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Open problems from the 12th International Conference on Fuzzy Set Theory and Its Applications

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Abstract

Eighteen open problems posed during FSTA 2014 (Liptovský Ján, Slovakia) are presented. These problems concern fuzzy logics, fuzzy partitions, copulas, triangular norms and related aggregation functions. Some open problems concerning effect and MV algebras are also included.

Keywords: Copula; Effect algebra; Fuzzy implication; Fuzzy logics; Fuzzy partition; Triangular norm

1. Introduction

A public announcement of open problems had a great impact on the development of several areas of science, including mathematics. It seems so that the most famous was the formulation of D. Hilbert's problems [20]. In the domain of fuzzy sets and related topics, several open problems were published in monographs [7,31,41,48]. There are several papers devoted purely to open problems concerning triangular norms [1,30]. Other collections of open problems are linked to problems posed at conferences; recall for example the collections summarizing the open problems posed at the 2nd, 8th and 10th FSTA conferences [28,39,37]. To illustrate the influence of these collections to the development of mathematics, observe that just within the field of fuzzy sets there are more than 40 papers devoted to the solution of some of the exposed problems. The aim of this paper is the presentation of open problems posed during the conference FSTA 2014 "The Twelfth International Conference on Fuzzy Set Theory and Applications" held from January 26 to January 31, 2014 in Liptovský Ján, Slovakia.

The paper is organized as follows. In each section a brief introduction to the area of summarized open problems prepared by persons introducing these problems is given. In the 2nd section triangular norms and related negations are discussed. Section 3 is devoted to open problems on fuzzy implications. Section 4 deals with copulas. Preorders induced by uninorms are studied in Section 5. Section 6 discusses the construction of aggregation functions by means

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of penalty functions. Fuzzy partitions are discussed in Section 7, while Section 8 concerns fuzzy logics. In Section 9, several open problems on the field of universal integrals are introduced. Section 10 deals with effect algebras. States on MV-algebras are studied in Section 11. Finally some concluding remarks with e-mail addresses of authors of presented open problems are added.

2. Are Archimedean t-norms with strong associated negations left-continuous?

T-norms can be categorized under different classes based on their analytic and algebraic properties, viz., continuity, left-continuity, Archimedeanness, nilpotence, cancellativity, etc. Already some interrelationships among them are known under some conditions – for instance, it is well known that under Archimedeanness left-continuity is equivalent to continuity [33]. Similarly, when one assumes continuity of a t-norm T, many properties, which are otherwise not equivalent, become equivalent, for instance strict monotonicity of a T is equivalent to strictness, which is further equivalent to conditional cancellativity, while existence of only trivial idempotent elements becomes equivalent to Archimedeanness. For more such interrelationships please see [31].

When one considers a t-norm T with an involutive associated negation, i.e., the function

$$N_T(x) = \sup\{t \in [0,1] \mid T(x,t) \le 0\}$$

is such that $N_T \circ N_T = \mathrm{id}_{[0,1]}$, the only known result is that nilpotence is equivalent to continuity.

Our study on the mutual equivalences among the above properties under this setting [23] resulted into the following problem:

Problem 2.1 *(B. Jayaram).* Does there exist any Archimedean t-norm T, whose N_T is involutive but is not conditionally cancellative or left-continuous? In other words, is an Archimedean t-norm T whose N_T is involutive necessarily conditionally cancellative or left-continuous?

3. Lattice of fuzzy implications and the exchange principle

The exchange principle, i.e. the equation of the form

$$I\big(x, I(y,z)\big) = I\big(y, I(x,z)\big), \quad x, y, z \in [0,1], \tag{EP}$$

where $I:[0,1]^2 \to [0,1]$, generalizes the classical tautology

$$p \to (q \to r) \equiv q \to (p \to r)$$

and is one of the most important properties of a fuzzy implication both from theoretical and applicational point of view (see [2]). Unfortunately, in general, (EP) is not preserved by standard lattice operations minimum and maximum. For a counterexample see [2, Remark 6.1.5], where it is shown that the Goguen implication

$$I_{\mathbf{GG}}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \frac{y}{x}, & \text{if } y > x, \end{cases}$$

and the Reichenbach implication

$$I_{\mathbf{RC}}(x,y) = 1 - x + xy$$

satisfy (EP), but fuzzy implications $I_{\mathbf{GG}} \vee I_{\mathbf{RC}}$ and $I_{\mathbf{GG}} \wedge I_{\mathbf{RC}}$ do not satisfy (EP). In particular, this implies that if I, J are two (S,N)-implications, then $I \vee J$ and $I \wedge J$ are not necessarily (S,N)-implications. One can easily check that the same holds for R-implications generated from left-continuous t-norms, or f- and g-implications.

Problem 3.1 *(M. Baczyński, B. Jayaram).* Characterize the subfamily of all fuzzy implications ((S,N)-implications, R-implications, etc.) which preserve the (EP) for lattice operations.

4. Constructions of copulas

The definition and importance of copulas can be found and is discussed in many publications, recall only lecture notes [40]. Just for the sake of self-containedness we recall the definition of n-ary copula.

Definition 4.1. Let $n \geq 2$ be fixed. A function $C : [0,1]^n \to [0,1]$ is called *a* (n-ary) *copula* whenever it is n – increasing and satisfies boundary conditions, i.e.,

(C1) for any $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $x_1 \leq y_1, \ldots, x_n \leq y_n$ it holds

$$V_C\big([\mathbf{x},\mathbf{y}]\big) = \sum_{\varepsilon\, \in\{-1,1\}^n} \left(C\big(\mathbf{z}^{(\varepsilon)}\big) \prod_{i=1}^{n} \varepsilon_i \right) \geq 0,$$

where $\mathbf{z}_{(\varepsilon)} = (z_1^{\varepsilon_1}, \ldots z_n^{\varepsilon_n})$, $z_i^1 = y_i$ and $z_i^{-1} = x_i$, $i = 1, \ldots, n$;

(C2) $C(\mathbf{x}) = 0$ whenever $0 \in \{x_1, \ldots, x_n\}$ and $C(\mathbf{x}) = x_i$ whenever $x_j = 1$ for each $j \neq i$.

Recently, an interesting method of constructing parametric families $(C_\lambda)_{\lambda \in [0,1]}$ of 2-copulas, starting from any copula $C : [0,1]^2 \to [0,1]$, $C \neq \mathrm{Min}$, was introduced in [38],

$$C_\lambda(x,y) = C(x,y) + \lambda\big(x - C(x,y)\big)\big(y - C(x,y)\big). \tag{1}$$

Observe that if $C = \mathrm{Min}$ then $C_\lambda = \mathrm{Min}$ for each $\lambda \in [0,1]$. For the product copula Π, $(\Pi_\lambda)_{\lambda \in [0,1]}$ is a subfamily of well known Fairley–Gumbel–Morgenstern family,

$$\Pi_\lambda(x,y) = xy + \lambda x(1-x)y(1-y).$$

Our first open problem concerns the possibility of extending the construction (1) to higher dimension.

Problem 4.1 *(Mesiar).* Let $C : [0,1]^n \to [0,1]$ be an n-ary copula. Define $C_1 : [0,1]^n \to [0,1]$ by

$$C_1(x_1,\ldots,x_n) = C(x_1,\ldots,x_n) - \prod_{i=1}^{n}\big(x_i - C(x_1,\ldots,x_n)\big). \tag{2}$$

Is the function C_1 n-increasing, i.e., is it an n-copula? If in general not, characterize all n-copulas C such that C_1 is an n-copula, too.

Observe that if, for a given n-copula C, C_1 given by (2) is n-copula, then also $C_\lambda : [0,1]^n \to [0,1]$, $\lambda \in [0,1]$, given by

$$C_\lambda(x_1,\ldots,x_n) = C(x_1,\ldots,x_n) - \lambda \prod_{i=1}^{n}\big(x_i - C(x_1,\ldots,x_n)\big)$$

is an n-copula.

For each 2-copula $C : [0,1]^2 \to [0,1]$, the function $C^* : [0,1]^2 \to [0,1]$ given by

$$C^*(x,y) = x + y - C(x,y)$$

is called a dual copula (to C).

It is immediate that for the lower Fréchet–Hoeffding bound $W : [0,1]^2 \to [0,1]$, $W(x,y) = \max\{x+y-1, 0\}$, the function $W^{(\lambda)} : [0,1]^2 \to [0,1]$, $\lambda \in [0,1]$, given by

$$W^{(\lambda)} = \frac{W}{1 + \lambda(1 - W^*)},$$

satisfies $W^{(\lambda)} = W$, i.e., it is a 2-copula.

For the product 2-copula Π, define

$$\Pi^{(\lambda)} = \frac{\Pi}{1+\lambda(1-\Pi^*)}, \quad \text{i.e.,} \quad \Pi^{(\lambda)}(x,y) = \frac{xy}{1+\lambda-\lambda x-\lambda y+\lambda xy}.$$

Recall that $\Pi^{(\lambda)}$ is Ali–Haq–Mikhail copula with parameter λ (i.e., a 2-copula), see [40]. Similarly, it is not difficult to check that $\mathrm{Min}^{(\lambda)}$ given by

$$\mathrm{Min}^{(\lambda)} = \frac{\mathrm{Min}}{1+\lambda(1-\mathrm{Min}^*)}$$

is a copula for each $\lambda \in [0,1]$.

We wonder whether this type of construction of parametric families of copulas can be based on an arbitrary 2-copula $C \neq W$.

Problem 4.2 *(Mesiar).* Let $C : [0,1]^2 \to [0,1]$ be a 2-copula. For $\lambda \in [0,1]$, define $C^{(\lambda)} : [0,1]^2 \to [0,1]$ by

$$C^{(\lambda)} = \frac{C}{1+\lambda-\lambda C^*}.$$

Is the function $C^{(\lambda)}$ a 2-copula, too? If not, in general, characterize the set of all parameters $\lambda \in [0,1]$ for which $C^{(\lambda)}$ is a 2-copula (obviously, $C^{(0)} = C$ is a 2-copula).

Observe that due to a recent result from [34], $C \cdot C^*$ is a 2-copula for any 2-copula C.

Consequently

$$C^{[\lambda]} = C \cdot \left(1-\lambda+\lambda C^*\right), \quad \lambda \in [0,1],$$

is a parametric family of 2-copulas.

For $C = \Pi$, $(\Pi^{[\lambda]})_{\lambda\in[0,1]}$ is a subfamily of Fairley–Gumbel–Morgenstern family of copulas, while for $C = \mathrm{Min}$, $(\mathrm{Min}^{[\lambda]})_{\lambda\in[0,1]}$ is a subfamily of Fréchet family.

5. Preorders induced by uninorms

In [27], the following order induced by t-norms was introduced:

Definition 5.1. Let L be a bounded lattice, T be a t-norm on L. Then the order

$$x \preceq_T y \quad \Leftrightarrow \quad (\exists \ell \in L) T(\ell, y) = x \tag{3}$$

is called a t-order for the t-norm T.

Uninorms were introduced by Yager and Rybalov in [54]. A complete characterization of representable uninorms can be found in [13].

Definition 5.2. A uninorm U is a function $U : [0,1]^2 \to [0,1]$ that is increasing, commutative, associative and has a neutral element $e \in [0,1]$.

A uninorm U_r with a neutral element $e \in]0,1[$ is said to be *representable* if there exists a continuous, strictly increasing function $g : [0,1] \to [-\infty,\infty]$ with $g(0) = -\infty$, $g(1) = \infty$ and $g(e) = 0$, and such that

$$U_r(x,y) = g^{-1}\big(g(x)+g(y)\big).$$

We say that a uninorm U with the neutral element $e \in]0,1[$ contains a *zoomed-out* representable uninorm, if there exist $0 \le a < e < b \le 1$ and a continuous strictly increasing function $f : [a,b] \to [-\infty,\infty]$ with $f(a) = -\infty$, $f(b) = \infty$ and $f(e) = 0$, and such that for all $(x,y) \in]a,b[^2$ we have

$$U(x,y) = f^{-1}\big(f(x)+f(y)\big).$$

In formula (3) we can replace the t-norm T by an arbitrary uninorm U. Of course, the relation $\preceq_U$ is not necessarily an ordering. The relation $\preceq_U$ was studied in [21]. Particularly, it was shown that if the uninorm U contains a zoomed-out representable uninorm U_r, then the relation $\preceq_U$ is not anti-symmetric.

Problem 5.1 *(M. Kalina).* Is the condition that U does not contain a zoomed-out representable uninorm sufficient for the relation $\preceq_U$ to be anti-symmetric?

6. Penalty functions over lattices

In many applied problems a crucial step is that of fusing data from several sources into one single output which provides a good representation of the fused information. When data are provided in a numerical way, one commonly used tool is an aggregation function. However, the choice of an appropriate aggregation function for a given problem can be a very difficult task. Usually, many authors just make use of an aggregation function (or a class of aggregation functions) regardless the specific data being considered. However, recently it has been shown that, at least for some specific decision making problems, the choice of one aggregation or another in a given problem leads to completely different solutions [5].

Since their introduction by Mesiar, Yager and Calvo [8], penalty functions have shown themselves very useful in this kind of problems where the choice of the appropriate fusion technique (aggregation function) for the considered problem is not clear [5,17]. Recall that a penalty function is defined as follows.

Definition 6.1. A penalty function is a mapping

$$P : [0,1]^{n+1} \to [0,\infty[$$

such that

1. $P(x_1,\dots,x_n,y) = 0$ if and only if $x_1 = \dots = x_n = y$;
2. P is quasiconvex in y; that is, for every $x_1,\dots,x_n,y_1,y_2,\lambda \in [0,1]$ it holds that:

$$P\big(x_1,\dots,x_n,\lambda y_1 + (1-\lambda)y_2\big) \le \max\big(P(x_1,\dots,x_n,y_1), P(x_1,\dots,x_n,y_2)\big)$$

Basically, these functions are able to determine the dissimilarity between the input y and each of the data x_i. The larger this dissimilarity is, the larger the output of P. So, by means of a minimization procedure, it is possible to find an output which is the least dissimilar to a given set of input in the sense defined by the penalty function itself. Moreover, such way of measuring the dissimilarity may be different for each of the components. For instance, recall the definition of restricted dissimilarity function.

Definition 6.2. A restricted dissimilarity function is a mapping $d : [0,1]^2 \to [0,1]$ such that, for every $x, y, z \in [0,1]$

1. $d(x,y) = d(y,x)$;
2. $d(x,y) = 0$ iff $x = y$;
3. $d(x,y) = 1$ iff $\{x,y\} = \{0,1\}$;
4. if $x \le y \le z$ then $d(x,y) \le d(x,z)$ and $d(y,z) \le d(x,z)$.

Then, if $d_1,\dots,d_n : [0,1]^2 \to [0,1]$ ($i \in \{1,\dots,n\}$) are n restricted dissimilarity functions which are quasiconvex in one variable and if K is a positive convex function of one variable with a unique minimum at $x = 0$, it follows that a penalty function can be defined as:

$$P(x_1,\dots,x_n,y) = K\big(d_1(x_1,y),\dots,d_n(x_n,y_n)\big).$$

The relevance of these functions lies in the fact that every averaging aggregation function can be recovered through the use of these penalty functions. That is, every aggregation function whose values are always between the minimum of all the considered inputs and the maximum of all the inputs can be obtained as a penalty-based function, where the latter, for a given penalty function P, is defined by:

$$f(x_1, \ldots, x_n) = \arg\min_y P(x_1, \ldots, x_n, y)$$

Note that quasiconvexity ensures that the minimum is attained either at a single point or in an interval. In the last case, we just define the output of f as the mid-point of such interval.

Penalty functions have been successfully used to improve the results of some classical algorithms in decision making problems. And, on the other hand, also extensions of fuzzy sets, and in particular, interval-valued fuzzy sets have allowed an improvement on the results of algorithms which make use of fuzzy techniques. The extension of penalty functions to these more general settings, however, is difficult, since the notion of quasiconvexity is not properly defined in general for lattices [4], nor does there exist an easy, natural procedure to carry out such extension.

Problem 6.1 *(H. Bustince, J. Fernandez, M. Pagola, D. Paternain, E. Barrenechea).* To define properly penalty functions and penalty-based functions in the lattice of closed subintervals of the unit interval [0, 1], $L([0,1])$, in such a way that averaging aggregation operators in this lattice can be recovered as penalty-based functions.

Note that this problem also implies the consideration of which is the order used for defining monotonicity in $L([0,1])$. So it is necessary to consider two possible situations:

1. To define penalty functions in $L([0,1])$ with respect to the partial order $[a,b] \le_L [c,d]$ iff $a \le c$ and $b \le d$.
2. To define penalty functions with respect to linear admissible orders; that is, with respect to linear orders that extend the partial order $\le_L$.

7. Generalized uniform fuzzy partitions

Ruspini was probably the first who suggested in [47] to relax the crisp borders of equivalence classes by membership functions and consider a partition of a set using overlapping fuzzy sets. Ruspini's idea initiated a deep research in the field of fuzzy partitions. In the literature, various definitions of fuzzy partitions have been proposed. The open problem concerns fuzzy partitions defined by an appropriate generalization of classical axioms – covering and disjointness property – see, for example, [6,10,32,42]. More specifically, we are interested in *uniform fuzzy partitions* of the real line, where all fuzzy sets of the partition are equal in size and shape and regularly shifted along the real line. The importance of this type of fuzzy partitions can be mainly seen in real applications for their simple form often providing a deeper insight into settings of parameters. Let us mention here fuzzy histogram estimation [36,49] or fuzzy transform [43,44], where the uniform fuzzy partitions have a prominent position, but we may also imagine their use in the application fields like fuzzy control or fuzzy relation equations.

Roughly speaking the open problem is about an effective construction of uniform fuzzy partitions with more than two overlapping fuzzy sets. The solution would help researches in the investigation of various techniques based on uniform fuzzy partitions.

Let $\mathbb{R}$ denote the set of real numbers. A function $K : \mathbb{R} \to [0,1]$ is said to be a *generating function* if K is an even function that is non-increasing on $[0, \infty)$ and $K(x) > 0$ iff $x \in (-1, 1)$ holds true. A generating function K is said to be *normal* if $K(0) = 1$.

Typical examples of normal generating functions are triangular and raised cosine functions:

$$K_T(x) = \max\big(1 - |x|, 0\big) \quad \text{and} \quad K_C(x) = \begin{cases} \frac{1}{2}(1 + \cos(\pi x)), & -1 \le x \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let K be a generating function (in general, not normal), h and r be positive real numbers and $x_0 \in \mathbb{R}$. A system of fuzzy sets $\{A_i \mid i \in \mathbb{Z}\}$ defined by

$$A_i(x) = K\left(\frac{x - x_0 - i\,r}{h}\right)$$

for any $i \in \mathbb{Z}$ is called a *generalized uniform fuzzy partition* (GUFP) of the real line determined by the quadruplet (K, h, r, x_0) if the following *Ruspini condition* is satisfied:

$$S(x) = \sum_{i \in \mathbb{Z}} A_i(x) = 1$$

holds for any $x \in \mathbb{R}$. The parameters h, r and x_0 are called a *bandwidth*, *shift*, and *central node*, respectively.

A full characterization of generalized uniform fuzzy partitions using the sum of suitable integrals has been proved in [22]. In contrast to the Ruspini condition, particular surfaces under the generating function are investigated here, which helps to understand in a better way the structure of GUFPs.

Theorem 7.1. *Put $\alpha = \frac{r}{h}$. A quadruplet (K, h, r, x_0) determines a generalized uniform fuzzy partition iff*

$$\sum_{i=1}^{\infty} \int_{i\alpha - y}^{y+(i-1)\alpha} K(x)dx = y - \frac{\alpha}{2} \tag{4}$$

holds for any $y \in [\frac{\alpha}{2}, \alpha]$.

A simple consequence of the preceding theorem is a necessary condition for GUFPs stating the important interrelation among the generating function K, the bandwidth h, and the shift r.

Corollary 7.1. *If (K, h, r, x_0) determines a generalized uniform fuzzy partition, then $\int_{-1}^{1} K(x)dx = \frac{r}{h}$.*

Let K be a normal generating function (i.e., $K(0) = 1$), and let $\alpha \in (0, 1]$. Define $K_\alpha(x) = \alpha \cdot K(x)$, where $\alpha \cdot K(x)$ is the common product of real numbers. In [22], Holčapek et al. have proved that the necessary and sufficient condition for GUFPs can be significantly simplified in the cases of triangular and raised cosine generating functions.

Theorem 7.2. *Let $K \in \{K_T, K_C\}$. Then, $(K_{\frac{r}{h}}, h, r, x_0)$ determines a GUFP iff $\frac{h}{r} \in \mathbb{N}$.*

The unquestionable benefit of this theorem is probably the simplest procedure how to construct all GUFPs based on triangular or raised cosine generating functions. Indeed, we need not verify if the parameters satisfy the Ruspini condition or (4), and it is sufficient to consider h and r such that the ratio of h to r is a natural number and set the generating function to $K_{\frac{r}{h}}$.

Studying particular cases one open question arises and we do not see a straightforward answer. The question is whether the previous particular result may be generalized for further normal generating functions, for example, defined by splines, Bernstein basis polynomials, or Shepard kernels (see [3]). It can be demonstrated that there are generating functions such that the assumption $h/r \in \mathbb{N}$ is too weak to guarantee a GUFP. On the other hand, we know that the triangular and raised cosine functions satisfy the following *symmetry condition*:

$$\int_{\frac{1}{2}-y}^{y+\frac{1}{2}} K(x)dx = y \tag{5}$$

holds true for any $y \in [0, \frac{1}{2}]$.

The open question is the confirmation or falsification of the following hypothesis.

Problem 7.1 *(M. Holčapek, I. Perfilieva, V. Novák and V. Kreinovich).* Let K be a normal generating function satisfying the symmetry condition (5). Then, $(K_{\frac{r}{h}}, h, r, x_0)$ determines a GUFP iff $\frac{h}{r} \in \mathbb{N}$.

If this hypothesis is not true, it would be very helpful to specify a condition (similarly to the symmetry condition) for generating functions under which the necessary and sufficient condition can be expressed in such a simple form.

8. Quantitative logic – what is a suitably general setting?

Quantitative logic was introduced by Guo-Jun Wang together with colleagues and students in papers like [50,51,55]. It is explained, e.g., in [52,53] and offers for its propositional part the following setting. There is a basic many-valued logic $\mathcal{L}_b$ which has as truth degree set W the real unit interval or a (finite) subset of it, and also suitable connectives. Typical examples are provided by the Łukasiewicz logics, and more generally by t-norm based logics.

But also classical two-valued logic is allowed for $\mathcal{L}_b$. The crucial idea is to connect with each formula φ of such a logic $\mathcal{L}$ a kind of meta-degree $\tau_{\mathcal{L}_b}(\varphi)$ in W which is determined by some sort of "averaging" over the $\mathcal{L}$-degrees which φ has in all the possible evaluations of its atomic parts, i.e. its propositional variables. To look a bit more into the technical details let φ contain n propositional variables, and let be $\widehat{\varphi}: W^n \to W$ the n-ary truth degree function determined by the formula φ. If $\mathcal{L}_b$ has $W = W_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \dots, 1\}$ as truth degree set, then quantitative logic considers the normed weighted sum

$$\tau_{\mathcal{L}_b}(\varphi) = \frac{1}{m^n} \sum_{w \in W} w \cdot \|\widehat{\varphi}^{-1}(w)\|$$

as this meta-degree, with $\|M\|$ denoting the standard cardinality of the set M.

If $\mathcal{L}_b$ has $W = W_\infty = [0,1]$ as truth degree set, then quantitative logic considers the integral value

$$\tau_{\mathcal{L}_b}(\varphi) = \int\limits_{[0,1]^n} \widehat{\varphi}(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n$$

as this meta-degree.

The first problem which has been discussed in this context then is to determine how the meta-degrees of combined formulas depend upon the meta-degrees of their constituents. Other problems concern degrees of similarity between formulas or degrees of consistency for sets of formulas.

Problem 8.1 *(S. Gottwald).* Which other types of aggregation operators instead of the integral or the weighted sum are suitable here? And how depends their choice from intended applications?

Problem 8.2 *(S. Gottwald).* Particularly, does one need – as in the case of the integral – a measure in the truth degree set, or can one do well without, too?

Problem 8.3 *(S. Gottwald).* How to generalize this setting for the case that the truth degree set is a lattice?

Problem 8.4 *(S. Gottwald).* Particularly, what about suitable measures on lattices; and what about aggregation operators on lattices?

9. Universal integrals, monotone measures and convergences

Let $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ is a σ-algebra of subsets of a non-empty set X, and let $\mathcal{S}$ be the family of all measurable spaces. A class of all $\mathcal{A}$-measurable functions $f: X \to [0,1]$ will be denoted by $\mathcal{F}_{(X,\mathcal{A})}$, and a class of all capacities on $\mathcal{A}$ (i.e., non-decreasing set functions $m: \mathcal{A} \to [0,1]$ with $m(\emptyset) = 0$ and $m(X) = 1$) is denoted by $\mathcal{M}_{(X,\mathcal{A})}$. Let $S: [0,1]^2 \to [0,1]$ be a semicopula, i.e., a non-decreasing function in both coordinates with the neutral element 1, and satisfying the inequality $S(x,y) \le \min(x,y)$ for all $(x,y) \in [0,1]^2$.

Problem 9.1 *(O. Hutník).* To characterize all the semicopulas S for which the inequality

$$S(x+y, z) \le S(x,z) + S(y,z) \tag{6}$$

holds for each $x, y \in [0,1]$ such that $x + y \in [0,1]$.

The above problem arises when studying properties of certain universal integrals, see [29]. Indeed, a class of the smallest semicopula-based universal integrals has the form

$$\mathbf{I}_S(m, f) := \sup_{t \in [0,1]} S\big(t, h_{m,f}(t)\big),$$

where $(X, \mathcal{A}) \in \mathcal{S}$, $(m, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$, and the function $h_{m,f}: [0,1] \to [0,1]$ is defined by $h_{m,f}(t) := m(\{x \in X;\ f(x) \ge t\})$. The inequality (6) plays an important role when considering transformation theorem as well as translatability of the integral $\mathbf{I}_S$, see [18].

Problem 9.2 *(O. Hutník).* Characterize a class of semicopulas S, for which the equality $\mathbf{I}_S(m, f+\alpha) = \mathbf{I}_S(m, f) + \alpha$ holds for each $(X, \mathcal{A}) \in \mathcal{S}$, each $(m, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$ and $\alpha \in [0, 1]$ such that $f + \alpha \in [0, 1]$.

Easily, for constant functions, an arbitrary semicopula S and monotone set functions m the above mentioned property of integral $\mathbf{I}_S$ holds trivially. Moreover, a class of semicopulas, for which the integral equality holds, is non-empty, because the Łukasiewicz t-norm T_L solves this open problem, see [18].

Problem 9.3 *(O. Hutník).* To characterize a class of semicopulas S for which the property

$$(\forall \alpha \in [0, 1]) \quad \mathbf{I}_S\big(m, S(\alpha, f)\big) = S\big(\alpha, \mathbf{I}_S(m, f)\big)$$

holds for all $(X, \mathcal{A}) \in \mathcal{S}$ and all $(m, f) \in \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}$.

We conjecture that the class of semicopulas solving Problem 9.3 will contain only the (semi)copulas minimum M and product Π.

Let $m \in \mathcal{M}_{(X,\mathcal{A})}$. A sequence $(f_n)_1^\infty \in \mathcal{F}_{(X,\mathcal{A})}$ *converges strictly in measure* m to a function $f \in \mathcal{F}_{(X,\mathcal{A})}$, if

$$\lim_{n\to\infty} m\big(\{x \in X; |f_n(x) - f(x)| \neq 0\}\big) = 0.$$

It is a well-known fact that strict convergence in measure is stronger than convergence in measure.

Problem 9.4 *(O. Hutník).* Characterize all the non-additive set functions for which strict convergence in measure is equivalent to convergence in measure on any measurable space.

For a fixed semicopula S we say that a sequence $(f_n)_1^\infty \in \mathcal{F}_{(X,\mathcal{A})}$ *converges in mean* (with respect to the integral $\mathbf{I}_S$) to a function $f \in \mathcal{F}_{(X,\mathcal{A})}$, if

$$\lim_{n\to\infty} \mathbf{I}_S\big(m, |f_n - f|\big) = 0.$$

It is shown in [19] that mean convergence does not imply strict convergence in measure (even for continuous semicopulas with trivial zero divisors). Therefore, we may naturally ask:

Problem 9.5 *(O. Hutník).* For which class of semicopulas (of capacities, eventually) is strict convergence in measure equivalent to mean convergence?

We conjecture that autocontinuous capacities may solve the latter open problem.

10. Sharpness and lattice order in effect algebras

Effect algebras [14] (discovered independently in [35,16]) were introduced by Foulis and Bennett to study the problems of unsharp measurements in quantum mechanics.

In [24], *homogeneous effect algebras* were introduced and studied. An effect algebra is homogeneous if and only if, for all u, u_1, u_2 such that $u \le v_1 \oplus v_2 \le u'$ there are v_1, v_2 such that $u = u_1 \oplus u_2$, $u_1 \le v_1$ and $u_2 \le v_2$. Homogeneous effect algebras are a common generalization of orthoalgebras [15], lattice effect algebras [46] and effect algebras satisfying the Riesz decomposition property [45].

An element a of an effect algebra is *sharp* iff $a \wedge a' = 0$. It is known that the set of all sharp elements in a homogeneous effect algebra E forms an orthoalgebra $S(E)$. It is known [26] that if E is a lattice, then $S(E)$ is a lattice.

Problem 10.1 *(G. Jenča).* Prove or disprove: if E is an orthocomplete (see [25]) homogeneous effect algebra E such that $S(E)$ is a lattice, then E is a lattice effect algebra.

We note that in the absence of orthocompleteness there is a counterexample: the set $P[0, 1]$ of all polynomial functions $[0, 1] \to [0, 1]$ equipped with the usual partial addition is a homogeneous effect algebra, $S(P[0, 1]) = \{0, 1\}$ is clearly a lattice, but $P[0, 1]$ is not a lattice.

The problem is open even for finite homogeneous effect algebras.

11. Generator of the variety of state MV-algebras

State MV-algebras were introduce in [12] in order to present an algebraic characterization of a notion of a state. We recall that according to [12], a *state MV-algebra* is a couple $(\mathbf{A}, \tau)$, where $\mathbf{A} = (A; \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and τ is a mapping from A into A, called an *internal state* or a *state operator*, such that

(i) $\tau(1) = 1$.
(ii) $\tau(x \oplus y) = \tau(x) \oplus \tau(y \ominus (x \odot y))$.
(iii) $\tau(x^*) = \tau(x)^*$.
(iv) $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$.

Basic properties of state MV-algebras are described in [12]. In particular, if τ is a state operator on $\mathbf{A}$, then $\tau \circ \tau = \tau$. If, in addition, τ satisfies $\tau(x \oplus y) = \tau(x) \oplus \tau(y)$, $x, y \in A$, $(\mathbf{A}, \tau)$ is said to be a *state morphism MV-algebra* and τ is a *state morphism operator*. It is possible to show, [9], that a mapping $\tau: A \to A$ is a state morphism operator on $\mathbf{A}$ iff τ is an MV-endomorphism on A such that $\tau \circ \tau = \tau$.

We denote by $\mathcal{SMV}$ and $\mathcal{SMMV}$ the variety of state MV-algebras and state morphism MV-algebras, respectively. Then $\mathcal{SMMV}$ is a proper subvariety of the variety $\mathcal{SMV}$. If $\mathbf{A} = (A; \oplus, \odot, ^*, 0, 1)$ is an MV-algebra, then the operator $\tau_A : A \times A \to A \times A$ defined by $\tau_A(a, b) = (a, a)$, $(a, b) \in A \times A$ is a state morphism operator, called a *diagonal operator* on $A \times A$ and $(A \times A; \tau_A)$ is a state morphism MV-algebra. It is well-known that the MV-algebra of the real interval [0, 1] generates the variety of MV-algebras. It was an open problem posed in [9] whether does the state-morphism MV-algebra $([0, 1]^2, \tau_{[0,1]})$ generates the variety of state morphism MV-algebras. This was answered in positive in [11, Theorem 5.4 (3)].

Inspired by this, we formulate the following open problem:

Problem 11.1 *(A. Dvurečenskij).* According to [11, Theorem 5.4 (3)], find a generator of the variety of state MV-algebras. Is this generator connected in some way with the MV-algebra of the real interval?

12. Concluding remarks

We have summarized 18 open problems coming from different areas covered by Fuzzy Sets and Systems journal. We believe that these problems will attract several researchers and that we will see soon most of these problems solved. Moreover, we believe, too, that in some cases our collection will initiate an intensive research in the discussed areas.

For interested scholars we add here the e-mail contact to persons presenting these problems:

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We wish all the possible solvers great success and satisfaction from their solutions.

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