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Ergodic Maximum Principle for Stochastic Systems

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Abstract We present a version of the stochastic maximum principle (SMP) for ergodic control problems. In particular we give necessary (and sufficient) conditions for optimality for controlled dissipative systems in finite dimensions. The strategy we employ is mainly built on duality techniques. We are able to construct a dual process for all positive times via the analysis of a suitable class of perturbed linearized forward equations. We show that such a process is the unique bounded solution to a backward SDE on infinite horizon from which we can write a version of the SMP.

Keywords Stochastic maximum principle · Stochastic ergodic control problems · Dissipative systems · Backward stochastic differential equation

Mathematics Subject Classification 60H15 · 93E20

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1 Introduction

We consider an optimal control problem with the following controlled *dissipative stochastic state equation*

$$\begin{cases} dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dW_t, \quad t \geq 0, \\ X_0 = x, \end{cases} \tag{1}$$

and an *ergodic cost functional* (e.g. a functional that depends only on the asymptotic behaviour of the state and of the control) such as:

$$J^{\inf}(u(\cdot)) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T f(X_t, u_t)dt, \tag{2}$$

$$J^{\sup}(u(\cdot)) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T f(X_t, u_t)dt. \tag{3}$$

In the above the state X is a $\mathbb{R}^n$-valued process and $(W_t)_{t\geq 0}$ is a d-dimensional Wiener process. Moreover the drift b and diffusion σ satisfy a joint monotonicity condition. Finally the control process (u_t) is progressively measurable and takes values in a non-empty convex subset $U \subset \mathbb{R}^l$. We refer to this setting as *ergodic control problem*. The choice of the functionals refers to "minmin" and "minmax" formulation. Our aim is to find a correct formulation of the stochastic maximum principle (SMP) in the sense of Pontryagin, by means of which we have at our disposal some necessary (and sufficient) condition for optimality.

Alternatively, under stronger regularity assumptions, one can use the dynamic programming and derive the Hamilton-Jacobi-Bellman equation whose solution gives the optimal cost and the optimal feedback control. In finite dimensions, the first result was obtained in the paper by Mandl [13], later generalized by Borkar and Gosh in [3]. For further generalizations of such an HJB approach, both in finite and infinite dimensional framework both by analytic and by probabilistic tools see e.g. [1,2,6–8,11,17]. We also refer to [9] for a survey on recent results obtained in this direction.

Nevertheless, it is by now well known that, even if it only provides necessary (only under strong convexity requirements also sufficient) optimality conditions, the SMP normally requires much less regularity and structural condition allowing for instance to easily include the case of control dependent diffusion. The first general formulation of the SMP for finite horizon controlled stochastic systems in finite dimensions was obtained by Peng in [16]. After this seminal paper, many directions have been followed by many authors. For what concerns ergodic costs, though, the theory is not yet fully developed. As far as we know, the only version of necessary and sufficient condition for optimality goes back to the paper by Kushner [12] in 1978, in which no backward stochastic equation appeared. In that framework the author adopted a martingale solution approach and considered only Markov feedback controls. The system is also assumed to be stable for each control. Under these assumptions, for each stationary Markov control there exists a unique invariant measure $\mu_u(\cdot)$ such that the initial cost functional can be rewritten in the following way

$$\lim_{T\to\infty} \mathbb{E}^u_{x_0} \frac{1}{T} \int_0^T f(X_t, u(X_t))dt = \int f(x, u(x))\mu_u(dx).$$

Using this formulation, Kushner derived a necessary and sufficient condition for $u(\cdot)$ to be optimal, which he called a "dynamic programming like" condition. Let us also mention a recent preprint [5] in which the authors give some sufficient conditions for optimality, studying the adjoint Backward SDE, as well as Feller property and exponential ergodicity of the controlled process. As in the present paper the adjoint BSDE is multidimensional in an infinite horizon. The point is that the approach chosen in [5] to prove well posedness of such an equation relies on Girsanov argument and seems to work under commutativity requirements that are satisfied, for instance, when $n = 1$ or when σ does not depend on x but are not satisfied when $n > 1$ and σ depends on x in a generic way. As a matter of fact, even though it is not explicitly stated, at the end of page 9 in [5], the Girsanov transform can be applied only if $\langle \delta Y, \nabla_x \sigma(X, u) Z\rangle_{\mathbb{R}^n}$ is proportional to $\langle \delta Y, Z\rangle_{\mathbb{R}^n}$ and this is not true, in general, if $\nabla_x \sigma(X, u)$ is a linear operator (in this last discussion we have for simplicity only considered one dimensional noise). Finally we also mention [10] for infinite horizon multidimensional BSDEs in the context of linear quadratic stationary optimal control.

Our formulation is fairly general. We do not impose the existence of a limit in the formulation of the cost functional and we consider general progressive controls. Moreover, notice that the convexity assumption on the control actions is a natural choice for the ergodic control problems. Indeed, due to the dissipativity of the system, a spike variation argument is not sufficient to extract useful information on the behaviour of the system at infinity. In the present paper we deduce a version of the maximum principle written in terms of the unique bounded solution to a multidimensional backward SDE on infinite horizon

$$-dp_t = \left[D_x b(X_t, u_t)^* p_t + D_x \sigma(X_t, u_t)^* q_t - D_x f(X_t, u_t)\right] dt - q_t dW_t. \quad (4)$$

The price to pay for a such a formulation is that we obtain a weak version of maximum principle that contains both an average with respect to time and an expectation (see equality (48)). In more standard situations it is possible to localize both in time and in Ω to get a $\mathbb{P}$-almost sure form that does not involve integrals or expectations. Instead, in our case, localization with respect to time is out of question since the problem deals, by its nature, with the asymptotic behaviour of the state. Localize the problem with respect to Ω seems difficult as well, due to the specific formulation of the cost that we have chosen. Roughly speaking this would be possible if we could interchange the lim and the expectation in (2) and (3) and consequently in (48), but this is not possible in the present general setting.

As far as we know, a well-posedness result for backward equations of this form is new. The major difficulty to overcome is the lack of integrability in time of the forcing term of the equation. Due to the hypothesis on the state equation we can guarantee that

$$\sup_{t\geq 0} \left(\mathbb{E}|D_x f(X_t, u_t)|^r\right)^{1/r} < \infty; \qquad \text{for some} \quad r > 1.$$

Similar equations are studied in the formulation of the SMP for discounted cost functionals in infinite horizon, see e.g. [14,15]. In that case, though, the spaces in which one is looking for a solution are weighted L^2-spaces, allowing the solution to explode at infinity in a controlled way. Here, due to the stability of the system, we expect the solution to be bounded up to infinity.

The strategy we employ is mainly built on duality techniques. Via the analysis of a suitable class of perturbed linearized forward equations, see Eq. (38) below, we are able, exploiting their dissipativity, to construct an adjoint process for all positive times. We introduce then a well-suited family of truncated equations and we show the consistency of the family with respect to the varying finite horizon $T > 0$, as $T \to \infty$.

We also propose a second version of maximum principle involving a family of backward equations on finite time horizon T with terminal condition $p_T^T = 0$ that could be verifiable in certain cases, see Proposition 2 below.

Once we have a necessary condition for optimality, it is natural to ask also for a sufficient counterpart of it. As in the classical setting, an extra convexity assumption on the Hamiltonian of the system guarantees the required sufficiency.

The paper is structured as follows. In Sect. 2 we fix the notation and we discuss the main assumptions on the state equation and on the control actions. In Sect. 3 we study the convex perturbation of the optimal control and we expand the optimal trajectory and cost functional with respect to the perturbation. Section 4 is the core of the paper. Here we introduce the adjoint equation and we present a well-posedness result for it. The main results concerning the necessary and sufficient versions of the SMP are contained in Sects. 5 and 6.

2 Preliminaries and Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(W_t)_{t\geq 0}$ a standard d-dimensional Brownian motion. Throughout the paper we use the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ associated to W, augmented in the usual way with the family of $\mathbb{P}$-null sets of $\mathcal{F}$. Given $t \in [0, T]$, we indicate by $L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$ the set of all $\mathcal{F}_t$-measurable square-integrable random variables with values in $\mathbb{R}^n$. By $|\cdot|$ we denote the Euclidean norm on $\mathbb{R}^n$ and $\|\cdot\|_2$ indicates the Hilbert-Schmidt norm on $\mathbb{R}^{n\times n}$. The tensor product between two vectors $u, v \in \mathbb{R}^n$ is denoted by $u \otimes v$: $(u \otimes v)_{ij} = u_i v_j$. Throughout the paper the value of a generic constant (usually denoted by C) may change from line to line.

For any $p \geq 1$ and $T > 0$ we define

- $L^p(\Omega \times [0, T]; \mathbb{R}^n)$, the set of all $(\mathcal{F}_t)$-progressive processes with values in $\mathbb{R}^n$ such that

$$\|X\|_{L^p(\Omega\times[0,T];\mathbb{R}^n)} = \left(\mathbb{E}\int_0^T |X_t|^p dt\right)^{1/p} < \infty;$$

- $L^p(\mathbb{R}_+; L^q(\Omega; \mathbb{R}^n))$ the set of all $(\mathcal{F}_t)$-progressive processes with values in $\mathbb{R}^n$ with $1 \le q < +\infty$ such that

$$\|X\|^p_{L^p(\mathbb{R}_+;L^q(\Omega;\mathbb{R}^n))} = \int_0^\infty (\ \mathbb{E}|X_t|^q\ )^{\frac{p}{q}}\, dt < \infty,$$

and

$$\|X\|_{L^\infty(\mathbb{R}_+;L^q(\Omega;\mathbb{R}^n))} = \sup_{t\ge 0} (\ \mathbb{E}|X_t|^q\ )^{\frac{1}{q}} < \infty.$$

The aim of this work is to give some necessary and sufficient conditions for optimality of a controlled system of the form

$$\begin{cases} dX_t = b(X_t, u_t)dt + \sigma(X_t, u_t)dW_t, \quad t \ge 0, \\ X_0 = x, \end{cases} \tag{5}$$

when a cost functional of ergodic type has to be minimized. The form of the cost functional slightly differs when considering a lim inf or a lim sup formulation. We define a truncated cost functional in the following form

$$J_T(u(\cdot)) = \mathbb{E}\int_0^T f(X_t, u_t)dt. \tag{6}$$

Let us denote the two forms in the following way

$$J^{\inf}(u(\cdot)) = \liminf_{T\to\infty} \frac{1}{T} J_T(u(\cdot)) = \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T f(X_t, u_t)dt, \tag{7}$$

$$J^{\sup}(u(\cdot)) = \limsup_{T\to\infty} \frac{1}{T} J_T(u(\cdot)) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T f(X_t, u_t)dt. \tag{8}$$

An control process $\bar{u}(\cdot)$ is said to be optimal either if

$$J^{\inf}(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}} J^{\inf}(u(\cdot)) \qquad \text{or} \qquad J^{\sup}(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}} J^{\sup}(u(\cdot)), \tag{9}$$

where $\mathcal{U}$ indicates a class of admissible controls. Now we give some assumptions on the state equation and on the control actions.

Hypothesis 1 Assumptions involve three constants m, p and k that we fix now and for the rest of the paper. We assume that m is a non-negative integer, $p > (4m+2) \vee 4$ and $k > (p-1)/2$
(**H1**) (*Controls*) Let U be a closed convex subset of $\mathbb{R}^l$ and $(u_t)_{t\in[0,T]}$ a progressively measurable U-valued process. We say that u is an *admissible control* if it satisfies:

$$\sup_{t\ge 0}\mathbb{E}|u_t|^p < +\infty. \tag{10}$$

We denote by $\mathcal{U}$ the set of admissible controls.
(**H2**) (*Drift term*) The vector field $b : \mathbb{R}^n \times U \to \mathbb{R}^n$ is $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U)$-measurable and $\mathcal{C}^2$ with respect to x and u. There exists $C_1 > 0$ such that

$$|D_u b(x,u)| \le C_1, \qquad \forall x \in \mathbb{R}^n, \forall u \in U.$$

Moreover:

$$\sup_{u\in U}\sup_{x\in\mathbb{R}^n} \frac{|D_x^\beta b(x,u)|}{1+|x|^{2m+1-\beta}+|u|^{1-\beta}} < +\infty, \qquad \beta \in \{0,1\}. \tag{11}$$

(**H3**) (*Diffusion term*) The mapping $\sigma : \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$ is measurable with respect to $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U)$. There exists $C_2 > 0$ such that

$$\|D_u\sigma(x,u)\|_2 \le C_2, \qquad \forall x \in \mathbb{R}^n, \forall u \in U.$$

Moreover it is $\mathcal{C}^2$ with respect to x, u and:

$$\sup_{u\in U}\sup_{x\in\mathbb{R}^n} \frac{\|D_x^\beta \sigma(x,u)\|_2}{1+|x|^{m-\beta}+|u|^{1-\beta}} < +\infty, \qquad \beta \in \{0,1\}. \tag{12}$$

(**H4**) (*Joint dissipativity*) There is $c_k < 0$ such that

$$\langle D_x b(x,u)y, y\rangle + k\|D_x\sigma(x,u)y\|_2^2 \le c_k|y|^2, \qquad \forall x, y \in \mathbb{R}^n, \forall u \in U. \tag{13}$$

(**H5**) (*Cost*) The function $f : \mathbb{R}^n \times U \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U)$-measurable, bounded from below by a constant f_0, it is differentiable in x and u and

$$|D_x f(x,u)| + |D_u f(x,u)| \le C(1+|x|+|u|),$$

for some $C > 0$.

Remark 1 We refer to [4,15] for a discussion on the joint monotonicity and on the relation between the growth of b and σ. In particular it easy to see that condition (H4) implies that for every $x, y \in \mathbb{R}^n$

$$\langle b(x,u)-b(y,u), x-y\rangle + k\|\sigma(x,u)-\sigma(y,u)\|_2^2 \le c_k|x-y|^2, \qquad \forall u \in U. \tag{14}$$

Indeed we have

$$\begin{aligned}
&\langle b(x,u)-b(y,u), x-y\rangle + k\|\sigma(x,u)-\sigma(y,u)\|_2^2 \\
&\quad = \int_0^1 \langle D_x b(x+\theta(y-x),u)(x-y), x-y\rangle d\theta \\
&\qquad + k\|\int_0^1 D_x\sigma(x+\theta(y-x),u)(x-y)d\theta\|_2^2
\end{aligned}$$

$$\leq \int_0^1 [\langle D_x b(x+\theta(y-x),u)(x-y), x-y\rangle$$
$$+k\|D_x\sigma(x+\theta(y-x),u)(x-y)\|_2^2\Big] d\theta$$
$$\leq c_k|x-y|^2$$

Remark 2 The choice of $p > (4m+2) \vee 4$ in (H1)–(H4) comes from the interplay between the dissipative behaviour of the system and polynomial growth of the coefficients. Actually this bound can be easily derived from the maximal moment of the state process that we need to estimate in the proofs (see Proposition 1). The condition for k is then the natural one.

We can state the following

Theorem 1 *Assume that Hypothesis 1 holds true. Then, for every $x \in \mathbb{R}^n$ and every admissible control $u(\cdot)$, Eq. (5) admits a unique progressively measurable solution for each admissible control. Moreover, the following estimate holds*

$$\mathbb{E}|X_t|^p \leq e^{-p\beta t}|x|^p + C\left(1 + \sup_{t\geq 0}\mathbb{E}|u_t|^p\right), \tag{15}$$

for some positive constants $C = C(p)$ and β.

Proof For a proof of existence and uniqueness of a strong solution to Eq. (5) see e.g. [4]. Let us prove the estimate (15). Define $\tilde{X}_t := e^{\beta t}X_t$ for a positive β. Then $\tilde{X}$ solves

$$\begin{cases} d\tilde{X}_t = \beta\tilde{X}_t + e^{\beta t}b\big(e^{-\beta t}\tilde{X}_t, u_t\big)dt + e^{\beta t}\sigma\big(e^{-\beta t}\tilde{X}_t, u_t\big)dW_t, & \forall t \geq 0, \\ \tilde{X}_0 = x. \end{cases} \tag{16}$$

If we call $\tilde{b}_t(x,u) = e^{\beta t}b(e^{-\beta t}x,u)$ and $\tilde{\sigma}_t(x,u) = e^{\beta t}\sigma(e^{-\beta t}x,u)$ then also $\tilde{b}_t, \tilde{\sigma}_t$ satisfy the joint dissipativity condition with the same constant

$$\langle\tilde{b}_t(x,u) - \tilde{b}_t(y,u), x-y\rangle + k\|\tilde{\sigma}_t(x,u) - \tilde{\sigma}_t(y,u)\|_2^2 \leq c_k|x-y|^2. \tag{17}$$

Moreover $|\tilde{b}(0,u)| \leq Ce^{\beta t}(1+|u|)$ and $|\tilde{\sigma}(0,u)| \leq Ce^{\beta t}(1+|u|)$. Denote $p = 2q$ and $\tilde{a} = \tilde{\sigma}(x,u)^*\tilde{\sigma}(x,u)$ (we omit the time dependence $\tilde{\sigma} = \tilde{\sigma}_t$ when it is clear). Now we apply the Itô formula to the function $f(x) = |x|^{2q}$. For simplicity's sake we omit the localization procedure of the martingale term. We refer the reader to [15, Thm. 4.1] for the details of this truncation argument. Hence

$$\mathbb{E}\big|\tilde{X}_t\big|^{2q} = |x|^{2q} + 2q\mathbb{E}\int_0^t \big|\tilde{X}_s\big|^{2(q-1)}\left(\big\langle\tilde{X}_s, \tilde{b}\big(\tilde{X}_s, u_s\big)\big\rangle + \frac{1}{2}\big\|\tilde{\sigma}\big(\tilde{X}_s, u_s\big)\big\|_2^2\right)ds$$
$$+ 2q\beta\mathbb{E}\int_0^t \big|\tilde{X}_s\big|^{2q}ds + 2q(q-1)\mathbb{E}\int_0^t \big|\tilde{X}_s\big|^{2(q-2)}Tr\left\{\tilde{a}_s\big(\tilde{X}_s \otimes \tilde{X}_s\big)\right\}ds$$

$$
\begin{aligned}
&\leq |x|^{2q} + 2q\mathbb{E}\int_0^t \left|\tilde{X}_s\right|^{2(q-1)} \left(\left\langle \tilde{X}_s, \tilde{b}(\tilde{X}_s, u_s)\right\rangle + \left(q-\frac{1}{2}\right)\|\tilde{\sigma}\left(\tilde{X}_s, u_s\right)\|_2^2\right) ds \\
&\quad +2q\beta\mathbb{E}\int_0^t \left|\tilde{X}_s\right|^{2q} ds \\
&\leq |x|^{2q} + 2q\mathbb{E}\int_0^t \left|\tilde{X}\right|_s^{2(q-1)} \left(\left\langle \tilde{X}_s, \tilde{b}\left(\tilde{X}_s, u_s\right)\right\rangle - \tilde{b}(0, u_s)\right. \\
&\quad \left. + \left(q-\frac{1}{2}\right)(1+\varepsilon)\left\|\tilde{\sigma}\left(\tilde{X}_s, u_s\right) - \tilde{\sigma}(0, u_s)\right\|_2^2\right) ds \\
&\quad + 2q\mathbb{E}\int_0^t \left|\tilde{X}\right|_s^{2(q-1)} \left(\left\langle \tilde{X}_s, \tilde{b}(0, u_s)\right\rangle + \left(1+\frac{1}{\varepsilon}\right)\|\tilde{\sigma}(0, u_s)\|_2^2\right) ds \\
&\quad + 2q\beta\mathbb{E}\int_0^t \left|\tilde{X}_s\right|^{2q} ds
\end{aligned}
$$

where we used weighted Young inequality. For $\varepsilon > 0$ small enoug we can always assume that $(q - 1/2)(1 + \varepsilon) = 2^{-1}(2p - 1))(1 + \varepsilon) \leq k$ and we can use the joint dissipativity in the form (17) to get:

$$
\begin{aligned}
\mathbb{E}\left|\tilde{X}_t\right|^{2q} &\leq |x|^{2q} + 2q\left(c_k + \beta + \frac{\delta}{2}\right)\mathbb{E}\int_0^t \left|\tilde{X}_s\right|^{2q} ds \\
&\quad + 2q\mathbb{E}\int_0^t \left|\tilde{X}_s\right|^{2(q-1)} \left(\frac{1}{2\delta}\left|\tilde{b}(0, u_s)\right|^2 + \left(1+\frac{1}{\varepsilon}\right)\|\tilde{\sigma}(0, u_s)\|_2^2\right) ds \\
&\leq |x|^{2q} + 2q\left(c_k + \beta + \frac{\delta}{2} + \frac{q-1}{q}\delta^{q/(q-1)}\right)\mathbb{E}\int_0^t \left|\tilde{X}_s\right|^{2q} ds \\
&\quad + C\mathbb{E}\int_0^t e^{2q\beta s}\left(1 + |u_s|^{2q}\right) ds,
\end{aligned}
$$

where we again used weighted Young inequality together with the growth condition on the coefficients (the constant C depends only on q, δ and ε). Choosing β and δ small enough and recalling that $c_k < 0$, we end up with the following estimate:

$$
\begin{aligned}
\mathbb{E}|X_t|^{2q} &\leq e^{-2q\beta t}|x|^{2q} + C\int_0^t e^{-2q\beta(t-s)}\mathbb{E}\left(1 + |u_s|^{2q}\right) ds \\
&\leq e^{-2q\beta t}|x|^{2q} + C\left(1 + \sup_{t\geq 0}\mathbb{E}|u_t|^{2q}\right).
\end{aligned} \tag{18}
$$

Notice that, taking the supremum on both sides we also have that

$$
\sup_{t\geq 0}\mathbb{E}|X_t|^{2q} \leq C(|x|^{2q} + 1), \tag{19}
$$

and the claim is proved.

3 Perturbation of the Controls

When considering ergodic control problems we can not expect to gain information by using local in time perturbations of the optimal control.

More precisely, let $u^i(\cdot)$, $i = 1, 2$ be some admissible controls with $u_t^1 = u_t^2$ for all $t > T_0$. If we denote by X^i, $i = 1, 2$ the corresponding states, then by the dissipativity assumption (H4) we get $\mathbb{E}|X_t^1 - X_t^2|^2 \to 0$ for $t > T_0$ exponentially fast (let us say with an exponential decay ε). If we assume for simplicity that f is Lipschitz and $J := J^{\inf} = J^{\sup}$ we get

$$\begin{aligned} \left|J\left(u^1(\cdot)\right) - J\left(u^2(\cdot)\right)\right| &= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \left|f\left(X_t^1, u_t^1\right) - f\left(X_t^2, u_t^2\right)\right| dt \\ &= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\int_{T_0}^T \left|f\left(X_t^1, u_t^1\right) - f\left(X_t^2, u_t^1\right)\right| dt \\ &\leq C \lim_{T\to\infty} \frac{1}{T}\int_{T_0}^T e^{-\varepsilon t} dt = 0. \end{aligned} \tag{20}$$

This is the reason for considering perturbations acting on the system up to infinity.

Let then $\bar{u}(\cdot)$ be an optimal control for the ergodic control problem (9) and denote the corresponding state process as $\bar{X}$. For $\theta \in (0, 1]$ and $u(\cdot)$ an admissible control process, we define u^θ as a convex combination $u^\theta(\cdot) := (1-\theta)\bar{u}(\cdot) + \theta u(\cdot) = \bar{u}(\cdot) + \theta v(\cdot)$, where $v(\cdot) := u(\cdot) - \bar{u}(\cdot)$. Then $u^\theta(\cdot)$ is admissible and the corresponding state process is denoted by X^θ. Notice that it is crucial to require that U is convex.

Lemma 1 *Under Hypothesis 1 the following holds*

$$\sup_{t\geq 0} \mathbb{E}|X_t^\theta - \bar{X}_t|^p \leq C\theta^2 \sup_{t\geq 0} \mathbb{E}|v_t|^p,$$

where C only depends on the constants appearing in Hypothesis 1.

Proof Denote $\Delta X_t^\theta := X_t^\theta - \bar{X}_t$ and write the corresponding equation

$$\begin{aligned} \Delta X_t^\theta &= \int_0^t \left[b\left(X_s^\theta, u_s^\theta\right) - b\left(\bar{X}_s, \bar{u}_s\right)\right] ds + \int_0^t \left[\sigma\left(X_s^\theta, u_s^\theta\right) - \sigma\left(\bar{X}_s, \bar{u}_s\right)\right] dW_s \\ &= \int_0^t \left[b\left(X_s^\theta, u_s^\theta\right) - b\left(\bar{X}_s, u_s^\theta\right)\right] ds + \int_0^t \left[b\left(\bar{X}_s, u_s^\theta\right) - b\left(\bar{X}_s, \bar{u}_s\right)\right] ds \\ &\quad + \int_0^t \left[\sigma\left(X_s^\theta, u_s^\theta\right) - \sigma\left(\bar{X}_s, u_s^\theta\right)\right] dW_s \\ &\quad + \int_0^t \left[\sigma\left(\bar{X}_s, u_s^\theta\right) - \sigma\left(\bar{X}_s, \bar{u}_s\right)\right] dW_s \end{aligned} \tag{21}$$

Following the technique developed in the proof of Theorem 1 we define $\tilde{X}_t^\theta := e^{\beta t} X_t^\theta$, $\tilde{\bar{X}}_t := e^{\beta t}\bar{X}_t$ and $\Delta\tilde{X}_t^\theta := e^{\beta t}\Delta X_t^\theta$ for a positive β. Denoting $p = 2q$, then the Itô formula gives

$$
\begin{aligned}
&\mathbb{E}\big|\Delta\tilde{X}_t^\theta\big|^{2q} \leq 2q\mathbb{E}\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2(q-1)} \\
&\Big[\big\langle\tilde{b}\big(\tilde{X}_s^\theta, u_s^\theta\big) - \tilde{b}\big(\tilde{\bar{X}}_s, u_s^\theta\big)\big\rangle, \Delta\tilde{X}_s^\theta + \left(q-\frac{1}{2}\right)\big\|\tilde{\sigma}\big(\tilde{X}_s^\theta, u_s^\theta\big) - \tilde{\sigma}\big(\tilde{\bar{X}}_s, u_s^\theta\big)\big\|_2^2\Big]ds \\
&\quad + 2q\beta\mathbb{E}\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2q}ds + 2q\mathbb{E}\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2(q-1)} \\
&\Big\langle\int_0^1 D_u\tilde{b}\big(\tilde{\bar{X}}_s, \bar{u}_s + \lambda\theta v_s\big)\theta v_s d\lambda, \Delta\tilde{X}_s^\theta\Big\rangle ds \\
&\quad + 2q\left(q-\frac{1}{2}\right)\mathbb{E}\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2(q-1)}\left\|\int_0^1 D_u\tilde{\sigma}\big(\tilde{\bar{X}}_s, \bar{u}_s + \lambda\theta v_s\big)\theta v_s d\lambda\right\|_2^2 ds \\
&\leq 2q\left(c_r + \beta + \frac{\delta}{2}\right)\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2q}ds + 2q\theta^2\mathbb{E}\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2(q-1)}\frac{e^{2\beta s}}{2\delta}|v_s|^2 ds \\
&\quad + 2q\left(q-\frac{1}{2}\right)\theta^2\mathbb{E}\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2(q-1)}e^{2\beta s}|v_s|^2 ds \\
&\leq 2q\left(c_r + \beta + \frac{\delta}{2} + \frac{2(q-1)}{q}\delta^{q/(q-1)}\right)\int_0^t \big|\Delta\tilde{X}_s^\theta\big|^{2q}ds \\
&\quad + C\theta^2\mathbb{E}\int_0^t e^{2q\beta s}|v_s|^{2q}ds,
\end{aligned}
$$

where we used the joint dissipativity in the form (17) with constant $r = (q - 1/2)$ and weighted Young inequality. The constant C depends only on q and δ. Choosing β, δ small enough, from the boundedness of $\sup_{s\geq 0}\mathbb{E}|v_s|^{2q}$ we get

$$
\mathbb{E}\big|\Delta X_t^\theta\big|^{2q} \leq C\theta^2\sup_{s\geq 0}\mathbb{E}|v_s|^{2q}\int_0^t e^{-2q\beta(t-s)}ds.
$$

The result follows by taking the supremum in time. □

Now we introduce the first variation equation of the system. Notice that in the equation appear the derivatives of the coefficients with respect to the control, which are bounded due to our assumptions.

$$
\begin{cases}
dY_t = \big[D_x b\big(\bar{X}_t, \bar{u}_t\big)Y_t + D_u b\big(\bar{X}_t, \bar{u}_t\big)v_t\big]dt \\
\qquad\quad + \big[D_x\sigma\big(\bar{X}_t, \bar{u}_t\big)Y_t + D_u\sigma\big(\bar{X}_t, \bar{u}_t\big)v_t\big]dW_t, \\
Y_0 = 0.
\end{cases}
\tag{22}
$$

Lemma 2 *Under Hypothesis 1, the first variation Eq. (22) admits a unique adapted solution. Moreover the following estimate holds true*

$$
\mathbb{E}\mid Y_t\mid^p \leq C\sup_{s\in[0,t]}\mathbb{E}|v_s|^p,
\tag{23}
$$

where again C only depends on the constants appearing in Hypothesis 1.

In particular, $\sup_{t\geq 0}\mathbb{E}\mid Y_t\mid^p \leq C\sup_{t\geq 0}\mathbb{E}\mid v_t\mid^p < +\infty$.

Proof The proof goes through by the same technique adopted in Theorem 1. What is crucial here is the uniform boundedness of $D_u b(x,u)$ and $D_u\sigma(x,u)$, along with Assumption (H1) on admissible controls. □

The following lemma is fundamental in order to obtain the right expansion of the cost functional with respect to the control.

Proposition 1 *Under our assumptions the process* $\hat{X}^\theta$ *defined as*

$$\hat{X}_t^\theta = \frac{X_t^\theta - \bar{X}_t}{\theta} - Y_t,$$

satisfies

$$\lim_{\theta\to 0_+}\sup_{t\geq 0}\mathbb{E}\big|\hat{X}_t^\theta\big|^2 = 0. \tag{24}$$

Proof The equation for $\hat{X}^\theta$ reads

$$\begin{aligned} d\hat{X}_t^\theta &= \frac{1}{\theta}\left[b\big(X_t^\theta, u_t^\theta\big) - b\big(\bar{X}_t,\bar{u}_t\big) - \theta D_x b\big(\bar{X}_t,\bar{u}_t\big)Y_t - \theta D_u b\big(\bar{X}_t,\bar{u}_t\big)v_t\right]dt \\ &\quad + \frac{1}{\theta}\left[\sigma\big(X_t^\theta, u_t^\theta\big) - \sigma\big(\bar{X}_t,\bar{u}_t\big) - \theta D_x \sigma\big(\bar{X}_t,\bar{u}_t\big)Y_t - \theta D_u \sigma\big(\bar{X}_t,\bar{u}_t\big)v_t\right]dW_t \\ &= \frac{1}{\theta}\Big[b\big(\bar{X}_t + \theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \theta v_t\big) - b\big(\bar{X}_t,\bar{u}_t\big) \\ &\quad - \theta D_x b\big(\bar{X}_t,\bar{u}_t\big)Y_t - \theta D_u b\big(\bar{X}_t,\bar{u}_t\big)v_t\Big]dt \\ &\quad + \frac{1}{\theta}\Big[\sigma\big(\bar{X}_t + \theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \theta v_t\big) - \sigma\big(\bar{X}_t,\bar{u}_t\big) \\ &\quad - \theta D_x \sigma\big(\bar{X}_t,\bar{u}_t\big)Y_t - \theta D_u \sigma\big(\bar{X}_t,\bar{u}_t\big)v_t\Big]dW_t, \end{aligned}$$

with $\hat{X}_0^\theta = 0$ as initial condition. Further, by Taylor expansion we have that

$$\begin{aligned} d\hat{X}_t^\theta = &\int_0^1 D_x b\big(\bar{X}_t + \lambda\theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \lambda\theta v_t\big)\hat{X}_t^\theta d\lambda dt \\ &+ \int_0^1 \Big[D_x b\big(\bar{X}_t + \lambda\theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \lambda\theta v_t\big) - D_x b\big(\bar{X}_t,\bar{u}_t\big)\Big]Y_t d\lambda dt \\ &+ \int_0^1 \Big[D_u b\big(\bar{X}_t + \lambda\theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \lambda\theta v_t\big) - D_u b\big(\bar{X}_t,\bar{u}_t\big)\Big]v_t d\lambda dt \\ &+ \int_0^1 D_x \sigma\big(\bar{X}_t + \lambda\theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \lambda\theta v_t\big)\hat{X}_t^\theta d\lambda dW_t \\ &+ \int_0^1 \Big[D_x \sigma\big(\bar{X}_t + \lambda\theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \lambda\theta v_t\big) - D_x \sigma\big(\bar{X}_t,\bar{u}_t\big)\Big]Y_t d\lambda dW_t \\ &+ \int_0^1 \Big[D_u \sigma\big(\bar{X}_t + \lambda\theta\big(Y_t + \hat{X}_t^\theta\big), \bar{u}_t + \lambda\theta v_t\big) - D_u \sigma\big(\bar{X}_t,\bar{u}_t\big)\Big]v_t d\lambda dW_t. \end{aligned}$$

To keep the notation simple, we rewrite the above equation as

$$d\hat{X}_t^\theta = \big(A_t^{\theta,x}\hat{X}_t^\theta + A_t^{\theta,y}Y_t + A_t^{\theta,v}v_t\big)dt + \big(B_t^{\theta,x}\hat{X}_t^\theta + B_t^{\theta,y}Y_t + B_t^{\theta,v}v_t\big)dW_t,$$

where we have kept the order of the terms from the previous equation. Now apply the Itô formula to $e^{\beta t}\big|\hat{X}_t^\theta\big|^2$ to get

$$\begin{aligned}\mathbb{E}\big(e^{\beta t}\big|\hat{X}_t^\theta\big|^2\big) = 2\mathbb{E}\int_0^t e^{\beta s}\big\langle A_s^{\theta,x}\hat{X}_s^\theta + A_s^{\theta,y}Y_s + A_s^{\theta,v}v_s, \hat{X}_s^\theta\big\rangle ds\\ + \mathbb{E}\int_0^t e^{\beta s}\big\| B_s^{\theta,x}\hat{X}_s^\theta + B_s^{\theta,y}Y_s + B_s^{\theta,v}v_s\big\|_2^2 ds\\ + \beta\mathbb{E}\int_0^t e^{\beta s}\big|\hat{X}_s^\theta\big|^2 ds. \end{aligned} \tag{25}$$

By the joint dissipativity assumption (H4) in Hypothesis 1 we have

$$2\big\langle A_s^{\theta,x}\hat{X}_s^\theta, \hat{X}_s^\theta\big\rangle + 2k\big\| B_s^{\theta,x}\hat{X}_s^\theta\big\|^2 + \beta\big|\hat{X}_s^\theta\big|^2 < 0,$$

for some $k > 1/2$ and β small enough.

Thus, repeating the same computations as in the proof of Theorem 1, we get the following intermediate estimate

$$\mathbb{E}\big|\hat{X}_t^\theta\big|^2 \le C\int_0^t e^{-\beta(t-s)}\mathbb{E}\big(\big|A_s^{\theta,y}Y_s\big|^2 + \big|A_s^{\theta,v}v_s\big|^2 + \big|B_s^{\theta,y}Y_s\big|^2 + \big|B_s^{\theta,v}v_s\big|^2\big)ds. \tag{26}$$

Now we show how to treat the first term in (26). The estimate of the remaining ones goes along similar lines.

We fix α with $p/(p-2) < \alpha < p/(4m)$, if $m \ge 1$, or $\alpha = 2$, if $m = 0$. Recall that $p > 4m + 2$ and notice that, this way, denoting by α' the conjugate of α (that is $1/\alpha + 1/\alpha' = 1$) then $2\alpha' < p$, $4m\alpha < p$ and $2\alpha < p$. First we start by observing that by Hölder inequality and by (23) we have that, for any $\alpha > 1$:

$$\begin{aligned}&\int_0^t e^{-\beta(t-s)}\mathbb{E}\big|A_s^{\theta,y}Y_s\big|^2 ds\\ &= \int_0^t e^{-\beta(t-s)}\mathbb{E}\Big|\int_0^1 \big[D_x b\big(\bar{X}_s + \lambda\theta\big(Y_s + \hat{X}_s^\theta\big), \bar{u}_s + \lambda\theta v_s\big)\\ &\quad - D_x b\big(\bar{X}_s, \bar{u}_s\big)\big]Y_s d\lambda\Big|^2 ds\\ &\le \int_0^t e^{-\beta(t-s)}\left(\int_0^1 \mathbb{E}\mid D_x b\big(\bar{X}_s + \lambda\theta\big(Y_s + \hat{X}_s^\theta\big), \bar{u}_s + \lambda\theta v_s\right) - D_x b\big(\bar{X}_s, \bar{u}_s\big) \mid^{2\alpha}\\ &\quad - D_x b\big(\bar{X}_s, \bar{u}_s\big) \mid^{2\alpha} d\lambda\big)^{\frac{1}{\alpha}}\cdot \big(\mathbb{E}|Y_s|^{2\alpha'}\big)^{\frac{1}{\alpha'}} ds. \end{aligned} \tag{27}$$

Since $2\alpha' < p$, using Lemma 2 to estimate $\sup_{s\in\mathbb{R}^+} \mathbb{E}|Y_s|^{2\alpha'} \leq C$, we get

$$\begin{aligned}
&\int_0^t e^{-\beta(t-s)}\mathbb{E}\big|A_s^{\theta,y} Y_s\big|^2 ds \\
&\quad\leq C \int_0^t e^{-\beta(t-s)} \left(\int_0^1 \mathbb{E}\mid D_x b\big(\bar{X}_s + \lambda\theta\big(Y_s + \hat{X}_s^\theta\big), \bar{u}_s + \lambda\theta v_s\big)\right. \\
&\quad\quad \left. - D_x b\big(\bar{X}_s, \bar{u}_s\big) \mid^{2\alpha} d\lambda\right)^{\frac{1}{\alpha}} ds \\
&\quad\leq C \int_0^t e^{-\beta(t-s)} \left(\int_0^1 \mathbb{E}\mid D_x b\big(\bar{X}_s + \lambda\theta\big(Y_s + \hat{X}_s^\theta\big), \bar{u}_s + \lambda\theta v_s\big)\right. \\
&\quad\quad \left. - D_x b\big(\bar{X}_s, \bar{u}_s + \lambda\theta v_s\big) \mid^{2\alpha} d\lambda\right)^{\frac{1}{\alpha}} ds \\
&\quad\quad + C \int_0^t e^{-\beta(t-s)} \left(\int_0^1 \mathbb{E}\mid D_x b\big(\bar{X}_s, \bar{u}_s + \lambda\theta v_s\big) - D_x b\big(\bar{X}_s, \bar{u}_s\big) \mid^{2\alpha} d\lambda\right)^{\frac{1}{\alpha}} ds. \qquad (28)
\end{aligned}$$

Now we prove the convergence of the first term, being the second one similar (and easier).

Due to Hypothesis 1, the gradient $D_x b$ is a locally Lipschitz function with respect to x, so that for all $R > 0$ there exists C_R such that $D_x b$ is Lipschitz with constant C_R in the ball of radius R. For each t and θ we define the sets

$$\mathcal{A}_{t,\theta}(R) = \big\{w \in \Omega : \big|\bar{X}_t\big| > R\big\} \cup \big\{w \in \Omega : \big|X_t^\theta\big| > R\big\}. \qquad (29)$$

By Chebyshev inequality we know that

$$\mathbb{P}(\mathcal{A}_{t,\theta}(R)) \leq \frac{\mathbb{E}|\bar{X}_t|^2}{R^2} + \frac{\mathbb{E}|X_t^\theta|^2}{R^2} \leq \frac{C}{R^2}, \qquad \forall\, t \in \mathbb{R}_+, \forall\, \theta \in (0,1]. \qquad (30)$$

Denoting for simplicity $X_s^\lambda = \bar{X}_s + \lambda\theta(Y_s + \hat{X}_s) = (1-\lambda)\bar{X}_s + \lambda X_s^\theta$ we have

$$\begin{aligned}
&\int_0^t e^{-\beta(t-s)} \left(\int_0^1 \mathbb{E}\mid D_x b\big(X_s^\lambda, \bar{u}_s + \lambda\theta v_s\big) - D_x b\big(\bar{X}_s, \bar{u}_s + \lambda\theta v_s\big) \mid^{2\alpha} d\lambda\right)^{\frac{1}{\alpha}} ds \\
&\quad\leq C \int_0^t e^{-\beta(t-s)} \left(\int_0^1 \int_{\mathcal{A}_{s,\theta}(R)} \mid D_x b\big(X_s^\lambda, \bar{u}_s + \lambda\theta v_s\big)\right. \\
&\quad\quad \left. - D_x b\big(\bar{X}_s, \bar{u}_s + \lambda\theta v_s\big) \mid^{2\alpha} d\mathbb{P} d\lambda\right)^{\frac{1}{\alpha}} ds \\
&\quad\quad + C \int_0^t e^{-\beta(t-s)} \left(\int_0^1 \int_{\mathcal{A}^c_{s,\theta}(R)} \mid D_x b\big(X_s^\lambda, \bar{u}_s + \lambda\theta v_s\big)\right. \\
&\quad\quad \left. - D_x b\big(\bar{X}_s, \bar{u}_s + \lambda\theta v_s\big) \mid^{2\alpha} d\mathbb{P} d\lambda\right)^{\frac{1}{\alpha}} ds
\end{aligned}$$

$$\leq C \int_0^t e^{-\beta(t-s)} \left(\int_0^1 \mathbb{P}(\mathcal{A}_{s,\theta}(R))^{\frac{\delta}{1+\delta}} \left(\mathbb{E} \mid D_x b\big(X_s^\lambda, \bar{u}_s + \lambda\theta v_s\big) - D_x b\left(\bar{X}_s, \bar{u}_s \right.\right.\right.$$
$$\left.\left.\left. + \lambda\theta v_s\right) \mid^{2\alpha(1+\delta)} \right)^{\frac{1}{1+\delta}} d\lambda \right)^{\frac{1}{\alpha}} ds$$
$$+ C \int_0^t e^{-\beta(t-s)} C_R^{\frac{1}{\alpha}} \left(\mathbb{E}\big|X_s^\theta - \bar{X}_s\big|^{2\alpha}\right)^{\frac{1}{\alpha}} ds, \tag{31}$$

where $\delta > 0$ is such that $4m\alpha(1+\delta) \leq p$ (we used Holder inequality with conjugate exponents $(1+\delta)$ and $\delta^{-1}(1+\delta)$.

Given $\varepsilon > 0$ we know by (30) that there exists R large enough so that $\mathbb{P}(\mathcal{A}_{s,\theta}(R)) \leq \varepsilon$. Moreover we choose $\delta > 0$ such that $4m\alpha(1+\delta) \leq p$. By Hypothesis 1 (H3) with $\beta = 1$, Theorem 1 and Lemma 1 we get:

$$\mathbb{E} \mid D_x b\big(X_s^\lambda, \bar{u}_s + \lambda\theta v_s\big) - D_x b\big(\bar{X}_s, \bar{u}_s + \lambda\theta v_s\big) \mid^{2\alpha(1+\delta)}$$
$$\leq C \left(\mathbb{E}\big|X_s^\lambda\big|^{4m\alpha(1+\delta)} + \mathbb{E}\big|\bar{X}_s\big|^{4m\alpha(1+\delta)} + 1 \right) \leq C,$$

(if $m = 0$ the above relation is straightforward). Thus the first of the two integrals in the last two lines in (31) can be estimated, for R large enough and all θ, λ in [0, 1], by $C\varepsilon^{\delta/[\alpha(1+\delta)]}$. Moreover, due to Lemma 1 we have that $\sup_{t\geq 0} \mathbb{E}|X_t^\theta - \bar{X}_t|^p \to 0$ as $\theta \to 0$.

Combining the two estimates above we have:

$$\sup_{t\geq 0} \int_0^t e^{-\beta(t-s)} \mathbb{E}\big|A_s^{\theta,y} Y_s\big|^2 ds \to 0 \text{ as } \theta \to 0. \tag{32}$$

Repeating the argument for all the terms in (26) we get the required result. □

Remark 3 Notice that we estimate only the second moment of the error term, uniformly in time. Nevertheless, estimates of higher moments of the state and first variation process are needed in order to complete the proof. More precisely, we can tune the value of α in (27) in order to minimize the maximal moment of the state equation we need to control. Indeed, the growth of the first term is

$$\mathbb{E}\big|D_x b\big(\bar{X}_s + \lambda\theta\big(Y_s + \hat{X}_s^\theta\big), \bar{u}_s + \lambda\theta v_s\big) - D_x b\big(\bar{X}_s, \bar{u}_s\big)\big|^{2\alpha}$$
$$\leq C\big(1 + \mathbb{E}\big|\bar{X}_s\big|^{4m\alpha} + \mathbb{E}\big|X_s^{\theta\,4m\alpha}\big).$$

So that, $4m\alpha = 2\alpha' = 2\frac{\alpha}{\alpha-1}$, from which $\alpha = \frac{2m+1}{2m}$. The maximal moment is then $p = 4m\alpha = 2(2m+1)$, which is the one appearing in Hypothesis 1. We also notice that , in this case, by Lemma 1 the term $\mathbb{E}|X_s^\theta|^{4m\alpha}$ can be estimated by the term $\mathbb{E}|\bar{X}_s|^{4m\alpha}$.

3.1 Perturbation of the Cost

Due to the hypotheses on the admissible controls and the estimate (15) the cost is well posed (recall that we have assumed that f is bounded from below):

$$\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T f(X_t,u_t)dt \le K\left[1+\sup_{t\ge 0}\mathbb{E}|X_t|^2+\sup_{t\ge 0}\mathbb{E}|u_t|^2\right]<\infty.$$

The same is true for the lim sup formulation. The expansion of the functional with respect to a convex perturbation of the control is given in the following

Lemma 3 *Let $\bar{u}$ be an optimal control and let u be any admissible control. If $v=u-\bar{u}$, using the above notation the following holds:*

$$\lim_{\theta\to 0_+}\frac{J^{\inf}(\bar{u}(\cdot)+\theta v(\cdot))-J^{\inf}(\bar{u}(\cdot))}{\theta}\le \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\Big[\big\langle D_x f\big(\bar{X}_t,\bar{u}_t\big),Y_t\big\rangle_{\mathbb{R}^n} + \big\langle D_u f\big(\bar{X}_t,\bar{u}_t\big),v_{t_{\mathbb{R}^l}}\Big]dt, \tag{33}$$

and

$$\lim_{\theta\to 0_+}\frac{J^{\sup}(\bar{u}(\cdot)+\theta v(\cdot))-J^{\sup}(\bar{u}(\cdot))}{\theta}\le \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T\Big[\big\langle D_x f\big(\bar{X}_t,\bar{u}_t\big),Y_t\big\rangle_{\mathbb{R}^n} + \big\langle D_u f\big(\bar{X}_t,\bar{u}_t\big),v_t\big\rangle_{\mathbb{R}^l}\Big]dt. \tag{34}$$

Proof We prove the first relation. The proof of the second one goes along the same lines. Let us compute

$$\begin{aligned}\frac{J_T(\bar{u}(\cdot)+\theta v(\cdot))-J_T(\bar{u}(\cdot))}{\theta} &= \frac{1}{\theta}\mathbb{E}\int_0^T\big[f\big(X_t^\theta,\bar{u}_t+\theta v_t\big)-f\big(\bar{X}_t,\bar{u}_t\big)\big]dt\\ &=\mathbb{E}\int_0^T\int_0^1 D_x f\big(\bar{X}_t+\lambda\big(X_t^\theta-\bar{X}_t\big),\bar{u}_t+\lambda\theta v_t\big)\big(\hat{X}_t^\theta+Y_t\big)d\lambda dt\\ &\quad+\mathbb{E}\int_0^T\int_0^1 D_u f\big(\bar{X}_t+\lambda\big(X_t^\theta-\bar{X}_t\big),\bar{u}_t+\lambda\theta v_t\big)v_t d\lambda dt.\end{aligned}$$

Passing to the ergodic lim inf cost functional (2) we have that

$$\begin{aligned}\frac{J^{\inf}(\bar{u}(\cdot)+\theta v(\cdot))-J^{\inf}(\bar{u}(\cdot))}{\theta} &= \frac{1}{\theta}\left[\liminf_{T\to\infty}\frac{1}{T}J_T(\bar{u}(\cdot)+\theta v(\cdot))\right.\\ &\quad\left.-\liminf_{T\to\infty}\frac{1}{T}J_T(\bar{u}(\cdot))\right]\\ &\le \limsup_{T\to\infty}\frac{1}{T}\left[\frac{J_T(\bar{u}(\cdot)+\theta v(\cdot))-J_T(\bar{u}(\cdot))}{\theta}\right]\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \int_0^1 \big\langle D_x f\big(\bar{X}_t + \lambda\big(X_t^\theta - \bar{X}_t\big), \bar{u}_t + \lambda\theta v_t\big), \hat{X}_t^\theta + Y_t\big\rangle d\lambda dt \\
&\quad + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \int_0^1 \big\langle D_u f\big(\bar{X}_t + \lambda\big(X_t^\theta - \bar{X}_t\big), \bar{u}_t + \lambda\theta v_t\big), v_t\big\rangle_U d\lambda dt,
\end{aligned}
$$

where we used that $\liminf(a_n) - \liminf(b_n) \le \limsup(a_n - b_n)$, for $(a_n)_{n\ge 1}$ and $(b_n)_{n\ge 1}$ two general real sequences. Let us concentrate on the term

$$
\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \int_0^1 \big\langle D_x f\big(\bar{X}_t + \lambda\big(X_t^\theta - \bar{X}_t\big), \bar{u}_t + \lambda\theta v_t\big), \hat{X}_t^\theta\big\rangle d\lambda dt \\
&\quad \le \limsup_{T\to\infty} \frac{1}{T}\int_0^T \int_0^1 (\mathbb{E}|D_x f\big(\bar{X}_t + \lambda\big(X_t^\theta - \bar{X}_t\big), \bar{u}_t \\
&\quad\quad + \lambda\theta v_t\big)|^2\big)^{1/2}\big(\mathbb{E}\big|\hat{X}_t^\theta\big|^2\big)^{1/2} d\lambda dt,
\end{aligned}
$$

which converges to zero, uniformly in T, as $\theta \to 0_+$. In fact, this follows from the linear growth of $D_x f(\cdot)$, the a priori estimates on X_t and Proposition 1. By identical argument, thanks to the uniform (in time) estimates (10) and (23) we can pass to the limit in the remaining terms. Namely we show that

$$
\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \int_0^1 \big\langle D_x f\big(\bar{X}_t + \lambda\big(X_t^\theta - \bar{X}_t\big), \bar{u}_t + \lambda\theta v_t\big), Y_t\big\rangle d\lambda dt \to 0 \\
&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \int_0^1 \big\langle D_u f\big(\bar{X}_t + \lambda\big(X_t^\theta - \bar{X}_t\big), \bar{u}_t + \lambda\theta v_t\big), v_t\big\rangle_U d\lambda dt \to 0,
\end{aligned}
$$

to get the required result. □

4 The Adjoint Equation

In this section we introduce the dual equation associated to the system, which is an infinite horizon Backward SDE in $\mathbb{R}^n$. Different approaches have been developed in the literature to study this class of equations, here we present a duality method built on the construction of a family of truncated equations along with a consistency argument. More precisely, the infinite horizon backward equation has the form

$$
-dp_t = \left[D_x b(X_t, u_t)^* p_t + \sum_{i=1}^d D_x \sigma^i(X_t, u_t)^* q_t^i - D_x f(X_t, u_t) \right] dt - \sum_{i=1}^d q_t^i dW_t^i, \tag{35}
$$

where, fixed any orthonormal basis $(e_i)_{i=1,\ldots,d}$ in $\mathbb{R}^d$ we set $W_s^i = \langle e_i, W_s\rangle$ and $\sigma^i(x,u) = \sigma(x,u)e_i$ moreover we denote by $(\cdot)^*$ the transposition operation in $\mathcal{L}(\mathbb{R}^n)$.

For every $T > 0$ fixed, its solution has to be understood as

$$p_t = p_T + \int_t^T \Big[D_x b(X_s, u_s)^* p_s + \sum_{i=1}^d D_x \sigma^i (X_s, u_s)^* q_s^i$$
$$-D_x f(X_s, u_s) \Big] ds - \sum_{i=1}^d \int_t^T q_s^i dW_s^i, \tag{36}$$

where p and q^i, $i = 1, \ldots, d$ take values in $\mathbb{R}^n$. Due to Hypothesis 1 and estimate (15) the forcing term in the driver is no better than bounded in time, i.e. $D_x f(X_s, u_s) \in L^\infty(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^n))$. Therefore we cannot expect the solution of (35) to be integrable up to infinity but only that $p \in L^\infty(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^n))$. Up to the authors' knowledge, there is not a general wellposendess result for such multidimensional BSDE's. Partial results have been obtained in [5] by a Girsanov argument that seems to work only if one knows a-priori that $\sum_{i=1}^d D_x \sigma^i (X_s, u_s)^* q_s^i$ can be written as $\sum_{i=1}^d q_s^i f^i$ for suitable adapted real process $(f_i)_{i=1,\ldots,d}$. In particular this is the case when $n = 1$ or the noise is additive.

Here the solution will be obtained via the introduction of a family of time truncations:

$$\begin{cases} -dp_t^{T,\nu} = \Big[D_x b(X_t, u_t)^* p_t^{T,\nu} + \sum_{i=1}^d D_x \sigma^i (X_t, u_t)^* q_t^{i,T,\nu} \\ \qquad -D_x f(X_t, u_t) \Big] dt - \sum_{i=1}^d q_t^{i,T,\nu} dW_t^i, \\ p_T^{T,\nu} = \nu, \end{cases} \tag{37}$$

which will be estimated by duality. For the approximating Eq. (37) a wellposedness result has been already adressed in [15].

To shorten the notation in the following paragraphs, let us denote

$$\Lambda_t := D_x b(X_t, u_t), \quad \Gamma_t^i := D_x \sigma^i (X_t, u_t), \quad \Psi_t := -D_x f(X_t, u_t);$$

moreover when $\nu = 0$ the solution of Eq. (37) will be denoted by $(p^T, q^{i,T})$.

Theorem 2 *For all $T \geq 0$ and all $\nu \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^n)$ there exists a unique $(d+1)$-tuple of $\mathbb{R}^n$-valued, adapted processes $(p^{T,\nu}, q^{1,T,\nu}, \ldots, q^{d,T,\nu})$ such that $p^{T,\nu}$ has continuous trajectories. Moreover $\sup_{t\in[0,T]} \mathbb{E}|p_t^{T,\nu}|^2 + \sum_{i=1}^d \mathbb{E} \int_0^T |q_t^{i,T,\nu}|^2 dt < \infty$ and, $\mathbb{P}$-almost surely, for all $t \in [0, T]$ it holds:*

$$p_t^{T,\nu} = \nu + \int_t^T \Lambda_s^* p_s^{T,\nu} ds + \sum_{i=1}^d \int_t^T (\Gamma_s^i)^* q_s^{i,T,\nu} ds$$
$$+ \int_t^T \Psi_s ds - \sum_{i=1}^d \int_t^T q_s^{i,T,\nu} dW_s^i.$$

Consider now the following affine forward SDE with general forcing term $(\gamma, \rho^i)_{i=1,\dots,d}$ with γ and ρ^i, $i = 1, \dots, d$ in $L^2([0, T]; L^2(\Omega; \mathbb{R}^n))$ and initial condition $\eta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$:

$$\begin{cases} d\mathcal{Y}_s^{t,\eta,\gamma,\rho} = \Lambda_s \mathcal{Y}_s^{t,\eta,\gamma,\rho} ds + \sum_{i=1}^d \Gamma_s^i \mathcal{Y}_s^{t,\eta,\gamma,\rho} dW_s^i + \gamma_s ds \\ \qquad\qquad + \sum_{i=1}^d \rho_s^i dW_s^i, \ s \geq t, \\ \mathcal{Y}_t^{t,\eta,\gamma,\rho} = \eta. \end{cases} \tag{38}$$

Then by the same technique we adopted in the proof of Theorem 1, the above equation admits a unique adapted solution and

$$\mathbb{E}|\mathcal{Y}_r^{t,\eta,\gamma,\rho}|^2 \leq e^{-2\beta(r-t)}\mathbb{E}|\eta|^2 + K \int_t^r e^{-2\beta(r-s)}\mathbb{E}\left[|\gamma_s|^2 + |\rho_s^1|^2 + \cdots + |\rho_s^d|^2\right] ds. \tag{39}$$

If $\gamma \equiv 0$ the solution to the above equation will be denoted by $\mathcal{Y}^{t,\eta,\rho}$ and analogously, if $\rho \equiv 0$ it will be denoted by $\mathcal{Y}^{t,\eta,\gamma}$.

The next result has been proven in [15] by computing the Itô formula for the differential of the product $d\langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, p_s^{T,\nu}\rangle$.

Lemma 4 *Given $(\rho^i)_{i=1,\dots,d}$ with $\gamma, \rho^i \in L^2([0, T]; L^2(\Omega; \mathbb{R}^n))$, $\eta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$, $\nu \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^n)$ it holds:*

$$\begin{aligned} &\mathbb{E}\int_t^T \langle p_s^{T,\nu}, \gamma_s\rangle ds + \sum_{i=1}^d \mathbb{E}\int_t^T \langle q_s^{i,T,\nu}, \rho_s^i\rangle ds + \mathbb{E}\langle p_t^{T,\nu}, \eta\rangle \\ &= \mathbb{E}\int_t^T \langle \mathcal{Y}_s^{t,\eta,\gamma,\rho}, \Psi_s\rangle ds + \mathbb{E}\langle \nu, \mathcal{Y}_T^{t,\eta,\gamma,\rho}\rangle. \end{aligned} \tag{40}$$

In the following, relation (40) will be the main instrument to get information on the behaviour of the BSDE. We will specifically choose the values of t, η, ρ according to our needs.

We are now in position to define what is a solution to the infinite horizon multidimensional BSDE (35) and prove its existence and uniqueness.

Definition 1 A solution to equation (35) is a $(d + 1)$-tuple of $\mathbb{R}^n$-valued, adapted processes $(p_t, q_t^1, \dots, q_t^d)_{t\in[0,\infty[}$ such that, for all $T > 0$ and all $i = 1, \dots, d$ it holds $\mathbb{E}\int_0^T |q_t^i|^2 dt < \infty$. Moreover p has continuous trajectories and $\sup_{t\in[0,\infty)} \mathbb{E}|p_t|^2 < \infty$. Finally, for all $0 \leq t \leq T$, (36) holds $\mathbb{P}$-almost surely.

The main result of this section is the following

Theorem 3 *Let Hypothesis 1 holds true. Then Eq. (35) admits a unique solution $(p^\infty, q^{1,\infty}, \dots, q^{d,\infty})$.*

Proof *Existence:* Let us take in (40) $\nu \equiv 0,\ \gamma \equiv 0,\ \rho \equiv 0, \eta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$ then

$$\mathbb{E}\langle p_t^T, \eta\rangle = \mathbb{E}\int_t^T \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds. \tag{41}$$

Since $\Psi \in L^\infty(\mathbb{R}_+; L^2(\Omega; \mathbb{R}^n))$ by (39) we deduce that

$$\mathbb{E}\int_t^T \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds \to \mathbb{E}\int_t^\infty \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds \qquad \text{as} \quad T \to \infty$$

and that the right hand side is a bounded linear operator from $L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n) \to \mathbb{R}$ defined as

$$\eta \mapsto \mathbb{E}\int_t^\infty \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds.$$

Hence, by Riesz representation theorem there exists an element $P_t \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$ such that

$$\mathbb{E}\langle P_t, \eta\rangle = \mathbb{E}\int_t^\infty \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds. \tag{42}$$

Moreover $p_t^T \rightharpoonup P_t$ in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$ and $\mathbb{E}|P_t|^2 \le \beta^{-1}\sup_{s\in[0,\infty[}(\mathbb{E}|\Psi_s|^2)^{1/2}$ for all $t > 0$.

Let now for all $N \in \mathbb{N}$, $(\tilde{p}_t^N, \tilde{q}_t^{1,N}, \ldots, \tilde{q}_t^{d,N})_{t\in[0,N]}$ be the solution of equation (37) with $T = N$ and $\nu = P_N$.

We claim that, for all $N, M \in \mathbb{N}$ with $0 \le N \le M$ and all $t \le N$ it holds

$$\tilde{p}_t^N = \tilde{p}_t^M, \mathbb{P}- \text{ a.s.}, \quad \tilde{q}^N = \tilde{q}^M, \mathbb{P}\otimes dt - \text{ a.s. in}\Omega \times [0, N]. \tag{43}$$

By definition and Lemma 4 we deduce that for all $\eta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$

$$\mathbb{E}\langle \tilde{p}_t^N, \eta\rangle = \mathbb{E}\int_t^N \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds + \mathbb{E}\langle P_N, \mathcal{Y}_N^{t,\eta}\rangle.$$

Plugging (42) with $t = N$ in the above relation we have

$$\mathbb{E}\langle \tilde{p}_t^N, \eta\rangle = \mathbb{E}\int_t^N \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds + \mathbb{E}\int_N^\infty \left\langle \mathcal{Y}_s^{N,\mathcal{Y}_N^{t,\eta}}, \Psi_s\right\rangle ds.$$

Finally, observing that by uniqueness of the solution to equation (38) we have $\mathcal{Y}_s^{N,\mathcal{Y}_N^{t,\eta}} = \mathcal{Y}_s^{t,\eta}$ $\mathbb{P}$-a.s., we conclude

$$\mathbb{E}\big\langle \tilde{p}_t^N, \eta\big\rangle = \mathbb{E}\int_t^\infty \langle \mathcal{Y}_s^{t,\eta}, \Psi_s\rangle ds = \mathbb{E}\langle \eta, P_t\rangle,$$

and our claim is proved since the right hand side does not depend on N.We also remark that by the above identity we deduce that

$$\sup_{t\in[0,N]} \left(\mathbb{E}\big|\tilde{p}_t^N\big|^2\right)^{1/2} \le \beta^{-1} \sup_{s\in[0,\infty[} \left(\mathbb{E}\big|\Psi_s\big|^2\right)^{1/2},$$

and that the right hand side does not depend neither on t nor on N. The consistency of the martingale term easily follows now from the uniqueness of the solution of BSDEs in finite horizon. Indeed both couples $(\tilde{p}^N, \tilde{q}^N)$ and $(\tilde{p}^M, \tilde{q}^M)$ are solutions to the finite horizon BSDE (37) with $T = N$ and the same final condition $\nu = \tilde{p}^N_T = \tilde{p}^M_T$.

Now we define

$$p^\infty_t = \sum_{N=1}^{\infty} \tilde{p}^N_t I_{[N-1,N[}(t), \quad q^{i,\infty}_t = \sum_{N=1}^{\infty} \tilde{q}^{i,N}_t I_{[N-1,N[}(t),$$

and claim that it is the desired solution. Indeed it satisfies the desired integrability and adaptedness conditions. Moreover fixed $0 \le t \le T$ then

$$\begin{aligned} p^\infty_t - p^\infty_T &= \left[p^\infty_t - p^\infty_{\lfloor t\rfloor+1}\right] + \left[p^\infty_{\lfloor T\rfloor} - p^\infty_T\right] + \sum_{n=\lfloor t\rfloor+1}^{\lfloor T\rfloor-1} \left[p^\infty_n - p^\infty_{n+1}\right] \\ &= \left[\tilde{p}^{\lfloor t\rfloor+1}_t - \tilde{p}^{\lfloor t\rfloor+2}_{\lfloor t\rfloor+1}\right] + \left[\tilde{p}^{\lfloor T\rfloor+1}_{\lfloor T\rfloor} - \tilde{p}^{\lfloor T\rfloor+1}_T\right] + \sum_{n=\lfloor t\rfloor+1}^{\lfloor T\rfloor-1} \left[\tilde{p}^{n+1}_n - \tilde{p}^{n+2}_{n+1}\right] \\ &= \left[\tilde{p}^{\lfloor t\rfloor+1}_t - \tilde{p}^{\lfloor t\rfloor+1}_{\lfloor t\rfloor+1}\right] + \left[\tilde{p}^{\lfloor T\rfloor+1}_{\lfloor T\rfloor} - \tilde{p}^{\lfloor T\rfloor+1}_T\right] + \sum_{n=\lfloor t\rfloor+1}^{\lfloor T\rfloor-1} \left[\tilde{p}^{n+1}_n - \tilde{p}^{n+1}_{n+1}\right], \quad (44) \end{aligned}$$

where in the last equality we have exploited (43) where it was needed. Recalling that $(\tilde{p}^N_t, \tilde{q}^{1,N}_t, \ldots, \tilde{q}^{d,N}_t)_{t\in[0,N]}$ solves Eq. (37); by the definition of $(p^\infty, q^{1,\infty}, \ldots, q^{d,\infty})$ the above equality can be rewritten as

$$\begin{aligned} p^\infty_t - p^\infty_T = &\int_t^{\lfloor t\rfloor+1} \Lambda^*_s p^\infty_s ds + \sum_{i=1}^d \int_t^{\lfloor t\rfloor+1} (\Gamma^i_s)^* q^{i,\infty}_s ds \\ &+ \int_t^{\lfloor t\rfloor+1} \Psi_s ds - \sum_{i=1}^d \int_t^{\lfloor t\rfloor+1} q^{i,\infty}_s dW^i_s \\ &+ \sum_{n=\lfloor t\rfloor+1}^{\lfloor T\rfloor-1} \left[\int_n^{n+1} \Lambda^*_s p^\infty_s ds \right. \\ &\left. + \sum_{i=1}^d \int_n^{n+1} (\Gamma^i_s)^* q^{i,\infty}_s ds + \int_n^{n+1} \Psi_s ds - \sum_{i=1}^d \int_n^{n+1} q^{i,\infty}_s dW^i_s \right] \\ &+ \int_{\lfloor T\rfloor}^T \Lambda^*_s p^\infty_s ds + \sum_{i=1}^d \int_{\lfloor T\rfloor}^T (\Gamma^i_s)^* q^{i,\infty}_s ds \\ &+ \int_{\lfloor T\rfloor}^T \Psi_s ds - \sum_{i=1}^d \int_{\lfloor T\rfloor}^T q^{i,\infty}_s dW^i_s \end{aligned}$$

$$= \int_t^T \Lambda_s^* p_s^\infty ds + \sum_{i=1}^d \int_t^T (\Gamma^i)_s^* q_s^{i,\infty} ds$$
$$+ \int_t^T \Psi_s ds - \sum_{i=1}^d \int_t^T q_s^{i,\infty} dW_s^i \tag{45}$$

and this completes the proof of existence of a solution to Eq. (36).

Uniqueness: Let $(p_t, q_t^1, \dots, q_t^d)_{t\geq 0}$ be a solution to Eq. (36). We choose $\rho \in L^2(\Omega \times [0, \infty[; \mathbb{R}^n)$ with support in the finite interval $[0, T]$ ($\rho_r = 0$, if $r \geq T$) and $\eta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$. Observe that $(p_t, q_t^1, \dots, q_t^d)_{t\geq 0}$ is, in particular, a solution to Eq. (37) in $[0, T]$ with $\nu = p_T$. Then, by Lemma 4 we get:

$$\mathbb{E}\int_t^T \big\langle \mathcal{Y}_s^{t,\eta,\rho}, \Psi_s \big\rangle ds + \mathbb{E}\big\langle p_T, \mathcal{Y}_T^{t,\eta,\rho} \big\rangle = \sum_{i=1}^d \mathbb{E}\int_t^T \langle q_s^i, \rho_s \rangle ds + \mathbb{E}\langle \eta, p_t \rangle.$$

Since $\rho_t = 0$ for $t > T$ then by (39) we have that $\mathbb{E}|\mathcal{Y}_s^{t,\eta,\rho}|^2 \leq Ce^{-2\beta(s-t)}$ for a suitable constant C. If we let $T \to \infty$ in the above equality we get (recall that $\sup_{t\geq 0} \mathbb{E}|p_t|^2 < \infty$ by definition of solution):

$$\mathbb{E}\int_t^\infty \big\langle \mathcal{Y}_s^{t,\eta,\rho}, \Psi_s \big\rangle ds = \sum_{i=1}^d \mathbb{E}\int_t^T \langle q_s^i, \rho_s \rangle ds + \mathbb{E}\langle \eta, p_t \rangle \tag{46}$$

and this completes the proof of uniqueness due to the arbitrariness of t, T, ρ and η. □

As a by-product of the above proof we have the following infinite-horizon version of the duality relation:

Corollary 4 *Let $(p_t, q_t^1, \dots, q_t^d)_{t\geq 0}$ be a solution to Eq. (36). Fix $\rho \in L^2(\Omega \times [0, \infty[; \mathbb{R}^n)$ with support in $[0, T]$, $t \in [0, T)$ and $\eta \in L^2(\Omega, \mathcal{F}_t; \mathbb{R}^n)$ then (46) holds.*

5 Necessary Ergodic SMP

We give two versions of the SMP in its necessary form. The first is based on the well-posedness result for the infinite horizon BSDE. The second one is written in terms of the family of truncated backward equations introduced in the previous section. The Hamiltonian associated to the system is

$$H\big(x, u, p, q^1, \dots, q^d\big) = \langle b(x, u), p \rangle + \sum_{i=1}^d \big\langle \sigma^i(x, u), q^i \big\rangle + f(x, u). \tag{47}$$

We are now able to formulate a necessary condition corresponding to the ergodic control problem.

Theorem 5 [SMP infinite horizon case] *Suppose that $(\bar{X}, \bar{u})$ is an optimal pair for the control problem $J^{\inf}$ or $J^{\sup}$ and let $(\bar{p}^{\infty}, \bar{q}^{\infty}) = (\bar{p}^{\infty}, \bar{q}^{\infty,1}, \ldots, \bar{q}^{\infty,d})$ be the solution of Eq. (35) corresponding to $(\bar{X}, \bar{u})$. Then under Hypothesis 1, the following variational inequality holds:*

$$0 \le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle D_u H\,(\,\bar{X}_t, \bar{u}_t, \bar{p}_t^{\infty}, \bar{q}_t^{\infty}\,)\,, u_t - \bar{u}_t\rangle_{\mathbb{R}^l} dt, \tag{48}$$

where $H(x, u, p, q)$ is the Hamiltonian of the system, and $u(\cdot)$ is an arbitrary admissible control.

Proof Let $v(\cdot) = u(\cdot) - \bar{u}(\cdot)$ and let Y_t be the solution to Eq. (22). Lemma 4 with $t = 0$, $\eta = 0$,$\nu = \bar{p}_T^{\infty}$,$\gamma = D_u b(\bar{X}, \bar{u})v$, $\rho^i = D_u\sigma^i(\bar{X}, \bar{u})v$ yields

$$\begin{aligned}
&\mathbb{E}\int_0^T \langle D_x f(\bar{X}_t, \bar{u}_t), Y_t\rangle dt \\
&\quad = \mathbb{E}\langle \bar{p}_T^{\infty}, Y_T\rangle + \mathbb{E}\int_0^T \langle \bar{p}_t^{\infty}, D_u b(\bar{X}_t, \bar{u}_t)v_t\rangle dt \\
&\quad + \sum_{i=1}^{d}\mathbb{E}\int_0^T \langle \bar{q}_t^{i,\infty}, D_u\sigma^i(\bar{X}_t, \bar{u}_t)v_t\rangle dt.
\end{aligned} \tag{49}$$

So that, from Lemma 3 and the relation above, we have

$$\begin{aligned}
0 &\le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \left[\left\langle D_x f\left(\bar{X}_t, \bar{u}_t\right), Y_t\right\rangle_{\mathbb{R}^n} + \left\langle D_u f\left(\bar{X}_t, \bar{u}_t\right), v_t\right\rangle_{\mathbb{R}^l}\right] dt \\
&\le -\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left\langle Y_T, \bar{p}_T^{\infty}\right\rangle \\
&\quad + \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \left[\left\langle D_u H\left(\bar{X}_t, \bar{u}_t, \bar{p}_t^{\infty}, \bar{q}_t^{\infty}\right), v_t\right\rangle_{\mathbb{R}^l}\right] dt.
\end{aligned}$$

Recalling that $\sup_{t\ge 0}\mathbb{E}|\bar{p}_t^{\infty}|^2 < +\infty$ by definition of solution to Eq. (35) and $\sup_{t\ge 0}\mathbb{E}|Y_t|^2 < +\infty$ by (39) we can conclude that

$$0 \le \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \left[\left\langle D_u H\left(\bar{X}_t, \bar{u}_t, \bar{p}_t^{\infty}, \bar{q}_t^{\infty}\right), v_t\right\rangle_{\mathbb{R}^l}\right] dt$$

and the claim is proved. □

Similarly we can prove a truncated version of the stochastic maximum principle that involves the solution $(\bar{p}^T, \bar{q}^{1,T}, \ldots, \bar{q}^{d,T})$ of Eq. (37) with $\nu = 0$. This is the content of the following

Proposition 2 [SMP truncated case] *Let $(\bar{X}, \bar{u})$ be an optimal pair for the control problem (9) and let $(\bar{p}^T, \bar{q}^{1,T}, \ldots \bar{q}^{d,T})$ be the solution to Eq. (37) with $\nu = 0$ and $(u, X) = (\bar{u}, \bar{X})$. Then, under Hypothesis 1, the following variational inequality holds*

$$0 \leq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle D_u H(\bar{X}_t, \bar{u}_t, \bar{p}_t^T, \bar{q}_t^T), u_t - \bar{u}_t\rangle_{\mathbb{R}^l} dt, \tag{50}$$

where $H(x, u, p, q)$ is the Hamiltonian of the system and $u(\cdot)$ is an arbitrary admissible control.

Proof Let $v_t = u_t - \bar{u}_t$, for every u_t admissible. The result easily follows combining Lemma 3 with a duality argument. Precisely, choose $\eta = 0$, $\gamma_t = D_u b(\bar{X}_t, \bar{u}_t) v_t$, $\rho_t^i = D_u\sigma^i(\bar{X}_t, \bar{u}_t) v_t$ and $\Psi_t = -D_x f(\bar{X}_t, \bar{u}_t)$ in the general formula (40). □

6 Sufficient SMP

In this part we prove that under some additional convexity assumption on the Hamiltonian function H, the variational inequality obtained in Theorem 5 (the same hold also for Theorem 2) is sufficient for optimality.

Theorem 6 [Sufficient SMP] *Let $u^\sharp(\cdot) \in \mathcal{U}_{ad}$ be an admissible control, $X^\sharp$ be the corresponding state process and $p^\sharp$ the first adjoint process on infinite time horizon solving (35) for the couple $(u^\sharp, X^\sharp)$. Further, let $(x, u) \mapsto H(x, u, p_t^\sharp, q_t^\sharp)$ be a convex function $d\mathbb{P} \times dt$−a.e. and the following minimality condition holds*

$$\limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle D_u H(X_t^\sharp, u_t^\sharp, p_t^\sharp, q_t^\sharp), u_t - u_t^\sharp\rangle_{\mathbb{R}^l} dt \geq 0, \tag{51}$$

for every $u(\cdot) \in \mathcal{U}_{ad}$. Then $u^\sharp(\cdot)$ is optimal both for lim inf *and* lim sup *formulations of the ergodic control problem.*

Proof Let $u(\cdot) \in \mathcal{U}_{ad}$ be arbitrary but fixed. Then the goal is to show that the difference $J(u^\sharp(\cdot)) - J(u(\cdot))$ is non-positive. Using the sub additivity of the lim sup we have

$$\begin{aligned} J^{\sup}(u^\sharp(\cdot)) - J^{\sup}(u(\cdot)) &\leq \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \left[f(X_t^\sharp, u_t^\sharp) - f(X_t, u_t)\right] dt \\ &= \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \left[H(X_t^\sharp, u_t^\sharp, p_t^\sharp, q_t^\sharp) - H(X_t, u_t, p_t^\sharp, q_t^\sharp)\right] dt \\ &+ \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle b(X_t, u_t) - b(X_t^\sharp, u_t^\sharp), p_t^\sharp\rangle dt \\ &+ \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle \sigma(X_t, u_t) - \sigma(X_t^\sharp, u_t^\sharp), q_t^\sharp\rangle dt = I_1 + I_2 + I_3. \end{aligned}$$

Now, due to convexity of H, the term I_1 can be estimated from above as follows

$$\begin{aligned} I_1 &\leq \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \langle D_x H\big(X_t^\sharp, u_t^\sharp, p_t^\sharp, q_t^\sharp\big), X_t^\sharp - X_t\rangle dt \\ &+ \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \big\langle D_u H\big(X_t^\sharp, u_t^\sharp, p_t^\sharp, q_t^\sharp\big), u_t^\sharp - u_t\big\rangle_U dt \\ &\leq \limsup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T \big\langle D_x H\big(X_t^\sharp, u_t^\sharp, p_t^\sharp, q_t^\sharp\big), X_t^\sharp - X_t\big\rangle dt, \end{aligned}$$

where in the last step we have used the minimality condition (51). Next,

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\big\langle p_T^\sharp, X_T^\sharp - X_T\big\rangle = 0, \tag{52}$$

due to the fact that $p^\sharp, X, X^\sharp \in L^\infty(\mathbb{R}_+; L^2(\Omega; H))$.

By applying the Itô formula to $\langle p_T^{\sharp,T}, X_T^\sharp - X_T\rangle_H$ and putting all the terms together we arrive at

$$J(u^\sharp(\cdot)) - J(u(\cdot)) \leq 0. \tag{53}$$

The above inequality means that $u^\sharp(\cdot)$ is an optimal control. □

The form of minimality condition (51) is related to our definition of the Hamiltonian. In fact, one could introduce an another sign convention for H, namely $H(x, u, p, q) = \langle b(t, x, u), p\rangle + \sum_{i=1}^d \langle \sigma^i(x, u), q^i\rangle - f(x, u)$ which would lead to the corresponding modification in the driver of the first adjoint equation, concavity assumption (instead of convexity) on H in (x, u) and the opposite inequality in (51). All these changes would lead to the maximality condition usually considered with stochastic maximum principle.

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