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Pseudo-fractional integral inequality of Chebyshev type

Hamzeh Agahi [a,*], Azizollah Babakhani [a], Radko Mesiar [b,c]

[a] Department of Mathematics, Faculty of Basic Sciences, Babol University of Technology, Babol 47148-71167, Iran
[b] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, SK-81368 Bratislava, Slovakia
[c] Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic

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ABSTRACT

In this paper, we give a general version of Chebyshev type inequality for pseudo-convolution integral on a semiring $([a,b],\oplus,\odot)$. Our result is flexible enough to support both pseudo-integral and convolution integral, (e.g., fractional integral), thus closing the series of papers. It includes the corresponding results of Agahi et al. [1] as a special case. Finally, some concluding remarks are drawn and some open problems for further investigations are given.

1. Introduction

Convolution integral plays an important role in several theoretical and applied fields. For instance, it is a useful tool in differential equations, probability, statistics, computer vision, image and signal processing and electrical engineering [3,5,6,8,12,21,26,27]. Let f and Φ be two real-valued functions on $[0,\infty)$. The convolution of these functions is defined by [22]

$$(\Phi * f)(t) = \int_0^t \Phi(t-x)f(x)dx$$

whenever the integral is defined. A main property of convolution integral is

$$(\Phi * f)(t) = (f * \Phi)(t).$$

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is defined on a real interval $[a,b] \subset [-\infty,\infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$, see [16,25]. Note that the pseudo-integrals have shown their usefulness in several applications, for example in the area of nonlinear partial differential equations occurring in different applied fields, see [15] as well as the edited volume [10].

The pseudo-convolution of the functions was introduced in [18], by means of the corresponding pseudo-integral [16,14,25]. The aim of this contribution is to give an inequality related to Chebyshev for pseudo-convolution integral (see Fig. 1). This inequality is flexible enough to support both pseudo-integral and convolution integral. Recently, there were obtained generalizations of the classical integral inequalities with respect to pseudo-integrals [1,20].

* Corresponding author.
E-mail addresses: h_agahi@nit.ac.ir (H. Agahi), mesiar@math.sk (R. Mesiar).

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The paper is organized as follows. In the next section, we briefly recall some preliminaries and summarization of some previous known results. In Section 3, we will focus on an inequality related to Chebyshev for pseudo-convolution integral. Finally, some concluding remarks are added.

2. Preliminaries

In this section, we recall some well known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [1,16,20]).

Let $[a,b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. A binary operation $\oplus$ on $[a,b]$ is pseudo-addition if it is commutative, non-decreasing (with respect to $\preceq$), continuous, associative, and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a,b]_+ = \{x | x \in [a,b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a,b]$ is pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a,b]_+$, associative and with a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$, $\mathbf{1} \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

The structure $([a,b], \oplus, \odot)$ is a *semiring* (see [7]).

Let X be a non-empty set. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X.

Definition 2.1 [18]. A set function $m : \mathcal{A} \to [a,b]_+$ (or semiclosed interval) is a $\oplus$-measure if there holds:

(i) $m(\phi) = \mathbf{0}$ (if $\oplus$ is not idempotent);
(ii) m is σ-$\oplus$-(decomposable) measure, i.e.

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$$

holds for any sequence $\{A_i\}_{i\in N}$ of pairwise disjoint sets from $\mathcal{A}$. If $\oplus$ is idempotent operation condition (i) can be left out and sets from sequence $\{A_i\}$ do not have to be pairwise disjointed.

We shall consider the semiring $([a,b], \oplus, \odot)$ for two important (with completely different behavior) cases. First class is when pseudo-operations are generated by a monotone and continuous function $g : [a,b] \to [0,\infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \text{ and } x \odot y = g^{-1}(g(x)g(y)).$$

Then the pseudo-integral for a function $f : [c,d] \to [a,b]$ reduces on the g-integral [16,17],

$$\int_{[c,d]}^{\oplus} f \odot dm = g^{-1}\left(\int_c^d g(f(x))dx\right).$$

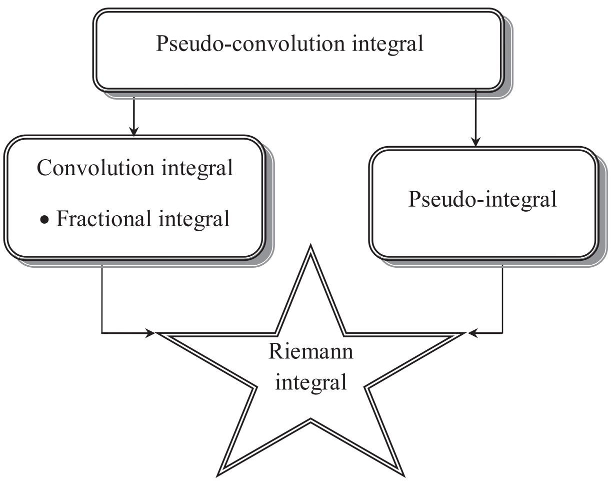

Fig. 1. Pseudo-convolution integral.

Since the generator g is an increasing function, then f is said to be integrable if $\int_{[c,d]}^{\oplus} f \odot dm < \infty$. More on this structure as well as on corresponding measures and integrals can be found in [16,17].

Recently, Agahi et al. [1] proved the Chebyshev type inequality for the first class of pseudo-integrals based on two comonotone functions. Notice that two functions $u, v : X \to \mathbb{R}$ are said to be comonotone if and only if $(u(x) - u(y))(v(x) - v(y)) \geqslant 0$ for each couple of elements $x, y \in X$, and u and v are said to be countermonotone if for all $x, y \in X$, $(u(x) - u(y))(v(x) - v(y)) \leqslant 0$.

Theorem 2.2 (*Chebyshev's inequality for the first class of pseudo-integrals*). *Let $u, v : [0,1] \to [a,b]$ be two measurable functions and let a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$\int_{[0,1]}^{\oplus} (u \odot v) \odot dm \geqslant \left(\int_{[0,1]}^{\oplus} u \odot dm \right) \odot \left(\int_{[0,1]}^{\oplus} v \odot dm \right)$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

The second class of pseudo-integrals is when $x \oplus y = \sup(x,y)$ and $x \odot y = g^{-1}(g(x)g(y))$, the pseudo-integral for a function $f : \mathbb{R} \to [a,b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f(x) \odot dm = \sup_{x \in \mathbb{R}} (f(x) \odot \psi(x)),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [11].

We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = ess \sup_{\mu} (x | x \in A) = \sup\{a | \mu(\{x | x \in A, x > a\}) > 0\}.$$

Theorem 2.3 [11]. *Let m be a sup-measure on $([0,\infty], \mathcal{B}([0,\infty]))$, where $\mathcal{B}([0,\infty])$ is the Borel σ-algebra on $[0,\infty]$, $m(A) = \text{esssup}_{\mu}(\psi(x) | x \in A)$, and $\psi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure on $([0,\infty), \mathcal{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0,\infty)$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.*

Theorem 2.4 [11]. *Let $([0,\infty], \sup, \odot)$ be a semiring with $\odot$ with a generator g, i.e.,we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a,b]$. Let m be the same as in Theorem 2.3. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by $g^\lambda, \lambda \in (0,\infty)$ such that for every continuous function $f : [0,\infty] \to [0,\infty]$*

$$\int^{\sup} f \odot dm = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda = \lim_{\lambda\to\infty} (g^\lambda)^{-1} \left(\int g^\lambda (f(x)) dx \right).$$

In [1], Agahi et al. proved the following result for the second class of pseudo-integral, when $\oplus = \sup$, and $\odot = g^{-1}(g(x)g(y))$.

Theorem 2.5 (*Chebyshev's inequality for the second class of pseudo-integrals*). *Let $u, v : [0,1] \to [a,b]$ be two continuous functions and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.3. If u and v are comonotone, then the inequality*

$$\int_{[0,1]}^{\sup} (u \odot v) \odot dm \geqslant \left(\int_{[0,1]}^{\sup} u \odot dm \right) \odot \left(\int_{[0,1]}^{\sup} v \odot dm \right)$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

3. Main results

This section provides an inequality related to Chebyshev type for pseudo-convolution integral. Now, our results can be stated as follows.

Theorem 3.1. *Let $b > a \geqslant 0$. Let $u, v, h_1, h_2 : [0,\infty) \to [a,b]$ be integrable functions and let a generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$\begin{aligned}[(h_1 * u)(t) \odot (h_2 * v)(t)] \oplus [(h_1 * v)(t) \odot (h_2 * u)(t)] \leqslant [(h_2 * \mathbf{1})(t) \odot (h_1 * (u \odot v))] \\ \oplus [(h_1 * \mathbf{1})(t) \odot (h_2 * (u \odot v))(t)] \end{aligned} \quad (3.1)$$

holds where the symbol $h_i * u, i = 1,2$ *denote the g- convolution of* h_i *and* u *that are defined by* [18]

$$(h_i * u)(t) = \int_{[0,t]}^{\oplus} [h_i(t-x) \odot u(x)] \odot dm, \quad i = 1,2$$

for all $t \in [0,\infty)$.

Proof. If u and v are comonotone, and g is an increasing function, then the composition $g \circ u$ and $g \circ v$ are also comonotone. So, for all $\tau \geqslant 0, p \geqslant 0$, we have

$$(g \circ u(\tau) - g \circ u(p))(g \circ v(\tau) - g \circ v(p)) \geqslant 0.$$

Then it is easy to see that

$$\begin{aligned}
&\left(\int_0^t (g \circ h_2(t-p))dp\right)\left(\int_0^t (g \circ h_1(t-\tau))(g \circ u(\tau))(g \circ v(\tau))d\tau\right) \\
&+ \left(\int_0^t (g \circ h_1(t-\tau))d\tau\right)\left(\int_0^t (g \circ h_2(t-p))(g \circ u(p))(g \circ v(p))dp\right) \\
&\geqslant \left(\int_0^t (g \circ h_1(t-\tau))(g \circ u(\tau))d\tau\right)\left(\int_0^t (g \circ h_2(t-p))(g \circ v(p))dp\right) \\
&+ \left(\int_0^t (g \circ h_1(t-\tau))(g \circ v(\tau))d\tau\right)\left(\int_0^t (g \circ h_2(t-p))(g \circ u(p))dp\right).
\end{aligned}$$

Since function g is an increasing function, then g^{-1} is also an increasing function and we have

$$\begin{aligned}
&g^{-1}\left(\begin{array}{c} \left(\int_0^t (g \circ h_2(t-p))dp\right)\left(\int_0^t (g \circ h_1(t-\tau))(g \circ u(\tau))(g \circ v(\tau))d\tau\right) \\ +\left(\int_0^t (g \circ h_1(t-\tau))d\tau\right)\left(\int_0^t (g \circ h_2(t-p))(g \circ u(p))(g \circ v(p))dp\right) \end{array}\right) \\
&\geqslant g^{-1}\left(\begin{array}{c} \left(\int_0^t (g \circ h_1(t-\tau))(g \circ u(\tau))d\tau\right)\left(\int_0^t (g \circ h_2(t-p))(g \circ v(p))dp\right) \\ +\left(\int_0^t (g \circ h_1(t-\tau))(g \circ v(\tau))d\tau\right)\left(\int_0^t (g \circ h_2(t-p))(g \circ u(p))dp\right) \end{array}\right).
\end{aligned} \tag{3.2}$$

Hence

$$\begin{aligned}
&g^{-1}\left[\left(\int_0^t g \circ h_2(t-p)dp\right) \cdot \left(\int_0^t (g \circ h_1(t-\tau))(g \circ u(\tau))(g \circ v(\tau))d\tau\right)\right] \\
&= g^{-1}\left[g\left(g^{-1}\left(\int_0^t g \circ h_2(t-p)dp\right)\right) \cdot g\left(g^{-1}\left(\int_0^t (g \circ h_1(t-\tau))(g \circ u(\tau))(g \circ v(\tau))d\tau\right)\right)\right] \\
&= g^{-1}\left(\int_0^t g \circ h_2(t-p)dp\right) \odot g^{-1}\left(\int_0^t (g \circ h_1(t-\tau))(g \circ u(\tau))(g \circ v(\tau))d\tau\right) \\
&= g^{-1}\left(\int_0^t g \circ h_2(t-p) \cdot g \circ g^{-1}(1)dp\right) \odot g^{-1}\left(\int_0^t (g \circ h_1(t-\tau)) \cdot g(u(\tau) \odot v(\tau))d\tau\right) \\
&= g^{-1}\left(\int_0^t g \circ \left(h_2(t-p) \odot g^{-1}(1)\right)dp\right) \odot g^{-1}\left(\int_0^t g \circ (h_1(t-\tau) \odot (u(\tau) \odot v(\tau)))d\tau\right) \\
&= \left(\int_{[0,t]}^{\oplus} (h_2(t-p) \odot g^{-1}(1)) \odot dm\right) \odot \left(\int_{[0,t]}^{\oplus} (h_1(t-\tau) \odot (u(\tau) \odot v(\tau))) \odot dm\right) \\
&= (h_2 * \mathbf{1})(t) \odot (h_1 * (u \odot v))(t).
\end{aligned} \tag{3.3}$$

Similarly,

$$g^{-1}\left[\left(\int_0^t (g \circ h_1(t-\tau))d\tau\right)\left(\int_0^t (g \circ h_2(t-p))(g \circ u(p))(g \circ v(p))dp\right)\right] = (h_1 * \mathbf{1})(t) \odot (h_2 * (u \odot v))(t). \tag{3.4}$$

In the other side we have:

$$
\begin{aligned}
&g^{-1}\left[\left(\int_0^t (g\circ h_1(t-\tau))(g\circ u(\tau))d\tau\right)\cdot\left(\int_0^t (g\circ h_2(t-p))(g\circ v(p))dp\right)\right]\\
&\quad = g^{-1}\left[g\left(g^{-1}\left(\int_0^t (g\circ h_1(t-\tau))(g\circ u(\tau))d\tau\right)\right)\cdot g\left(g^{-1}\left(\int_0^t (g\circ h_2(t-p))(g\circ v(p))dp\right)\right)\right]\\
&\quad = g^{-1}\left(\int_0^t (g\circ h_1(t-\tau))(g\circ u(\tau))d\tau\right)\odot g^{-1}\left(\int_0^t (g\circ h_2(t-p))(g\circ v(p))dp\right)\\
&\quad = g^{-1}\left(\int_0^t g\circ (h_1(t-\tau)\odot u(\tau))d\tau\right)\odot g^{-1}\left(\int_0^t g\circ (h_2(t-p)\odot v(p))dp\right)\\
&\quad = \left(\int_{[0,t]}^{\oplus} (h_1(t-\tau)\odot u(\tau))\odot dm\right)\odot\left(\int_{[0,t]}^{\oplus} (h_2(t-p)\odot v(p))\odot dm\right) = (h_1 * u)(t)\odot(h_2 * v)(t).
\end{aligned}
\tag{3.5}
$$

Similarly,

$$
g^{-1}\left[\left(\int_0^t (g\circ h_1(t-\tau))(g\circ v(\tau))d\tau\right)\cdot\left(\int_0^t (g\circ h_2(t-p))(g\circ u(p))dp\right)\right] = (h_1 * v)(t)\odot(h_2 * u)(t). \tag{3.6}
$$

Now, (3.2)–(3.5) and (3.6) imply that

$$
\begin{aligned}
&g^{-1}(g[(h_2 * \mathbf{1})(t)\odot(h_1 * (u\odot v))(t)] + g[(h_1 * \mathbf{1})(t)\odot(h_2 * (u\odot v))(t)])\\
&\quad \geqslant g^{-1}(g[(h_1 * u)(t)\odot(h_2 * v)(t)] + g[(h_1 * v)(t)\odot(h_2 * u)(t)]).
\end{aligned}
$$

I.e.,

$$
[(h_2 * \mathbf{1})(t)\odot(h_1 * (u\odot v))]\oplus[(h_1 * \mathbf{1})(t)\odot(h_2 * (u\odot v))(t)] \geqslant [(h_1 * u)(t)\odot(h_2 * v)(t)]\oplus[(h_1 * v)(t)\odot(h_2 * u)(t)],
$$

which completes the proof. □

Remark 3.2. The reverse inequality (3.1) holds whenever u and v are countermonotone functions.

Let $h_1 = h_2 = h$ in Theorem 3.1. Then we obtain the following result.

Corollary 3.3. *Let $b > a \geqslant 0$. Let $u, v, h : [0,\infty) \to [a,b]$ be integrable functions and let a generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$
\begin{aligned}
&\left(\int_{[0,t]}^{\oplus} [h(t-x)\odot u(x)]\odot dm\right)\odot\left(\int_{[0,t]}^{\oplus} [h(t-x)\odot v(x)]\odot dm\right)\\
&\quad\leqslant \left(\int_{[0,t]}^{\oplus} [h(t-x)\odot(u(x)\odot v(x))]\odot dm\right)\odot\left(\int_{[0,t]}^{\oplus} h(t-x)\odot dm\right)
\end{aligned}
$$

holds for all $t \in [0,\infty)$.

Corollary 3.4. *Let $u, v : [0,\infty) \to [a,b]$ be two integrable functions and let a generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$
J^{\alpha}_{\oplus,\odot}u(t)\odot J^{\alpha}_{\oplus,\odot}v(t) \leqslant K(t)\odot J^{\alpha}_{\oplus,\odot}(u\odot v)(t),
$$

holds where $K(t) = g^{-1}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)$ and the symbol $\mathbb{J}^{\alpha}_{\oplus,\odot}f$ denotes pseudo-fractional integrals of the order $\alpha > 0$ that is defined by

$$
\mathbb{J}^{\alpha}_{\oplus,\odot}u(t) = \int_{[0,t]}^{\oplus}\left(g^{-1}\left((\Gamma(\alpha))^{-1}(t-x)^{\alpha-1}\right)\odot u(x)\right)\odot dm
$$

for all $t \in [0,\infty)$. Here $\Gamma(\alpha)$ is the gamma function.

Proof. Let $h(t-x) = g^{-1}\left((\Gamma(\alpha))^{-1}(t-x)^{\alpha-1}\right), \alpha > 0$ in Corollary 3.3, then we get the desired result. □

Example 3.5. Let $g(x) = x^{\beta}$ for some $\beta \in (0,\infty)$. The corresponding pseudo-operations are $x\oplus y = (x^{\beta}+y^{\beta})^{\frac{1}{\beta}}$ and $x\odot y = xy$. If u and v are comonotone, then it holds

$$
\frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-x)^{\alpha-1}u^{\beta}(x)dx\right)\left(\int_0^t (t-x)^{\alpha-1}v^{\beta}(x)dx\right) \leqslant \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(\int_0^t (t-x)^{\alpha-1}u^{\beta}(x)v^{\beta}(x)dx\right),
$$

and the reverse inequality holds whenever u and v are countermonotone functions.

Using Corollary 3.4 for $\alpha = 1$, we have [1, Theorem 3.3] on $[0, t]$.

Corollary 3.6. *Let $u, v : [0,\infty) \to [a,b]$ be two integrable functions and let a generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then for all $t \in [0,\infty)$, the inequality*

$$g^{-1}(t) \odot \int_{[0,t]}^{\oplus} (u \odot v) \odot dm \geqslant \left(\int_{[0,t]}^{\oplus} u \odot dm\right) \odot \left(\int_{[0,t]}^{\oplus} v \odot dm\right)$$

holds.

Note 3.7. Under conditions of Theorem 3.1, inequality (3.1) holds where the symbol $h_i * u, i = 1,2$ are defined by [18]

$$(h_i * u)(t) = \int_{[1,t]}^{\oplus} [h_i\left(\frac{t}{x}\right) \odot u(x)] \odot dm, \quad i = 1,2$$

for all $t \in [1,\infty)$.

Now we consider the second case, when $\oplus = \sup$, and $\odot = g^{-1}(g(x)g(y))$.

Theorem 3.8. *Let $u, v, h_1, h_2 : [0,\infty) \to [a,b]$ be continuous functions and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.3. If u and v are comonotone, then the inequality*

$$\sup([(h_1*u)(t)\odot(h_2*v)(t)],[(h_1*v)(t)\odot(h_2*u)(t)]) \leqslant \sup([(h_2*\mathbf{1})(t)\odot(h_1*(u\odot v))],[(h_1*\mathbf{1})(t)\odot(h_2*(u\odot v))(t)]) \quad (3.7)$$

holds where

$$(h_i * u)(t) = \int_{[0,t]}^{\sup} [h_i(t-x) \odot u(x)] \odot dm, \quad i = 1,2$$

for all $t \in [0,\infty)$.

Proof. Since u and v are comonotone functions, then proof is obtained immediately from Theorems 3.1 and 2.4. □

Remark 3.9. The reverse inequality (3.7) holds whenever u and v are countermonotone functions.

Let $h_1 = h_2 = h$ in Theorem 3.8. Then we obtain the following result.

Corollary 3.10. *Let $u, v, h : [0,\infty) \to [a,b]$ be continuous functions and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.3. If u and v are comonotone, then the inequality*

$$\begin{aligned}&\left(\int_{[0,t]}^{\sup} [h(t-x) \odot u(x)] \odot dm\right) \odot \left(\int_{[0,t]}^{\sup} [h(t-x) \odot v(x)] \odot dm\right)\\ &\leqslant \left(\int_{[0,t]}^{\sup} [h(t-x) \odot (u(x) \odot v(x))] \odot dm\right) \odot \left(\int_{[0,t]}^{\sup} h(t-x) \odot dm\right)\end{aligned}$$

holds for all $t \in [0,\infty)$.

Corollary 3.11. *Let $u, v : [0,\infty) \to [a,b]$ be two continuous functions and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.3. If u and v are comonotone, then for all $\alpha > 0$, the inequality*

$$J^{\alpha}_{\sup,\odot}\, u(t) \odot J^{\alpha}_{\sup,\odot}\, v(t) \leqslant K(t) \odot J^{\alpha}_{\sup,\odot}(u \odot v)(t),$$

holds where $K(t) = g^{-1}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)$ and

$$\mathbb{J}^{\alpha}_{\sup,\odot}\, u(t) = \int_{[0,t]}^{\sup} \left(g^{-1}\left((\Gamma(\alpha))^{-1}(t-x)^{\alpha-1}\right) \odot u(x)\right) \odot dm$$

for all $t \in (0,\infty)$.

Proof. Let $h = g^{-1}\left((\Gamma(\alpha))^{-1}(t-x)^{\alpha-1}\right), \alpha > 0$ in Corollary 3.10, then we have the desired result. □

Example 3.12. Let $g^{\lambda}(x) = e^{\lambda x}$ and $\psi(x)$ be from Theorem 2.3. Then

$$x \odot_{\lambda} y = x + y$$

and

$$\lim_{\lambda\to\infty}\left(\frac{1}{\lambda}\ln\left(e^{\lambda x}+e^{\lambda y}\right)\right)=\max(x,y).$$

If u and v are comonotone, then the inequality

$$\begin{aligned}&\sup_{0\leqslant x\leqslant t}\left(\frac{1}{\lambda}\ln\left(\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right)+u(x)+\psi(x)\right)+\sup_{0\leqslant x\leqslant t}\left(\frac{1}{\lambda}\ln\left(\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right)+v(x)+\psi(x)\right)\\&\quad\leqslant\frac{1}{\lambda}\ln\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)+\sup_{0\leqslant x\leqslant t}\left(\frac{1}{\lambda}\ln\left(\frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\right)+u(x)+v(x)+\psi(x)\right),\end{aligned}$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

Using Corollary 3.11 for $\alpha=1$, we have [1, Theorem 3.3] on $[0,t]$.

Corollary 3.13. *Let $u,v:[0,\infty)\to[a,b]$ be two continuous functions and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.3. If u and v are comonotone, then for all $t\in(0,\infty)$, the inequality*

$$g^{-1}(t)\odot\int_{[0,t]}^{\sup}(u\odot v)(x)\odot dm\geqslant\left(\int_{[0,t]}^{\sup}u(x)\odot dm\right)\odot\left(\int_{[0,t]}^{\sup}v(x)\odot dm\right)$$

holds.

Note 3.14. Under conditions of Theorem 3.8, inequality (3.7) holds where

$$(h_i*u)(t)=\int_{[1,t]}^{\sup}\left[h_i\left(\frac{t}{x}\right)\odot u(x)\right]\odot dm,\quad i=1,2$$

for all $t\in[1,\infty)$.

Remark 3.15. We note that the third important case $\oplus=\sup$ and $\odot=\min$ has been studied in [4,9,13], where the pseudo-convolution integral in such a case yield the Sugeno integral [24] when the considered measure is maxitive. Observe that the results in the quoted references are valid also in the more general setting of monotone measures, i.e., for the standard Sugeno integral.

Remark 3.16. Observe that generalizations of our convolutions results can be obtained when considering $h(k(t,x))$ instead of $h(t-x)$ when defining a pseudo-convolution, where k is an appropriate 2-place function, for example $k(t,x)=t/x$.

4. Concluding remarks

We have introduced a general version of Chebyshev type inequality for pseudo-convolution integral. This inequality is flexible enough to support both pseudo-integral and convolution integral. In the case of generated pseudo-operations, we recover the g-integral of Pap [16], which can be seen as a generalization of the Lebesgue integral. This type of integrals was shown to be extremely useful in the advanced investigation and applications of nonlinear partial differential equations [15], and just there our results concerning integral inequalities are expected to play an important role for proving convergences, finding good estimations of solutions, etc., similar to that one of classical integral inequalities in the domain of linear partial differential equations. For more details about possible applications of pseudo-convolutions and related inequalities in the area of partial differential equations, we recommend the discussion on Bellman differential equation for multicriteria decision problems in [19, Section 4.2], where our inequalities allow to estimate the optimal solution.

For further investigations we propose to consider the following problems:

Open Problem: What can be told for the pseudo-convolution integral inequality of Chebyshev type (3.1) when set-valued functions [23] are considered?

Open Problem: Is there a version of convolution integral when Choquet integral [2] is considered?

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