Bimodality testing of the stochastic cusp model

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Abstract. Multimodal distributions are popular in many areas: biology (fish and shark population), engineering (material collapse under pressure, stability of ships), psychology (attitude transitions), physics (freezing of water) etc. There were a few attempts to utilize multimodal distributions in financial mathematics as well (e.g. [2], [6], [5]).

Cobb et al. [4] described a class of multimodal distributions belonging to the exponential family, which has unique maximum likelihood estimators and showed a connection to the stationary distribution of the stochastic cusp catastrophe model. Moreover was shown, how to identify bimodality for given parameters of the stochastic cusp model using the sign of Cardans discriminant.

A statistical test for bimodality of the stochastic cusp model using maximum likelihood estimates is proposed in the paper as well as the necessary condition for bimodality which can be used for simplified testing to reject bimodality. By proposed methods is tested the bimodality of exchange rate between USD and GBP in the periods within the years 1975 - 2014.

Keywords: multimodal distributions, stochastic cusp model, statistical bimodality test.

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1 Introduction

Stochastic cusp model has its name thanks to classification of singularities by Vladimir Arnold [1] within deterministic models of catastrophe theory proposed by Rene Thom [9], where cusp model is the most used one. Its deterministic version was popularized in the 1970s and in the 1980s was developed the theory of the stochastic version by Lauren Cobb [3]. He also proposed the numerical maximum likelihood method and methods of moments for estimation of parameters. There are other methods for estimation of catastrophe models, however the paper will utilize the properties of maximum likelihood estimators for the statistical testing.

The specification of the stochastic cusp model is such that the probability density function belongs to the class of generalized exponential distributions. It is convenient that probability density function of the stochastic cusp model accommodates variable skewness, kurtosis, and even bimodality. Bimodality of the model is accessed by Cardan's discriminant, which is negative, when the probability density function has two modes. These properties as well as the existence of estimating methodology encouraged empirical research especially in behavioural science and psychology. The distinct perception of agents in the market was the motivation for the usage of stochastic cusp model in the financial mathematics (eg. by Zeeman [10], Creedy and Martin [5] or Fernandes [6]).

Despite the frequent usage of the stochastic cusp model the statistical test for negativeness of Cardan's discriminant, which implies bimodality, calculated from estimated parameters, was not proposed. This paper tries to fill this gap by approximating the distribution of estimated Cardan's discriminant using delta method. The approximate distribution is than used to propose a statistical test, where the null hypotheses is rejected, if there is enough statistical evidence against bimodality of the distribution.

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The rest of paper is organized as follows. Section 2.1 describes a cusp model and its connection to the multimodal distributions, section 2.2 shows different approaches to estimate parameters and the statistical test of bimodality is proposed in the Section 2.3. Section 3 utilizes the methodology for the real data example of USD, GBP exchange rate and demonstrates different situations, where the bimodality is rejected. Section 4 summarizes the results and concludes.

2 Cusp model

2.1 Connection to multimodal distributions

This section summarizes the connection of the stochastic cusp model to the exponential class of multimodal distributions. The standard parametrization of both approaches is shown as well as the transformation from one to another following Cobb et al. [4].

The generalized exponential family of distributions is characterized by probability density function

$$f_k(x) = \xi(\beta) \exp\left[\int_a^x \frac{g(s)}{v(s)} ds\right], \tag{1}$$

where $g(x) = \sum_{i=0}^{k} \beta_i x^k$ is polynomial function of order $k > 0$ and function $v(x)$ has one of the form:

$$\begin{array}{rll} \text{typeN}: & v(x) = 1 & -\infty < x < \infty \\ \text{typeG}: & v(x) = x & 0 < x < \infty \\ \text{typeI}: & v(x) = x^2 & 0 < x < \infty \\ \text{typeB}: & v(x) = x(1-x) & 0 < x < 1. \end{array} \tag{2}$$

Let (a, b) be an open interval, where $v(x)$ is positive and $\xi(\beta)$ is normalizing constant for $\int_a^b f_x(x)dx$ to be unity.

Cusp model in polynomial parametrization is given by the polynomial:

$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3, \tag{3}$$

where $b_3 < 0$ and $v(x) = 1$. Then the probability density function will be

$$f_p(x) = \xi_p(\mathbf{b}) \exp\left[b_0 x + \frac{b_1}{2} x^2 + \frac{b_2}{3} x^3 + \frac{b_3}{4} x^4\right], \tag{4}$$

where $\xi_p(\mathbf{b})$ is normalizing constant depending on parameters b_0, b_1, b_2 and b_3.

To get the probability density function of standard cusp model parametrization:

$$f_c(z) = \xi_c(\theta) \exp\left[\alpha z + \frac{\beta}{2} z^2 - \frac{1}{4} z^4\right], \tag{5}$$

where $z = \frac{x-\lambda}{\sigma}$, $\xi_c(\theta)$ is the normalizing constant and θ stands for parameters $\alpha, \beta, \lambda, \sigma$, one needs to utilize the following substitutions:

$$\begin{aligned} &\sigma = (-b_3)^{-1/4}, \\ &\lambda = -b_2/(3b_3), \\ &\beta = (b_1 + b_2\lambda)\sigma^2, \\ &\alpha = \sigma g(\lambda). \end{aligned} \tag{6}$$

The cusp distribution in both types of parametrization may also be characterized by nonlinear diffusion processes. Let $\sigma^2(x) := v(x)$ and $\mu(x) := \frac{1}{2}\left(g(x) + \sigma^2(x)'\right)$. Then $f_p(x)$ is the stationary density function of a process x_t that is driven by the stochastic differential equation:

$$dx_t = \mu(x_t)dt + \sigma(x_t)dW_t, \tag{7}$$

where W_t is a standard Wiener process.

For identifying the bimodality (or unimodality) of the cusp probability density function (5), one needs to calculate the Cardan's discriminant

$$\delta_C = \left(\frac{\alpha}{2}\right)^2 - \left(\frac{\beta}{3}\right)^3, \tag{8}$$

which is negative, when the probability density function is bimodal and positive in unimodality case. The parameters α (*asymetry*) and β (*bifurcation*) are invariant with respect to changes in λ (*location*) and σ (*scale*), as is δ_C, and they have following approximate interpretations. If $\delta_C \geq 0$ then α measures skewness and β kurtosis and when $\delta_C < 0$ then α indicates the relative height of the two modes and β their relative separations.

2.2 Estimation of parameters

For the estimation of the polynomial parametrization (4) Cobb et al. [4] proposed a moment recursion relations which connects $k+1$ parameters to the first $2k$ moments of the probability density function. However, these estimates can not be utilized for the testing of bimodality, but could serve after the transformation (6) as a starting values for the numerical search for the maximum likelihood estimators $\hat{\theta}$ of the cusp probability density function (5). By following the theory of exponential families (e.g. Lehman [8]) we know that maximum likelihood estimators of the polynomial form exist, are unique can be found for example by a Newton-Raphson search. Other possible way is to use an R-package called "cusp" [7], which uses R build in function *optim* for maximizing the log likelihood of observed values. The variance matrix of the MLE is estimated using the Hessian matrix of the log likelihood function. Then the asymptotic distribution of the MLE is:

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right) \xrightarrow{d} N\left(0, I^{-1}\right), \tag{9}$$

where I is the Fisher information matrix.

2.3 Formulation of the test

The necessary condition for Cardan's discriminant to be negative is the positivity of parameter β. This condition could be statistically tested at first using the MLE result $\hat{\beta} \xrightarrow{d} N(\beta_0, \hat{\sigma}^2_\beta)$, where $\hat{\sigma}^2_\beta$ denotes the estimate of $\hat{\beta}$ variance from the Hessian matrix of the log likelihood function. The null hypothesis would be $H0 : \beta = 0$ with the one sided alternative $H1 : \beta < 0$. Rejection of the null hypothesis implies rejection of the negativity of Cardan's discriminant, which means rejection of bimodality.

For the direct testing of bimodality serves the approximate distribution of Cardan's discriminant δ_C (10) derived using delta method from (9). The delta method yields

$$\sqrt{n}\left(\hat{\delta}_C - \delta_0\right) \xrightarrow{d} N\left(0, \nabla h(\theta)^T I^{-1} \nabla h(\theta)\right), \tag{10}$$

where the function $h(\theta) = \left(\alpha^2/4 - \beta^3/27\right)$ transforms parameters of cusp probability density function (5) into Cardan's discriminant. This result allows to propose a statistical test for bimodality: we would like to test $H0 : \delta_C = 0$ against $H1 : \delta_C > 0$. Similarly to testing only parameter β, the test statistics has asymptotically normal distribution and rejecting of null hypothesis means the rejection of bimodality.

3 Empirical testing of bimodality

	1975-2014	1993	2003	2011
mean	1.699	1.502	1.635	1.603
median	1.639	1.497	1.624	1.607
std. dev.	0.235	0.036	0.052	0.030
skewness	0.859	-0.057	0.759	-0.379
kurtosis	3.914	2.680	2.983	2.207
minimum	1.042	1.418	1.550	1.534
maximum	2.455	1.593	1.668	1.627

Table 1 Descriptive statistics.

The performance of the statistical test will be illustrated using the exchange rate of USD and GBP, which is inspired by article [5], which suggested using multiple equilibria model for exchange rate. The data represents indicative middle market (mean of spot buying and selling) rates as observed by the Bank's Foreign Exchange Desk in the London interbank market around 4pm from the beginning of the year 1974 till the end of 2014. The basic statistical characteristic are presented in Table 1.

	1975-2014	1993	2003	2011
$\hat{\alpha}$	-1.760	0.353	-0.857	0.402
$\hat{\beta}$	0.108	-2.766	0.706	0.682
$\hat{\delta}_C$	0.775	0.815	0.171	0.029
p-val δ_C	0.000	0.314	0.021	0.193
sd $\hat{\delta}_C$	0.064	1.686	0.084	0.033
p-val β	0.916	0.087	0.988	0.988
sd $\hat{\beta}$	0.078	2.036	0.311	0.302

Table 2 Descriptive statistics.

The years 1993, 2003 and 2011 (Figure 2, 3 and 4, histogram and estimated probability density function) were chosen for demonstrative purposes of statistical testing of bimodality proposed in section 2.3. The results of the testing are summarized in Table 2, where *p-val* denotes p-value of corresponding test (i.e. test of Cardan's discriminant or parameter β alone). From the results we see, that we reject the null hypothesis (bimodality) for the year 1993 due to test of parameter β (at the significance level 0.1) and for the year 2003 due to the test of δ_C (at the significance level 0.05) as well as for the whole sample period. The rejecting of bimodality in the year 1993 is caused by negative estimate of beta parameter. On the other hand, in the year 2003 is Cardan's discriminant made statistically higher than zero by the estimate $\hat{\alpha}$ more distant from zero compared to the β. By the same reason we reject bimodality of the whole sample period. Year 2011 demonstrates the lack of statistical evidence against bimodality, hence we can not reject bimodality by any test.

4 Conclusion

In this paper is proposed the statistical test for bimodality of the stochastic cusp model using Cardan's discriminant and a simpler test builded on parameter beta and a necessary condition for bimodality, which can be used for a rejection of the bimodality as well. In more formal words, tests allow to reject null hypothesis of bimodality of the probability density function of the stochastic cusp model. The tests were performed on the USD, GBP exchange rate in chosen years to demonstrate the abilities of the tests to reject bimodality and different reasons for rejection of bimodality were pointed out. In general, these test could be used for testing of bimodality, whenever the estimated parameters has multivariate normal distribution as is the case of maximum likelihood estimation.

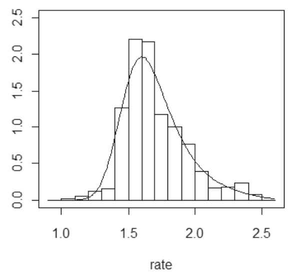

Figure 1 Data 1975-2014

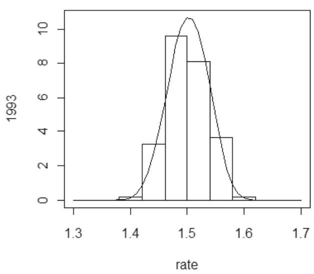

Figure 2 Data 1993

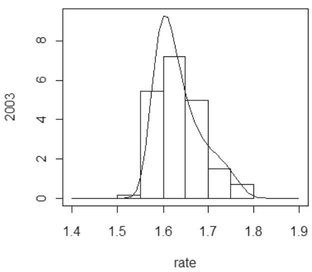

Figure 3 Data 2003

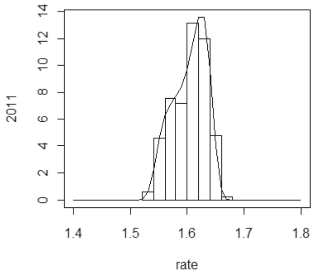

Figure 4 Data 2011

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