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CHOQUET-LIKE INTEGRALS WITH RESPECT TO LEVEL-DEPENDENT CAPACITIES AND φ-ORDINAL SUMS OF AGGREGATION FUNCTION

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Dedicated to the memory of Ivan Kramosil

In this study we merge the concepts of Choquet-like integrals and the Choquet integral with respect to level dependent capacities. For finite spaces and piece-wise constant level-dependent capacities our approach can be represented as a φ-ordinal sum of Choquet-like integrals acting on subdomains of the considered scale, and thus it can be regarded as extension method. The approach is illustrated by several examples.

Keywords: Choquet integral, Choquet-like integral, level-dependent capacity, φ-ordinal sum of aggregation functions

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1. INTRODUCTION

Based on the problem of integration with respect to inner and outer measures, Vitali [14] proposed to merge the information hidden in a monotone measure m (not necessarily σ-additive) and in a non-negative measurable function f into one source, namely a real function $h_{m,f}\colon [0,\infty] \to [0,\infty]$ given by

$$h_{m,f}(t) = m(\{f \geq t\}),$$

where $\{f \geq t\}$ stands for the set of all arguments where the function f attains a value which is at least t, i. e., $\{f \geq t\} = \{\omega \in \Omega | f(\omega) \geq t\}$. Note that this is an idea related to the probability theory approach, when survival functions, i. e., complementary functions to distribution functions, are considered. Note that a survival function $S_{P,X}$ is given by $S_{P,X}(t) = P(\{X \geq t\})$, where $(\Omega, \mathcal{A}, P)$ is a given probability space and X a non-negative random variable on $(\Omega, \mathcal{A}, P)$. Recall that then the expected value of X can be computed by means of the (improper) Riemann integral

$$E_P(X) = \int_0^\infty S_{P,X}(t)\,\mathrm{d}t, \tag{1}$$

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independently of the type of random variable X (discrete, with density, etc.). Considering capacities, Choquet [1] introduced an integral, which is now called the Choquet integral

$$Ch_m(f) = \int_0^\infty h_{m,f}(t)\,\mathrm{d}t. \tag{2}$$

A deep study and discussion concerning the Choquet integral can be found in Denneberg's monograph [2], Pap's handbook [7], see also [15], and also in many scientific papers. From among several generalizations of the Choquet integral, we will consider the concept of Choquet-like integrals [5] and the concept of the Choquet integral with respect to level-dependent capacities [4]. The main aim of this paper is the introduction of Choquet-like integrals with respect to level-dependent capacities and the study of representation of these integrals by means of special ordinal sums introduced in [6], see also [3].

The paper is organized as follows: In Section 2, we recall the concept of Choquet-like integrals. Section 3 explains the concept of level-dependent capacities and the related Choquet integral. Then, in Section 4, these two concepts are merged into Choquet-like integrals with respect to level-dependent capacities. In Section 5, φ-ordinal sums are recalled, and Section 6 is devoted to finite spaces and piece-wise constant level-dependent capacities. In this section, Choquet-like integrals with respect to level-dependent capacities are represented as φ-ordinal sums of Choquet-like integrals. Finally, some concluding remarks are provided.

2. CHOQUET-LIKE INTEGRALS

Let $(\Omega, \mathcal{A})$ be a fixed measurable space. A set function $v\colon \mathcal{A} \to [0,1]$ is called a *capacity* if it is monotone (i. e., $v(A) \le v(B)$ whenever $A \subseteq B$), and $v(\emptyset) = 0$, $v(\Omega) = 1$. The set of all $\mathcal{A}$-measurable functions $f\colon \Omega \to [0,1]$ will be denoted by $\mathcal{F}_{\mathcal{A}}$.

Definition 2.1. (Choquet, Denneberg [1, 2]) Let v be a capacity on $(\Omega, \mathcal{A})$. Then the functional $Ch_v\colon \mathcal{F}_{\mathcal{A}} \to [0,1]$ given by

$$Ch_v(f) = \int_0^1 h_{v,f}(t)\,\mathrm{d}t = \int_0^1 v(\{f \ge t\})\,\mathrm{d}t \tag{3}$$

is called the Choquet integral.

Remark 2.2.

(i) Having in mind aggregation functions on the interval $[0,1]$ [3], we have constrained the range of considered functions to be contained in $[0,1]$ and the boundary value of the set function v to be $v(\Omega) = 1$. However, all the next results also stay valid without these constraints, if we suppose the range of functions to be a subset of $[0,\infty]$ and ask the positivity of $v(\Omega)$ only.

(ii) Due to Schmeidler [10, 11], we also have an axiomatic characterization of the Choquet integral. Recall that two functions f, $g \in \mathcal{F}_\mathcal{A}$ are comonotone whenever

$$(f(\omega_1) - f(\omega_2))(g(\omega_1) - g(\omega_2)) \geq 0$$

for any $\omega_1, \omega_2 \in \Omega$ (i. e., the orderings on Ω induced by f and g, respectively, are not contradictory). Then every comonotone additive functional $I\colon \mathcal{F}_\mathcal{A} \to [0,1]$, $I(1_\Omega) = 1$ and $I(f+g) = I(f) + I(g)$ whenever $f, g, f+g \in \mathcal{F}_\mathcal{A}$ and f and g are comonotone, is just the Choquet integral, $I = Ch_v$, where $v(A) = I(1_A)$ for every $A \in \mathcal{A}$.

The standard arithmetical operations $+$ and $\cdot$ acting on the interval $[0,\infty]$ are the background of several integrals, including the Lebesgue and Choquet integrals. Many attempts to generalize these classical integrals were based on a generalization of these operations into a pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ [8, 9, 13]. In this paper, we will consider a distinguished kind of couples $(\oplus, \odot)$ generated by an automorphism $\varphi\colon [0,\infty] \to [0,\infty]$.

Definition 2.3. Let $\varphi\colon [0,\infty] \to [0,\infty]$ be an increasing bijection. Then the couple $(\oplus_\varphi, \odot_\varphi)$ of binary operations on $[0,\infty]$ given by

$$\begin{aligned} x \oplus_\varphi y &= \varphi^{-1}\left(\varphi(x) + \varphi(y)\right), \\ x \odot_\varphi y &= \varphi^{-1}\left(\varphi(x)\varphi(y)\right) \end{aligned}$$

is called a φ-generated couple of pseudo-arithmetical operations.

A typical example of a φ-generated couple $(\oplus_\varphi, \odot_\varphi)$ is generated by a power function $\varphi\colon \varphi(x) = x^p$, $p \in]0,\infty[$, and denoted by $(\oplus_p, \odot_p)$. Clearly, for each p, $\odot_p$ is the standard multiplication, while $x \oplus_p y = (x^p + y^p)^{1/p}$ is well known from L_p-spaces theory (when $p \geq 1$). Note that $\odot_\varphi$ has a neutral element $e = \varphi^{-1}(1)$, and thus we will often require $\varphi(1) = 1$. Note also that for each automorphism φ on $[0,\infty]$, $\frac{\varphi}{\varphi(1)} = \varphi^*$ is an automorphism that satisfies $\varphi^*(1) = 1$, $\oplus_\varphi = \oplus_{\varphi^*}$ and $1 \odot_{\varphi^*} x = x \odot_{\varphi^*} 1 = x$ for every $x \in [0,1]$. This automorphism is called a normed automorphism.

Theorem 2.4. Let $\varphi\colon [0,\infty] \to [0,\infty]$ be a normed automorphism and let $I\colon \mathcal{F}_\mathcal{A} \to [0,1]$ be a comonotone $\oplus_\varphi$-additive functional (i. e., for any $f, g \in \mathcal{F}_\mathcal{A}$ such that $f \oplus_\varphi g \in \mathcal{F}_\mathcal{A}$, and f, g are comonotone, it holds that $I(f \oplus_\varphi g) = I(f) \oplus_\varphi I(g)$), which satisfies $I(1_\Omega) = 1$. Then

$$I(f) = \varphi^{-1}\left(Ch_{\varphi\circ v}(\varphi \circ f)\right) = \varphi^{-1}\left(\int_0^1 \varphi\left(v(\{f \geq \varphi^{-1}(t)\})\right)\,\mathrm{d}t\right), \tag{4}$$

where $v\colon \mathcal{A} \to [0,1]$ is a capacity given by $v(A) = I(1_A)$.

Proof. We first note that $f, g \in \mathcal{F}_\mathcal{A}$ are comonotone if and only if $\varphi \circ f$, $\varphi \circ g \in \mathcal{F}_\mathcal{A}$ are comonotone. For comonotone functions $f, g \in \mathcal{F}_\mathcal{A}$ such that $f \oplus_\varphi g$ is also in $\mathcal{F}_\mathcal{A}$, the comonotone $\oplus_\varphi$-additivity of I can be written as

$$I(f \oplus_\varphi g) = I \circ \varphi^{-1}(\varphi \circ f + \varphi \circ g) = I(f) \oplus_\varphi I(g) = \varphi^{-1}\left(\varphi(I(f)) + \varphi(I(g))\right),$$

i. e., with the notation $\varphi \circ I \circ \varphi^{-1} = J$, we have

$$J(\varphi \circ f + \varphi \circ g) = J(\varphi \circ f) + J(\varphi \circ g).$$

Hence, $J\colon \mathcal{F}_{\mathcal{A}} \to [0,1]$ is a comonotone additive functional and by Remark 2.2 (ii),

$$J(h) = Ch_m(h),$$

where the capacity $m\colon \mathcal{A} \to [0,1]$ is given by

$$m(A) = J(1_A) = \varphi\left(I\left(\varphi^{-1}(1_A)\right)\right) = \varphi\left(I(1_A)\right) = \varphi\left(v(A)\right),$$

with $v(A) = I(1_A)$. Now, the representation (4) of I follows. □

Recall that for a general monotone measure $m\colon \mathcal{A} \to [0,1]$ (i. e., with the properties $m(\emptyset) = 0$, $m(\Omega) > 0$, and $m(A) \le m(B)$ whenever $A \subseteq B$), a measurable function $f\colon \Omega \to [0,\infty]$ and an automorphism $\varphi\colon [0,\infty] \to [0,\infty]$, the formula

$$\varphi^{-1}\left(Ch_{\varphi \circ m}(\varphi \circ f)\right) = \varphi^{-1}\left(\int_0^{\infty} \varphi\left(m(\{f \ge \varphi^{-1}(t)\})\right) \mathrm{d}t\right) \tag{5}$$

defines a φ-generated Choquet-like integral introduced in [5]. If we denote this integral by Ch_m^{φ}, then the functional I given by (4) satisfies $I = Ch_m^{\varphi}$. Thus, Theorem 2.4 provides an axiomatic characterization of Choquet-like integrals related to a normed automorphism, i. e., such an automorphism φ, for which it holds that

$$Ch_m^{\varphi}(1_A) = v(A), \quad \text{for any capacity } v\colon \mathcal{A} \to [0,1].$$

3. LEVEL-DEPENDENT CAPACITIES AND THE CHOQUET INTEGRAL

Let $X = \{c_1, \ldots, c_n\}$ be a set of criteria. A capacity $v\colon \mathcal{A} \to [0,1]$ can be regarded as a weighting function assigning a weight to a group of criteria $A \in \mathcal{A}$. The idea of such weight being dependent on the level of criteria satisfaction degrees to be aggregated led Greco et al. [4] to the introduction of level-dependent capacities.

Definition 3.1. A mapping $M\colon [0,1] \times \mathcal{A}$ is called a level-dependent capacity whenever $M(t,\cdot)\colon \mathcal{A} \to [0,1]$ is a capacity for each $t \in [0,1]$.

It is obvious that a level-dependent capacity M can be written in the form $M = (m_t)_{t \in [0,1]}$, i. e., as a system of capacities m_t, $t \in [0,1]$. Inspired by Vitali's idea to assign to any capacity v on $(\Omega, \mathcal{A})$ and any function $f \in \mathcal{F}_{\mathcal{A}}$ the function $h_{v,f}$, one can introduce the function $h_{M,f}\colon [0,1] \to [0,1]$ as follows:

$$h_{M,f}(t) = m_t(\{f \ge t\}). \tag{6}$$

Note that the mentioned function $h_{v,f}$ is decreasing and thus Riemann integrable for any capacity v and any $f \in \mathcal{F}_{\mathcal{A}}$, but this need not be true in the case of $h_{M,f}$ (neither monotonicity nor Riemann integrability is guaranteed). A function $f \in \mathcal{F}_{\mathcal{A}}$, such that for a given level-dependent capacity M the function $h_{M,f}$ is Lebesgue integrable, is called an M-integrable function.

Definition 3.2. (Greco et al. [4]) Let M be a fixed level-dependent capacity and let $f \in \mathcal{F}_{\mathcal{A}}$ be M-integrable. Then the Choquet integral of f with respect to M (with the notation $Ch_M(f)$) is defined by

$$Ch_M(f) = \int_0^1 h_{M,f}(t)\,\mathrm{d}t = \int_0^1 m_t(\{f \geq t\})\,\mathrm{d}t, \tag{7}$$

where the Lebesgue integral with respect to the standard Lebesgue measure on $[0,1]$ is applied.

To ensure the M-integrability of every $f \in \mathcal{F}_{\mathcal{A}}$, we introduce the notion of piece-wise constant level-dependent capacities.

Definition 3.3. For a fixed $k \in \mathbb{N}$, let $0 = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1$ and let, for $i = 1, \ldots, k, v_1, \ldots, v_k \colon \mathcal{A} \to [0,1]$ be capacities. Put $M = (m_t)_{t \in [0,1]}$, where

$$m_t = v_i \text{ if } a_{i-1} \leq t < a_i, \text{ and } m_1 = v_k.$$

Then M is called a piece-wise constant level-dependent capacity.

Proposition 3.4. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $\mathcal{A} = 2^{\Omega}$. Let M be a piece-wise constant level-dependent capacity as is described in Definition 3.3. Then each $f \in \mathcal{F}_{\mathcal{A}}$ is M-integrable.

Proof. The result follows from the fact that the function $h_{M,f}$ is piece-wise constant. □

Remark 3.5. Note that in general, the finitness of Ω does not guarantee the M-integrability of each $f \in \mathcal{F}_{\mathcal{A}}$. Consider $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $f(\omega_i) = (i-1)/2$ and let, for a Borel non-measurable set $E \subset [0,1], 0 \notin E$,

$$m_t(A) = \begin{cases} 0 & \text{for each } t \in E, A \neq \Omega, \\ 1 & \text{for each } t \notin E, A \neq \emptyset. \end{cases}$$

Then

$$h_{M,f}(t) = \begin{cases} 0 & \text{if } t \in E, \\ 1 & \text{if } t \notin E, \end{cases}$$

is not Borel measurable and thus not Lebesgue integrable.

4. CHOQUET-LIKE INTEGRALS WITH RESPECT TO LEVEL-DEPENDENT CAPACITIES

In what follows, we merge the concepts discussed in Sections 2 and 3.

Definition 4.1. Let M be a level-dependent capacity, $f \in \mathcal{F}_{\mathcal{A}}$, and let φ be a normed automorphism. Let the function $h^{\varphi}_{M,f} \colon [0,1] \to [0,1]$ given by $h^{\varphi}_{M,f}(t) = \varphi\left(m_t(\{f \geq \varphi^{-1}(t)\})\right)$ be Lebesgue integrable. Then f is called $\varphi - M$-integrable and the value

$$Ch^{\varphi}_M(f) = \varphi^{-1}\left(Ch_{\varphi \circ M}(\varphi \circ f)\right)$$

is called a φ-based Choquet-like integral of f with respect to M.

The next result is obvious.

Corollary 4.2. Let Ω be a finite space, $\mathcal{A} = 2^{\Omega}$, and let M be a piece-wise constant level-dependent capacity. Then, for any normed automorphism φ, any function $f \in \mathcal{F}_{\mathcal{A}}$ is $\varphi - M$-integrable.

Example 4.3. Let $\Omega = \{\omega_1, \omega_2\}$. Then each $f \in \mathcal{F}_{\mathcal{A}}$ can be identified with a couple $(x, y) \in [0,1]^2$, $x = f(\omega_1)$, $y = f(\omega_2)$. Define two capacities $v_1, v_2 \colon 2^{\Omega} \to [0,1]$, by

$$\begin{aligned} v_1(\{\omega_1\}) &= a, \quad v_2(\{\omega_1\}) = c, \\ v_1(\{\omega_2\}) &= b, \quad v_2(\{\omega_2\}) = d, \end{aligned}$$

with $a, b, c, d \in [0,1]$ (and obviously, $v_i(\emptyset) = 0$ and $v_i(\Omega) = 1$), and also define a piece-wise constant level-dependent capacity $M = (m_t)_{t \in [0,1]}$, where, for $\alpha \in]0,1[$,

$$m_t = \begin{cases} v_1 & \text{if } t \leq \alpha, \\ v_2 & \text{if } t > \alpha. \end{cases}$$

For an arbitrary normed automorphism φ (i. e., $\varphi|_{[0,1]}$ is an automorphism of $[0,1]$), consider $\varphi(x) \leq \alpha < \varphi(y)$. Then

$$h^{\varphi}_{M,(x,y)}(t) = \begin{cases} 1 & \text{if } t \leq \varphi(x), \\ \varphi(b) & \text{if } \varphi(x) < t \leq \alpha, \\ \varphi(d) & \text{if } \alpha < t \leq \varphi(y), \\ 0 & \text{else,} \end{cases}$$

and

$$\begin{aligned} Ch^{\varphi}_{M}((x,y)) &= \varphi^{-1}\left(\varphi(x) + (\alpha - \varphi(x))\varphi(b) + (\varphi(y) - \alpha)\varphi(d)\right) \\ &= \varphi^{-1}\left(\varphi(x)(1 - \varphi(b)) + \varphi(y)\varphi(d) + \alpha(\varphi(b) - \varphi(d)\right). \end{aligned}$$

It is not difficult to check that if $(x, y) \in [0, \varphi^{-1}(a)]^2$, then

$$Ch^{\varphi}_{M}((x,y)) = Ch^{\varphi}_{v_1}((x,y)),$$

while if $(x, y) \in [\varphi^{-1}(a), 1]^2$, then

$$Ch^{\varphi}_{M}((x,y)) = Ch^{\varphi}_{v_2}((x,y)).$$

5. φ-ORDINAL SUMS OF AGGREGATION FUNCTIONS

Ordinal sums are well known for t-norms, copulas, semicopulas (the same formula based on Min), as well as for t-conorms (a dual formula based on Max). In order to unify all these formulae in a unique general formula, in [6], the concept of φ-ordinal sums of aggregation functions was introduced. Before recalling this notion, we still note that an (n-ary) aggregation function $A \colon [a,b]^n \to [a,b]$ is defined as an increasing function in each coordinate, which satisfies the properties $A(a, \ldots, a) = a$ and $A(b, \ldots, b) = b$.

Definition 5.1. For $n, k \in \mathbb{N}$, let $0 = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1$, and let $A_i \colon [a_{i-1}, a_i]^n \to [a_{i-1}, a_i]$ be given aggregation functions. Let $\varphi \colon [0,1] \to [0,1]$ be an automorphism. Then the function $A \colon [0,1]^n \to [0,1]$, denoted by $\varphi - (\langle a_{i-1}, a_i, A_i \rangle, i \in \{1, \ldots, k\})$ and given by

$$A(x_1, \ldots, x_n) = \varphi^{-1}\left(\sum_{i=1}^{k} \left(\varphi\left(A_i(/x_1/_i, \ldots, /x_n/_i)\right) - \varphi(a_{i-1})\right)\right),$$

with $/x/_i = \min\{a_i, \max\{a_{i-1}, x\}\}$ for every $i \in \{1, \ldots, k\}$ and every $x \in [0,1]$, is called a φ-ordinal sum (of summands $\langle a_{i-1}, a_i, A_i \rangle$, $i \in \{1, \ldots, k\}$).

Note that if $(x_1, \ldots, x_n) \in [a_{i-1}, a_i]^n$, then $A(x_1, \ldots, x_n) = A_i(x_1, \ldots, x_n)$, and thus A is an extension of aggregation functions A_i acting on subdomains $[a_{i-1}, a_i]^n$ to the full domain $[0,1]^n$. Note that φ-ordinal sums preserve continuity and symmetry of the $A_i's$. Moreover, if all aggregation functions A_i are t-norms (copulas, semicopulas, t-conorms), then for an arbitrary automorphism φ of $[0,1]$ the corresponding φ-ordinal sum is also a t-norm (copula, semicopula, t-conorm) coinciding with the above mentioned ordinal sum of t-norms (copulas, semicopulas, t-conorms).

6. CHOQUET-LIKE INTEGRALS AND φ-ORDINAL SUMS

For a fixed finite space $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $\mathcal{A} = 2^\Omega$, the Choquet integral as well as Choquet-like integrals with respect to a fixed capacity v can be seen as n-ary aggregation functions on $[0,1]$. Note that they are idempotent, i.e., for a constant function $f = c$, $c \in [0,1]$, $Ch_v(c) = Ch_v^\varphi(c) = c$ for any normed automorphism φ. However, this means that for any subinterval $[a_{i-1}, a_i] \subseteq [0,1]$, $Ch_v|_{[a_{i-1}, a_i]}$ and $Ch_v^\varphi|_{[a_{i-1}, a_i]}$ are also (idempotent) n-ary aggregation functions on $[a_{i-1}, a_i]$. When these integrals are considered with respect to a piece-wise constant level-dependent capacity M, then the following representation by means of φ-ordinal sums holds. Let us still note that $Ch_v = Ch_v^{id}$, where $id(x) = x$, $x \in [0, \infty]$.

Theorem 6.1. Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $\mathcal{A} = 2^\Omega$. For $k \in \mathbb{N}$, let $0 = a_0 < a_1 < \ldots < a_{k-1} < a_k = 1$, and let $M = (m_t)_{t \in [0,1]}$ be a piece-wise constant level-dependent capacity with $m_t = v_i$ whenever $a_{i-1} \le t < a_i$. Let $\varphi \colon [0,\infty] \to [0,\infty]$ be a normed automorphism. By abuse of notation we use the same letter φ for $\varphi|_{[0,1]}$. Let $A \colon [0,1]^n \to [0,1]$ be an aggregation function. Then the following are equivalent.

(i) $A = Ch_M^\varphi$.

(ii) $A = \varphi - \left(\langle \varphi^{-1}(a_{i-1}), \varphi^{-1}(a_i), Ch_{v_i}^\varphi \rangle, i \in \{1, \ldots, k\}\right)$.

Proof. It is not difficult to check that it is enough to prove the equivalence (i) $\Leftrightarrow$ (ii) for one fixed normed automorphism only, in particular, for $\varphi = id$. Note that then $Ch_M = Ch_M^{id}$. It is enough to define φ-ordinal sums for $k = 2$ only, and then, the general case can be obtained by induction. Thus, it is enough to prove the result for $k = 2$.

For a finite space $\Omega = \{\omega_1, \ldots, \omega_n\}$, consider two capacities $v_1, v_2 \colon 2^\Omega \to [0,1]$ and a treshold value $\alpha \in]0,1[$. Let $M = (m_t)_{t\in[0,1]}$ be given by

$$m_t = \begin{cases} v_1 & \text{if } t \le \alpha, \\ v_2 & \text{if } t > \alpha. \end{cases}$$

Each $f \in \mathcal{F}_{\mathcal{A}}$ can be represented in the form of an n-dimensional vector $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$, $x_i = f(\omega_i)$. Let $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation such that $x_{\sigma(1)} \le x_{\sigma(2)} \le \ldots \le x_{\sigma(n)}$, and let $x_{\sigma(j-1)} \le \alpha \le x_{\sigma(j)}$. Then

$$h_{M,f}(t) = \begin{cases} v_1(\{\sigma(i), \ldots, \sigma(n)\}) & \text{if } \ i < j, \ t \in]x_{\sigma(i-1)}, x_{\sigma(i)}], \\ v_1(\{\sigma(j), \ldots, \sigma(n)\}) & \text{if } \ t \in]x_{\sigma(j-1)}, \alpha], \\ v_2(\{\sigma(j), \ldots, \sigma(n)\}) & \text{if } \ t \in]\alpha, x_{\sigma(j)}], \\ v_2(\{\sigma(i), \ldots, \sigma(n)\}) & \text{if } \ i > j, \ t \in]x_{\sigma(i-1)}, x_{\sigma(i)}], \end{cases}$$

and for $Ch_M(f)$ we have

$$\begin{aligned} Ch_M(f) &= \sum_{i=1}^{j-1} \left(x_{\sigma(i)} - x_{\sigma(i-1)}\right) v_1(\{\sigma(i), \ldots, \sigma(n)\}) \\ + \quad & \left(\alpha - x_{\sigma(j-1)}\right) v_1(\{\sigma(j), \ldots, \sigma(n)\}) + \left(x_{\sigma(j)} - \alpha\right) v_2(\{\sigma(j), \ldots, \sigma(n)\}) \\ + \quad & \sum_{i=j+1}^{n} \left(x_{\sigma(i)} - x_{\sigma(i-1)}\right) v_2(\{\sigma(i), \ldots, \sigma(n)\}). \end{aligned}$$

On the other hand, the *id*-ordinal sum is given by:

$$\begin{aligned} & id - (\langle 0, \alpha, Ch_{v_1}\rangle, \langle \alpha, 1, Ch_{v_2}\rangle)\,(f) = Ch_{v_1}(f \wedge \alpha) + Ch_{v_2}(f \vee \alpha) - \alpha \\ = & \left(\sum_{i=1}^{j-1} \left(x_{\sigma(i)} - x_{\sigma(i-1)}\right) v_1(\{\sigma(i), \ldots, \sigma(n)\}) + \left(\alpha - x_{\sigma(j-1)}\right) v_1(\{\sigma(j), \ldots, \sigma(n)\})\right) \\ + & \left(\alpha + \left(x_{\sigma(j)} - \alpha\right) v_2(\{\sigma(j), \ldots, \sigma(n)\}) + \sum_{i=j+1}^{n} \left(x_{\sigma(i)} - x_{\sigma(i-1)}\right) v_2(\{\sigma(i), \ldots, \sigma(n)\})\right) - \alpha. \end{aligned}$$

Hence both formulae coincide, i. e.,

$$Ch_M(f) = id - (\langle 0, \alpha, Ch_{v_1}\rangle, \langle \alpha, 1, Ch_{v_2}\rangle)\,(f),$$

which proves the theorem. □

Recall that if a capacity v is additive, i. e., v is a discrete probability measure, then the Choquet integral on $\Omega = \{\omega_1, \ldots, \omega_n\}$ is just the weighted arithmetic mean, $Ch_v = W_w$, where $W_w(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$ with $w_i = v(\{\omega_i\})$. Then, if a piece-wise constant level dependent capacity M is linked to additive capacities $v_1, \ldots v_k$, the corresponding Choquet integral Ch_M can be seen as an ordinal sum of weighted arithmetic means $W_1, \ldots, W_k$. A similar consideration can be applied to Choquet-like integrals Ch_M^{φ}, φ

being a normalized automorphism and $v_1, \dots v_k$ being $\oplus_\varphi$-additive. Observe that then each integral $Ch^\varphi_{v_j}$ is a weighted quasi-arithmetic mean,

$$Ch^\varphi_{v_j}(x_1, \dots, x_n) = \varphi^{-1}\left(\sum_{i=1}^{n} \varphi\left(w_i^{(j)}\right)\varphi(x_i)\right),$$

where $w_i^{(j)} = v_j(\{\omega_i\})$.

Example 6.2. Consider $\Omega = \{1, 2\}$, $f\colon \Omega \to [0, 1]$ such that $f(1) = x$, $f(2) = y$, and define capacities $v_1, v_2, v_3 \colon 2^\Omega \to [0, 1]$ as follows:
$v_1(\emptyset) = 0$, $v_2(\emptyset) = 0$, $v_3(\emptyset) = 0$,
$v_1(\{1\}) = 0.3$, $v_2(\{1\}) = 0.5$, $v_3(\{1\}) = 0.7$,
$v_1(\{2\}) = 0.7$, $v_2(\{2\}) = 0.5$, $v_3(\{2\}) = 0.3$,
$v_1(\Omega) = 1$, $v_2(\Omega) = 1$, $v_3(\Omega) = 1$.
Define the system $M = (m_t)_{t\in[0,1]}$ of capacities m_t by

$$m_t = \begin{cases} v_1 & \text{if } t \in [0, \frac{1}{3}], \\ v_2 & \text{if } t \in]\frac{1}{3}, \frac{2}{3}], \\ v_3 & \text{if } t \in]\frac{2}{3}, 1]. \end{cases} \tag{8}$$

Consider an aggregation function A known on subintervals depending on M and the related probability measures v_i as follows:

$$A(x, y) = \begin{cases} 0.3x + 0.7y & \text{if } (x, y) \in [0, 1/3]^2, \\ 0.5x + 0.5y & \text{if } (x, y) \in]1/3, 2/3]^2, \\ 0.7x + 0.3y & \text{if } (x, y) \in]2/3, 1]^2. \end{cases} \tag{9}$$

The task is to extend A to the whole domain $[0, 1]^2$. It can be made by means of the formula (7), i. e.,

$$A(x, y) = Ch_M(f) = \int_0^1 h_{M,f}(t)\,\mathrm{d}t.$$

The related function $h_{M,f}$ is piece-wise constant but not monotone, in general.

For example, if $(x, y) \in [\frac{2}{3}, 1] \times [0, \frac{1}{3}]$ there are 5 possible values for $h_{M,f}(t)$:

1. $t \le y \Rightarrow x > y \ge t \Rightarrow f(1) > t, f(2) \ge t \Rightarrow m_t(\{f \ge t\}) = m_t(\{1, 2\}) = 1$,
2. $y < t \le \frac{1}{3} \Rightarrow m_t(\{f \ge t\}) = m_t(\{1\}) = v_1(\{1\}) = 0.3$,
3. $y < \frac{1}{3} \le t < \frac{2}{3} < x \Rightarrow m_t(\{f \ge t\}) = m_t(\{1\}) = v_2(\{1\}) = 0.5$,
4. $\frac{2}{3} < t \le x \Rightarrow m_t(\{f \ge t\}) = m_t(\{1\}) = v_3(\{1\}) = 0.7$,
5. $x < t \Rightarrow m_t(\{f \ge t\}) = m_t(\{\emptyset\}) = 0$.

Thus for $0 \le y \le \frac{1}{3}$ and $\frac{2}{3} < x \le 1$ we have

$$h_{M,f}(t) = \begin{cases} 1 & \text{if } t \le y, \\ 0.3 & \text{if } y < t \le \frac{1}{3}, \\ 0.5 & \text{if } t < \frac{2}{3} \le x, \\ 0.7 & \text{if } \frac{2}{3} < t \le x, \\ 0 & \text{if } x < t. \end{cases} \tag{10}$$

In this case the Choquet integral $Ch_M(f)$ is

$$Ch_M(f) = y \cdot 1 + \left(\frac{1}{3} - y\right) \cdot 0.3 + \left(\frac{2}{3} - \frac{1}{3}\right) \cdot 0.5 + \left(x - \frac{2}{3}\right) \cdot 0.7 = 0.7x + 0.7y - 0.2,$$

which gives the corresponding values $A(x, y)$.

The results obtained by this approach for all remaining subdomains are in Table 1.

$A(x,y)$	$x \in [0, \frac{1}{3}]$	$x \in [\frac{1}{3}, \frac{2}{3}]$	$x \in [\frac{2}{3}, 1]$
$y \in [\frac{2}{3}, 1]$	$0.3x + 0.3y + 0.2$	$0.5x + 0.3y + \frac{0.4}{3}$	$0.7x + 0.3y$
$y \in [\frac{1}{3}, \frac{2}{3}]$	$0.3x + 0.5y + \frac{0.2}{3}$	$0.5x + 0.5y$	$0.7x + 0.5y - \frac{0.4}{3}$
$y \in [0, \frac{1}{3}]$	$0.3x + 0.7y$	$0.5x + 0.7y - \frac{0.2}{3}$	$0.7x + 0.7y - 0.2$

Tab. 1. Results of Example 6.2.

Observe, that the obtained aggregation function $A\colon [0,1]^2 \to [0,1]$ described in Table 1 is continuous, idempotent and piece-wise linear on $[0,1]^2$.

Example 6.3. Consider $\Omega = \{1,2\}$, $f\colon \Omega \to [0,1]$, where $f(1) = x$, $f(2) = y$, and for $i \in \{1,2\}$ define capacities $v_i\colon 2^\Omega \to [0,1]$ as follows:
$v_1(\{1\}) = 0.2$, $v_1(\{2\}) = 0.4$,
$v_2(\{1\}) = 0.6$, $v_2(\{2\}) = 0.3$,
$v_i(\emptyset) = (0)$, $v_i(\Omega) = 1, i = 1,2$.
Both v_1 and v_2 are nonadditive capacities. Define $M = (m_t)_{t\in[0,1]}$ by

$$m_t = \begin{cases} v_1 & \text{if } t \le 1/2, \\ v_2 & \text{otherwise.} \end{cases} \tag{11}$$

Then M is a level dependent capacity. In this case, if $x, y \in [0, 1/2]$ (or if $x, y \in]1/2, 1]$), we have to distinguish the cases $x \le y$ and $y < x$. Then the resulting aggregation function A is the Choquet integral with respect to v_1 (v_2). Extension of these Choquet integrals to full domain $[0,1]^2$ can be computed by formula (7) and the obtained results are in Table 2.

$A(x,y)$	$x \in [0,1/2]$	$x \in [1/2,1]$
$y \in [1/2,1]$	$0.6x + 0.3y + 0.05$	$0.7x + 0.3y$ if $x < y$ $0.6x + 0.4y$ if $y \le x$
$y \in [0,1/2]$	$0.6x + 0.4y$ if $x < y$ $0.2x + 0.8y$ if $y \le x$	$0.6x + 0.8y - 0.2$

Tab. 2. Results of Example 6.3.

Observe, that aggregation function $A\colon [0,1]^2 \to [0,1]$ described in Table 2 is again continuous, idempotent and piece-wise linear on $[0,1]^2$.

7. CONCLUDING REMARKS

We have discussed Choquet-like integrals with respect to (piece-wise constant) level-dependent capacities and shown their relation to φ-ordinal sums of aggregation functions. We expect applications of our results in several decision problems, especially when a different approach to evaluating the utility (aggregation of score vector) is expected, when only low (middle, high) values are to be aggregated. Note also that for a capacity v, the dual capacity v^d is given by $v^d(A) = 1 - v(A^c)$. Similarly, we can introduce a dual M^d to a level-dependent capacity M by $M^d(t,A) = 1 - M(1-t, A^c)$. Note that if $M = (m_t)_{t\in[0,1]}$, then $M^d = \left(m^d_{1-t}\right)_{t\in[0,1]}$. If the Choquet integral Ch_v is considered as an aggregation function, $Ch_v\colon [0,1]^n \to [0,1]$, its dual is given by $Ch_v^d(\mathbf{x}) = 1 - Ch_v(\mathbf{1}-\mathbf{x})$. Then $Ch_v^d = Ch_{v^d}$, see [3]. It can be shown that a similar claim is valid for the level-dependent capacities-based Choquet integral, i. e., $Ch_M^d = Ch_{M^d}$.

To illustrate the above mentioned facts consider the extremal capacities $v_*, v^*\colon \mathcal{A} \to [0,1]$, $v_*(A) = 0$ for all $A \neq \Omega$ and $v^*(A) = 1$ for all $A \neq \emptyset$. Then $v_*^d = v^*$. For a fixed $\alpha \in]0,1[$, let $M_\alpha = (m_t)_{t\in[0,1]}$ be given by

$$m_t = \begin{cases} v^* & \text{if } t \le \alpha, \\ v_* & \text{if } t > \alpha. \end{cases}$$

Then, representing $f \in \mathcal{F}_\mathcal{A}$ in the form $\mathbf{x} = (x_1, \ldots, x_n)$, we have

$$Ch_{M_\alpha}(f) = \operatorname{med}\left(\min\{x_1,\ldots,x_n\}, \alpha, \max\{x_1,\ldots,x_n\}\right) = \begin{cases} Ch_{v^*}(f) & \text{if } f \le \alpha, \\ Ch_{v_*}(f) & \text{if } f \ge \alpha, \\ \alpha & \text{else.} \end{cases}$$

The corresponding dual $M_\alpha^d = (\mu_t)_{t\in[0,1]}$ is given by

$$\mu_t = \begin{cases} v^* & \text{if } t < 1-\alpha, \\ v_* & \text{if } t \ge 1-\alpha. \end{cases}$$

Then

$$Ch^d_{M_\alpha} = Ch_{M^d_\alpha} = Ch_{M_{1-\alpha}}.$$

On the other hand, if $M_{(\alpha)} = (m_t)_{t\in[0,1]}$ is given by

$$m_t = \begin{cases} v_* & \text{if } t \le \alpha, \\ v^* & \text{if } t > \alpha, \end{cases}$$

it holds that

$$Ch_{M_{(\alpha)}}(f) = \begin{cases} \min\{x_1, \dots, x_n\} & \text{if } f \le \alpha, \\ \max\{x_1, \dots, x_n\} & \text{if } f \ge \alpha, \\ \min\{x_1, \dots, x_n\} + \max\{x_1, \dots, x_n\} - \alpha & \text{else.} \end{cases}$$

In this case it also holds that $Ch^d_{M_{(\alpha)}} = Ch_{M^d_{(\alpha)}} = Ch_{M_{(1-\alpha)}}$.

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