KYBERNETIKA — VOLUME *55* (2019), NUMBER 2, PAGES 422–434

RANDOM NOISE AND PERTURBATION OF COPULAS

Radko Mesiar, Ayyub Sheikhi and Magda Komorníková

For a random vector (X, Y) characterized by a copula $C_{X,Y}$ we study its perturbation $C_{X+Z,Y}$ characterizing the random vector $(X + Z, Y)$ affected by a noise Z independent of both X and Y. Several examples are added, including a new comprehensive parametric copula family $(\mathcal{C}_k)_{k\in[-\infty,\infty]}$.

Keywords: copula, noise, perturbation of copula, random vector

Classification: 60E05, 62H20

1. INTRODUCTION AND PRELIMINARIES

Observation of random variables is often polluted by a random noise. On the other hand, the stochastic dependence of a couple of random variables is described by means of copulas, e. g., Sklar (1959) [6], Joe (1997) [4], Nelsen (2006) [5] and Durante & Sempi (2016) [2].

The main aim of this contribution is the study of perturbation of a copula $C_{X,Y}$ related to a random vector (X, Y) in the case when the first coordinate X is polluted by some noise Z, i. e., we are interested in copula $C_{X+Z,Y}$ when Z is random variable independent of both X and Y.

Recall that due to Sklar theorem [6] for any random vector (X, Y) such that

$$F_{X,Y} : \mathbb{R}^2 \to [0, 1]$$

is its joint distribution function and

$$F_X,\ F_Y : \mathbb{R} \to [0, 1]$$

are distribution functions of random variables X, Y respectively, there exists a copula

$$C : [0, 1]^2 \to [0, 1]$$

such that

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)). \tag{1}$$

Copula C is unique if both X and Y are continuous random variables.

DOI: 10.14736/kyb-2019-2-0422

We denote the copula coupling X and Y as $C_{X,Y}$. Note also that we have an axiomatic characterization of copulas, see [2, 4, 5]:

A function $C : [0,1]^2 \to [0,1]$ is a copula if and only if it satisfies the next three axioms:

C1 $C(x,0) = C(0,x) = 0$ for each $x \in [0,1]$ (C is grounded);

C2 $C(x,1) = C(1,x) = x$ for each $x \in [0,1]$ (C has a neutral element $e = 1$);

C3 for all $x_1, x_2, y_1, y_2 \in [0,1]$, $x_1 \le x_2$, $y_1 \le y_2$ it holds $C(x_1,y_1) + C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) \ge 0$ (C is 2–increasing).

A similar axiomatization describe n–ary copulas characterizing the stochastic dependence structure of n–ary random vectors $(X_1, ..., X_n)$. In this paper, we consider only binary copulas, and for readers interested in n–ary copulas, $n \ge 2$ we recommend books [2, 4, 5].

There are many copula families available in the literature, in which, the upper Fréchet-Hoeffding, the lower Fréchet-Hoeffding and the product copula are known as the three basic copulas and denoted by M, $M(u,v) = \min(u,v)$, W, $W(u,v) = \max(0,\ u+v-1)$ and Π, $\Pi(u,v) = uv$, respectively. It is well known that for any copula C it holds

$$W \le C \le M.$$

More, it is known that when X and Y are linked by a copula $C_{X,Y}$, the distribution function of $X+Y$ of their sum is given by

$$F_{X+Y}(s) = \int\int_{x+y\le s} \mathrm{d}H_{X,Y}(x,y) \quad s \in \mathbb{R}. \tag{2}$$

Where $H_{X,Y}(x,y) = C_{X,Y}(F_X(x), F_Y(y))$ and $\mathrm{d}H_{X,Y}(x,y)$ is its derivative. When $C = \Pi$, the formula (2) is referred as convolution of two independent random variables X and Y. Williamson and Downs (1990) [8] discussed the distribution of some arithmetic convolution function of X, Y where their marginal distributions is fixed. One may refer to Cherubini et al. (2016) [1], for more details of distributions of convolution of variables.

Also, the distribution of linear combination of two (or more) random variables in the case when their marginal distributions as well as their copula are available, has been investigated in the literature. Durante and Sempi (2016) [2] have obtained the distribution of $X+Y$ as well as the copula of $X+Y$ and Y in this situation. Gijbels and Herrmann (2014) [3] have found the distribution of sums of random variables with copula-induced dependence. See also Williamson and Downs (1990) [8], Wang (2014) [7] and Durante and Sempi (2016) [2].

As already mentioned, the main aim of this paper is related to the description of copula $C_{X+Z,Y}$ which can be seen as a perturbation of the copula $C_{X,Y}$. Here we suppose the independence of Z on X and Y, and the main results are given in the next section. In Section 3, we study a particular case when X and Y are uniformly distributed on $[0,1]$, and Z_ε is uniformly distributed on $[0,\varepsilon]$, where $\varepsilon \in]0,\infty[$ is a positive parameter charactering the considered noise. Our approach allows to construct a new parametric comprehensive family of copulas [5], i. e., family containing all three basic copulas M, W and Π. Finally, some concluding remarks are added.

2. PERTURBATION OF COPULAS UNDER NOISE

Consider two random variables X and Y linked by a copula $C_{X,Y}$. Let Z be a random variable Z, independent of both X and Y. Evidently, the knowledge of $C_{X,Y}$ and of distribution function F_Z is not sufficient to determination of the copula $C_{X+Z,Y}$. This fact follows from the well known facts that $C_{X,Y} = C_{\varphi(X),\eta(Y)}$ for any transformations $\varphi, \eta : \mathbb{R} \to \mathbb{R}$ which are strictly increasing on *Ran* X and *Ran* Y, respectively. Therefore, to describe the perturbed copula $C_{X+Z,Y}$, we need to know the original copula $C_{X,Y}$ and distribution functions F_X, F_Y and F_Z (possible modifications of values of F_Y which do not influence the form of copula $C_{X,Y}$ do not play a role in the next considerations). More, we will consider in the rest of the paper continuous random variables X, Y and Z. This is caused not only by the fact that then the copulas $C_{X,Y}$ and $C_{X+Z,Y}$ are unique, but also by the formula inverse to (1) valid in this case, namely that then

$$C_{X,Y}(u,v) = F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)), \tag{3}$$

where F_X^{-1}, F_Y^{-1} $:]0,1[\to \mathbb{R}$ are the related quantile functions for some bijective functions F_X and F_Y respectively (for more details see [2, 4, 5]).

Theorem 1. If $C_{X,Y}$ is the copula function of X and Y and the random variable Z is independent of X and Y, then the copula function $C_{X+Z,Y}$ is given by

$$C_{X+Z,Y}(u,v) = \int\int_0^v D_2C_{X,Y}\left(F_X(F_{X+Z}^{-1}(u) - z), r\right) \,\mathrm{d}r\mathrm{d}F_Z(z) \tag{4}$$

where $D_2C_{X,Y}(u,v) = \frac{\partial F_{X,Y}(u,v)}{\partial v}$ if this derivative exists, and $D_2C_{X,Y}(u,v) = 0$ otherwise. More, the bijective function F_{X+Z} is the standard convolution of F_X and F_Z.

Proof. Using the theorem of total probability we have

$$\begin{aligned}
F_{X+Z,Y}(s,t) &= P(X+Z<s, Y<t) = \int P(X+Z<s, Y<t|Z=z)\,\mathrm{d}F_Z(z) \\
&= \int P(X+z<s, Y<t|Z=z)\,\mathrm{d}F_Z(z) = \int P(X<s-z, Y<t)\,\mathrm{d}F_Z(z) \\
&= \int\int_{-\infty}^t P(X<s-z|Y=y)\,\mathrm{d}F_Y(y)\,\mathrm{d}F_Z(z) \\
&= \int\int_{-\infty}^t D_2C_{X,Y}(F_X(s-z), F_Y(y))\,\mathrm{d}F_Y(y)\,\mathrm{d}F_Z(z).
\end{aligned}$$

Defining $r = F_Y(y)$, it becomes

$$F_{X+Z,Y}(s,t) = \int\int_0^{F_Y(t)} D_2C_{X,Y}(F_X(s-z), r)\,\mathrm{d}r\,\mathrm{d}F_Z(z).$$

So, the perturbed copula $C_{X+Z,Y}$ is given by

$$C_{X+Z,Y}(u,v) = \int\int_0^v D_2C_{X,Y}\left(F_X(F_{X+Z}^{-1}(u) - z), r\right)\mathrm{d}r\mathrm{d}F_Z(z).$$

□

Example 1. Let X and Y be two random variables uniformly distributed on $[0,1]$ linked with minimum copula, $C_{X,Y} = M$, and let the random variable Z independent of X and Y has uniform distribution on $[0,1]$. Clearly, then $X = Y$, and

$$C_{X+Z,Y}(u,v)) = \begin{cases} u & \text{if } 0 \le u \le \frac{1}{2},\ \sqrt{2u} \le v \\ v\sqrt{2u} - \frac{v^2}{2} & \text{if } 0 \le u \le \frac{1}{2},\ \sqrt{2u} \ge v \\ v - \frac{(\sqrt{2(1-u)}+v-1)^2}{2} & \text{if } \frac{1}{2} \le u \le 1,\ 1 - \sqrt{2(1-u)} \le v \\ v & \text{if } \frac{1}{2} \le u \le 1,\ 1 - \sqrt{2(1-u)} \ge v. \end{cases} \tag{5}$$

Proof. Let $F_{X+Z}(s)$ is the CDF of Irwin-Hall distribution for $n = 2$, Since the support of the joint density of X, Z is $[0,1]^2$ and is finite so $F_{X+Z}(s)$ is a strictly increasing function which yields the bijectivity proprerty of $F_{X+Z}(s)$. By inverting the $F_{X+Z}(s)$ we obtain

$$F^{-1}_{X+Z}(s) = \begin{cases} 0 & \text{if } s \le 0 \\ \sqrt{2s} & \text{if } 0 \le s \le \frac{1}{2} \\ 2 - \sqrt{2(1-s)} & \text{if } \frac{1}{2} \le s \le 1 \\ 2 & \text{if } 1 \le s. \end{cases}$$

Also, since $D_2 C_{X,Y}(u,v) = \frac{\partial M(u,v)}{\partial v} = \mathbf{1}_{(u<v)}$, where $\mathbf{1}_{(u<v)}$ is an indicator function which equals 1 if $u < v$ and zero otherwise, we have

$$D_2 C_{X,Y}\left(F_X(F^{-1}_{X+Y}(u) - z), r\right) = \mathbf{1}_{(r<F^{-1}_{X+Y}(u)-z)}$$

and based on (4) we have the following four parts:

1. if $0 \le u \le \frac{1}{2},\ \sqrt{2u} \le v$
   $C_{X+Z,X}(u,v) = \int_0^{\sqrt{2u}} \int_0^{\sqrt{2u}-z} \mathrm{d}r\mathrm{d}z = u,$

2. if $0 \le u \le \frac{1}{2},\ \sqrt{2u} \ge v$
   $C_{X+Z,X}(u,v) = \int_0^{\sqrt{2u}-v} \int_0^{v} \mathrm{d}r\mathrm{d}z + \int_{\sqrt{2u}-v}^{\sqrt{2u}} \int_0^{\sqrt{2u}-z} \mathrm{d}r\mathrm{d}z = v\sqrt{2u} - \frac{v^2}{2},$

3. for the sake of notational simplicity, define $g(u) = 2 - \sqrt{2(1-u)}$, if $\frac{1}{2} \le u \le 1,\ g(u) - 1 \le v,$

   $C_{X+Z,X}(u,v) = \int_0^{g(u)-v} \int_0^{v} \mathrm{d}r\mathrm{d}z + \int_{g(u)-v}^{1} \int_0^{g(u)-z} \mathrm{d}r\mathrm{d}z = v - \frac{(\sqrt{2(1-u)}+v-1)^2}{2},$

4. if $\frac{1}{2} \le u \le 1,\ g(u) - 1 \ge v$

   $C_{X+Z,X}(u,v) = \int_0^{1} \int_0^{v} \mathrm{d}r\mathrm{d}z = v,$

which completes the proof. □

Figure 1 depicts this copula function.

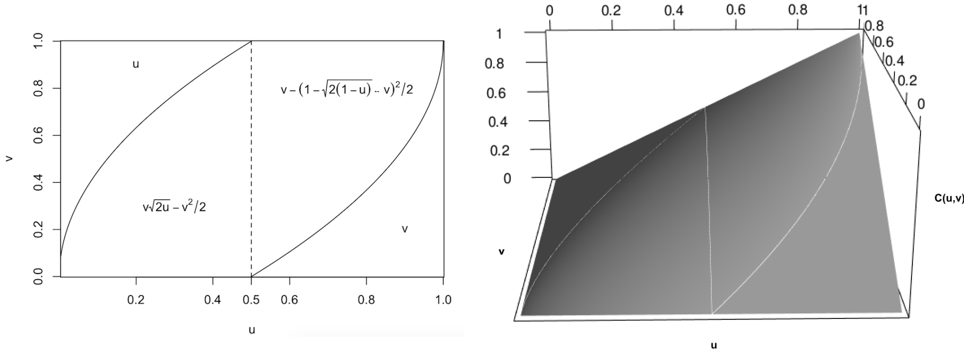

Fig. 1. 2D and 3D view of Copula function $C_{X+Z,X}$.

It should be noticed the L_∞ norm of the difference between $C_{X,X}$ and $C_{X+Z,X}$ can be obtained as follows.

Corollary 1. Under the assumption of Example 1,

$$L_\infty(C_{X,X}, C_{X+Z,X}) = \frac{1}{8}.$$

Proof. We have two cases: $0 \le u \le \frac{1}{2}$ and $\frac{1}{2} \le u \le 1$.
If $0 \le u \le \frac{1}{2}$, using (5) and $C_{X,Y}(u,\ v) = M(u,v)$ we have

$$\begin{aligned} L_\infty(C_{X,X}, C_{X+Z,X}) &= \max|C_{X,X}(u,v) - C_{X+Z,X}(u,v)| \\ &= \begin{cases} \max|u-u| & \text{if } u \le v \ \&\ v \le v\sqrt{2u} - \frac{v^2}{2} \\ \max\left|u - v\sqrt{2u} + \frac{v^2}{2}\right| & \text{if } u \le v \ \&\ v \ge v\sqrt{2u} - \frac{v^2}{2} \\ \max\left|v - v + \frac{(\sqrt{2(1-u)}+v-1)^2}{2}\right| & \text{if } u \ge v \ \&\ v \le v\sqrt{2u} - \frac{v^2}{2} \\ \max|v-v| & \text{if } u \ge v \ \&\ v \ge v\sqrt{2u} - \frac{v^2}{2} \end{cases} \\ &= \begin{cases} 0 & \text{if } u \le v \ \&\ v \le v\sqrt{2u} - \frac{v^2}{2} \\ \max\left|u - \frac{1}{2}\sqrt{2u} + \frac{1}{8}\right| = \frac{1}{8} & \text{if } u \le v \ \&\ v \ge v\sqrt{2u} - \frac{v^2}{2} \\ \max\left|\frac{v^2}{2}\right| = \frac{1}{8} & \text{if } u \ge v \ \&\ v \le v\sqrt{2u} - \frac{v^2}{2} \\ 0 & \text{if } u \ge v \ \&\ v \ge v\sqrt{2u} - \frac{v^2}{2} \end{cases} \end{aligned}$$

and similarly, if $\frac{1}{2} \le u \le 1$, we have four parts; so

$$L_\infty(C_{X,X}, C_{X+Z,X}) = \max|C_{X,X}(u,v) - C_{X+Z,X}(u,v)|$$

$$= \begin{cases} 0 & \text{if } u \le v \ \&\ v \ge v(2-\sqrt{2u}) - \frac{v^2-(1-\sqrt{2(1-u)})^2}{2} \\ \max\left|u - \sqrt{2u} - \frac{1}{2}\right| = \frac{1}{8} & \text{if } u \le v \ \&\ v \le v(2-\sqrt{2u}) - \frac{v^2-(1-\sqrt{2(1-u)})^2}{2} \\ \max\left|v - (v - \frac{v^2}{2})\right| = \frac{1}{8} & \text{if } u \ge v \ \&\ v \le v(2-\sqrt{2u}) - \frac{v^2-(1-\sqrt{2(1-u)})^2}{2} \\ 0 & \text{if } u \ge v \ \&\ v \ge v(2-\sqrt{2u}) - \frac{v^2-(1-\sqrt{2(1-u)})^2}{2} \end{cases}.$$

□

Example 2. Let X and Y be two random variables uniformly distributed on $[0,1]$ with the Fréchet copula W, $C_{X,Y}(u,\ v) = \max(0, u+v-1)$, (i. e., $Y = 1-X$) and random variable Z, independent of X and Y, has uniform distribution on $[0,1]$, then

$$C_{X+Z,Y}(u,v) = \begin{cases} 0 & \text{if } 0 \le u \le \frac{1}{2},\ \sqrt{2u} \le 1-v \\ \frac{(\sqrt{2u}+v-1)^2}{2} & \text{if } 0 \le u \le \frac{1}{2},\ \sqrt{2u} \ge 1-v \\ u+v-1 & \text{if } \frac{1}{2} \le u \le 1,\ \sqrt{2(1-u)} \le v \\ v(1-\sqrt{2(1-u)}) + \frac{v^2}{2} & \text{if } \frac{1}{2} \le u \le 1\ ,\ \sqrt{2(1-u)} \ge v. \end{cases} \tag{6}$$

P r o o f. Similar to Example 1 we have the same $F^{-1}_{X+Z}(s)$ and since $D_2 C_{X,Y}(u,v) = \frac{\partial W(u,v)}{\partial v} = \mathbf{1}_{(u>1-v)}$, then

1. if $0 \le u \le \frac{1}{2},\ \sqrt{2u} \ge 1-v$

$$C_{X+Z,1-X}(u,v) = \int_{1-v}^{\sqrt{2u}} \int_0^{\sqrt{2u}-z} \mathrm{d}r\mathrm{d}z = \frac{(\sqrt{2u}+v-1)^2}{2}$$

2. if $0 \le u \le \frac{1}{2},\ \sqrt{2u} \le 1-v$

$$C_{X+Z,1-X}(u,v) = 0,$$

3. if $\frac{1}{2} \le u \le 1,\ v \ge \sqrt{2(1-u)}$, by defining $g(u) = 2 - \sqrt{2(1-u)}$,

$$C_{X+Z,1-X}(u,v) = v - \int_{g(u)-1}^{1} \int_{g(u)-z}^{1} \mathrm{d}r\mathrm{d}z = v+u-1,$$

4. if $\frac{1}{2} \le u \le 1,\ v \le \sqrt{2(1-u)}$

$$\begin{aligned} C_{X+Z,1-X}(u,v) &= \int_0^{g(u)-1} \int_{1-v}^{1} \mathrm{d}r\mathrm{d}z + \int_{g(u)-1}^{g(u)-v+1} \int_{1-v}^{g(u)-z} \mathrm{d}r\mathrm{d}z \\ &= v(1-\sqrt{2(1-u)}) + \frac{v^2}{2}. \end{aligned}$$

Hence,

$$C_{X+Z,Y}(u,v)) = \begin{cases} 0 & \text{if } 0 \le u \le \frac{1}{2},\ \sqrt{2u} \le 1-v \\ \frac{(\sqrt{2u}+v-1)^2}{2} & \text{if } 0 \le u \le \frac{1}{2},\ \sqrt{2u} \ge 1-v \\ u+v-1 & \text{if } \frac{1}{2} \le u \le 1,\ \sqrt{2(1-u)} \le v \\ v(1-\sqrt{2(1-u)}) + \frac{v^2}{2} & \text{if } \frac{1}{2} \le u \le 1\ ,\ \sqrt{2(1-u)} \ge v. \end{cases}$$

□

Figure 2 presents this copula function and similar to Corollary 1, after a simple computation, we can obtain that

$$L_\infty(C_{X,1-X}, C_{X+Z,1-X}) = \max |C_{X,1-X}(u,v) - C_{X+Z,1-X}(u,v)| = \frac{1}{8}.$$

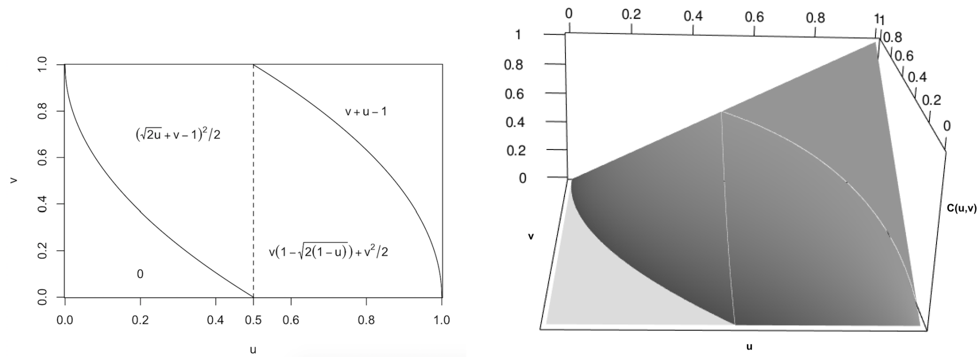

Fig. 2. 2D and 3D view of copula function $C_{X+Z,1-X}$.

The proof of the following example is trivial.

Example 3. If X, Y and Z are independent random variables with uniform distribution on $[0,1]$ then

$$C_{X+Z,Y}(u,v)=uv.$$

Remark 1. (i) Note that Theorem 1 can be applied also in case when Z is not independent of X and Y. Obviously, then the distribution function F_{X+Z} should be computed by means of formula

$$F_{X+Y}(s)=\int\int_{x+y\leq s}\mathrm{d}H_{X,Y}(x,y))\qquad s\in R.$$

In particular, if $Z=Y$, then the copula $C_{X+Y,Y}$ is given by

$$C_{X+Y,Y}(u,v)=\int_0^v D_2C_{X,Y}\left(F_X(F_{X+Y}^{-1}(u)-F_Y^{-1}(r)),r\right)\mathrm{d}r.$$

Recall that this result was obtained as Theorem (3.4.2) already in book [2].

(ii) Observe that copulas $C_{X+Z,X}$ and $C_{X+Z,1-X}$ discussed in Example 1 and Example 2, respectively, are related by the flipping relations,

$$C_{X+Z,1-X}(u,v)=u-C_{X+Z,X}(u,1-v)=v-C_{X+Z,X}(1-u,v).$$

Consequently, both these copulas are survival invariant, i.e.,

$$C_{X+Z,X}(u,v)=1-u-v+C_{X+Z,X}(1-u,1-v)$$

and

$$C_{X+Z,1-X}(u,v)=1-u-v+C_{X+Z,1-X}(1-u,1-v).$$

3. A PARAMETRIC COMPREHENSIVE FAMILY OF COPULAS

In this section, a new parametric comprehensive family of copulas is introduced. It is based on the subsequent Examples 4 and 5.

Example 4. Let X and Y be two random variables with uniform distribution on $[0,1]$ with minimum copula, $C_{X,Y}(u,\ v) = \min(u,v)$, and random variable Z, independent of X and Y, has uniform distribution on $[0,\varepsilon]$, where $\varepsilon \in]0,\infty[$. Then

a) if $\varepsilon \leq 1$,

$$C_{X+Z,Y}(u,v)) = \begin{cases} u & \text{if } 0 \leq u \leq \frac{\varepsilon}{2} \ \& \ \sqrt{2\varepsilon u} \leq v \\ & \quad \text{or } \frac{\varepsilon}{2} \leq u \leq 1-\frac{\varepsilon}{2} \ \& \ u+\frac{\varepsilon}{2} \leq v \leq 1 \\ \frac{1}{\varepsilon}[v\sqrt{2\varepsilon u} - \frac{v^2}{2}] & \text{if } 0 \leq u \leq \frac{\varepsilon}{2} \ \& \ v \leq \sqrt{2\varepsilon u} \\ u - \frac{1}{\varepsilon}\frac{(u+\frac{\varepsilon}{2}-v)^2}{2} & \text{if } \frac{\varepsilon}{2} \leq u \leq 1-\frac{\varepsilon}{2} \ \& \ u-\frac{\varepsilon}{2} \leq v \leq u+\frac{\varepsilon}{2} \\ v - \frac{1}{\varepsilon}\frac{(v-g(u))^2}{2} & \text{if } 1-\frac{\varepsilon}{2} \leq u \leq 1 \ \& \ g(u) \leq v \\ v & \text{if } \frac{\varepsilon}{2} \leq u \leq 1-\frac{\varepsilon}{2} \ \& \ 0 \leq v \leq u-\frac{\varepsilon}{2} \\ & \quad \text{or } 1-\frac{\varepsilon}{2} \leq u \leq 1 \ \& \ v \leq g(u) \end{cases} \tag{7}$$

b) if $\varepsilon > 1$,

$$C_{X+Z,Y}(u,v)) = \begin{cases} u & \text{if } 0 \leq u \leq \frac{1}{2\varepsilon} \ \& \ \sqrt{2\varepsilon u} \leq v \\ \frac{1}{\varepsilon}[v\sqrt{2\varepsilon u} - \frac{v^2}{2}] & \text{if } 0 \leq u \leq \frac{1}{2\varepsilon} \ \& \ v \leq \sqrt{2\varepsilon u} \\ \frac{1}{\varepsilon}[v(\varepsilon u + \frac{1}{2} - v) + \frac{v^2}{2}] & \text{if } \frac{1}{2\varepsilon} \leq u \leq 1-\frac{1}{2\varepsilon} \ \& \ 0 \leq v \leq 1 \\ v - \frac{(v-g(u))^2}{2\varepsilon} & \text{if } 1-\frac{1}{2\varepsilon} \leq u \leq 1 \ \& \ v \geq g(u) \\ v & \text{if } 1-\frac{1}{2\varepsilon} \leq u \leq 1 \ \& \ v \leq g(u) \end{cases} \tag{8}$$

where $g(u) = 1 - \sqrt{2\varepsilon(1-u)}$.

Proof.

a) Similar to Example 1 we have the bijective function

$$F_{X+Z}(s) = \begin{cases} \frac{s^2}{2\varepsilon} & \text{if } 0 \leq s \leq \varepsilon \\ s - \frac{\varepsilon}{2} & \text{if } \varepsilon \leq s \leq 1 \\ 1 - \frac{(1+\varepsilon-s)^2}{2\varepsilon} & \text{if } 1 \leq s \leq \varepsilon+1 \end{cases}$$

and equal zero for $s < 0$ and 1 for $s > \varepsilon + 1$. So, its inverse in the corresponding intervals is

$$F^{-1}_{X+Z}(s) = \begin{cases} \sqrt{2\varepsilon s} & \text{if } 0 \leq s \leq \frac{\varepsilon}{2} \\ s + \frac{\varepsilon}{2} & \text{if } \frac{\varepsilon}{2} \leq s \leq 1-\frac{\varepsilon}{2} \\ \varepsilon + 1 - \sqrt{2\varepsilon(1-s)} & \text{if } 1-\frac{\varepsilon}{2} \leq s \leq 1 \end{cases}$$

and similar to the Example 1, $D_2 C_{X,Y}(u,v) = \mathbf{1}_{(r<F^{-1}_{X+Z}(s)-z)}$. So, we have the following cases:

1. if $0 \le u \le \frac{\varepsilon}{2}$, $\sqrt{2\varepsilon u} \le v$
   $C_{X+Z,X}(u,v) = \int_0^{\sqrt{2\varepsilon u}} \int_0^{\sqrt{2\varepsilon u}-z} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z = u,$
2. if $0 \le u \le \frac{\varepsilon}{2}$, $\sqrt{2\varepsilon u} \ge v$
   $C_{X+Z,X}(u,v) = \int_0^{\sqrt{2\varepsilon u}-v} \int_0^{v} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z + \int_{\sqrt{2\varepsilon u}-v}^{\sqrt{2\varepsilon u}} \int_0^{\sqrt{2\varepsilon u}-z} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z = \frac{1}{\varepsilon}(v\sqrt{2\varepsilon u} - \frac{v^2}{2}),$
3. if $\frac{\varepsilon}{2} \le u \le 1 - \frac{\varepsilon}{2}$ and
   i) if $0 \le v \le u - \frac{\varepsilon}{2}$
   $C_{X+Z,X}(u,v) = \int_0^{\varepsilon} \int_0^{v} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z = v$
   ii) if $u - \frac{\varepsilon}{2} \le v \le u + \frac{\varepsilon}{2}$
   $C_{X+Z,X}(u,v) = \int_0^{u+\frac{\varepsilon}{2}-v} \int_0^{v} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z + \int_{u+\frac{\varepsilon}{2}-v}^{\varepsilon} \int_0^{u+\frac{\varepsilon}{2}-z} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z = u - \frac{1}{\varepsilon}\frac{(u+\frac{\varepsilon}{2}-v)^2}{2}$
   iii) if $u + \frac{\varepsilon}{2} \le v \le 1$
   $C_{X+Z,X}(u,v) = \int_0^{\varepsilon} \int_0^{u-\frac{\varepsilon}{2}} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z + \int_0^{\varepsilon} \int_{u-\frac{\varepsilon}{2}}^{u+\frac{\varepsilon}{2}-z} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z = u$
4. if $1 - \frac{\varepsilon}{2} \le u \le 1, v \le g(u)$,
   $C_{X+Z,X}(u,v) = \int_0^{\varepsilon} \int_0^{v} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z = v$
5. if $1 - \frac{\varepsilon}{2} \le u \le 1, g(u) \le v$
   $C_{X+Z,X}(u,v) = \int_0^{\varepsilon} \int_0^{v} \frac{1}{\varepsilon}\,\mathrm{d}r\mathrm{d}z - \int_{g(u)+\varepsilon-v}^{v} \int_{g(u)+\varepsilon-z}^{v} \frac{1}{\varepsilon}\,\mathrm{d}z\mathrm{d}r = v - \frac{1}{\varepsilon}\frac{(v-g(u))^2}{2}.$

which is (7). Figure 3 presents this copula function for $\varepsilon = \frac{1}{2}$.

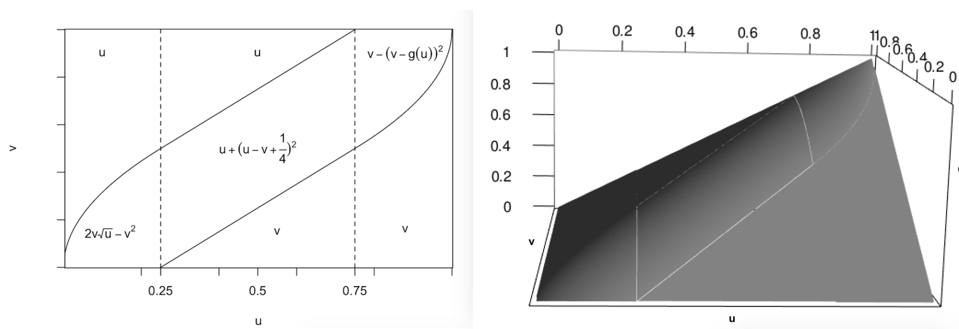

Fig. 3. 2D and 3D view of copula function $C_{X+Z,X}$ from Example 4 for $\varepsilon = \frac{1}{2}$.

b) Again, similar to to Example 1 we have the bijective function

$$F_{X+Z}(s) = \begin{cases} \frac{s^2}{2\varepsilon} & \text{if } 0 \le s \le 1 \\ \frac{2s-1}{2\varepsilon} & \text{if } 1 \le s \le \varepsilon \\ 1 - \frac{(1+\varepsilon-s)^2}{2\varepsilon} & \text{if } \varepsilon \le s \le \varepsilon + 1 \end{cases}$$

and equals zero for $s < 0$ and 1 for $s > \varepsilon + 1$. So, its inverse in the corresponding intervals is

$$F^{-1}_{X+Z}(s) = \begin{cases} \sqrt{2\varepsilon s} & \text{if } \ 0 \le s \le \frac{1}{2\varepsilon} \\ \varepsilon s + \frac{1}{2} & \text{if } \ \frac{1}{2\varepsilon} \le s \le 1 - \frac{1}{2\varepsilon} \\ \varepsilon + 1 - \sqrt{2\varepsilon(1-s)} & \text{if } \ 1 - \frac{1}{2\varepsilon} \le s \le 1 \end{cases}$$

and similar to the Example 1, $D_2 C_{X,Y}(u,v) = \mathbf{1}_{(r < F^{-1}_{X+Z}(s) - z)}$. So, we have the following five cases:

1. if $0 \le u \le \frac{1}{2\varepsilon}$, $\sqrt{2\varepsilon u} \le v$
   $C_{X+Z,X}(u,v) = \int_0^{\sqrt{2\varepsilon u}} \int_0^{\sqrt{2\varepsilon u} - z} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z = u,$
2. if $0 \le u \le \frac{1}{2\varepsilon}$, $\sqrt{2\varepsilon u} \ge v$
   $C_{X+Z,X}(u,v) = \int_0^{\sqrt{2\varepsilon u} - v} \int_0^{v} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z + \int_{\sqrt{2\varepsilon u} - v}^{\sqrt{2\varepsilon u}} \int_0^{\sqrt{2\varepsilon u} - z} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z = \frac{1}{\varepsilon}(v\sqrt{2\varepsilon u} - \frac{v^2}{2}),$
3. if $\frac{1}{2\varepsilon} \le u \le 1 - \frac{1}{2\varepsilon}$,
   $C_{X+Z,X}(u,v) = \int_0^{\varepsilon u + \frac{1}{2} - v} \int_0^{v} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z + \int_{\varepsilon u + \frac{1}{2} - v}^{\varepsilon} \int_0^{\varepsilon u + \frac{1}{2} - z} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z = \frac{1}{\varepsilon}(v(\varepsilon u + \frac{1}{2} - v) + \frac{v^2}{2}),$
4. if $1 - \frac{1}{2\varepsilon} \le u \le 1$, $v \ge g(u)$,
   $C_{X+Z,X}(u,v) = \int_0^{\varepsilon + g(u) - v} \int_0^{v} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z + \int_{\varepsilon + g(u) - v}^{\varepsilon} \int_0^{\varepsilon + g(u) - z} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z = v - \frac{(v - g(u))^2}{2\varepsilon},$
5. if $1 - \frac{1}{2\varepsilon} \le u \le 1$, $v \le g(u)$
   $C_{X+Z,X}(u,v) = \int_0^{\varepsilon} \int_0^{v} \frac{1}{\varepsilon} \, \mathrm{d}r \mathrm{d}z = v,$

which is (8).

□

Figure 4 presents this copula function for $\varepsilon = 2$.

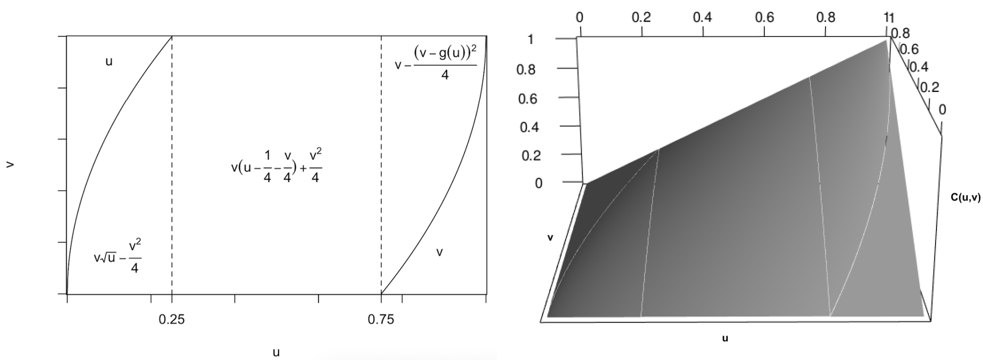

Fig. 4. 2D and 3D view of copula function $C_{X+Z,X}$ from Example 4 for $\varepsilon = 2$.

Using a similar computation, we will obtain the following results when $Y = 1 - X$.

Example 5. Let X and Y be two random variables uniformly distributed on $[0,1]$ linked with copula W, $C_{X,Y}(u,\ v) = \max(0, u+v-1)$ (i. e., $Y = 1-X$), and let the random variable Z independent of X and Y has uniform distribution on $[0,\varepsilon]$, then

a) if $\varepsilon \leq 1$,

$$C_{X+Z,Y}(u,v)) = \begin{cases} 0 & \text{if } \begin{array}{c} 0 \leq u \leq \frac{\varepsilon}{2} \ \&\ \sqrt{2\varepsilon u} \leq 1-v \\ \text{or} \\ \frac{\varepsilon}{2} \leq u \leq 1-\frac{\varepsilon}{2} \ \&\ u+\frac{\varepsilon}{2} \leq 1-v \end{array} \\ \frac{1}{2\varepsilon}(\sqrt{2\varepsilon u}+v-1)^2 & \text{if } 0 \leq u \leq \frac{\varepsilon}{2} \ \&\ \sqrt{2\varepsilon u} \geq 1-v \\ \frac{1}{\varepsilon}\frac{(u+\frac{\varepsilon}{2}-v-1)^2}{2} & \text{if } \frac{\varepsilon}{2} \leq u \leq 1-\frac{\varepsilon}{2} \ \&\ u-\frac{\varepsilon}{2} \leq 1-v \leq u+\frac{\varepsilon}{2} \\ \frac{1}{\varepsilon}[v(g(u)+\varepsilon-1)+\frac{v^2}{2}] & \text{if } 1-\frac{\varepsilon}{2} \leq u \leq 1 \ \&\ g(u) \leq 1-v \\ u+v-1 & \text{if } \begin{array}{c} \frac{\varepsilon}{2} \leq u \leq 1-\frac{\varepsilon}{2} \ \&\ 1-v \leq u-\frac{\varepsilon}{2} \\ \text{or} \\ 1-\frac{\varepsilon}{2} \leq u \leq 1 \ \&\ g(u) \geq 1-v \end{array} \end{cases} \tag{9}$$

b) if $\varepsilon > 1$,

$$C_{X+Z,Y}(u,v)) = \begin{cases} 0 & \text{if } 0 \leq u \leq \frac{1}{2\varepsilon} \ \&\ \sqrt{2\varepsilon u} \leq 1-v \\ \frac{1}{2\varepsilon}(\sqrt{2\varepsilon u}+v-1)^2 & \text{if } 0 \leq u \leq \frac{1}{2\varepsilon} \ \&\ \sqrt{2\varepsilon u} \geq 1-v \\ \frac{(\varepsilon u+v-\frac{1}{2})^2-(\varepsilon u-\frac{1}{2})^2}{2\varepsilon} & \text{if } \frac{1}{2\varepsilon} \leq u \leq 1-\frac{1}{2\varepsilon} \\ \frac{(h(u)+v)^2-h^2(u)}{2\varepsilon} & \text{if } 1-\frac{1}{2\varepsilon} \leq u \leq 1 \ \&\ h(u) \leq \varepsilon-v \\ u+v-1 & \text{if } 1-\frac{1}{2\varepsilon} \leq u \leq 1 \ \&\ h(u) \geq \varepsilon-v \end{cases} \tag{10}$$

where $g(u)$ is defined as in Example 4 and $h(u) = \varepsilon - \sqrt{2\varepsilon(1-u)}$.

It should be noted that the copula functions (7) and (9) can be reduced to (5) as well as (8) and (10) reduce to (6) when $\varepsilon = 1$. The following corollaries describe the asymptotic behaviors of these copula functions.

Corollary 2. Under the assumptions of Example 4 we have the following results

a) $C_{X+Z,X}(u,v)) \to M(u,v)$, as $\varepsilon \to 0$

b) $C_{X+Z,X}(u,v)) \to \Pi(u,v)$, as $\varepsilon \to \infty$.

Proof.

a) When $\varepsilon \to 0$, the third and the fifth part of (7) respectively will be

$$\begin{aligned} \lim_{\varepsilon \to 0} C_{X+Z,X}(u,v)) I_{(\frac{\varepsilon}{2},\ 1-\frac{\varepsilon}{2})}(u) I_{(0,u-\frac{\varepsilon}{2})}(v) &= v \lim_{\varepsilon \to 0} I_{(\frac{\varepsilon}{2},\ 1-\frac{\varepsilon}{2})}(u) I_{(0,u-\frac{\varepsilon}{2})}(v) \\ &= v I_{(0,1)}(u) I_{(0,u)}(v) \end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} C_{X+Z,X}(u,v))I_{(\frac{\varepsilon}{2},\ 1-\frac{\varepsilon}{2})}(u)I_{(u+\frac{\varepsilon}{2},1)}(v) &= u\lim_{\varepsilon\to 0} I_{(\frac{\varepsilon}{2},\ 1-\frac{\varepsilon}{2})}(u)I_{(u+\frac{\varepsilon}{2},1)}(v)\\ &= uI_{(0,1)}(u)I_{(u,1)}(v).\end{aligned}$$

Hence $C_{X+Z,X}(u,v)) \to \min(u,v) = M(u,v)$ as $\varepsilon \to 0$.

b) When $\varepsilon \to \infty$, the third part of (8) will be

$$\begin{aligned}&\lim_{\varepsilon\to\infty} C_{X+Z,X}(u,v))I_{(\frac{1}{2\varepsilon},1-\frac{1}{2\varepsilon})}(u)I_{(0,1)}(v)\\ =\ &\lim_{\varepsilon\to\infty}\frac{1}{\varepsilon}[v(\varepsilon u+\frac{1}{2}-v)+\frac{v^2}{2}]I_{(\frac{1}{2\varepsilon},1-\frac{1}{2\varepsilon})}(u)I_{(0,1)}(v)uvI_{(0,1)}(u)I_{(0,1)}(v)\\ =\ &\Pi(u,v).\end{aligned}$$

□

Similar to Corollary 2 we readily can prove the following corollary.

Corollary 3. Under the assumptions of Example 5 we have the following results

a) $C_{X+Z,1-X}(u,v)) \to W(u,v)$, as $\varepsilon \to 0$,

b) $C_{X+Z,1-X}(u,v)) \to \Pi(u,v)$, as $\varepsilon \to \infty$.

For $k \in]0,\infty[$, let us denote by $\mathcal{C}_k$ the copula $C_{X+Z,X}$ from Example 4 in the case when $k = \frac{1}{\varepsilon}$. Similarly for $k \in]-\infty, 0[$, we denote by $\mathcal{C}_k$ the copula $C_{X+Z,1-X}$ from Example 5 in the case when $k = -\frac{1}{\varepsilon}$. Moreover, let $C_0 = \Pi$, $C_\infty = M$ and $C_{-\infty} = W$.

Theorem 2. The parametric copula family $(\mathcal{C}_k)_{k\in[-\infty,\infty]}$ is a comprehensive family of copulas continuous and strictly increasing in parameter k.

The proof of the above theorem follows from Examples 4, 5 and Corollaries 2, 3. Evidently, for any dependence parameter like Kendal's tau, Spearmann's rho, Gini's gamma, Blomquist beta, see [2, 4, 5], the range of our comprehensive family $(\mathcal{C}_k)_{k\in[-\infty,\infty]}$ is full, i. e., it is $[-1,1]$, and any of these dependence parameters is increasing in parameter k.

More, any copula $\mathcal{C}_k$ is survival invariant,

$$\mathcal{C}_k(u,v) = 1-u-v+\mathcal{C}_k(1-u,1-v),$$

and $\mathcal{C}_k$ and $\mathcal{C}_{-k}$ are linked by the flipping relations,

$$\mathcal{C}_{-k}(u,v) = u-\mathcal{C}_k(u,1-v) = v-\mathcal{C}_k(1-u,v).$$

4. CONCLUDING REMARKS

We have discussed the form of copula $C_{X+Z,Y}$ when the first coordinate X of a random vector (X,Y) is affected by a noise Z (independent of both X and Y). Obviously, similar results for the case when the second coordinate is affected, i. e., for $C_{X,Y+Z}$, can be obtained in a straight way. Repeating our approach twice, one can obtain the copula $C_{X+Z_1,Y+Z_2}$ dealing with affection of both coordinates. A deeper study of this case, including the dropping of the independence of Z $(Z_1,\ Z_2)$ and X,Y, is a topic for the further study. More, in the next study the general case of n–dimensional random vectors affected by noise should be considered.

ACKNOWLEDGEMENT

This work was partially supported by grant VEGA 1/0891/17 and grant APVV-14-0013. We thank the anonymous reviewers for their helpful comments.

(Received October 11, 2018)

REFERENCES

[1] U. Cherubini, F. Gobbi, and S. Mulinacci: Convolution Copula Econometrics. Springer–Briefs in Statistics 2016. DOI:10.1007/978-3-319-48015-2

[2] F. Durante and C. Sempi: Principles of Copula Theory. CRC Chapman and Hall, Boca Raton 2016. DOI:10.1201/b18674

[3] I. Gijbels and K. Herrmann: On the distribution of sums of random variables with copula-induced dependence. Insurance Math. Econom. *59* (2014), C, 27–44. DOI:10.1016/j.insmatheco.2014.08.002

[4] H. Joe: Multivariate Models and Dependence Concepts. Chapman and Hall 1997. DOI:10.1201/b13150

[5] R. B. Nelsen: An introduction to copulas. Second edition. Springer Series in Statistics, Springer-Verlag, New York 2006. DOI:10.1007/0-387-28678-0

[6] A. Sklar: Fonctions de répartition a n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris *8 (1959), 229–231.*

[7] R. Wang: Sum of arbitrarily dependent random variables. Electron. J. Probab. *19* (2014), 84, 1–18. DOI:10.1214/ejp.v19-3373

[8] R. C. Williamson and R. C. Downs: Probabilistic arithmetic I: numerical methods for calculating convolutions and dependency bounds. Int. J. Approx. Reason. *4* (2014), 89–158. DOI:10.1016/0888-613x(90)90022-t

Radko Mesiar, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava, Slovak Republic and Institute of Information Theory and Automation, The Czech Academy of Sciences, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: radko.mesiar@stuba.sk

Ayyub Sheikhi, Corresponding author. Department of Statistics, Faculty of mathematics and computer, Shahid Bahonar University of Kerman, Kerman. Iran.
e-mail: sheikhy.a@uk.ac.ir

Magda Komorníková, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 810 05 Bratislava. Slovak Republic.
e-mail: magdalena.komornikova@stuba.sk