Discrete Dynamic Endogenous Growth Model: Derivation, Calibration and Simulation

Jan Kodera[1], Quang Van Tran[1] , Miloslav Vošvrda[2]

Abstract. Endogenous economic growth model were developed to improve traditional growth models with exogenous technological changes. There are several approaches how to incorporate technological progress into a growth model. Romer was the first author who has introduced it by expanding the variety of intermediate goods. Overall, the growth models are often continuous. In our paper we formulate a discrete version of Romer's model with endogenous technological change based on expanding variety of intermediates, both in the final good sector and in the research-development sector, where the target is to maximize present value of the returns from discovering of intermediate goods which should prevail introducing costs. These discrete version then will be calibrated by a numerical example. Our aim is to find the solution and analyse the development of economic variables with respect to external changes.

Keywords: growth model, endogenous technological progress, Romer's model, discrete optimization problem, impulse response analysis

JEL classification: C51, E32
AMS classification: 62P20, 91B62

1 Introduction

In our paper we study the features of a specific model of economic growth based on the structural approach. In growth models, technological progress is a very important determinant for modelling of economic growth. In Sollow-Swan's model technological progress is treated exogenously and independently of factors of production [1]. However, the role of technological progress as a process resulting from internal causes is more natural and more relevant as we can often observe. Taking this fact into account, endogenous growth model links technological progress with growth models with optimal consumer behaviour first proposed by Ramsey [6]. There are several ways how to introduce endogenous technological progress. One of them is to introduce it as an expanding variety of intermediate products suggested by Romer [7] and developed in [3], [4] and [5]. In this approach the incorporation of technological progress into the model differs from the Schumpeterian quality ladders approach [2]. Both approaches can be used to analyze the behavior of firms, their production, markup dynamics and the implications for endogenous fluctuations and growth. We stick to the traditional Romer's approach, but we derive a discrete time version of an endogenous growth model with expanding variety of intermediates as an alternative to the continuous time approach. The reason for this approach is that continuous time models are resistent to calibrations, simulations and verifications. On the other hand, a discrete time version of an endogenous growth model allows us to apply all DSGE modeling technique to investigate its behavior which is a DSGE endogenous growth model symbiosis. Then the calibration and impulse response analysis will be carried out on the derived model to examine and quantify the effect of exogenous factors on the development of endogenous variables in the model.

2 The discrete model with expanding variety of intermediates

In order to derive the discrete version of the endogenous growth model we choose the traditional procedure similar to the one for the continuous version. There are three sectors in the model. The first one is the

[1]University of Economics in Prague, Department of Statistics and Probability, nám. W. Churchilla 4, Praha 3 - 130 67, {kodera,tran}@vse.cz
[2]Institute of Information Theory and Automation, Department of Econometrics, Pod Vodárenskou věží 4, Praha 8, 182 08, vosvrda@utia.cas.cz

final production sector represented by one aggregated firm. The second sector is a sector of Research and Development firms which consists of a continuum of entities. The sector of households is filled with unique household. All three sectors interact to construe a market equilibrium.

2.1 Modelling of the Final Product Producers

Let us assume that final goods are produced by one aggregate firm. Its production depends on constant amount of labour L and on intermediate products produced by N_t different Research and Development firms

$$Y_t = A_t L^{1-\alpha} \sum_{j=1}^{N_t} X_t^{\alpha}(j). \tag{1}$$

where Y_t is the output of final goods. The variable $X_t(j)$ denotes the amount of j-th intermediate product employed for final goods production. Technological progress takes the form of expansion of N_t which is a number of specialised intermediate products. To better understand the effect of the extension of N for the increase of final production let us assume that for the production of final goods Y_t the same amount of $X_t(j) = X_t$ is used. Using this assumption in expression (1) we get

$$Y_t = A_t L^{1-\alpha} N_t X_t^{\alpha} = A_t L^{1-\alpha} (N_t X_t)^{\alpha} N_t^{1-\alpha} \tag{2}$$

The production function (2) exhibits constant returns to scale with respect to $N_t X_t$ and L. Notice that if X_t increases the marginal product of intermediate goods decreases. If N_t increases the marginal product of intermediate goods is constant. It can be observed that the marginal product of intermediate goods depends on which component of $N_t X_t$ changes. The most intensive factor of growth therefore is the number of intermediate goods.

Expression (1) requires N_t to be an integer, but N_t is a quantity which denotes technological complexity what positive number describes more properly. From that reason we re-formulate expression (1) to

$$Y_t = A_t L^{1-\alpha} \int_0^{N_t} X_t^{\alpha}(j) \mathrm{d}j \tag{3}$$

and we will use it in the following text. Firms producing final goods maximize their profit. The profit for final goods produced by final product firm at time t is given by

$$Y_t - w_t L - \int_0^{N_t} P_t(j) X_t(j) \mathrm{d}j = A_t L^{1-\alpha} \int_0^{N_t} X_t^{\alpha}(j) \mathrm{d}j - W_t L - \int_0^{N_t} P_t(j) X_t(j) \mathrm{d}j \tag{4}$$

The firm producing final goods Y_t maximizes its profit with respect to intermediates and labour. Derived necessary conditions are used for the derivation of demand function for intermediates. Necessary conditions for the maximization with respect to intermediates are the Euler equations for degenerate problem of variation calculus

$$A_t \alpha L^{1-\alpha} X_t^{\alpha-1}(j) - P(j) = 0 \tag{5}$$

The demand for $X_t(j)$ depending on $P_t(j)$ is given by

$$X_t(j) = L \left(\frac{A_t \alpha}{P_t(j)} \right)^{1/(1-\alpha)} \tag{6}$$

The necessary condition for profit maximization with respect to labour is

$$(1-\alpha) A_t L^{-\alpha} \int_0^{N_t} X_t^{\alpha}(j) \mathrm{d}j - W_t = (1-\alpha) \frac{Y_t}{L} - W_t = 0 \tag{7}$$

Hence

$$(1-\alpha) \frac{Y_t}{L} = W_t \tag{8}$$

2.2 Research firms

Research firms transform one unit of final product to one unit of intermediate product of type j. The profit of a research firm at time τ is given by

$$\pi_\tau(j) = (P_\tau(j) - 1)X_\tau(j) \tag{9}$$

Let's choose $t \geq 0$ fixed. Then the present value of profits of Research and Development firms at time t is given by

$$V_t(j) = \sum_{\upsilon=0}^{\infty} \pi_{t+\upsilon}(j) Q_{t,\upsilon} \tag{10}$$

where

$$\begin{aligned} Q_{t,0} &= 1 \\ Q_{t,1} &= \tfrac{1}{(1+r_t)} \\ Q_{t,\upsilon} &= \tfrac{1}{(1+r_t)\times\cdots\times(1+r_{t+\upsilon-1})},\ \upsilon = 2,3,\ldots \end{aligned} \tag{11}$$

The research and development firm maximizes its present value subject to demand functions (6). To solve present value maximization problem we substitute equation (9) into equation (10) and using expression (6) we get for the profit of j-th firm

$$\pi_\tau(j) = (P_\tau(j) - 1)L\left(\frac{A_t\alpha}{P_\tau(j)}\right)^{1/(1-\alpha)} \tag{12}$$

Substituting the equation displayed above into (10) we get

$$V_t(j) = \sum_{\upsilon=0}^{\infty}(P_{t+\upsilon}(j) - 1)L\left(\frac{A_t\alpha}{P_{t+\upsilon}(j)}\right)^{1/(1-\alpha)} Q_{t,\upsilon} \tag{13}$$

We obtain the necessary condition for the maximum of present value by taking the derivative of elements of infinite sum with respect to $P_{t+\upsilon}(j)$ which we will put equal to zero:

$$\frac{\mathrm{d}\pi_t(j)}{\mathrm{d}P_t(j)} = L\left[\left(\frac{A_t\alpha}{P_t(j)}\right)^{1/1-\alpha} + \frac{1}{1-\alpha}[1 - P_t(j)]\left(\frac{A_t\alpha}{P_t(j)}\right)^{1/1-\alpha}\frac{1}{P_t(j)}\right] = 0 \tag{14}$$

Having solved it, we get $P_t(j) = 1/\alpha$. Each Research and Development firm quotes the same optimum monopolistic price which is constant over time. The production of the j-th firm providing that the price $P_t(j) = 1/\alpha$ is given by

$$X_t(j) = L(A_t\alpha^2)^{1/1-\alpha} \tag{15}$$

which means that intermediate firms produce the same quantity of product. Then aggregate production of intermediates is

$$X_t = \int_0^{N_t} X_t(j)\mathrm{d}j = \int_0^{N_t} L(A_t\alpha^2)^{1/1-\alpha}\mathrm{d}j = LN_t(A_t\alpha^2)^{1/1-\alpha} \tag{16}$$

Substituting (16) into (3) $X_t(j)$ we have:

$$Y_t = A_tL^{1-\alpha}N_tL^\alpha(A_t\alpha^2)^{\alpha/(1-\alpha)} = A_t^{\frac{1}{1-\alpha}}\alpha^{2\alpha/(1-\alpha)}LN_t \tag{17}$$

Finally we will express present value of Research and Development firm replacing $1/\alpha$ for $P_t(j)$ into (13).

$$V_t = \frac{1-\alpha}{\alpha}L(A_t\alpha^2)^{1/1-\alpha}\sum_{\upsilon=0}^{\infty} Q_{t,\upsilon} \tag{18}$$

It is clear that that present value doesn't depend on j, so we omit it.

Let us assume that there is free entry into sector of Research and Development firms. Under this condition $V_t = \eta$, where η denotes constant costs of starting business in the Research and Development sector. If $V_t < \eta$ no one starts activity in Research and Development. If $V_t > \eta$ anybody can enter the sector and the price of intermediary and profits declines. Decreasing profits will give decline to present values. The process stops as soon as the equilibrium between present value and starting cost is restored.

In the equilibrium the present value of the Research and Development firm is given by (18). To express the present value of the firm at time $t+1$, we write

$$V_{t+1} = \frac{1-\alpha}{\alpha} L(A_t \alpha^2)^{1/1-\alpha} \sum_{\upsilon=0}^{\infty} Q_{t+1,\upsilon} \tag{19}$$

Notice that $V_t = V_{t+1} = \eta$ in equilibrium, so we have

$$V_t - \frac{1}{1+r_t} V_{t+1} = \frac{r_t}{1+r_t}\eta = \frac{1-\alpha}{\alpha} L(A\alpha^2)^{1/1-\alpha} \frac{1}{1+r_t} \tag{20}$$

After a small rearrangement we get

$$r_t = \frac{1-\alpha}{\eta\alpha} L(A_t\alpha^2)^{\frac{1}{1-\alpha}} \tag{21}$$

Using equation (17), then equation (21) can be written as

$$r_t = \alpha(1-\alpha)\frac{1}{\eta}\frac{Y_t}{N_t} \tag{22}$$

2.3 Households

Households maximize utility in infinite time horizon and receive wage rate W_t for constant amount of labour L supplied to economy. Households' utility functional is given by

$$U = \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta} - 1}{1-\theta} \tag{23}$$

where C_t denotes consumption of the household, β subjective discount factor and $1-\theta$ is elasticity of the utility function. Budget equations of households generally describe the dynamics of assets owned by household. The households are the owners of Research and Development firms whose assets are evaluated as ηN_t.

$$\eta(N_{t+1} - N_t) = W_t L + r_t \eta N_t - C_t \tag{24}$$

Rearranging it, we have

$$N_{t+1} = \frac{1}{\eta}(W_t L - C_t) + (1+r_t)N_t \tag{25}$$

To obtain necessary conditions for maximum of utility functional (23) we use Lagrange functional

$$\mathfrak{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{C_t^{1-\theta} - 1}{1-\theta} + \mu_t[\eta(N_{t+1} - N_t) - W_t L + \eta r_t N_t - C_t] \right\}$$

Taking a derivative of Lagrange functional with respect to C_t and N_{t+1} and put them equal zero. After excluding μ_t we get so called Euler equation (we leave out the expectation operator)

$$C_{t+1} = [(1+r_t)\beta]^{(1/\theta)} C_t \tag{26}$$

which is the first necessary condition of the consumer maximization problem. The second necessary condition is budget equation (25).

2.4 Market Equilibrium of the Model

Market equilibrium is given by the following equations. The first equation is the necessary condition for final product firm. It was derived in section 2 as equation (8), for better readability we display it again.

$$W_t = (1-\alpha)\frac{Y_t}{L} \tag{27}$$

The second equation is derived in the Section 3 as equation (22) and it is derived from equilibrium condition for Research and Development firms. Let's display it again.

$$r_t = (1-\alpha)\alpha\frac{1}{\eta}\frac{Y_t}{N_t} \tag{28}$$

The expression for Y_t is derived from production function in the Section 2 in the form

$$Y_t = A_t L^{1-\alpha} X_t^{\alpha} N^{1-\alpha} = A_t^{1/(1-\alpha)} \alpha^{2/(1-\alpha)} L N_t \tag{29}$$

The model hence consists of equations (25)-(29).

3 Calibration and Simulation

First we log-linearize the system of equations (25)-(29) describing the economy behaviour. As the model like the AK model exhibits no actual steady state, but only a quasi-steady state, in which variables C, Y, W and N growth at the same rate denoted g. Let $\hat{C}, \hat{Y}, \hat{W}, \hat{N}$ be the values of C_t, Y_t, W_t and N_t respectively at the beginning of the steady state. Let define $z_t = \ln Z_t - \ln \hat{Z}$ as the deviation of an endogenous variable from its corresponding steady state value. Then c_t, y_t, w_t and n_t the deviations of C_t, Y_t, W_t and N_t from $\hat{C}, \hat{Y}, \hat{W}, \hat{N}$. Let $\bar{a} = \ln A, \bar{r}$ be the values of a_t and r_t at the steady state and these values are unchanged along the balanced growth trajectory. We keep the labor force constant and normalize it to one. The log-linearized system is as follows

$$\begin{aligned}
&(1+g)^{\theta}\theta(c_{t+1} - c_t) = \bar{r}\beta r_t \\
&(1+g)\hat{N}n_{t+1} = \tfrac{1}{\eta}(L\hat{W}w_t - \hat{C}c_t) + \bar{r}r_t + (1+\bar{r}r_t)\hat{N}n_t \\
&y_t = \tfrac{1}{1-\alpha}a_t + n_t \\
&r_t = y_t - n_t \\
&x_t = y_t \\
&w_t = y_t \\
&a_t = \rho a_{t-1} + \epsilon_t.
\end{aligned} \tag{30}$$

The system has only one exogenous variable which is the productivity level a_t. We set $\alpha = 0.5$, $\beta = 0.91$, $\theta = 0.5$, $\rho = 0.9$, $\bar{a} = 0$, $g = 0.01$. The size of the productivity shock is set at size of 0.1. The system is solved in Dynare together with impulse response analysis. The results of impulse response analysis are shown in Figure 1. The results of impulse response analysis show that a positive productivity shock leads to a jump of values of production and wage whose size is twice of the size of the shock due to terms $\frac{1}{1-\alpha}$. The size of consumption growth is even higher while r_t and n_t grow slower than y_t. The reason is that as r_t rises, the value of monopoly profit decreases which demotivate the research firms at the beginning. When interest rate goes down, the number of intermediate goods grows again. The economy as the whole then forms a new higher steady state level then it was before the shock. This implication of the impulse response analysis is consistent with common economic wisdom indicating the applicability of our model.

4 Conclusion

In this paper we have derived a simple endogenous growth discrete time model with expanding variety of intermediate goods in the discrete time fashion. The problem of growth models is that steady state does not exist. In our model it is replaced by steady state growth. Then the gap variables in the model are defined as the deviation of model variables from their corresponding steady state growth variables. The model in this form can be easily solved and the follow-up simulation can be performed on it. We have conducted this task with the help of Dynare and the results we have obtained using our model are quite consistent with economic theory. Our model can be extended in many aspects and make it more complicated to study the effect of various factors on numerous economic variables.

Acknowledgements

The support from the Czech Science Foundation under Grant P402/12/G097 is gratefully acknowledged.

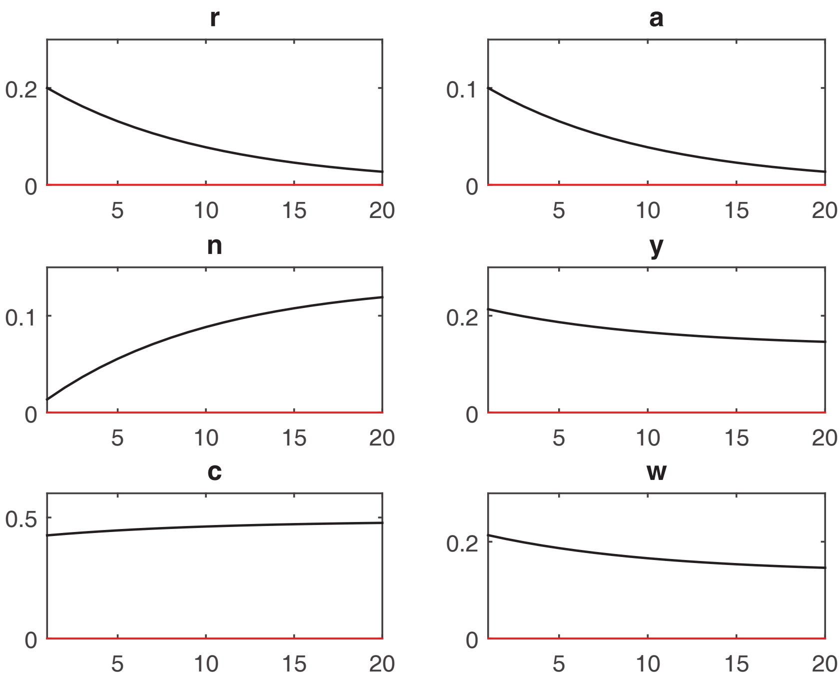

Figure 1 The responses of endogenous variables to productivity shock

References

[1] Barro, R. J., and Sala-i-Martin, X.: *Economic Growth*. MIT Press Cambridge, 2004.

[2] Barrau, N. G.: Schumpeterian foundations of real business cycles. *MPRA Paper* 9430, University Library of Munich, 2008.

[3] Holden, T.: Products, patents and productivity persistence: A DSGE model of endogenous growth. *Dynare Working Papers Series*, WP No. **4**, May 2011.

[4] Jaimovich, N.: Firm dynamics and markup variations: Implications for sunspot equilibria and endogenous economic fluctuations. *Journal of Economic Theory* **137**, 1 (2007), 300-325.

[5] Jones, C. I.: R&D-Based Models of Economic Growth. *Journal of Political Economy* **103**, August (1995), 759–784.

[6] Ramsey, F. P.: A mathematical theory of saving. *The Economic Journal* **38**, 152 (1928), 543—559.

[7] Romer, P. M.: Endogenous technological change. *The Journal of Political Economy* **98**, 5 (1990), 71–102.