Soft Comput (2015) 19:1627–1634
DOI 10.1007/s00500-014-1578-0

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On Cauchy–Schwarz's inequality for Choquet-like integrals without the comonotonicity condition

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Published online: 6 January 2015

Abstract Cauchy-Schwarz's inequality is one of the most important inequalities in probability, measure theory and analysis. The problem of finding a sharp inequality of Cauchy–Schwarz type for Sugeno integral without the comonotonicity condition based on the multiplication operator has led to a challenging and an interesting subject for researchers. In this paper, we give a Cauchy–Schwarz's inequality without the comonotonicity condition based on pseudo-analysis for two classes of Choquet-like integrals as generalizations of Choquet integral and Sugeno integral. In the first class, pseudo-operations are defined by a continuous strictly increasing function g. Another class concerns the Choquet-like integrals based on the operator "sup" and a pseudo-multiplication $\otimes$. When working on the second class of Choquet-like integrals, our results give a new version of Cauchy–Schwarz's inequality for Sugeno integral without the comonotonicity condition based on the multiplication operator.

Communicated by L. Spada.

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Keywords Monotone probability · Choquet expectation · Sugeno integral · Choquet-like integrals · Cauchy–Schwarz's inequality · Hölder's inequality · Pseudo-analysis

1 Introduction

Recently, much research has been done on the connection of the inequalities and the nonadditive integrals such as the Sugeno integral and the Choquet integral (Agahi et al. 2010, 2013; Caballero and Sadarangani 2010; Wu et al. 2010). These studies are often based on the concept of comonotonicity. For example, Caballero and Sadarangani (2010) obtained the Cauchy–Schwarz inequality for the Sugeno integral. However, the lack of comonotonicity condition in many problems in statistics, probability and engineering is a disadvantage of these results. On the other hands, many classical inequalities are free from this condition.

Recall that two functions $X, Y : \Omega \to \mathbb{R}$ are said to be comonotone if and only if

$$\Big(X(\omega_1) - X(\omega_2)\Big)\Big(Y(\omega_1) - Y(\omega_2)\Big) \geqslant 0$$

for each couple of elements $\omega_1, \omega_2 \in \Omega$. First, let us look at the right side of inequality (1.1). The authors found that the special operator $\vee$ should be used instead of the multiplication operator. Note that some of the next notions will be properly defined later in Sect. 2.

Theorem 1.1 (Caballero and Sadarangani 2010, Theorem 1) (Cauchy–Schwarz's inequality) *Let $X, Y : \Omega \to [0, \infty)$ be two comonotone functions and μ a real monotone measure such that $\mathrm{Su}_\mu[XY] \leq 1$. Then the inequality*

$$\mathrm{Su}_\mu[XY] \leq \sqrt{\Big(\mathrm{Su}_\mu[X^2]\Big) \vee \Big(\mathrm{Su}_\mu[Y^2]\Big)} \qquad (1.1)$$

holds.

Recently, Wu et al. (2010) studied a Hölder-type inequality for Sugeno integral based on the comonotonicity condition and a binary operator $\star$. We see that in general, we cannot put the multiplication operator instead of $\star$.

Theorem 1.2 (Hölder's inequality) *Let $X, Y : \Omega \to [0, 1]$ be two comonotone functions and μ a monotone probability. Let $\star\colon [0, 1]^2 \to [0, 1]$ be continuous and non-decreasing in both arguments and bounded from below by maximum. Then the inequality*

$$\mathrm{Su}_\mu [X \star Y] \le \left(\mathrm{Su}_\mu \left[X^p\right]\right)^{\frac{1}{p}} \star \left(\mathrm{Su}_\mu \left[Y^q\right]\right)^{\frac{1}{q}} \tag{1.2}$$

holds for all $p, q \in [1, \infty)$.

Therefore, finding a Cauchy–Schwarz's inequality (or a Hölder's inequality) for Sugeno integral without the comonotonicity condition based on the multiplication operator is both a challenging and an interesting subject.

In 1995, Mesiar (1995) introduced two classes of Choquet-like integrals as generalizations of Choquet integral (expectation) and Sugeno integral. The first class is called "Choquet-like expectation" which generalizes the Choquet expectation (see Definition 2.6) and the second class is an extension of Sugeno integral (see Definition 2.8). In this paper, we give a Cauchy–Schwarz's inequality without the comonotonicity condition for two classes of Choquet-like integrals. Notice that when working on the second class of Choquet-like integrals, one of our results (Theorem 3.3) gives us a new version of Cauchy–Schwarz's inequality for Sugeno integral without the comonotonicity condition based on the multiplication operator.

The paper is organized as follows: Sect. 2 recalls the concepts of Choquet-like integrals while Sect. 3 presents our main results. Finally, some concluding remarks are added.

2 Preliminaries

To prove our results, we shall first recall some basic definitions and previous results. For details, we refer to Mesiar (1995) [see also Sheng et al. (2011)].

Definition 2.1 (Sugeno and Murofushi 1987) An operation $\oplus\colon [0, \infty]^2 \to [0, \infty]$ is called a pseudo-addition if the following properties are satisfied:

(P1) $a \oplus 0 = 0 \oplus a = a$ (neutral element);
(P2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(P3) $a \le c$ and $b \le d$ imply that $a \oplus b \le c \oplus d$ (monotonicity);
(P4) $a_n \to a$ and $b_n \to b$ imply that $a_n \oplus b_n \to a \oplus b$ (continuity).

Definition 2.2 (Mesiar 1995; Sugeno and Murofushi 1987) Let $\oplus$ be a given pseudo-addition on $[0, \infty]$. Another binary operation $\otimes$ on $[0, \infty]$ is said to be a pseudo-multiplication corresponding to $\oplus$ if the following properties are satisfied:

(M1) $a \otimes (x \oplus y) = (a \otimes x) \oplus (a \otimes y)$ (left distributivity);
(M2) $a \le b$ implies $(a \otimes x) \le (b \otimes x)$ and $(x \otimes a) \le (x \otimes b)$ (monotonicity);
(M3) $a \otimes x = 0 \Leftrightarrow a = 0$ or $x = 0$ (absorbing element and no zero divisors);
(M4) $\exists\, e \in (0, \infty]$ (i.e., there exist the neutral element e) such that $e \otimes x = x \otimes e = x$ for any $x \in [0, \infty]$ (neutral element);
(M5) $a_n \to a \in (0, \infty)$ and $x_n \to x$ imply $(a_n \otimes x_n) \to (a \otimes x)$ and $\infty \otimes x = \lim_{a \to \infty}(a \otimes x)$ (continuity);
(M6) $a \otimes x = x \otimes a$ (commutativity);
(M7) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ (associativity).

Theorem 2.3 (Mesiar 1995) *Let $\otimes$ be a pseudo-multiplication corresponding to a given pseudo-addition $\oplus$ fulfilling axioms (M1)–(M7).*

(I) *If its identity element e is not an idempotent of $\oplus$, then there is a unique continuous strictly increasing function $g\colon [0, \infty] \to [0, \infty]$ with $g(0) = 0$ and $g(\infty) = \infty$, such that $g(e) = 1$ and*

$$a \oplus b = g^{-1}\big(g(a) + g(b)\big) \oplus \text{ is called a } g\text{-addition},$$

$$a \otimes b = g^{-1}\big(g(a) \cdot g(b)\big) \otimes \text{is called a } g\text{-multiplication}.$$

(II) *If the identity element e of the pseudo-multiplication is also an idempotent of $\oplus$ (i.e., $e \oplus e = e$), then $\oplus = \vee$ ($=$ sup, i.e., the logical addition).*

For $x \in [0, \infty]$ and $p \in (0, \infty)$, we will introduce the pseudo-power $x_\otimes^{(p)}$ as follows: If $p = n$ is a natural number, then $x_\otimes^{(n)} = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n-\text{times}}$. If p is not a natural number, then the corresponding power is defined by $x_\otimes^{(p)} = \sup\left\{y_\otimes^{(m)} | y_\otimes^{(n)} \le x, \text{ where } m, n \text{ are natural numbers such that } \frac{m}{n} \le p\right\}$. Evidently, if $x \otimes y = g^{-1}\big(g(x) \cdot g(y)\big)$, then $x_\otimes^{(p)} = g^{-1}\big(g^p(x)\big)$.

Definition 2.4 (Klement et al. 2010) A monotone measure μ on a measurable space $(\Omega, \mathcal{F})$ is a set function $\mu : \mathcal{F} \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$;
(ii) $\mu(\Omega) > 0$;
(iii) $\mu(A) \le \mu(B)$ whenever $A \subseteq B$;

moreover, μ is called real if $||\mu|| = \mu(\Omega) < \infty$.The triple $(\Omega, \mathcal{F}, \mu)$ is also called a monotone measure spaceif μ is a monotone measure on $\mathcal{F}$.

We call μ a monotone probability, if $||\mu|| = 1$. When μ is a monotone probability, the triple $(\Omega, \mathcal{F}, \mu)$ is called a monotone probability space.

Definition 2.5 Let $(\Omega, \mathcal{F}, \mu)$ be a monotone measure space and $X : \Omega \to [0, \infty)$ be an $\mathcal{F}-$measurable function. The Choquet expectation of X over $A \in \mathcal{F}$ w.r.t. the real monotone measure μ is defined by Choquet (1954), Mesiar (1995)

$$\mathbb{E}_C^{\mu}[X\mathbb{I}_A] = \int_0^{\infty} \mu\Big(A \cap \{X \geq t\}\Big)\mathrm{d}t. \tag{2.1}$$

where the integral on the right-hand side is the (improper) Riemann integral. In particular, if $A = \Omega$, then

$$\mathbb{E}_C^{\mu}[X] = \int_0^{\infty} \mu\Big(\{X \geq t\}\Big)\mathrm{d}t.$$

Mesiar (1995) has shown that there are two classes of Choquet-like integral: the Choquet-like integral (denoted by $\mathbb{E}_{Cl,g}^{\mu}$) based on a g-addition and a g-multiplication and the Choquet-like integral based on $\vee$ and a corresponding pseudo-multiplication $\otimes$.

Definition 2.6 (Mesiar 1995) Let $(\Omega, \mathcal{F})$ be a measurable space and $\mu\colon \mathcal{F} \to [0, \infty]$ be a monotone measure. Let $\oplus$ and $\otimes$ be generated by a generator g. The Choquet-like expectation of a non-negative measurable function X over $A \in \mathcal{F}$ w.r.t. the real monotone measure μ can be represented as

$$\begin{aligned}\mathbb{E}_{Cl,g}^{\mu}[X\mathbb{I}_A] &= g^{-1}\left(\mathbb{E}_C^{g(\mu)}[g(X)\mathbb{I}_A]\right)\\ &= g^{-1}\left(\int_0^{\infty} g\mu\left(A \cap \{g(X) \geq t\}\right)\mathrm{d}t\right).\end{aligned}$$

In particular, if $A = \Omega$, then

$$\mathbb{E}_{Cl,g}^{\mu}[X] = g^{-1}\left(\mathbb{E}_C^{g(\mu)}[g(X)]\right). \tag{2.2}$$

Remark 2.7 Notice that we sometimes call this kind of Choquet-like integral a g-Choquet integral ($g-C$–integral for short). It is plain that the $g-C-$ integral is the original Choquet integral (expectation) whenever $g = i$ (the identity mapping).

Definition 2.8 (Mesiar 1995) Let $\otimes$ be a pseudo-multiplication corresponding to $\vee$ and fulfilling (M1)–(M7). Then the Choquet-like integral (so-called $\mathbb{S}_{\mu}^{\otimes}$ integral) of a measurable function $X : \Omega \to [0, \infty)$ w.r.t. a real monotone measure μ can be represented as

$$\mathbb{S}_{\mu}^{\otimes}[X] = \sup_{a\in[0,\infty]} \Big(a \otimes \mu(\{X \geq a\})\Big). \tag{2.3}$$

It is plain that the $\mathbb{S}_{\mu}^{\otimes}$ integral is the Sugeno integral (denoted by $Su_{\mu}[.]$) whenever $\otimes = \wedge$ (Wang and Klir 2008). If $\otimes$ is the standard product, then we have the Shilkret integral (Shilkret 1971) (denoted by $Sh_{\mu}[.]$). During this paper, we always consider the existence of all $\mathbb{S}_{\mu}^{\otimes}[.]$.

Remark 2.9 Though the second class of Choquet-like integrals introduced in Mesiar (1995) deals with pseudo-multiplications $\otimes$ satisfying (M1)–(M7) when $\oplus = \vee$ (then left-distributivity is just the monotonicity in the second coordinate), formula (2.3) can be applied considering any increasing function $\otimes\colon [0, \infty]^2 \to [0, \infty]$ as a pseudo-multiplication. When working on $[0, 1]$ (i.e., considering function with range contained in $[0, 1]$ only—they can be seen as membership functions of fuzzy events, and considering monotone probabilities), we mostly deal with $e = 1$. Then we have to deal with $\otimes = \circledast$ only, where $\circledast$ is a semicopula (t-seminorm), i.e., a binary operation $\circledast\colon [0, 1]^2 \to [0, 1]$ which is non-decreasing in both components and has 1 as neutral element. Then $\otimes = \circledast$ satisfies $a \otimes b \leq \min(a, b)$ for all $(a, b) \in [0, 1]^2$, see Durante and Sempi (2005).

Definition 2.10 The $\mathbb{S}_{\mu}^{\otimes}$ integral on the $[0, 1]$ scale related to the semicopula $\circledast$ w.r.t. the monotone probability μ is given by

$$\mathbb{S}_{\mu}^{\circledast}[X] = \sup_{a\in[0,1]} \big(a \circledast \mu(\{X \geq a\})\big). \tag{2.4}$$

This type of integral was called seminormed integral in Suárez García and Gil Álvarez (1986).

Remark 2.11 For a fixed strict t-norm T (Klement et al. 2000), the corresponding $\mathbb{S}_{\mu}^{T}$ integral is the so-called Sugeno–Weber integral (Weber 1986). If $\circledast$ is the standard product, then the original Shilkret integral (Shilkret 1971) can be recognized. Notice that the original Sugeno integral which was introduced by Sugeno (1974) in 1974 is a special seminormed integral when $\circledast = \min$.

The following Lemma helps us to reach the main results.

Lemma 2.12 *Let $\beta \in (0, \infty)$. Let X be a non-negative measurable function. If $\otimes$ is a pseudo-multiplication satisfying*

$$(a \otimes b) \leq \frac{1}{\beta}\Big(\beta a \otimes b\Big) \tag{2.5}$$

then for the $\mathbb{S}_{\mu}^{\otimes}$ integral (2.3), the inequality

$$\mathbb{S}_{\mu}^{\otimes}[\beta X] \geqslant \beta\mathbb{S}_{\mu}^{\otimes}[X]$$

holds.

Proof Let $\mathbb{S}_{\mu}^{\otimes}[X] = q$. Then for any $\varepsilon > 0$, there exist q_{ε} such that $M = \mu\big(\{X \geq q_{\varepsilon}\}\big)$, where $(q_{\varepsilon} \otimes M) \geqslant q - \varepsilon$. So, by (2.5), we have

$$\mathbb{S}_\mu^{\otimes}[\beta X] = \sup_{a\in[0,\infty]} \Big(a \otimes \mu(\{\beta X \geq a\})\Big)$$
$$\geqslant \beta q_\varepsilon \otimes \mu\Big(\{\beta X \geq \beta q_\varepsilon\}\Big)$$
$$\geqslant \beta(q_\varepsilon \otimes M) \geqslant \beta(q-\varepsilon).$$

Whence $\mathbb{S}_\mu^{\otimes}[\beta X] \geq \beta(q-\varepsilon)$ follows from the arbitrariness of ε. This completes the Proof. □

Notice that if $\otimes$ is minimum (i.e., for Sugeno integral) in Lemma 2.12, then the following result holds.

Corollary 2.13 *Let X be a non-negative measurable function. Then the inequality*

$$Su_\mu[\beta X] \geqslant \beta Su_\mu[X]$$

holds where $0 < \beta \leq 1$.

3 Cauchy–Schwarz's inequality

In this section, we prove the Cauchy–Schwarz's inequality for two classes of Choquet-like integrals. Theorem 3.1 gives us the Cauchy–Schwarz's inequality for the first class of Choquet-like integrals, i.e., for Choquet-like expectation. Then, we will prove the Cauchy–Schwarz's inequality for the second class in Theorem 3.3.

Theorem 3.1 (Cauchy–Schwarz's inequality) *Let X, Y be two non-negative measurable functions and let the pseudo-operations be generated by a generator g. If*

$$mg\Big(Y(\omega)\Big) \leq g\Big(X(\omega)\Big) \leq Mg\Big(Y(\omega)\Big) \quad \forall\omega\in\Omega,$$

where $0 < m \leq M$ then, independently of a monotone measure μ, for the Choquet-like expectation given by (2.2), the following inequalities

$$\Big(\mathbb{E}^{\mu}_{Cl,g}\Big[X_\otimes^{(2)}\Big]\Big)_\otimes^{\left(\frac{1}{2}\right)} \otimes \Big(\mathbb{E}^{\mu}_{Cl,g}\Big[Y_\otimes^{(2)}\Big]\Big)_\otimes^{\left(\frac{1}{2}\right)} \geqslant K_1 \otimes \mathbb{E}^{\mu}_{Cl,g}[X\otimes Y], \tag{3.1}$$

$$\Big(\mathbb{E}^{\mu}_{Cl,g}\Big[X_\otimes^{(2)}\Big]\Big)_\otimes^{\left(\frac{1}{2}\right)} \otimes \Big(\mathbb{E}^{\mu}_{Cl,g}\Big[Y_\otimes^{(2)}\Big]\Big)_\otimes^{\left(\frac{1}{2}\right)} \leq K_2 \otimes \mathbb{E}^{\mu}_{Cl,g}[X\otimes Y] \tag{3.2}$$

hold where $K_1 = g^{-1}\left(\sqrt{\frac{m}{M}}\right)$ and $K_2 = g^{-1}\left(\sqrt{\frac{M}{m}}\right)$.

Proof We will prove (3.1) and the other case is similar. Since $g\big(X(\omega)\big) \geqslant mg\big(Y(\omega)\big)$, then

$$\big(g\big(X(\omega)\big)\big)^2 \geqslant m\big(g\left(X(\omega)\right)g\big(Y(\omega)\big)\big),$$

and

$$\Big(\mathbb{E}_C^{g(\mu)}\big[(g(X))^2\big]\Big)^{\frac{1}{2}} \geqslant \sqrt{m}\Big(\mathbb{E}_C^{g(\mu)}\big[g(X)\,g(Y)\big]\Big)^{\frac{1}{2}}. \tag{3.3}$$

Also, by $M\big(g(Y(\omega))\big) \geqslant g\big(X(\omega)\big)$, we have

$$\Big(g(Y(\omega))\Big)^2 \geqslant \frac{1}{M}\Big(g(X(\omega))\,g(Y(\omega))\Big),$$

and then

$$\Big(\mathbb{E}_C^{g(\mu)}\big[(g(Y))^2\big]\Big)^{\frac{1}{2}} \geqslant \sqrt{\frac{1}{M}}\Big(\mathbb{E}_C^{g(\mu)}\left[(g(X)\,g(Y))\right]\Big)^{\frac{1}{2}}. \tag{3.4}$$

Therefore, (3.3) and (3.4) give us the following result:

$$\Big(\mathbb{E}_C^{g(\mu)}\big[(g(X))^2\big]\Big)^{\frac{1}{2}}\Big(\mathbb{E}_C^{g(\mu)}\big[(g(Y))^2\big]\Big)^{\frac{1}{2}} \geqslant \sqrt{\frac{m}{M}}\,\mathbb{E}_C^{g(\mu)}\Big[g(X)\,g(Y)\Big]. \tag{3.5}$$

Let $K_1 = g^{-1}\left(\sqrt{\frac{m}{M}}\right)$. Now, observe that

$$K_1 \otimes \mathbb{E}^{\mu}_{Cl,g}\big[(X\otimes Y)\big] = g^{-1}\Big(g(K_1)\,g\Big(\mathbb{E}^{\mu}_{Cl,g}\left[(X\otimes Y)\right]\Big)\Big)$$
$$= g^{-1}\left(\sqrt{\frac{m}{M}}\,\mathbb{E}_C^{g(\mu)}\left[g\left((X\otimes Y)\right)\right]\right)$$
$$= g^{-1}\Big(g(K_1)\,\mathbb{E}_C^{g(\mu)}\left[(g\circ X)\cdot(g\circ Y)\right]\Big). \tag{3.6}$$

Using (3.5) and (3.6), we have

$$g^{-1}\left(\sqrt{\frac{m}{M}}\Big(\mathbb{E}_C^{g(\mu)}\big[\left((g\circ X)\cdot(g\circ Y)\right)\big]\Big)\right)$$
$$\leqslant g^{-1}\left(\Big(\mathbb{E}_C^{g(\mu)}\big[(g\circ X)^2\big]\Big)^{\frac{1}{2}}\cdot\Big(\mathbb{E}_C^{g(\mu)}\big[(g\circ Y)^2\big]\Big)^{\frac{1}{2}}\right)$$
$$= g^{-1}\left(\left[g\left(g^{-1}\left(\Big(\mathbb{E}_C^{g(\mu)}\big[(g\circ X)^2\big]\Big)^{\frac{1}{2}}\right)\right)\cdot g\left(g^{-1}\left(\Big(\mathbb{E}_C^{g(\mu)}\big[(g\circ Y)^2\big]\Big)^{\frac{1}{2}}\right)\right)\right]\right)$$
$$= g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\Big[g\Big(g^{-1}\left((g\circ X)^2\right)\Big)\Big]\right)^{\frac{1}{2}}\right)\otimes g^{-1}\left(\left(\mathbb{E}_C^{g(\mu)}\Big[g\Big(g^{-1}\left((g\circ Y)^2\right)\Big)\Big]\right)^{\frac{1}{2}}\right)$$
$$= g^{-1}\left(\Big(\mathbb{E}_C^{g(\mu)}[g(X_\otimes^{(2)})]\Big)^{\frac{1}{2}}\right)\otimes g^{-1}\left(\Big(\mathbb{E}_C^{g(\mu)}\big[g\big(Y_\otimes^{(2)}\big)\big]\Big)^{\frac{1}{2}}\right)$$
$$= g^{-1}\left(\left(g\left(g^{-1}\left(\mathbb{E}_C^{g(\mu)}\big[g\big(X_\otimes^{(2)}\big)\big]\right)\right)\right)^{\frac{1}{2}}\right) \otimes g^{-1}\left(\left(g\left(g^{-1}\left(\mathbb{E}_C^{g(\mu)}\big[g\big(Y_\otimes^{(2)}\big)\big]\right)\right)\right)^{\frac{1}{2}}\right)$$
$$= g^{-1}\left(\left(g\left(\mathbb{E}^{\mu}_{Cl,g}[X_\otimes^{(2)}]\right)\right)^{\frac{1}{2}}\right)\otimes g^{-1}\left(\left(g\left(\mathbb{E}^{\mu}_{Cl,g}[Y_\otimes^{(2)}]\right)\right)^{\frac{1}{2}}\right)$$
$$= \left(\mathbb{E}^{\mu}_{Cl,g}[X_\otimes^{(2)}]\right)_\otimes^{\left(\frac{1}{2}\right)} \otimes \left(\mathbb{E}^{\mu}_{Cl,g}[Y_\otimes^{(2)}]\right)_\otimes^{\left(\frac{1}{2}\right)}.$$

This completes the Proof. □

Example 3.2 (i) Let $g(x) = x^\alpha, \alpha > 0$. The corresponding pseudo-operations are $x \oplus y = \sqrt[\alpha]{x^\alpha + y^\alpha}$ and

$x \otimes y = xy$. Then (3.1) and (3.2) reduce on the following inequalities

$$K_1 \sqrt[\alpha]{\mathbb{E}_C^{\mu^\alpha}\left[(XY)^\alpha\right]} \leq \sqrt[2\alpha]{\mathbb{E}_C^{\mu^\alpha}\left[X^{2\alpha}\right]}\ \sqrt[2\alpha]{\mathbb{E}_C^{\mu^\alpha}\left[Y^{2\alpha}\right]},$$

$$K_2 \sqrt[\alpha]{\mathbb{E}_C^{\mu^\alpha}\left[(XY)^\alpha\right]} \geqslant \sqrt[2\alpha]{\mathbb{E}_C^{\mu^\alpha}\left[X^{2\alpha}\right]}\ \sqrt[2\alpha]{\mathbb{E}_C^{\mu^\alpha}\left[Y^{2\alpha}\right]}$$

where $K_1 = \sqrt[2\alpha]{\frac{m}{M}}$ and $K_2 = \sqrt[2\alpha]{\frac{m}{M}}$.

(ii) Let $g(x) = \ln(x+1)$. The corresponding pseudo-operations are $x \oplus y = x + y + xy$ and $x \otimes y = (x+1)^{\ln(y+1)} - 1$. Then (3.1) and (3.2) reduce on the following inequalities

$$(K_1+1)^{\left(\mathbb{E}_C^{\ln(\mu+1)}[((\ln(X+1))\cdot(\ln(Y+1)))]\right)} \leq e^{\left(\mathbb{E}_C^{\ln(\mu+1)}[(\ln(X+1))^2]\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}_C^{\ln(\mu+1)}[(\ln(Y+1))^2]\right)^{\frac{1}{2}}},$$

$$(K_2+1)^{\left(\mathbb{E}_C^{\ln(\mu+1)}[((\ln(X+1))\cdot(\ln(Y+1)))]\right)} \geqslant e^{\left(\mathbb{E}_C^{\ln(\mu+1)}[(\ln(X+1))^2]\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}_C^{\ln(\mu+1)}[(\ln(Y+1))^2]\right)^{\frac{1}{2}}},$$

where $K_1 = e^{\sqrt{\frac{m}{M}}} - 1$ and $K_2 = e^{\sqrt{\frac{M}{m}}} - 1$.

Now we consider the second class of Choquet-like integrals where is based on $\vee$ and a corresponding pseudo-multiplication $\otimes$.

Theorem 3.3 (Cauchy–Schwarz's inequality) *Fix a real monotone measure μ. Let X, Y be two non-negative measurable functions such that*

$$0 < t \leq \frac{X(\omega)}{Y(\omega)} \leq T,$$

for any $\omega \in \Omega$. If $\otimes$ is a pseudo-multiplication satisfying

$$(a \otimes b) \leq \min\left\{\frac{\|\mu\|}{t}\left(\frac{at}{\|\mu\|} \otimes b\right), T\,\|\mu\|\left(\frac{a}{\|\mu\|\,T} \otimes b\right)\right\} \tag{3.7}$$

then for the $\mathbb{S}_\mu^\otimes$ integral (2.3), the inequality

$$K\mathbb{S}_\mu^\otimes[XY] \leq \sqrt{\|\mu\|\,\mathbb{S}_\mu^\otimes\left[\frac{1}{\|\mu\|}X^2\right]}\sqrt{\|\mu\|\,\mathbb{S}_\mu^\otimes\left[\frac{1}{\|\mu\|}Y^2\right]}$$

holds where $K = \sqrt{\frac{t}{T}}$.

Proof Since $t \leq \frac{X(\omega)}{Y(\omega)}$, we have

$$t\Big(X(\omega)\,Y(\omega)\Big) \leq \Big(X(\omega)\Big)^2.$$

The monotonicity of $\mathbb{S}_\mu^\otimes$ integral, and Lemma 2.12 imply that

$$\sqrt{\mathbb{S}_\mu^\otimes\left[\frac{1}{\|\mu\|}X^2\right]} \geqslant \sqrt{\mathbb{S}_\mu^\otimes\left[\frac{t}{\|\mu\|}XY\right]} \geqslant \sqrt{\frac{t}{\|\mu\|}\mathbb{S}_\mu^\otimes[XY]}. \tag{3.8}$$

Also, since $T \geqslant \frac{X(\omega)}{Y(\omega)}$ we have

$$\big(Y(\omega)\big)^2 \geqslant \frac{1}{T}X(\omega)\,Y(\omega),$$

and then,

$$\sqrt{\mathbb{S}_\mu^\otimes\left[\frac{1}{\|\mu\|}Y^2\right]} \geqslant \sqrt{\mathbb{S}_\mu^\otimes\left[\frac{1}{\|\mu\|\,T}XY\right]} \geqslant \sqrt{\frac{1}{\|\mu\|\,T}\mathbb{S}_\mu^\otimes[XY]}. \tag{3.9}$$

By multiplying (3.8) and (3.9), we have

$$K\mathbb{S}_\mu^\otimes[XY] \leq \sqrt{\|\mu\|\,\mathbb{S}_\mu^\otimes\left[\frac{1}{\|\mu\|}X^2\right]}\sqrt{\|\mu\|\,\mathbb{S}_\mu^\otimes\left[\frac{1}{\|\mu\|}Y^2\right]}.$$

□

Let $\otimes$ be the standard product (i.e., Shilkret integral) in Theorem 3.3. Then the following result holds.

Corollary 3.4 *Let X, Y be two non-negative measurable functions such that*

$$0 < t \leq \frac{X(\omega)}{Y(\omega)} \leq T,$$

for any $\omega \in \Omega$. The inequality

$$K\,\mathrm{Sh}_\mu[XY] \leq \sqrt{\|\mu\|\,\mathrm{Sh}_\mu\left[\frac{1}{\|\mu\|}X^2\right]}\sqrt{\|\mu\|\,\mathrm{Sh}_\mu\left[\frac{1}{\|\mu\|}Y^2\right]}$$

holds where $K = \sqrt{\frac{t}{T}}$.

Notice that if $\otimes$ is minimum (i.e., for Sugeno integral) and $\|\mu\| \geqslant \max\left\{\frac{1}{T}, t\right\}$ in Theorem 3.3 then (3.7) holds readily. Then the following result holds.

Corollary 3.5 *Let X, Y be two non-negative measurable functions such that*

$$0 < t \leq \frac{X(\omega)}{Y(\omega)} \leq T,$$

for any $\omega \in \Omega$. Then, for any monotone measure μ such that $\|\mu\| \geqslant \max\left\{\frac{1}{T}, t\right\}$, the inequality

$$K\,\mathrm{Su}_\mu[XY] \leq \sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}X^2\right]}\sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}Y^2\right]} \tag{3.10}$$

holds where $K = \sqrt{\frac{t}{T}}$ and $\|\mu\| \geqslant \max\left\{\frac{1}{T}, t\right\}$.

The following example proves that the inequality of Corollary 3.5 is sharp.

Example 3.6 Let $\Omega = [0, 2], X(\omega) = Y(\omega) \equiv 1$. Let $\mu(A) = \lambda(A)$ where λ is the Lebesgue measure on $\mathbb{R}$. Clearly $\frac{X(\omega)}{Y(\omega)} = 1$. Then $t = T = 1$. So, by Corollary 3.5, $K = 1$ and

$$\mathrm{Su}_\mu[XY] = 1, \quad \mathrm{Su}_\mu\left[\frac{X^2}{\|\mu\|}\right] = \mathrm{Su}_\mu\left[\frac{Y^2}{\|\mu\|}\right] = \frac{1}{2}.$$

Therefore,

$$K\,\mathrm{Su}_\mu[XY] = 1 = \sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}X^2\right]} \times\sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}Y^2\right]}.$$

Example 3.7 Let $\Omega = [0, 2], X(\omega) = 2e^{\omega}, Y(\omega) = e^{-\omega}$. Let $\mu(A) = \lambda(A)$ where λ is the Lebesgue measure on $\mathbb{R}$. Clearly $t = 2 \le \frac{X(\omega)}{Y(\omega)} = 2e^{2\omega} \le 2e^4 = T$. So, by Corollary 3.5, $K = e^{-2}$ and

$$\mathrm{Su}_\mu[XY] = 2, \quad \mathrm{Su}_\mu\left[\frac{X^2}{2}\right] = 2, \quad \mathrm{Su}_\mu\left[\frac{Y^2}{2}\right] = 0.2836.$$

Therefore,

$$0.27067 = K\,Su_\mu[XY] \le \sqrt{\|\mu\|\,Su_\mu\left[\frac{1}{\|\mu\|}X^2\right]} \times\sqrt{\|\mu\|\,Su_\mu\left[\frac{1}{\|\mu\|}Y^2\right]} = 1.5063.$$

Note 3.8 It is easy to see that

$$\|\mu\|\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}X^2\right] \le (\|\mu\| \vee 1)\,\mathrm{Su}_\mu\left[X^2\right], \tag{3.11}$$

$$\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}Y^2\right] \le (\|\mu\| \vee 1)\,\mathrm{Su}_\mu\left[Y^2\right]. \tag{3.12}$$

Therefore, by (3.10), (3.11) and (3.12), we have

$$K\,\mathrm{Su}_\mu[XY] \le \sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}X^2\right]}\sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}Y^2\right]} \le (\|\mu\| \vee 1)\sqrt{\mathrm{Su}_\mu\left[X^2\right]\mathrm{Su}_\mu\left[Y^2\right]}.$$

The following example shows that the condition $\|\mu\| \geqslant \max\left\{\frac{1}{T}, t\right\}$ in Corollary 3.5 cannot be omitted.

Example 3.9 Let $\Omega = [0, 4], X(\omega) = 4, Y(\omega) = \frac{1}{2}$. Let $\mu(A) = \sqrt{\lambda(A)}$ where λ is the Lebesgue measure on $\mathbb{R}$. Clearly $3 \le \frac{X(\omega)}{Y(\omega)} = 8 \le 9$ for all ω. Put $t = 3$ and $T = 9$. So, by Corollary 3.5, $K = \sqrt{\frac{t}{T}} = \frac{1}{3}\sqrt{3}$. Then

$$\mathrm{Su}_\mu[XY] = 2, \quad \mathrm{Su}_\mu\left[\frac{X^2}{2}\right] = 2, \quad \mathrm{Su}_\mu\left[\frac{Y^2}{2}\right] = \frac{1}{8}.$$

Therefore,

$$\frac{2}{3}\sqrt{3} = K\,\mathrm{Su}_\mu[XY] > \sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}X^2\right]} \times\sqrt{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}Y^2\right]} = 1.$$

Notice that when working on [0, 1] in Theorem 3.3, then $\otimes = \circledast$ is semicopula (t-seminorm) and the following result holds.

Corollary 3.10 *Let $X, Y : \Omega \to [0, 1]$ be two non-negative measurable functions such that*

$$0 < t \le \frac{X(\omega)}{Y(\omega)} \le T,$$

for any $\omega \in \Omega$. If semicopula $\circledast$ satisfying

$$(a \circledast b) \le \min\left\{\frac{1}{t}(ta \circledast b),\, T\left(\frac{a}{T} \circledast b\right)\right\} \tag{3.13}$$

for all $a, b \in [0, 1]$ such that $a \le \min\left\{\frac{1}{t}, T\right\}$, then for the $\mathbb{S}_\mu^\circledast$ integral (2.4), the inequality

$$K\mathbb{S}_\mu^\circledast[XY] \le \sqrt{\mathbb{S}_\mu^\circledast\left[X^2\right]}\sqrt{\mathbb{S}_\mu^\circledast\left[Y^2\right]}$$

holds where $K = \sqrt{\frac{t}{T}}$.

Remark 3.11 If semicopula $\circledast$ is the standard product, then an inequality for the original Shilkret integral is recaptured.

Let semicopula $\circledast$ be minimum (i.e., for the original Sugeno integral). Then we have the following result:

Corollary 3.12 *Let $X, Y : \Omega \to [0, 1]$ be two non-negative measurable functions such that*

$$0 < t \le \frac{X(\omega)}{Y(\omega)} \le T,$$

for any $\omega \in \Omega$. Then the inequality

$$K\,\mathrm{Su}_\mu[XY] \le \sqrt{\mathrm{Su}_\mu\left[X^2\right]}\sqrt{\mathrm{Su}_\mu\left[Y^2\right]}$$

holds where $K = \sqrt{\frac{t}{T}}$ and $1 \geqslant \max\left\{\frac{1}{T}, t\right\}$.

The following example shows that the condition (3.13) in Corollary 3.10 cannot be omitted.

Example 3.13 Let $\Omega = [0, 1], X(\omega) = \frac{\omega+1}{2}, Y(\omega) = \frac{1}{2}$. Let semicopula $\circledast$ be the Łukasiewicz t-norm T_L(i.e.,

$T_L(x, y) = \max\{(x + y - 1), 0\}$ and $\mu(A) = \lambda(A)$ where λ is the Lebesgue measure on $\mathbb{R}$. Clearly $1 \le \frac{X(\omega)}{Y(\omega)} = \omega + 1 \le 2$. Put $t = 1$ and $T = 2$. So, by Corollary 3.10, $K = \frac{1}{2}\sqrt{2}$ and

$$\begin{aligned}
\mathbb{S}_\mu^{T_L}[XY] &= \bigvee_{\alpha\in[0,1]} T_L(\alpha, \mu(\{XY \ge \alpha\})) \\
&= \bigvee_{\alpha\in[0,1]} T_L(\alpha, 2 - 4\alpha) = \bigvee_{a\in[0,1]} (1 - 3\alpha) = 1, \\
\mathbb{S}_\mu^{T_L}\left[X^2\right] &= \bigvee_{\alpha\in[0,1]} T_L\left(\alpha, \mu(\left\{X^2 \ge \alpha\right\}\right) \\
&= \bigvee_{\alpha\in[0,1]} T_L\left(\alpha, 2 - 2\sqrt{\alpha}\right) \\
&= \bigvee_{\alpha\in[0,1]} \left(\sqrt{\alpha} - 1\right)^2 = 1, \\
\mathbb{S}_\mu^{T_L}\left[Y^2\right] &= \bigvee_{\alpha\in[0,1]} T_L\left(\alpha, \mu(\left\{Y^2 \ge \alpha\right\}\right) \\
&= \bigvee_{a\in\left[0,\frac{1}{4}\right]} T_L(\alpha, 1) = \frac{1}{4}.
\end{aligned}$$

Therefore,

$$K\mathbb{S}_\mu^{T_L}[XY] = \frac{1}{2}\sqrt{2} > \frac{1}{2} = \sqrt{\mathbb{S}_\mu^{T_L}[X^2]}\sqrt{\mathbb{S}_\mu^{T_L}[Y^2]}.$$

We can prove Hölder's inequality for the second class of Choquet-like integrals, i.e., for $\mathbb{S}_\mu^{\otimes}$ integral.

Theorem 3.14 (Hölder's inequality) *Fix a real monotone measure μ. Let X, Y be two non-negative measurable functions such that*

$$0 < t \le \frac{(X(\omega))^{p-1}}{Y(\omega)} \text{ and } 0 < \frac{X(\omega)}{(Y(\omega))^{q-1}} \le T$$

for any $\omega \in \Omega$. If $\otimes$ is a pseudo-multiplication satisfying

$$(a \otimes b) \le \min\left\{\frac{\|\mu\|}{t}\left(\frac{ta}{\|\mu\|} \otimes b\right), T\|\mu\|\left(\frac{a}{\|\mu\| T} \otimes b\right)\right\},$$

then for the $\mathbb{S}_\mu^{\otimes}$ integral (2.3), the inequality

$$C\mathbb{S}_\mu^{\otimes}[XY] \le \sqrt[p]{\|\mu\|\,\mathbb{S}_\mu^{\otimes}\left[\frac{1}{\|\mu\|}X^p\right]}\sqrt[q]{\|\mu\|\,\mathbb{S}_\mu^{\otimes}\left[\frac{1}{\|\mu\|}Y^q\right]}$$

holds where $C = \frac{t^{\frac{1}{p}}}{T^{\frac{1}{q}}}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Since $t \le \frac{(X(\omega))^{p-1}}{Y(\omega)}$, we have

$$t\,(X(\omega)\,Y(\omega)) \le (X(\omega))^p.$$

The monotonicity of $\mathbb{S}_\mu^{\otimes}$ integral, Lemma 2.12 imply that

$$\sqrt[p]{\mathbb{S}_\mu^{\otimes}\left[\frac{1}{\|\mu\|}X^p\right]} \geqslant \sqrt[p]{\mathbb{S}_\mu^{\otimes}\left[\frac{t}{\|\mu\|}XY\right]} \geqslant \sqrt[p]{\frac{t}{\|\mu\|}\mathbb{S}_\mu^{\otimes}[XY]}. \tag{3.14}$$

Also, since $\frac{X(\omega)}{(Y(\omega))^{q-1}} \le T$, we have

$$\left(Y(\omega)\right)^q \geqslant \frac{1}{T}X(\omega)\,Y(\omega),$$

and then,

$$\sqrt[q]{\mathbb{S}_\mu^{\otimes}\left[\frac{1}{\|\mu\|}Y^q\right]} \geqslant \sqrt[q]{\mathbb{S}_\mu^{\otimes}\left[\frac{1}{\|\mu\|\,T}XY\right]} \geqslant \sqrt[q]{\frac{1}{\|\mu\|\,T}\mathbb{S}_\mu^{\otimes}[XY]}. \tag{3.15}$$

By multiplying (3.14) and (3.15), we have

$$C\mathbb{S}_\mu^{\otimes}[XY] \le \sqrt[p]{\|\mu\|\,\mathbb{S}_\mu^{\otimes}\left[\frac{1}{\|\mu\|}X^p\right]}\sqrt[q]{\|\mu\|\,\mathbb{S}_\mu^{\otimes}\left[\frac{1}{\|\mu\|}Y^q\right]}.$$

□

Corollary 3.15 *Let X, Y be two non-negative measurable functions such that*

$$t \le \frac{(X(\omega))^{p-1}}{Y(\omega)} \text{ and } \frac{X(\omega)}{(Y(\omega))^{q-1}} \le T$$

for any $\omega \in \Omega$. Then the inequality

$$CSh_\mu[XY] \le \sqrt[p]{\|\mu\|\,Sh_\mu\left[\frac{1}{\|\mu\|}X^p\right]}\sqrt[q]{\|\mu\|\,Sh_\mu\left[\frac{1}{\|\mu\|}Y^q\right]}$$

holds where $C = \frac{t^{\frac{1}{p}}}{T^{\frac{1}{q}}}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 3.16 *Let X, Y be two non-negative measurable functions such that*

$$0 < t \le \frac{(X(\omega))^{p-1}}{Y(\omega)} \text{ and } 0 < \frac{X(\omega)}{(Y(\omega))^{q-1}} \le T$$

for any $\omega \in \Omega$. Then the inequality

$$\begin{aligned}
C\mathrm{Su}_\mu[XY] &\le \sqrt[p]{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}X^p\right]}\sqrt[q]{\|\mu\|\,\mathrm{Su}_\mu\left[\frac{1}{\|\mu\|}Y^q\right]} \\
&\le (\|\mu\| \vee 1)\,\sqrt[p]{\mathrm{Su}_\mu[X^p]}\sqrt[q]{\mathrm{Su}_\mu[Y^q]}
\end{aligned}$$

holds where $C = \frac{t^{\frac{1}{p}}}{T^{\frac{1}{q}}}$, and $\|\mu\| \geqslant \max\left\{\frac{1}{T}, t\right\}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 3.17 *Let $X, Y : \Omega \to [0, 1]$ be two non-negative measurable functions such that*

$$0 < t \leq \frac{\left(X(\omega)\right)^{p-1}}{Y(\omega)} \text{ and } 0 < \frac{X(\omega)}{\left(Y(\omega)\right)^{q-1}} \leq T$$

for any $\omega \in \Omega$. Then for the $\mathbb{S}_\mu^{\circledast}$ integral (2.4), the inequality

$$(a \circledast b) \leq \min\left\{\frac{1}{t}\big(ta \circledast b\big), T\big(\frac{a}{T} \circledast b\big)\right\}$$

for all $a, b \in [0, 1]$ such that $a \leq \min\left\{\frac{1}{t}, T\right\}$, then for the $\mathbb{S}_\mu^{\circledast}$ integral (2.4), the inequality

$$C\mathbb{S}_\mu^{\circledast}[XY] \leq \sqrt[p]{\mathbb{S}_\mu^{\circledast}[X^p]}\sqrt[q]{\mathbb{S}_\mu^{\circledast}[Y^q]}$$

holds where $C = \frac{t^{\frac{1}{p}}}{T^{\frac{1}{q}}}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

4 Conclusions

We have shown a version of Cauchy–Schwarz's inequality without the comonotonicity condition for two classes of Choquet-like integrals. At first, two classes of Choquet-like integrals were introduced. Then, we prepared extensions of these inequalities from the Choquet expectation and the Sugeno integral to the two classes of Choquet-like integrals. Recently, Agahi and Mesiar (2014) proved a new version of Minkowski's inequality for Sugeno integral without the comonotonicity condition.

Acknowledgments The second author was supported by Grant VEGA 1/0171/12.

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