On Expected Utility Under Ambiguity

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Abstract. The paper introduces a new approach to constructing models exhibiting the ambiguity aversion. The level of ambiguity aversion is described by a subjective parameter from the unit interval with the semantics: the higher the aversion, the higher the coefficient. On three examples, we illustrate the approach is consistent with the experimental results observed by Ellsberg and other authors.

Keywords: Belief function · Credal set · Probability transform · Decision-making · Vagueness

1 Introduction

It is well known, and it has also been confirmed by our experiments that people prefer lotteries, in which they know the content of a drawing drum to situations when the constitution of the drum's content is unknown. In our experiments, the participants were asked to choose one from six predetermined colors and they got the prize when the color of a randomly drawn ball coincided with their choice. It appeared that the participants were willing to pay in average by 90% more to take part in games when they knew that the urn contained the same number of balls of all six colors in comparison with the situation when they knew only that the urn contained balls of the specified colors and their proportion was unknown. This well known, seemingly paradoxical phenomenon, can hardly be explained by different subjective utility functions or by different subjective probability distributions. To explain this fact, we accepted a hypothesis that humans do not use their personal probability distributions but just *capacity functions* that do not sum up to one [13]. Roughly speaking, the subjective probability of drawing a red ball is $\frac{1}{6}$ in the case that the person knows that all colors are in the same amount in the drum. However, the respective "subjective probability" in the case of lack of knowledge is $\varepsilon < \frac{1}{6}$. *The lack of knowledge psychologically decreases the subjective chance of drawing the selected color – it decreases the subjective chance of success.*

This paper is one of many studying the so-called *ambiguity aversion*, which is used to model the fact that human behavior violates Savage's expected utility

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theory [17]. We present one possible way how to find a personal weight function (the above-mentioned capacity) that can be used, similarly to probability function, to compute the personal subjective expected value of a reward in case that the description of the situation is ambiguous. It is clear from the literature [6–8,15] that it cannot be a probability function. It cannot be normalized because our experiments show that people usually expect smaller reward under total ignorance than in case they know that all alternatives are of equal probabilities. As we will see later (when discussing the Ellsberg's experiments), this function is neither additive. Thus, the considered function will belong to the class of *superadditive capacities*.

To find a way, how to compute this personal weight function we will take advantage of the fact that situations with ambiguity are well described by tools of a theory of belief functions. This theory distinguishes between two types of uncertainty: the uncertainty connected with the fact that we do not know the result of a random experiment (a result of a random lottery) and the ignorance arising when we do not know the content of a drawing urn. In this paper, we start with describing the situation by belief functions that can be interpreted as *generalized probability* [9], i.e., each belief function corresponds to a set of probability functions, which is called a *credal set* [9]. Then, we adopt a decision-theoretic framework used also by other authors based on the transformation of the belief function into a probability function. However, we do not use the achieved probabilistic representative directly to decision, we add one additional step. Before computing the expected reward, we reduce the probabilities to account for ambiguity aversion. This is the only point in which our approach differs from Smets' decision-making framework [20], which is based on the Dempster-Shafer theory of belief functions [5,18].

Before describing the process in more details, let us stress that our aim is not as ambitious as developing a mathematical theory describing the ambiguity aversion within the theory of belief functions. In fact, it was already done by Jaffray [12], who shows how to compute generalized expected utility for belief function. We do not even consider all elements from a credal set with all the preference relations as, for example, in [3]. The ambition of our approach is to provide tools making it possible to assign a personal coefficient of ambiguity to experimental persons. Then, we will have a possibility to study its stability with respect to different decision tasks and/or its stability in time. Such a coefficient of ambiguity is considered also by Srivastava [22] and the suggested approach repeats some of his basic ideas. For example, we use almost the same idea to identify the amount of ambiguity connected with individual states of the considered state space.

2 Belief Functions

The basic concepts and notations are taken over from [13], where the described approach was introduced for the first time. We consider only a finite *state space* Ω. In the examples described below, Ω is the set of six considered colors: $\Omega = \{red, black, white, yellow, green, azure\}$ ($\Omega = \{r, b, w, y, g, a\}$ for short in the sequel). Similar to probability theory, where a probability measure is a set function defined on some *algebra* of the considered events, belief functions are represented by functions defined on the set of all nonempty subsets of Ω [5,18]. Let 2^Ω denote the set of all subsets of Ω.

The fundamental notion is that of a *basic probability assignment* (bpa), which describes all the information we have about the considered situation. It is a function $m : 2^\Omega \to [0,1]$, such that $\sum_{\mathbf{a}\in 2^\Omega} m(\mathbf{a}) = 1$ and $m(\emptyset) = 0$.

For bpa m, $\mathbf{a} \in 2^\Omega$ is said to be a *focal element* of m if $m(\mathbf{a}) > 0$. This enables us to distinguish the following two special classes of bpa's representing the extreme situations:

(1) m is said to be *vacuous* if $m(\Omega) = 1$, i.e., it has only one focal element, Ω. A vacuous bpa is denoted by m_ι. It represents total ignorance. In our examples, m_ι represents situations when we do not have any information as for the proportion of colors in the drawing urn.
(2) m is said to be *Bayesian*, if all its focal elements are singletons, i.e., for Bayesian bpa m, $m(\mathbf{a}) > 0$ implies $|\mathbf{a}| = 1$. Bayesian bpa's represents exactly the same knowledge as probability functions. As all focal elements of a Bayesian bpa m are singletons, we can define probability distribution P_m for Ω such that
$$P_m(x) = m(\{x\}) \tag{1}$$
for all $x \in \Omega$. Thus, Bayesian bpa's represent in our examples situations when the proportion of colors in a drawing ball is known.

The same knowledge that is expressed by a bpa m can also be expressed by a *belief function*, and by *plausibility function*.

$$Bel_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega :\, \mathbf{b}\subseteq \mathbf{a}} m(\mathbf{b}). \tag{2}$$

$$Pl_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega :\, \mathbf{b}\cap \mathbf{a}\neq\emptyset} m(\mathbf{b}). \tag{3}$$

We have already mentioned that we interpret the belief function theory as a generalization of the probability theory. It means that for each bpa we consider its *credal set*, which is a convex set of probability distributions P on Ω defined as follows ($\mathcal{P}$ denote the set of all probability distributions on Ω):

$$\mathcal{P}(m) = \left\{ P \in \mathcal{P} : \sum_{x\in \mathsf{a}} P(x) \geq Bel_m(\mathsf{a}) \text{ for } \forall \mathsf{a} \in 2^\Omega \right\}.$$

Notice that P_m defined by Eq. (1) for a Bayesian bpa m is such that $\mathcal{P}(m) = \{P_m\}$, and that $\mathcal{P}(m_\iota) = \mathcal{P}$. It is also easy to show that for all $P \in \mathcal{P}(m)$

$$Bel_m(\mathsf{a}) \leq P(\mathsf{a}) \leq Pl_m(\mathsf{a}),$$

for all $\mathsf{a} \in 2^\Omega$. Thus, if $Bel(\mathbf{a}) = Pl(\mathbf{a})$ then we are sure that the probability of $\mathbf{a}$ equals $Bel(\mathbf{a})$. Otherwise, the larger the difference $Pl(\mathbf{a}) - Bel(\mathbf{a})$, the more uncertain we are about the value of the probability of $\mathbf{a}$.

In this paper, we use belief functions only to represent the knowledge regarding the content of a drawing drum. How can we model the computation of a subjective expected gain if we know that in situation $x \in \Omega$ our reward will be $g(x)$? Since we want to reduce the expected value on the account of ambiguity we do not apply any direct formula (e.g., Choquet integral [2], Shenoy expectation [19]). We propose to use some of the probability transforms suggested to find a probabilistic representation of a belief function [4]. In this paper, we take advantage of the fact that for the examples presented in the next section it was shown in [14] that several probabilistic transforms yield the same results. Therefore we choose the simplest of them, the famous *pignistic transform*, which was for this purpose strongly advocated by Smets [20,21]):

$$Bet\_P_m(x) = \sum_{\mathsf{a} \in 2^{\Omega}: x \in \mathsf{a}} \frac{m(\mathsf{a})}{|\mathsf{a}|}. \tag{4}$$

3 Experimental Lotteries

In our experiments, we considered 12 simple lotteries described below. For each lottery, the subjects were asked how much they are maximally willing to pay to be allowed to take part in the specified lottery. The considered lotteries should reveal the behavior of subjects in the following three situations.

Ellsberg's Example. First, we wanted to verify whether the behavior of our subjects corresponds to what was observed by many other authors. Therefore we included a simple modification of the original Ellsberg's example ([6], pp. 653–654) with an urn containing 30 red balls and 60 black or yellow balls, the latter in an unknown proportion. With this urn, Ellsberg considered two experiments. The first experiment (Ellsberg's Actions I and II) studied whether people prefer betting on the red or black ball, in which case they get the reward (\$100) if the ball of the respective color is drawn at random. In the second experiment (Ellsberg's Actions III and IV), a person has a possibility to bet on red and yellow, or, alternatively, on black and yellow. Again, the participant gets the reward (\$100) in case that the randomly drawn ball is of one of the selected colors.

Following the Ellsberg's idea we included two lotteries:

E1 The drawing urn contains 15 red, black and yellow balls, you know that exactly 5 of them are red, you do not know the proportion of the remaining black and yellow balls. How much you are maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball has the color of your choice?

E2 The drawing urn contains 15 red, black and yellow balls, you know that exactly 5 of them are red, you do not know the proportion of the remaining black and yellow balls. How much you are maximally willing to pay to take part at the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is either yellow or of the color of your choice?

One Red Ball Example. This example is designed to test the decrease of a subjective "probability" in comparison with the combinatorial probability. For this, we included eight lotteries, which differ from each other just in the total number of balls in the drawing urn: the number n. We included lotteries with $n = 5, 6, 7, 8, 9, 10, 11, 12$:

Rn The drawing urn contains n balls, each of which is either red, or black, or yellow, or white, or green, or azure. You know that one and only one of them is red, nothing more. You even do not know how many colors are present in the urn. How much you are maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is of the color of your choice?

6-Color Example. This example concerns situations, in which six colors are considered and we do not have any reason to prefer one of them to others. Such situations occur in two completely different setting: *fair distribution of colors* and *total ignorance*. Thus, the following two lotteries considered:

F1 The drawing urn contains 30 balls, five of each of the following colors: red, black, yellow, white, green, and azure. How much you are maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is of the color of your choice?

F2 The drawing urn contains 30 balls, they may be of the following colors: red, black, yellow, white, green, and azure. You know nothing more, you even do not know how much colors are present in the urn. How much you are maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is of the color of your choice?

4 Decision Models

As said in the introduction, to describe the considered situations we define the respective bba's, and belief and plausibility functions. These belief function models are further transformed into probabilistic ones. As we have already mentioned in Sect. 2, for the specified simple situations we consider only the pignistic transform Bet_P_m defined by Eq. (4). However, the resulting probability distribution is not directly used to compute an expected reward. Before computing the subjective expected reward, the considered probabilities are reduced using a coefficient of ambiguity α, and the subjective expected reward is computed using the resulting capacity function $r_{m,\alpha}$. Let us stress again that $r_{m,\alpha}$ is not a probability distribution because it does not sum up to one. Now, we describe this process in more details.

Denote m the bpa describing the situation under consideration. Let Bet_P_m be the corresponding probability distribution obtained by the pignistic transform. Denote by Bel_m and Pl_m belief and plausibility functions corresponding to bpa m. Let us recall that the higher $Pl_m(\{x\})-Bel_m(\{x\})$, the higher ambiguity about the probability of state $x \in \Omega$. Our intuition says, the higher ambiguity about the probability of a state x, the greater reduction of the respective probability should be done. Therefore we define a *reduced capacity function* $r_{m,\alpha}$ for all $x \in \Omega$ as follows:

$$r_{m,\alpha}(x) = (1-\alpha)Bet\_P_m(x) + \alpha Bel_m(\{x\}), \tag{5}$$

where $\alpha \in [0,1]$ denote a subjective *coefficient of ambiguity aversion* $\alpha \in [0,1]$. Its introduction is inspired by the Hurwicz's *optimism-pessimism* coefficient [10,11]. In contrary to Hurwicz, who suggests that everybody *can choose* a personal coefficient expressing her optimism, we assume that each person *has* a personal coefficient of ambiguity aversion. The higher the aversion the higher the coefficient α. The detection of this coefficient for experimental persons is one of the goals why do we propose the described approach.

Notice that the amount of reduction realized in Formula (5) depends on the ambiguity aversion coefficient α, and the amount of ignorance associated with the state x. If we are certain about the probability of state x, it means that $Bet\_P_m(x) = Bel_m(\{x\})$, then the corresponding probability is not reduced: $r_{m,\alpha}(x) = Bet\_P_m(x)$. On the other hand, the maximum reduction is achieved for the states connected with maximal ambiguity, i.e., for the states for which $Bel_m(\{x\}) = 0$.

Some trivial properties of function $r_{m,\alpha}$ (we will call it *r-weight function*, or simply r-weight, in the sequel) are as follows:

1. $\sum_{x\in\Omega} r_{m,\alpha}(x) \le 1$; and
2. m is Bayesian if and only if $m(\{x\}) = Bet\_P_m(x) = r_{m,\alpha}(x)$ for all $x \in \Omega$, and $\alpha \in [0,1]$.

This r-weight function is then used to compute *expected subjective reward*, which is computed similarly to expected value, but the probabilities are substituted by the respective r-weights.

$$R_{m,\alpha} = \sum_{x\in\Omega} r_{m,\alpha}(x)g(x), \tag{6}$$

where $g(x)$ denote the reward (gain) one expects in case $x \in \Omega$ occurs. Thus, $R_{m,\alpha}$ does not express a mathematical expected reward, but a *subjectively reduced expectation* of a decision maker, whose subjectivity, i.e., level of ambiguity aversion, is described by α. Let us note that for $\alpha > 0$, betting the amount $R_{m,\alpha}$ guarantees a sure gain [1,15].

Let us now apply this computational process to the situations considered in the preceding section. To proceed from simpler models to more complex ones, let us consider the respective examples in reverse order.

6-Color Example. For this example, $\Omega = \{r, b, y, w, g, a\}$. The knowledge about the content of the drawing urn differs; in case of lottery F1, the situation is described by a Bayesian bpa defined $m_\phi(\{x\}) = \frac{1}{6}$ for all $x \in \Omega$; in case of lottery F2, the situation is described by the vacuous bpa m_ι.

For both the lotteries, the pignistic transforms coincide: $Bet\_P_{m_\phi}(x) = Bet\_P_{m_\iota}(x) = \frac{1}{6}$ for all colors $x \in \Omega$. However, the respective subjective r-weight functions differ because the respective belief functions differ: $Bel_{m_\phi}(\{x\}) = \frac{1}{6}$ for all $x \in \Omega$, whilst $Bel_{m_\iota}(\{x\}) = 0$ for all $x \in \Omega$. Therefore, using Formula (5), $r_{m_\phi,\alpha}(x) = \frac{1}{6}$, and $r_{m_\iota,\alpha}(x) = \frac{1-\alpha}{6}$ for all $x \in \Omega$.

Consider that a player chose, let us say, red color. Let $g(x)$ denote the gain received in case when color x is drawn, i.e., $g(r) = 100$, and for $x \neq r$, $g(x) = 0$. The expected subjective rewards are as follows:

$$R_{m_\phi,\alpha} = \sum_{x\in\Omega} r_{m_\phi,\alpha}(x)g(x) = \sum_{x\in\Omega} \frac{1}{6}\; g(x) = \frac{100}{6},$$

$$R_{m_\iota,\alpha} = \sum_{x\in\Omega} r_{m_\iota,\alpha}(x)g(x) = \sum_{x\in\Omega} \frac{1-\alpha}{6} g(x) = \frac{100\cdot(1-\alpha)}{6},$$

for F1 and F2, respectively. This can be interpreted as follows. If there were not for the subjective utility functions and for a different subjective risk attitude, a person should be willing to pay a maximum amount of $\frac{100}{6}$ CZK and $\frac{100\cdot(1-\alpha)}{6}$ CZK for taking part at lottery F1 and F2, respectively. The fact that in case of lottery F1 the person is willing to pay maximally $b \neq \frac{100}{6}$ CZK is explained by her personal risk attitude and utility functions. Nevertheless, the difference between the amounts the person is willing to pay for F1 and F2 can be explained only by her ambiguity aversion measured by the coefficient α. Assuming a linear dependence, it gives us a possibility to estimate the value of a personal coefficient of aversion. If a person is willing to pay a CZK for taking part at lotteries F1/F2 and b CZK for taking part at I1/I2 one can assume that her personal coefficient of ambiguity is about

$$\alpha = \frac{a-b}{a}. \tag{7}$$

One Red Ball Example. For this example, again $\Omega = \{r, b, y, w, g, a\}$, and the uncertainty is described by the bpa m_ϱ as follows:

$$m_\varrho(\mathbf{a}) = \begin{cases} \frac{1}{n}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{n-1}{n}, & \text{if } \mathbf{a} = \{b, g, o, y, w\}; \\ 0, & \text{otherwise.} \end{cases}$$

Using the pignistic transform, we get:

$$Bet\_P_{m_\varrho}(x) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ \frac{n-1}{5n}, & \text{for } x \in \{b, g, o, y, w\}. \end{cases}$$

Since $Bel_{m_\varrho}(\{x\}) = 0$ for all $x \in \{b, g, o, y, w\}$, and $Bel_{m_\varrho}(\{r\}) = \frac{1}{n}$ we get the following reduced weights:

$$r_{m_\varrho,\alpha}(x) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ (1-\alpha)\cdot\frac{n-1}{5n}, & \text{for } x \in \{b, g, o, y, w\}. \end{cases}$$

Considering (for the sake of simplicity just two) gain functions $g^{\mathrm{r}}(x)$, and $g^{\mathrm{w}}(x)$, the total subjective rewards are as follows. When betting on red it equals

$$R_{m_\varrho,\alpha}(\mathrm{r}) = \frac{1}{n}g^{\mathrm{r}}(\mathrm{r}) + \sum_{x\in\Omega:x\neq\mathrm{r}} \frac{(1-\alpha)(n-1)}{5n}g^{\mathrm{r}}(x) = \frac{100}{n},$$

and analogously, for betting on white

$$R_{m_\varrho,\alpha}(\mathrm{w}) = \frac{1}{n}g^{\mathrm{w}}(\mathrm{r}) + \sum_{x\in\Omega:x\neq r} \frac{(1-\alpha)(n-1)}{5n}g^{\mathrm{w}}(x) = \frac{100(1-\alpha)(n-1)}{5n}.$$

Table 1. One Red Ball Example: Total subjective reward as a function of the coefficient of ambiguity aversion α, and the number of balls n.

n	$R_{m_\varrho,\alpha}(\mathrm{r})$	$R_{m_\varrho,\alpha}(\mathrm{w})$						
		$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.28$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
5	20.00	16.00	14.40	12.80	11.52	11.20	9.60	8.00
6	16.67	$\boxed{16.67}$	15.00	13.33	12.00	11.67	10.00	8.33
7	14.29	17.14	$\boxed{15.43}$	13.71	12.34	12.00	10.29	8.57
8	12.50	17.50	15.75	$\boxed{14.00}$	$\boxed{12.60}$	12.25	10.50	8.75
9	11.11	17.78	16.00	14.22	12.80	$\boxed{12.44}$	10.67	8.89
10	10.00	18.00	16.20	14.40	12.96	12.60	$\boxed{10.80}$	9.00

Some of the values of these functions are tabulated in Table 1. From this table we see that, for example, a person with $\alpha = 0.28$ should bet on red color for $n \leq 7$, because for these $R_{m_\varrho,\alpha}(\mathrm{r}) > R_{m_\varrho,\alpha}(x)$ $(x \neq \mathrm{r})$, and bet on any other color for $n \geq 8$, because for these n, $R_{m_\varrho,\alpha}(\mathrm{r}) \leq R_{m_\varrho,\alpha}(x)$ $(x \neq \mathrm{r})$. This means that for $n \leq 7$, it is subjectively more advantageous to bet on the red color.

Ellsberg's Example. Before showing how the idea of reduced weights is applied to Ellsberg's experiment, let us confess that to clear the main idea to the reader, we have purposely simplified the exposition. The computation of a r-weight function by Formula (5) and its application to computation of a total subjective reward by Formula (6) can be used only in simple situations when the gain function $g : \Omega \to \mathbb{R}$ does not assign the same positive value to two different states from Ω, i.e.,

$$x_1, x_2 \in \Omega, x_1 \neq x_2, g(x_1) > 0 \implies g(x_1) \neq g(x_2). \tag{8}$$

This condition was obviously met by the gain functions considered above because the gain function was positive just for one state from Ω. Let us now introduce a proper general belief function approach that can be used for any gain function.

Generally, we have to consider distribution Bet_P_m that is got from bpa m by the pignistic transform as a set function, and, analogously, also the r-weight function must be defined for all nonempty subsets a of Ω

$$r_{m,\alpha}(\mathsf{a}) = (1-\alpha)P_m(\mathsf{a}) + \alpha Bel_m(\mathsf{a}), \tag{9}$$

with the same subjective coefficient of ambiguity aversion α. The reader can easily show that this r-weight is monotonous and superadditive

1. for $\mathsf{a} \subseteq \mathsf{b}$, $r_{m,\alpha}(\mathsf{a}) \leq r_{m,\alpha}(\mathsf{b})$;
2. for $\mathsf{a} \cap \mathsf{b} = \emptyset$, $r_{m,\alpha}(\mathsf{a} \cup \mathsf{b}) \geq r_{m,\alpha}(\mathsf{a}) + r_{m,\alpha}(\mathsf{b})$.

Realize also that we can use the same symbol to denote it, because for singletons it coincide with Formula (5).

As it can be expected, this r-weight set function is used to compute the expected subjective reward. For this, denote $\Gamma = \{g(x) : x \in \Omega\} \setminus \{0\}$, then

$$R_{m,\alpha} = \sum_{\gamma \in \Gamma} \gamma \, r_{m,\alpha}(g^{-1}(\gamma)), \tag{10}$$

where $g^{-1}(\gamma) = \{x \in \Omega : g(x) = \gamma\}$. Notice that most of authors use for this purpose Choquet integral [3,16], which is not, in our opinion, as intuitive as the proposed formula, and which can be shown to be always less or equal to the introduced $R_{m,\alpha}$.

Now, let us apply this general approach to the belief function model corresponding to E1 and E2 lotteries. For this, $\Omega = \{r, b, y\}$ and the bpa m_ε is as follows:

$$m_\varepsilon(\mathbf{a}) = \begin{cases} \frac{1}{3}, \text{ if } \mathbf{a} = \{r\}; \\ \frac{2}{3}, \text{ if } \mathbf{a} = \{b, y\}; \\ 0, \text{ otherwise.} \end{cases}$$

Its pignistic transform yields a uniform distribution $Bet\_P_{m_\varepsilon}(x) = \frac{1}{3}$ for all $x \in \Omega$. The corresponding belief function is $Bel_{m_\varepsilon}(\{r\}) = \frac{1}{3}$, and $Bel_{m_\varepsilon}(\{b\}) = Bel_{m_\varepsilon}(\{y\}) = 0$, $Bel_{m_\varepsilon}(\{r, b\}) = Bel_{m_\varepsilon}(\{r, y\}) = \frac{1}{3}$, $Bel_{m_\varepsilon}(\{b, y\}) = \frac{2}{3}$, and $Bel_{m_\varepsilon}(\Omega) = 1$. Therefore,

$$r_{m_\varepsilon,\alpha}(\mathbf{a}) = \begin{cases} \frac{1}{3}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{(1-\alpha)}{3}, & \text{for } \mathbf{a} = \{b\}, \{y\}; \\ \frac{(2-\alpha)}{3}, & \text{for } \mathbf{a} = \{r, b\}, \{r, y\}; \\ \frac{2}{3}, & \text{if } \mathbf{a} = \{b, y\}. \end{cases}$$

For E1, we have to consider two gain functions: $g^r(x)$, and $g^b(x)$ for betting on red and black balls, respectively. These functions are as follows:

$$g^r(r) = 100, g^r(b) = g^r(y) = 0,$$
$$g^b(b) = 100, g^b(r) = g^b(y) = 0.$$

Using Formula (10), the total subjective reward for betting on red ball is

$$R_{m_\varepsilon,\alpha}(r) = 100\ r_{m_\varepsilon,\alpha}((g^{\mathrm{r}})^{-1}(100)) = 100\ r_{m_\varepsilon,\alpha}(\{r\}) = \frac{100}{3},$$

and analogously, for betting on black ball is as follows:

$$R_{m_\varepsilon,\alpha}(b) = 100\ r_{m_\varepsilon,\alpha}((g^{b})^{-1}(100)) = 100\ r_{m_\varepsilon,\alpha}(\{b\}) = \frac{100(1-\alpha)}{3}.$$

Thus, for positive α, we get $R_{m_\varepsilon,\alpha}(r) > R_{m_\varepsilon,\alpha}(b)$, which is consistent with the Ellsberg's observation that "very frequent pattern of response is that betting on red is preferred to betting on black."

Let us consider the lottery E2, which involves betting on a couple of colors. In comparison with the first experiment, the situation changes only in the respective gain functions; denote them $g^{\mathrm{ry}}(x)$ and $g^{by}(x)$ for betting on red and yellow, and for betting on black and yellow balls, respectively.

$$g^{\mathrm{ry}}(\mathrm{r}) = g^{\mathrm{ry}}(\mathrm{y}) = 100, g^{\mathrm{ry}}(b) = 0,$$
$$g^{b\mathrm{y}}(b) = g^{b\mathrm{y}}(\mathrm{y}) = 100, g^{b\mathrm{y}}(\mathrm{r}) = 0.$$

Thus, the expected subjective rewards are as follows:

$$R_{m_\varepsilon,\alpha}(r\mathrm{y}) = 100\ r_{m_\varepsilon,\alpha}((g^{\mathrm{ry}})^{-1}(100)) = 100\ r_{m_\varepsilon,\alpha}(\{ry\}) = 100\ \frac{(2-\alpha)}{3},$$

$$R_{m_\varepsilon,\alpha}(b\mathrm{y}) = 100\ r_{m_\varepsilon,\alpha}((g^{b\mathrm{y}})^{-1}(100)) = 100\ r_{m_\varepsilon,\alpha}(\{by\}) = 100\ \frac{2}{3}.$$

Thus, we observe that, for positive α, $R_{m_\varepsilon,\alpha}(b\mathrm{y}) > R_{m_\varepsilon,\alpha}(r\mathrm{y})$, which is consistent with Ellsberg's observations that "betting on black and yellow is preferred to betting on red and yellow balls."

5 Conclusions

In the paper, we have introduced a belief function model manifesting a similar ambiguity aversion as human decision-makers. The intensity of this aversion is expressed by the subjective coefficient $\alpha \in [0, 1]$ with the semantics: the higher the aversion, the higher the coefficient. In the time of submitting the paper for the conference, we have data about the behavior of 32 experimental subjects (university and high school students), who were offered a possibility to take part at the lotteries described in Sect. 3. Thus, one can hardly make serious conclusions. Nevertheless, it appears that computing the ambiguity aversion coefficient as suggested in Formula (7), the experimental subjects show a great variety of the intensity of ambiguity aversion; in fact, the individual coefficients are from the whole interval $[0, 1]$, including both extreme values. The average value of this coefficient is about 0.36.

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