varphi(y), \\
  0 & \text{else},
\end{cases}
\]

and

\[
Ch_{M}^{\varphi}((x,y)) = \varphi^{-1}(\varphi(x) + (\alpha - \varphi(x))\varphi(b) + (\varphi(y) - \alpha)\varphi(d))
\]

\[
= \varphi^{-1}(\varphi(x)(1 - \varphi(b)) + \varphi(y)\varphi(d) + \alpha(\varphi(b) - \varphi(d))).
\]

It is not difficult to check that if \((x, y) \in [0, \varphi^{-1}(a)]^2\), then

\[
Ch_{M}^{\varphi}((x,y)) = Ch_{v_1}^{\varphi}((x,y)),
\]

while if \((x, y) \in [\varphi^{-1}(a), 1]^2\), then

\[
Ch_{M}^{\varphi}((x,y)) = Ch_{v_2}^{\varphi}((x,y)).
\]

5. \(\varphi\)-ORDINAL SUMS OF AGGREGATION FUNCTIONS

Ordinal sums are well known for t-norms, copulas, semicopulas (the same formula based on Min), as well as for t-conorms (a dual formula based on Max). In order to unify all these formulae in a unique general formula, in [5], the concept of \(\varphi\)-ordinal sums of aggregation functions was introduced. Before recalling this notion, we still note that an \((n\text{-ary})\) aggregation function \(A : [a, b]^n \to [a, b]\) is defined as an increasing function in each coordinate, which satisfies the properties \(A(a, \ldots, a) = a\) and \(A(b, \ldots, b) = b\).
Definition 5.1. For \( n, k \in \mathbb{N} \), let \( 0 = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1 \), and let \( A_i : [a_{i-1}, a_i]^n \rightarrow [a_{i-1}, a_i] \) be given aggregation functions. Let \( \varphi : [0, 1] \rightarrow [0, 1] \) be an automorphism. Then the function \( A : [0, 1]^n \rightarrow [0, 1] \), denoted by \( \varphi - \langle (a_{i-1}, a_i, A_i), i \in \{1, \ldots, k\} \rangle \) and given by

\[
A(x_1, \ldots, x_n) = \varphi^{-1}\left( \sum_{i=1}^{k} (\varphi(A_i(/x_1/i, \ldots, /x_n/i)) - \varphi(a_{i-1})) \right),
\]

with \( /x/i = \min \{a_i, \max \{a_{i-1}, x\}\} \) for every \( i \in \{1, \ldots, k\} \) and every \( x \in [0, 1] \), is called a \( \varphi \)-ordinal sum (of summands \( \langle a_{i-1}, a_i, A_i \rangle, i \in \{1, \ldots, k\} \)).

Note that if \( (x_1, \ldots, x_n) \in [a_{i-1}, a_i]^n \), then \( A(x_1, \ldots, x_n) = A_i(x_1, \ldots, x_n) \), and thus \( A \) is an extension of aggregation functions \( A_i \) acting on subdomains \([a_{i-1}, a_i]^n\) to the full domain \([0, 1]^n\). Note that \( \varphi \)-ordinal sums preserve continuity and symmetry of the \( A_i \)’s. Moreover, if all aggregation functions \( A_i \) are t-norms (copulas, semicopulas, t-conorms), then for an arbitrary automorphism \( \varphi \) of \([0, 1]\) the corresponding \( \varphi \)-ordinal sum is also a t-norm (copula, semicopula, t-conorm) coinciding with the above mentioned ordinal sum of t-norms (copulas, semicopulas, t-conorms).

6. CHOQUET-LIKE INTEGRALS AND \( \varphi \)-ORDINAL SUMS

For a fixed finite space \( \Omega = \{\omega_1, \ldots, \omega_n\} \) and \( \mathcal{A} = 2^\Omega \), the Choquet integral as well as Choquet-like integrals with respect to a fixed capacity \( \nu \) can be seen as \( n \)-ary aggregation functions on \([0, 1]\). Note that they are idempotent, i.e., for a constant function \( f = c, c \in [0, 1], Ch_v(c) = Ch_v^\nu(c) = c \) for any normed automorphism \( \varphi \). However, this means that for any subinterval \([a_{i-1}, a_i] \subseteq [0, 1], Ch_v|_{[a_{i-1}, a_i]} \) and \( Ch_v^\varphi|_{[a_{i-1}, a_i]} \) are also (idempotent) \( n \)-ary aggregation functions on \([a_{i-1}, a_i]\). When these integrals are considered with respect to a piece-wise constant level-dependent capacity \( M \), then the following representation by means of \( \varphi \)-ordinal sums holds. Let us still note that \( Ch_v = Ch_v^{id} \), where \( id(x) = x, x \in [0, \infty]. \)

Theorem 6.1. Let \( \Omega = \{\omega_1, \ldots, \omega_n\} \) and \( \mathcal{A} = 2^\Omega \). For \( k \in \mathbb{N} \), let \( 0 = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1 \), and let \( M = (m_t)_{t \in [0, 1]} \) be a piece-wise constant level-dependent capacity with \( m_t = v_i \) whenever \( a_{i-1} \leq t < a_i \). Let \( \varphi : [0, \infty] \rightarrow [0, \infty] \) be a normed automorphism. By abuse of notation we use the same letter \( \varphi \) for \( \varphi|_{[0, 1]} \). Let \( A : [0, 1]^n \rightarrow [0, 1] \) be an aggregation function. Then the following are equivalent.

(i) \( A = Ch_M^\varphi \).

(ii) \( A = \varphi - \langle (\varphi^{-1}(a_{i-1}), \varphi^{-1}(a_i), Ch_M^\varphi), i \in \{1, \ldots, k\} \rangle \).

Proof. It is not difficult to check that it is enough to prove the equivalence (i) \( \Leftrightarrow \) (ii) for one fixed normed automorphism only, in particular, for \( \varphi = id \). Note that then \( Ch_M = Ch_M^{id} \). It is enough to define \( \varphi \)-ordinal sums for \( k = 2 \) only, and then, the general case can be obtained by induction. Thus, it is enough to prove the result for \( k = 2 \).
For a finite space $\Omega = \{\omega_1, \ldots, \omega_n\}$, consider two capacities $v_1, v_2: 2^\Omega \to [0, 1]$ and a threshold value $\alpha \in [0, 1]$. Let $M = (m_t)_{t \in [0, 1]}$ be given by

$$m_t = \begin{cases} v_1 & \text{if } t \leq \alpha, \\ v_2 & \text{if } t > \alpha. \end{cases}$$

Each $f \in \mathcal{F}_A$ can be represented in the form of an $n$-dimensional vector $x = (x_1, \ldots, x_n) \in [0, 1]^n$, $x_i = f(\omega_i)$. Let $\sigma: \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)}$, and let $x_{\sigma(j-1)} \leq \alpha \leq x_{\sigma(j)}$. Then

$$h_{M,f}(t) = \begin{cases} v_1(\{\sigma(i), \ldots, \sigma(n)\}) & \text{if } i < j, \ t \in [x_{\sigma(i-1)}, x_{\sigma(i)}], \\ v_1(\{\sigma(j), \ldots, \sigma(n)\}) & \text{if } t \in [x_{\sigma(j-1)}, \alpha], \\ v_2(\{\sigma(j), \ldots, \sigma(n)\}) & \text{if } t \in [\alpha, x_{\sigma(j)}], \\ v_2(\{\sigma(i), \ldots, \sigma(n)\}) & \text{if } i > j, \ t \in [x_{\sigma(i-1)}, x_{\sigma(i)}], \end{cases}$$

and for $Ch_M(f)$ we have

$$Ch_M(f) = \sum_{i=1}^{j-1} (x_{\sigma(i)} - x_{\sigma(i-1)}) v_1(\{\sigma(i), \ldots, \sigma(n)\}) + (\alpha - x_{\sigma(j-1)}) v_1(\{\sigma(j), \ldots, \sigma(n)\}) + \sum_{i=j+1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)}) v_2(\{\sigma(i), \ldots, \sigma(n)\}).$$

On the other hand, the $id$-ordinal sum is given by:

$$id \cdot 1 = (0, \alpha, Ch_{v_1}) + (\alpha, 1, Ch_{v_2})(f) = Ch_{v_1}(f \land \alpha) + Ch_{v_2}(f \lor \alpha) - \alpha$$

$$= \left( \sum_{i=1}^{j-1} (x_{\sigma(i)} - x_{\sigma(i-1)}) v_1(\{\sigma(i), \ldots, \sigma(n)\}) + (\alpha - x_{\sigma(j-1)}) v_1(\{\sigma(j), \ldots, \sigma(n)\}) \right)$$

$$+ \left( \alpha + (x_{\sigma(j)} - \alpha) v_2(\{\sigma(j), \ldots, \sigma(n)\}) + \sum_{i=j+1}^{n} (x_{\sigma(i)} - x_{\sigma(i-1)}) v_2(\{\sigma(i), \ldots, \sigma(n)\}) \right) - \alpha.$$

Hence both formulae coincide, i.e.,

$$Ch_M(f) = id \cdot 1 = (0, \alpha, Ch_{v_1}) + (\alpha, 1, Ch_{v_2})(f),$$

which proves the theorem. \qed

Recall that if a capacity $v$ is additive, i.e., $v$ is a discrete probability measure, then the Choquet integral on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is just the weighted arithmetic mean, $Ch_v = W_v$, where $W_v(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$ with $w_i = v(\{\omega_i\})$. Then, if a piece-wise constant level dependent capacity $M$ is linked to additive capacities $v_1, \ldots, v_k$, the corresponding Choquet integral $Ch_M$ can be seen as an ordinal sum of weighted arithmetic means $W_1, \ldots, W_k$. A similar consideration can be applied to Choquet-like integrals $Ch_M^\varphi$, $\varphi$
being a normalized automorphism and \( v_1, \ldots, v_k \) being \( \oplus_\varphi \)-additive. Observe that then each integral \( Ch^\varphi_{\nu_j} \) is a weighted quasi-arithmetic mean,

\[
Ch^\varphi_{\nu_j}(x_1, \ldots, x_n) = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi\left(w_i^{(j)}\right) \varphi(x_i)\right),
\]

where \( w_i^{(j)} = v_j(\{\omega_i\}) \).

**Example 6.2.** Consider \( \Omega = \{1, 2\} \), \( f : \Omega \to [0, 1] \) such that \( f(1) = x, \ f(2) = y \), and define capacities \( v_1, v_2, v_3 : 2^\Omega \to [0, 1] \) as follows:

\[
\begin{align*}
    v_1(\emptyset) &= 0, \quad v_2(\emptyset) = 0, \quad v_3(\emptyset) = 0, \\
v_1(\{1\}) &= 0.3, \quad v_2(\{1\}) = 0.5, \quad v_3(\{1\}) = 0.7, \\
v_1(\{2\}) &= 0.7, \quad v_2(\{2\}) = 0.5, \quad v_3(\{2\}) = 0.3, \\
v_1(\Omega) &= 1, \quad v_2(\Omega) = 1, \quad v_3(\Omega) = 1.
\end{align*}
\]

Define the system \( M = (m_t)_{t \in [0, 1]} \) of capacities \( m_t \) by

\[
m_t = \begin{cases} 
    v_1 & \text{if } t \in [0, \frac{1}{3}], \\
v_2 & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\
v_3 & \text{if } t \in \left[\frac{2}{3}, 1\right].
\end{cases}
\]

Consider an aggregation function \( A \) known on subintervals depending on \( M \) and the related probability measures \( v_i \) as follows:

\[
A(x, y) = \begin{cases} 
    0.3x + 0.7y & \text{if } (x, y) \in [0, 1/3]^2, \\
    0.5x + 0.5y & \text{if } (x, y) \in [1/3, 2/3]^2, \\
    0.7x + 0.3y & \text{if } (x, y) \in [2/3, 1]^2.
\end{cases}
\]

The task is to extend \( A \) to the whole domain \([0, 1]^2\). It can be made by means of the formula \([7]\), i.e.,

\[
A(x, y) = Ch_M(f) = \int_0^1 h_{M,f}(t) \, dt.
\]

The related function \( h_{M,f} \) is piece-wise constant but not monotone, in general.

For example, if \( (x, y) \in [\frac{2}{3}, 1] \times [0, \frac{1}{3}] \) there are 5 possible values for \( h_{M,f}(t) \):

1. \( t \leq y \Rightarrow x > y \geq t \Rightarrow f(1) > t, f(2) \geq t \Rightarrow m_t(\{f \geq t\}) = m_t(\{1, 2\}) = 1, \)
2. \( y < t \leq \frac{1}{3} \Rightarrow m_t(\{f \geq t\}) = m_t(\{1\}) = v_1(\{1\}) = 0.3, \)
3. \( y < \frac{1}{3} \leq t < \frac{2}{3} \Rightarrow x \Rightarrow m_t(\{f \geq t\}) = m_t(\{1\}) = v_2(\{1\}) = 0.5, \)
4. \( \frac{2}{3} < t \leq x \Rightarrow m_t(\{f \geq t\}) = m_t(\{1\}) = v_3(\{1\}) = 0.7, \)
5. \( x < t \Rightarrow m_t(\{f \geq t\}) = m_t(\{\emptyset\}) = 0. \)
Thus for $0 \leq y \leq \frac{1}{3}$ and $\frac{2}{3} < x \leq 1$ we have

$$h_{M,f}(t) = \begin{cases} 1 & \text{if } t \leq y, \\ 0.3 & \text{if } y < t \leq \frac{1}{3}, \\ 0.5 & \text{if } \frac{1}{3} < t \leq x, \\ 0.7 & \text{if } \frac{2}{3} < t \leq x, \\ 0 & \text{if } x < t. \end{cases}$$ (10)

In this case the Choquet integral $Ch_M(f)$ is

$$Ch_M(f) = y \cdot 1 + \left( \frac{1}{3} - y \right) \cdot 0.3 + \left( \frac{2}{3} - \frac{1}{3} \right) \cdot 0.5 + \left( x - \frac{2}{3} \right) \cdot 0.7 = 0.7x + 0.7y - 0.2,$$

which gives the corresponding values $A(x, y)$.

The results obtained by this approach for all remaining subdomains are in Table 1.

<table>
<thead>
<tr>
<th>$A(x, y)$</th>
<th>$x \in [0, \frac{1}{3}]$</th>
<th>$x \in [\frac{1}{3}, \frac{2}{3}]$</th>
<th>$x \in [\frac{2}{3}, 1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y \in [\frac{2}{3}, 1]$</td>
<td>$0.3x + 0.3y + 0.2$</td>
<td>$0.5x + 0.3y + \frac{0.4}{3}$</td>
<td>$0.7x + 0.3y$</td>
</tr>
<tr>
<td>$y \in [\frac{1}{3}, \frac{2}{3}]$</td>
<td>$0.3x + 0.5y + \frac{0.2}{3}$</td>
<td>$0.5x + 0.5y$</td>
<td>$0.7x + 0.5y - \frac{0.4}{3}$</td>
</tr>
<tr>
<td>$y \in [0, \frac{1}{3}]$</td>
<td>$0.3x + 0.7y$</td>
<td>$0.5x + 0.7y - \frac{0.2}{3}$</td>
<td>$0.7x + 0.7y - 0.2$</td>
</tr>
</tbody>
</table>

**Tab. 1.** Results of Example 6.2

Observe, that the obtained aggregation function $A : [0, 1]^2 \rightarrow [0, 1]$ described in Table 1 is continuous, idempotent and piece-wise linear on $[0, 1]^2$.

**Example 6.3.** Consider $\Omega = \{1, 2\}$, $f : \Omega \rightarrow [0, 1]$, where $f(1) = x$, $f(2) = y$, and for $i \in \{1, 2\}$ define capacities $v_i : 2^\Omega \rightarrow [0, 1]$ as follows:

- $v_1(\{1\}) = 0.2$, $v_1(\{2\}) = 0.4$,
- $v_2(\{1\}) = 0.6$, $v_2(\{2\}) = 0.3$,
- $v_i(\emptyset) = (0)$, $v_i(\Omega) = 1$, $i = 1, 2$.

Both $v_1$ and $v_2$ are nonadditive capacities. Define $M = (m_t)_{t \in [0, 1]}$ by

$$m_t = \begin{cases} v_1 & \text{if } t \leq 1/2, \\ v_2 & \text{otherwise.} \end{cases}$$ (11)

Then $M$ is a level dependent capacity. In this case, if $x, y \in [0, 1/2]$ (or if $x, y \in [1/2, 1]$), we have to distinguish the cases $x \leq y$ and $y < x$. Then the resulting aggregation function $A$ is the Choquet integral with respect to $v_1$ ($v_2$). Extension of these Choquet integrals to full domain $[0, 1]^2$ can be computed by formula (7) and the obtained results are in Table 2.
<table>
<thead>
<tr>
<th>$A(x, y)$</th>
<th>$x \in [0, 1/2]$</th>
<th>$x \in [1/2, 1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y \in [1/2, 1]$</td>
<td>$0.6x + 0.3y + 0.05$</td>
<td>$0.7x + 0.3y$ if $x &lt; y$ $0.6x + 0.4y$ if $y \leq x$</td>
</tr>
<tr>
<td>$y \in [0, 1/2]$</td>
<td>$0.6x + 0.4y$ if $x &lt; y$ $0.2x + 0.8y$ if $y \leq x$</td>
<td>$0.6x + 0.8y - 0.2$</td>
</tr>
</tbody>
</table>

**Tab. 2.** Results of Example 6.3

Observe, that aggregation function $A: [0, 1]^2 \to [0, 1]$ described in Table 2 is again continuous, idempotent and piece-wise linear on $[0, 1]^2$.

7. CONCLUDING REMARKS

We have discussed Choquet-like integrals with respect to (piece-wise constant) level-dependent capacities and shown their relation to $\varphi$-ordinal sums of aggregation functions. We expect applications of our results in several decision problems, especially when a different approach to evaluating the utility (aggregation of score vector) is expected, when only low (middle, high) values are to be aggregated. Note also that for a capacity $v$, the dual capacity $v^d$ is given by $v^d(A) = 1 - v(A^c)$. Similarly, we can introduce a dual $M^d$ to a level-dependent capacity $M$ by $M^d(t, A) = 1 - M(1 - t, A^c)$. Note that if $M = (m_t)_{t \in [0,1]}$, then $M^d = (m^d_{1-t})_{t \in [0,1]}$. If the Choquet integral $Ch_v$ is considered as an aggregation function, $Ch_v: [0, 1]^n \to [0, 1]$, its dual is given by $Ch^d_v(x) = 1 - Ch_v(1 - x)$. Then $Ch^d_v = Ch_{v^d}$, see [3]. It can be shown that a similar claim is valid for the level-dependent capacities-based Choquet integral, i.e., $Ch^d_M = Ch_{M^d}$.

To illustrate the above mentioned facts consider the extremal capacities $v_*, v^*: A \to [0, 1]$, $v_*(A) = 0$ for all $A \neq \Omega$ and $v^*(A) = 1$ for all $A \neq \emptyset$. Then $v^d_ = v^*$. For a fixed $\alpha \in [0, 1]$, let $M_\alpha = (m_t)_{t \in [0,1]}$ be given by

$$m_t = \begin{cases} v^* & \text{if } t \leq \alpha, \\ v_* & \text{if } t > \alpha. \end{cases}$$

Then, representing $f \in \mathcal{F}_A$ in the form $x = (x_1, \ldots, x_n)$, we have

$$Ch_{M_\alpha}(f) = \text{med} (\min\{x_1, \ldots, x_n\}, \alpha, \max\{x_1, \ldots, x_n\}) = \begin{cases} Ch_{v^*}(f) & \text{if } f \leq \alpha, \\ Ch_{v^*}(f) & \text{if } f \geq \alpha, \\ \alpha & \text{else}. \end{cases}$$

The corresponding dual $M^d_\alpha = (\mu_t)_{t \in [0,1]}$ is given by

$$\mu_t = \begin{cases} v^* & \text{if } t < 1 - \alpha, \\ v_* & \text{if } t \geq 1 - \alpha. \end{cases}$$
Then

\[ Ch^d_{M_\alpha} = Ch_{M_{d_\alpha}} = Ch_{M_{1-\alpha}}. \]

On the other hand, if \( M_{(\alpha)} = (m_t)_{t \in [0,1]} \) is given by

\[ m_t = \begin{cases} v^* & \text{if } t \leq \alpha, \\ v^* & \text{if } t > \alpha, \end{cases} \]

it holds that

\[ Ch_{M_{(\alpha)}}(f) = \begin{cases} \min\{x_1, \ldots, x_n\} & \text{if } f \leq \alpha, \\ \max\{x_1, \ldots, x_n\} & \text{if } f \geq \alpha, \\ \min\{x_1, \ldots, x_n\} + \max\{x_1, \ldots, x_n\} - \alpha & \text{else.} \end{cases} \]

In this case it also holds that \( Ch^d_{M_{(\alpha)}} = Ch^d_{M_{(1-\alpha)}} \).

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Radko Mesiar, Department of Mathematics and Descriptive Geometry Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 813 68 Bratislava 1, Slovak Republic and Institute of Information Theory and Automation — Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.

e-mail: radko.mesiar@stuba.sk

Peter Smrek, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Radlinského 11, 813 68 Bratislava 1, Slovak Republic.

e-mail: peter.smrek@stuba.sk