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Chapter 1

Truss topology design by linear conic optimization

1.1 ▪ Introduction

This chapter can be viewed as a complement to Chapter ?? of this book *Truss topology design by linear optimization*, reference [15]. We will use the same mechanical model of trusses and, whenever possible, the same notation. In [15] the truss topology design problem is formulated and solved as a linear optimization problem. In this chapter, we will introduce alternative formulations using linear conic optimization. In particular, we will present linear SOCO and linear SDO formulations of the minimum volume and minimum compliance problem. All formulations will be developed in the "primal" variables (bar cross-sectional areas) and the "dual" variables (displacements).

We will start with the nonlinear (and nonconvex) formulation of the basic truss topology problem, prove the existence of a solution and show that the Lagrangian dual to this problem is a convex quadratically constrained quadratic problem. Then we introduce the SOCO formulations of the problem, both primal and dual, and the SDO formulations, again primal and dual. In the last section, we will demonstrate why we need these conic formulations, when we already have the LO formulations from [15]. In particular, we will show that by adding new important constraints to the basic problem, the conic formulations will prove to be very useful.

1.2 ▪ Truss notation

By truss we understand an assemblage of pin-jointed uniform straight bars. The bars can only carry axial tension and compression when the truss is loaded at the joints. We denote by m the number of bars and by N the number of joints. The positions of the joints are collected in a vector y of dimension $\tilde{n} := dim \cdot N$ where dim is the spatial dimension of the truss. The material properties of bars are characterized by their Young's moduli E_i, the bar lengths are denoted by ℓ_i and bar cross-sectional areas by a_i, $i = 1, \ldots, m$.

Let $dim = 2$, to shorten the notation. For an i-th bar, let

$$r_i = \frac{1}{\ell_i}\Big(-(y_1^{(k)} - y_1^{(j)}), \quad -(y_2^{(k)} - y_2^{(j)}), \quad y_1^{(k)} - y_1^{(j)}, \quad y_2^{(k)} - y_2^{(j)}\Big)^T,$$

where $y^{(j)}$ and $y^{(k)}$ are the end-points of the bar. The compatibility matrix is then defined by $R = (r_1, r_2, \ldots, r_m)$.

Let $f \in \mathbb{R}^{\tilde{n}}$ be a load vector of nodal forces. The response of the truss to the load

f is measured by nodal displacements collected in a displacement vector $u \in \mathbb{R}^{\tilde{n}}$. Some of the displacement components may be restricted: a node can be fixed in a wall, then the corresponding displacements are prescribed to be zero. The number of free nodes multiplied by the spatial dimension will be denoted by n and we will assume that $f \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$.

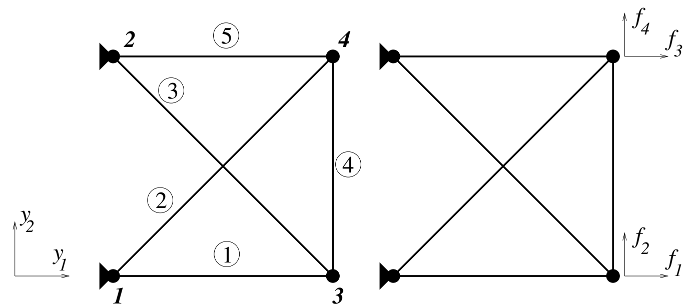

Figure 1.1. *A five-bar truss with four nodes, two of them fixed. Here* $y^{(1)} = (0,0)$, $y^{(2)} = (0,1)$, $y^{(3)} = (1,0)$, $y^{(4)} = (1,1)$, $dim = 2$, $N = 4$, $\tilde{n} = 8$ *and* $n = 4$.

In agreement with [15], we denote by

$$q_i = \frac{E_i a_i}{\ell_i} r_i^T u$$

the axial force in i-th bar, introduce bar stiffness matrices and assemble them in the global stiffness matrix of the truss

$$K(a) = \sum_{i=1}^{m} a_i K_i = \sum_{i=1}^{m} a_i \frac{E_i}{\ell_i} r_i r_i^T, \quad i = 1, \ldots, m$$

and introduce the equilibrium equation

$$K(a)u = f\,. \tag{1.1}$$

Finally, to simplify the notation, we define $b_i = \sqrt{\frac{E_i}{\ell_i}}\, r_i$.

Assumption 1.1. $K(1)$ *is positive definite and the load vector* f *is in the range space of* $K(1)$.

1.3 ▪ Nonlinear optimization formulation

The most natural objective is to minimize the volume of the structure. The minimal requirement on the optimal structure is that it should satisfy equilibrium equation (1.1). The lightest structure satisfying equilibrium tends to be no structure at all, so it is reasonable to introduce another constraint that would control the stiffness (or weakness) of the optimal structure. Commonly used is the *compliance* of the truss $f^T u$ where f and u satisfy the equilibrium equation (1.1). The smaller the compliance, the smaller the displacement at the loaded nodes and thus the stiffer the truss.

1.3.1 ▪ Primal problem

Let $\gamma \in \mathbb{R}$ and $\underline{a} \in \mathbb{R}^m$, $\overline{a} \in \mathbb{R}^m$ such that $\gamma > 0$, $0 \le \underline{a}_i \le \overline{a}_i$, $i = 1, \dots, m$ and the following assumption is satisfied.

Assumption 1.2. *Let $\overline{u} = K(\overline{a})^{-1} f$. Then $f^T \overline{u} < \gamma$.*

The simplest, yet meaningful, truss design problem is the single load topology optimization minimum volume problem:

$$
\begin{aligned}
\min_{a \in \mathbb{R}^m,\, u \in \mathbb{R}^n} \quad & \sum_{i=1}^m a_i \ell_i \\
\text{s.t.} \quad & K(a)u = f \\
& f^T u \le \gamma \\
& \underline{a}_i \le a_i \le \overline{a}_i, \quad i = 1, \dots, m\,.
\end{aligned}
\tag{1.2}
$$

Now let $V \in \mathbb{R}$ be a given maximal volume satisfying the following assumption.

Assumption 1.3. *The maximum volume V must satisfy $\sum_{i=1}^m \underline{a}_i \ell_i < V < \sum_{i=1}^m \overline{a}_i \ell_i$.*

Alternatively to (1.2), we can maximize the stiffness of the truss (minimize compliance) subject to equilibrium and resources constraints:

$$
\begin{aligned}
\min_{a \in \mathbb{R}^m,\, u \in \mathbb{R}^n} \quad & f^T u \\
\text{s.t.} \quad & K(a)u = f \\
& \sum_{i=1}^m a_i \ell_i \le V \\
& \underline{a}_i \le a_i \le \overline{a}_i, \quad i = 1, \dots, m\,.
\end{aligned}
\tag{1.3}
$$

Theorem 1.4 ([1]). *Problems (1.2) and (1.3) are equivalent. Any solution (a^*, u^*) of (1.2) is also a solution of (1.3) with $V = \sum_{i=1}^m a_i^* \ell_i$. Any solution (a^*, u^*) of (1.3) is a also solution of (1.2) with $\gamma = f^T u^*$.*

Problems (1.2) and (1.3) are nonlinear optimization (NLO) problems that are rather difficult to solve. The reason for this is that the equilibrium constraint does not satisfy the Mangasarian-Fromowitz constraint qualification required by most NLO algorithms.

1.3.2 ▪ Existence of solution

In this section, we closely follow [7]. The minimum compliance problem (1.3) can be equivalently written as

$$
\begin{aligned}
\min_{a \in \mathbb{R}^m} \sup_{u \in \mathbb{R}^n} \quad & 2 f^T u - u^T K(a) u \\
\text{s.t.} \quad & \sum_{i=1}^m a_i \ell_i \le V \\
& \underline{a}_i \le a_i \le \overline{a}_i, \quad i = 1, \dots, m\,.
\end{aligned}
\tag{1.4}
$$

Remark 1.5. Formulation (1.4) is, in a sense, the basic and “most natural" formulation of the truss design problem, and also a general structural optimization problem. Searching equilibrium by minimization of the potential energy is more general than just the equilibrium equation, as it allows for more general physical laws (nonlinear material and geometry) and more general constraints (e.g., unilateral contact conditions). We can say that nature minimizes the potential energy, while the designer tries to bring this minimum as close to zero as possible.

Let us define the *compliance function* as follows

$$c(a) := \sup_{u\in\mathbb{R}^n} (2f^T u - u^T K(a)u)\,.$$

Theorem 1.6. *The compliance function is convex and lower semi-continuous on* $\mathbb{R}^m$.

Proof. $c(\cdot)$ is a pointwise supremum of linear and thus closed functions. Hence it is convex and closed ([14, Thms.5.5,9.4]), and so lower semi-continuous ([14, p.52]). □

Now assume that $\underline{a}_i < a_i < \overline{a}_i$, $i = 1,\dots,m$, which, in particular, means that $a > 0$ and thus $K(a)$ is invertible. Then the "sup" in the definition of the compliance is attained (Assumption 1.1) and we can write the compliance function as $c(a) = f^T K^{-1}(a) f$. Consider the following problem

$$\begin{aligned} \min_{\alpha\in\mathbb{R},\ a\in\mathbb{R}^m} \quad & \alpha \\ \text{s.t.} \quad & (a,\alpha)\in \operatorname{cl}\Omega \end{aligned} \tag{1.5}$$

where Ω is defined as

$$\Omega := \{(a,\alpha)\in\mathbb{R}^m\times\mathbb{R} \mid f^T K^{-1}(a) f < \alpha,\ \sum_{i=1}^m a_i\ell_i \le V,\ \underline{a}_i < a_i < \overline{a}_i,\ i=1,\dots,m\}\,.$$

In addition to Assumptions 1.1–1.3 we further assume that $\operatorname{int}\Omega \neq \emptyset$.

Proposition 1.7 ([14, Thm. 7.6]). *Let* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *be a convex proper function, and let* $\alpha\in\mathbb{R}$, $\alpha > \inf\varphi$. *The convex level sets* $\{x \mid \varphi(x)\le\alpha\}$ *and* $\{x \mid \varphi(x) < \alpha\}$ *then have the same closure and the same relative interior, namely*

$$\{x \mid (\operatorname{cl}\varphi)(x) \le \alpha\}, \qquad \{x \in \operatorname{ri}(\operatorname{dom}\varphi) \mid f(x) < \alpha\},$$

respectively.

Corollary 1.8. *Let* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *be a closed proper convex function and let* $\omega\subset\mathbb{R}^n$ *be closed and convex. Then for* $\inf\varphi < \alpha < +\infty$ *one has*

$$\{x \mid \varphi(x)\le\alpha,\ x\in\omega\} = \operatorname{cl}\{x \mid \varphi(x) < \alpha,\ x\in\operatorname{ri}\omega\}$$

Hence problem (1.5) is equivalent to problem (1.4).

Theorem 1.9. *Problem (1.5) has at least one solution.*

Proof. $\operatorname{cl}\Omega$ is a closed convex set due to convexity of the compliance function. We are thus minimizing a linear function over a closed convex set $\operatorname{cl}\Omega$. Furthermore, the level sets $\{(a,\alpha)\in\operatorname{cl}\Omega \mid \alpha < c\}$ are clearly bounded. Hence, (1.5) is solvable. □

Corollary 1.10. *Problem (1.4) has at least one solution.*

Above we have also proved the following proposition.

Proposition 1.11. *Let $a^* \in \mathbb{R}^m$ be a solution of the minimum volume problem (1.2) with $\underline{a}_i = 0$, $i = 1,\dots,m$. There exists a sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \in \mathbb{R}^m$, with the following properties*

- *a_k is a solution of (1.2) with $\underline{a}_i = \varepsilon_k$, $i = 1,\dots,m$;*
- *$\varepsilon_k \to 0$ and $a_k \to a^*$ as $k \to \infty$.*

1.3.3 ▪ Dual problem (QCQO formulation)

We now introduce a reformulation of (1.3) that has a great impact on the numerical solution of truss design problems. It is the following quadratically constrained quadratic optimization (QCQO) problem:

$$\begin{aligned}
\min_{u\in\mathbb{R}^n,\ \alpha\in\mathbb{R},\ \underline{\rho}\in\mathbb{R}^m,\ \overline{\rho}\in\mathbb{R}^m} \quad & \alpha V - f^T u - \underline{a}^T\underline{\rho} + \overline{a}^T\overline{\rho} \\
\text{s.t.} \quad & \frac{1}{2}u^T K_i u \le \alpha \ell_i - \underline{\rho}_i + \overline{\rho}_i, \quad i = 1,\dots,m \\
& \underline{\rho} \ge 0 \\
& \overline{\rho} \ge 0\,.
\end{aligned} \tag{1.6}$$

Theorem 1.12 ([1]). *Problems (1.3) and (1.6) are equivalent in the following sense:*

(i) If one problem has a solution then also the other problem has a solution and the optimal objective values of the two problems are equal.

(ii) Let $(u^*, \alpha^*, \underline{\rho}^*, \overline{\rho}^*)$ be a solution to (1.6). Let further τ^* be the vector of Lagrangian multipliers for the inequality constraints associated with this solution. Then (u^*, τ^*) is a solution of (1.3). Moreover, $\underline{\rho}_i^*\overline{\rho}_i^* = 0$, $i = 1,\dots,m$.

(iii) Let (u^*, a^*) be a solution of (1.3). Let further $\underline{r}^*$ and $\overline{r}^*$ be the Lagrangian multipliers associated with the lower and upper bounds on t, respectively, and let α^* be the multiplier for the volume constraint. Then $(u^*, \alpha^*, \underline{r}^*, \overline{r}^*)$ is a solution of (1.6).

1.4 ▪ SOCO formulation

1.4.1 ▪ Primal SOCO problem

We start with a simple but useful lemma that shows the relation between convex quadratic constraints and second-order conic optimization (SOCO) constraints.

Lemma 1.13. *Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \in \mathbb{R}$, $s > 0$. Then*

$$\frac{xx^T}{s} \le t \quad \Longleftrightarrow \quad \left\| \begin{pmatrix} x \\ \frac{t-s}{2} \end{pmatrix} \right\|_2 \le \frac{t+s}{2}\,.$$

As (1.6) is a convex quadratic problem, it can now be immediately re-written as an SOCO problem

$$\begin{aligned}
\min_{u\in\mathbb{R}^n,\ \alpha\in\mathbb{R},\ \underline{\rho}\in\mathbb{R}^m,\ \overline{\rho}\in\mathbb{R}^m} \quad & \alpha V - f^T u - \underline{a}^T \underline{\rho} + \overline{a}^T \overline{\rho} \\
\text{s.t.} \quad & \left\| \begin{pmatrix} \frac{\sqrt{2}}{2} b_i^T u \\ \frac{\alpha \ell_i - \underline{\rho}_i + \overline{\rho}_i - 1}{2} \end{pmatrix} \right\|_2 \le \frac{\alpha \ell_i - \underline{\rho}_i + \overline{\rho}_i + 1}{2}, \quad i = 1,\dots,m \\
& \underline{\rho} \ge 0 \\
& \overline{\rho} \ge 0 .
\end{aligned} \tag{1.7}$$

1.4.2 ▪ Dual SOCO problem

Proposition 1.14. *The dual problem to (1.7) can be written as*

$$\begin{aligned}
\min_{a\in\mathbb{R}^m,\ \tau\in\mathbb{R}^m,\ q\in\mathbb{R}^m} \quad & \frac{1}{2}\sum_{i=1}^m \tau_i \\
\text{s.t.} \quad & \sum_{i=1}^m a_i \ell_i = V \\
& \sum_{i=1}^m q_i b_i = f \\
& \left\| \begin{pmatrix} \sqrt{2} q_i \\ \frac{2a_i - \tau_i}{2} \end{pmatrix} \right\|_2 \le \frac{2a_i + \tau_i}{2}, \quad i = 1,\dots,m \\
& \underline{a} \le a_i \le \overline{a}, \quad i = 1,\dots,m ,
\end{aligned} \tag{1.8}$$

where a are the bar areas, q the bar axial forces and the objective function is equal to the compliance.

Proof. Let μ_i, ν_i be Lagrangian multipliers to the conic constraints, where $\mu_i = (\mu_{i,1}, \mu_{i,2}) \in \mathbb{R}^2$ and $\nu_i \in \mathbb{R}$, $i = 1,\dots,m$. Further, let $\varkappa \in \mathbb{R}^m$ be the multiplier to the bound constraint on ρ. The Lagrangian dual to (1.7) reads as

$$\begin{aligned}
\max_{\mu\in\mathbb{R}^{m\times 2},\ \nu\in\mathbb{R}^m,\ \underline{\varkappa}\in\mathbb{R}^m,\ \overline{\varkappa}\in\mathbb{R}^m} \quad & \frac{1}{2}\sum_{i=1}^m \mu_{i,2} - \nu_i \\
\text{subject to} \quad & \frac{1}{2}\sum_{i=1}^m (\mu_{i,2} + \nu_i)\ell_i = V \\
& \sum_{i=1}^m \mu_{i,1} \frac{\sqrt{2}}{2} b_i = f \\
& \|\mu_i\|_2 \le \nu_i, \quad i = 1,\dots,m \\
& \frac{1}{2}\mu_{i,2} + \frac{1}{2}\nu_i - \underline{\varkappa}_i = \underline{a}_i, \quad i = 1,\dots,m \\
& \frac{1}{2}\mu_{i,2} + \frac{1}{2}\nu_i + \overline{\varkappa}_i = \overline{a}_i, \quad i = 1,\dots,m \\
& \underline{\varkappa}_i \ge 0,\ \overline{\varkappa}_i \ge 0, \quad i = 1,\dots,m .
\end{aligned}$$

Setting $q_i = \frac{\sqrt{2}}{2}\mu_{i,1}$, $a_i = \frac{1}{2}(\mu_{i,2} + \nu_i)$ and $\tau = \nu_i - \mu_{i,2}$, we see from the last three constraints that $\underline{a}_i \le a_i \le \overline{a}$. Consequently, we get (1.8). ☐

Similarly, we would get the minimum volume SOCO formulation, just by replacing the objective in (1.8) by $\sum_{i=1}^m a_i\ell_i$ and the first constraint by $\sum_{i=1}^m \tau_i\ell_i \le \gamma$ where γ is an upper bound on compliance as in (1.2).

1.5 ▪ SDO formulation

1.5.1 ▪ Primal SDO problem

The SDO formulation of the primal problem is based on the Schur complement theorem [6]. Because we allow the stiffness matrix to be singular, we need a minor generalization of the standard theorem. The proof can be found, e.g., in [2].

Proposition 1.15 ([2]). *Let $a \in \mathbb{R}^m$, $a \ge 0$, and $\gamma \in \mathbb{R}$ be fixed. Then there exists $u \in \mathbb{R}^n$ satisfying*

$$K(a)u = f \qquad \textit{and} \qquad f^T u \le \gamma$$

if and only if

$$\begin{pmatrix} \gamma & -f^T \\ -f & K(a) \end{pmatrix} \succeq 0 .$$

Using this proposition, we get equivalent formulations of problems (1.2) and (1.3), respectively:

$$\begin{aligned} \min_{a\in\mathbb{R}^m} \quad & \sum_{i=1}^m a_i\ell_i \\ \text{s.t.} \quad & \begin{pmatrix} \gamma & -f^T \\ -f & K(a) \end{pmatrix} \succcurlyeq 0 \\ & \underline{a}_i \le a_i \le \overline{a}_i, \quad i = 1,\dots,m \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} \min_{a\in\mathbb{R}^m,\ \gamma\in\mathbb{R}} \quad & \gamma \\ \text{s.t.} \quad & \begin{pmatrix} \gamma & -f^T \\ -f & K(a) \end{pmatrix} \succcurlyeq 0 \\ & \sum_{i=1}^m a_i\ell_i \le V \\ & \underline{a}_i \le a_i \le \overline{a}_i, \quad i = 1,\dots,m . \end{aligned} \tag{1.10}$$

Theorem 1.16. *Problems (1.2) and (1.9) are equivalent. If (a^*, u^*) is a solution of (1.2), then a^* is a solution of (1.9). If a^* is a solution of (1.9), then there exists $u^* \in \mathbb{R}^n$ such that $K(a^*)u^* = f$ and that (a^*, u^*) is a solution of (1.2). The same holds for problems (1.3) and (1.10).*

1.5.2 ▪ Dual SDO problem

Let us now write down a dual to the semidefinite optimization problem (1.9). It is the problem

$$
\begin{aligned}
\max_{W\in\mathbb{S}^{n+1},\, \overline{\rho}\in\mathbb{R}^m,\underline{\rho}\in\mathbb{R}^m} \quad & \left\langle \begin{pmatrix} -\gamma & f^T \\ f & 0 \end{pmatrix}, W\right\rangle - \sum_{i=1}^m \overline{\rho}_i \overline{a}_i + \sum_{i=1}^m \underline{\rho}_i \underline{a}_i \\
\text{s.t.} \quad & \left\langle \begin{pmatrix} 0 & 0 \\ 0 & K_i \end{pmatrix}, W\right\rangle - \overline{\rho}_i + \underline{\rho}_i = \ell_i, \quad i = 1,\dots,m \\
& W \succcurlyeq 0 \\
& \overline{\rho}_i \geq 0,\ \underline{\rho}_i \geq 0, \quad i = 1,\dots,m\,.
\end{aligned} \tag{1.11}
$$

Proposition 1.17. *There exists a solution $(W^*, \overline{\rho}^*, \underline{\rho}^*) \in \mathbb{S}^{n+1} \times \mathbb{R}^m \times \mathbb{R}^m$ of the dual SDO problem (1.11) such that the rank of W^* is one.*

Proof. Let us first show that both the primal and the dual SDO problems (1.9) and (1.11) satisfy Slater condition. Indeed, by Assumption 1.2, $\gamma - f^T K^{-1}(\overline{a}) f > 0$, so, by the Schur complement theorem, $\overline{a}$ is a Slater point for (1.9). Now let $\widehat{W}$ be the identity matrix. Then $\widehat{W} \succ 0$ and $\left\langle \begin{pmatrix} 0 & 0 \\ 0 & K_i \end{pmatrix}, \widehat{W}\right\rangle = \operatorname{trace}(K_i) > 0$. For any $i = 1,\dots,m$ we can now always find $\overline{\rho}_i \geq 0$ and $\underline{\rho}_i \geq 0$ satisfying $\operatorname{trace}(K_i) - \overline{\rho}_i + \underline{\rho}_i = \ell_i$, so $\widehat{W}$ is a Slater point for (1.11). Hence the assumptions of the conic duality theorem from Chapter **??** are satisfied ([4, Theorem 2]).

Let a^* be a solution of the primal problem (1.9). Denote by $S^* \in \mathbb{S}^{n+1}$ the primal slack matrix variable

$$
S^* = \begin{pmatrix} \gamma & -f^T \\ -f & K(a^*) \end{pmatrix}.
$$

From SDO duality, S^* is complementary to any solution W of the dual problem (1.11), $\langle W, S^*\rangle = 0$ and the pair satisfies the following rank condition:

$$
\operatorname{rank}(S^*) + \operatorname{rank}(W) \leq n+1\,.
$$

If $\underline{a}_i > 0$, $i = 1,\dots,m$, the matrix $K(a^*)$ is positive definite and, due to Assumption 1.1, has a full rank n. Hence, by the above condition, the rank of any dual solution W is at most one. Excluding trivial solutions, the rank of W is then equal to one.

Assume now that $\underline{a}_i = 0$, $i = 1,\dots,m$, i.e., the matrix $K(a^*)$ can be rank deficient. Due to Proposition 1.11, there is a sequence of solutions $\{S_k\}$ to the primal problem with $(\underline{a}_i)_k = \varepsilon_k$, $\varepsilon_k \to 0$ as $k \to \infty$, such that $S_k \to S^*$. Associated with $\{S_k\}$ there is a sequence of dual solutions $\{W_k\}$, such that any pair (S_k, W_k) satisfies the complementarity and thus the rank condition. Using the same argument as above, the rank of matrices W_k is one. Hence there is a sequence of vectors $\{w_k\}$ such that $W_k = w_k w_k^T$ for each k and $w_k \to w^*$ as $k \to \infty$. By construction, $W^* = w^* w^{*T}$ is a rank-one matrix which is attained due to continuity of the Frobenius inner product. Finally, W^* is a solution to (1.11) as it satisfies the necessary and sufficient optimality conditions: due to continuity of the Frobenius inner product, it is complementary to S^* and feasible in (1.11). □

Analogously, we can write down a dual to the semidefinite formulation of the minimum compliance problem (1.10). It is the problem

$$\begin{aligned}
\max_{W\in\mathbb{S}^{n+1},\ \overline{\rho}\in\mathbb{R}^m,\underline{\rho}\in\mathbb{R}^m,\ \omega\in\mathbb{R}} \quad & \langle\begin{pmatrix}0 & f^T\\ f & 0\end{pmatrix}, W\rangle - \sum_{i=1}^m \overline{\rho}_i\overline{a}_i + \sum_{i=1}^m \underline{\rho}_i\underline{a}_i - \omega V \\
\text{s.t.} \quad & \langle\begin{pmatrix}0 & 0\\ 0 & K_i\end{pmatrix}, W\rangle - \overline{\rho}_i + \underline{\rho}_i - \omega\ell_i = 0, \quad i=1,\dots,m \\
& W_{11} = 1 \\
& W \succcurlyeq 0 \\
& \omega \ge 0 \\
& \overline{\rho}_i \ge 0,\ \underline{\rho}_i \ge 0, \quad i=1,\dots,m\,.
\end{aligned} \tag{1.12}$$

Proposition 1.18. *There exists a solution* $(W^*,\overline{\rho}^*,\underline{\rho}^*,\omega^*)\in\mathbb{S}^{n+1}\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}$ *of the dual SDO problem (1.12) such that the rank of* W^* *is one.*

1.5.3 ▪ Closing the circle

Theorem 1.19. *The dual SDO problem (1.12) is equivalent to the problem (1.6):*

(i) Let $(u^*,\alpha^*,\underline{\rho}^*,\overline{\rho}^*)$ *be a solution of (1.6). Then*

$$(W^+,\underline{\rho}^+,\overline{\rho}^+,\omega^+) := (u^*u^{*T},\underline{\rho}^*,\overline{\rho}^*,\frac{2\alpha^*}{V})$$

is a solution of (1.12).

(ii) Let $(W^*,\underline{\rho}^*,\overline{\rho}^*,\omega^*)$ *be a solution to (1.12) such that* $\operatorname{rank} W^* = 1$*. Then there exists* $w^*\in\mathbb{R}^{n+1}$ *with* $W^* = w^*w^{*T}$ *and such that*

$$(u^+,\alpha^+,\underline{\rho}^+,\overline{\rho}^+) := (w^*_{2:n+1},\frac{1}{2}\omega V,\underline{\rho}^*,\overline{\rho}^*)$$

is a solution of (1.6).

Proof. Let $(W,\overline{\rho},\underline{\rho},\omega)\in\mathbb{S}^{n+1}\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}$ be a solution of (1.12). By the above proposition, assume that $W=\tilde{w}\tilde{w}^T$ with some $\tilde{w}\in\mathbb{R}^{n+1}$. Denote $\tilde{w}=\begin{pmatrix}c\\ w\end{pmatrix}$ with $c\in\mathbb{R}$. The dual SDO problem (1.12) can thus be written as

$$\begin{aligned}
\max_{W\in\mathbb{S}^{n+1},\ \overline{\rho}\in\mathbb{R}^m,\underline{\rho}\in\mathbb{R}^m,\ \omega\in\mathbb{R}} \quad & 2cf^Tw - \sum_{i=1}^m \overline{\rho}_i\overline{a}_i + \sum_{i=1}^m \underline{\rho}_i\underline{a}_i - \omega V \\
\text{s.t.} \quad & w^TK_iw - \overline{\rho}_i + \underline{\rho}_i = \omega\ell_i, \quad i=1,\dots,m \\
& c = 1 \\
& \omega \ge 0 \\
& \overline{\rho}_i \ge 0,\ \underline{\rho}_i \ge 0, \quad i=1,\dots,m\,,
\end{aligned}$$

which is just the problem (1.6) with $\omega = 2\alpha/V$ and $w=u$. □

This closes the circle of equivalences.

1.6 ▪ Applications

All the formulations of the truss topology design problem introduced above can be seen as a nice exercise in conic optimization. When it comes to the numerical solution of the problem, however, they appear to be virtually useless. We know from [15] that the problem without upper bounds on a can be equivalently formulated as a linear optimization problem and the modern LO solvers will certainly beat linear SOCO and SDO solvers for problems of the same size. Introduction of upper bounds on a disallows the use of LO reformulation [15]; however, the (convex) QCQO formulation (1.6) can still be efficiently solved by interior point methods [8]. So is this chapter anything else than an exercise? Below we will demonstrate that the conic formulations of the problem can be extremely useful, as soon as we add more (important) constraints to the basic problem. Other problems benefiting from the conic formulation include problems with stability constraints [12], multiple load problems [5], problems with local and global stress constraints, and related problems, such as cable networks [9] or free material optimization [11].

1.6.1 ▪ Vibration constraints

We may be required to set an additional constraint on free vibrations of the optimal structure. The free vibrations are the squares of the eigenvalues of the following generalized eigenvalue problem

$$K(a)w = \lambda(M(a)+M_0)w\,. \tag{1.13}$$

Here $M(a) = \sum_{i=1}^{m} a_i M_i$ is the so-called mass matrix that collects information about the mass distribution in the truss. The matrices M_i are positive semidefinite and have the same sparsity structure as K_i. The non-structural mass matrix M_0 is a constant, typically diagonal matrix with very few nonzero elements.

Low vibrations are dangerous and may lead to structural collapse. Hence we typically require the smallest free vibration to be bigger than some threshold, that is:

$$\lambda_{\min} \geq \overline{\lambda} \quad \text{for a given } \overline{\lambda} > 0$$

where $\lambda_{\min}$ is the smallest eigenvalue of (1.13). This constraint can be equivalently written as a linear matrix inequality

$$K(a) - \overline{\lambda}(M(a)+M_0) \succcurlyeq 0 \tag{1.14}$$

which is to be added to the basic truss topology problem. As (1.14) is a linear matrix inequality in variable a, it is natural to add this constraint to the primal SDO formulation (1.9). We will thus get the following linear SDO formulation of the truss topology design with a vibration constraint:

$$\begin{aligned}
\min_{a\in\mathbb{R}^m} \quad & \sum_{i=1}^{m} \ell_i a_i \\
\text{s.t.} \quad & \begin{pmatrix} \gamma & -f^T \\ -f & K(a) \end{pmatrix} \succcurlyeq 0 \\
& K(a) - \overline{\lambda}(M(a)+M_0) \succcurlyeq 0 \\
& \underline{a}_i \leq a_i \leq \overline{a}_i, \quad i = 1,\dots,m\,.
\end{aligned}$$

Example 1.20. Consider a 7×3 nodal grid with the ground-structure, boundary conditions and the load as depicted in Figure 1.2 top-left. The result of the standard minimum volume problem with no vibration constraints with $\gamma = 10$ is shown in Figure 1.2 top-right—two independent horizontal bars. The volume of this structure is $V^* = 5.0$. Figure 1.2 bottom shows the result of the problem with vibration constraints for $\gamma = 20$ and $\overline{\lambda} = 1.0 \cdot 10^{-3}$; the optimal structure has volume $V^* = 7.6166$. ∎

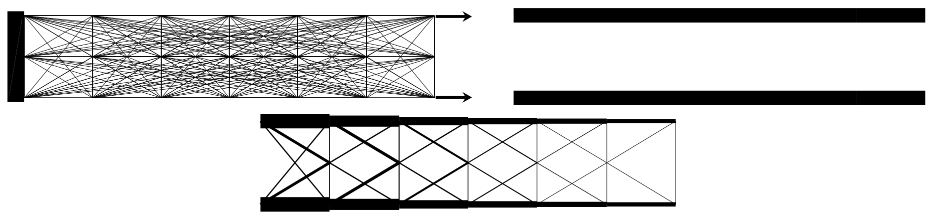

Figure 1.2. *A medium size example (Ex. 1.20): initial layout (top-left); optimal topology without (top-right) and with (bottom) vibration constraints.*

1.6.2 ▪ Problems with integer variables

In truss topology design we try to find an optimum structural design of the truss by finding optimal cross-sectional areas of the bars. Often, from the manufacturing point of view it is highly desirable that variables attain only few given discrete values, for instance 0,1,2,3. Then the problem becomes an optimization problem with integer variables.

As we are adding constraints on the bar areas a, we can only consider the primal formulations of the problem. Our basic formulation (1.2) is a nonlinear and nonconvex optimization problem, where the nonlinearity is due to equilibrium conditions. When searching for binary or integer design, the resulting problem is then a nonconvex mixed integer nonlinear optimization (MINLO) problem. These problems are, typically, extremely difficult to solve, both due to nonconvexity and the integer nature of some of the variables. There have been many attempts to solve these problems, most of them based on heuristic optimization methods that cannot give any guarantees about the solution. A few articles have recently appeared in the literature that are based on mathematical optimization approach to the problem and that deliver a guaranteed global minimum; see, e.g., [3, 16].

However, we know from Section 1.4 that the basic problem is equivalent to a linear SOCO problem (1.8). Using that, we can reformulate the nonconvex MINLO problem as a linear conic problem with integer variables. As such, it is much easier to solve than the original formulation. In particular, we can directly apply available software such as MOSEK or Gurobi to its solution.

The problem formulation is obvious:

$$\begin{aligned}
\min_{a\in\mathbb{R}^m,\ \tau\in\mathbb{R}^m,\ q\in\mathbb{R}^m} \quad & \frac{1}{2}\sum_{i=1}^m \tau_i \\
\text{s.t.} \quad & \sum_{i=1}^m a_i \ell_i = V \\
& \sum_{i=1}^m q_i b_i = f \\
& \left\| \begin{pmatrix} \sqrt{2} q_i \\ \frac{2a_i - \tau_i}{2} \end{pmatrix} \right\|_2 \le \frac{2a_i + \tau_i}{2}, \quad i = 1,\ldots,m \\
& a_i \in \{0,1,\ldots,T\}, \quad i = 1,2,\ldots,m\,,
\end{aligned} \tag{1.15}$$

where $T > 0$ is a given integer number.

Example 1.21 ([16]). Consider the minimum volume problem (1.15). The initial layout is shown in Figure 1.3. The dimensions are $m = 72$ an $n = 27$. We have solved two

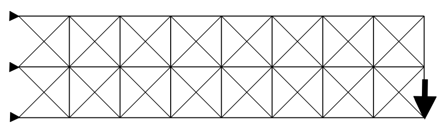

Figure 1.3. *Integer variables (Ex. 1.21): initial design.*

instances of the problem, one with binary variables $a_i \in \{0,1\}$ and compliance bound $\gamma = 50.0$ and one with integer variables $a_i \in \{0,1,2,3\}$ and compliance bound $\gamma = 25.0$. The mixed-integer SOCO problem (1.15) was solved by Gurobi 5.62 with default setting of parameters on an Intel Core i7-620M (2.66 GHz) processor with 4GB memory. The solution of the first problem (288 continuous and 72 binary variables) required visit of 3959 nodes and 14 sec. of CPU time. The optimal objective value of the relaxed problem ($a_i \in [0,1]$) was 14.7342, and of the binary problem 16.9853.To solve the second problem (288 continuous and 72 integer variables), Gurobi visited 25273 nodes and needed 28 sec. of CPU time. The optimal objective value of the relaxed problem ($a_i \in [0,3]$) was 23.3415, and of the integer problem 24.6924.

The optimal solutions for the binary and the integer problems, together with solutions of the relaxed problems, are shown in Figures 1.4. ∎

Another major advantage of this approach, apart from linearity, lies in the fact that we can easily add more conic constraints, such as the vibration constraint introduced in the previous section. This constraint amounts to a linear matrix inequality, so we may as well involve the linear SDO formulation of the truss problem (1.9). The resulting problem reads as

$$\begin{aligned}
\min_{a\in\mathbb{R}^m} \quad & \sum_{i=1}^m \ell_i a_i \\
\text{s.t.} \quad & \begin{pmatrix} \gamma & -f^T \\ -f & K(a) \end{pmatrix} \succcurlyeq 0 \\
& K(a) - \overline{\lambda}(M(a) + M_0) \succcurlyeq 0 \\
& a_i \in \{0,1,\ldots,T\}, \quad i = 1,2,\ldots,m\,.
\end{aligned} \tag{1.16}$$

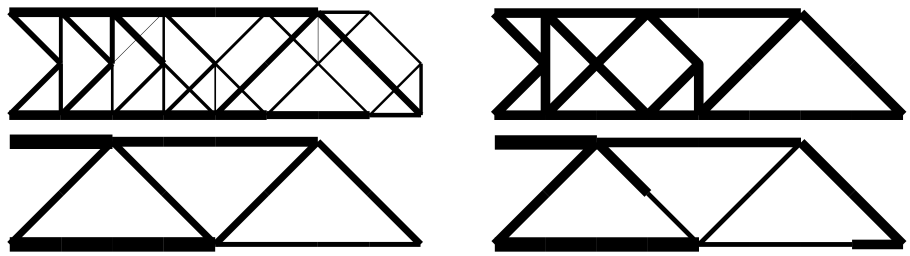

Figure 1.4. *Ex. 1.21: TOP: binary variables $a_i \in \{0,1\}$ and $\gamma = 50.0$, relaxed solution (left) and binary solution (right); BOTTOM: integer variables $a_i \in \{0,1,2,3\}$ and $\gamma = 25.0$, relaxed solution (left) and integer solution (right).*

The drawback, as compared to (1.15), is that (1.16) is a mixed integer linear SDO and, at the moment of writing, it is not supported by any "mainstream" optimization software, unlike mixed integer linear SOCO. The next example was solved by the branch and bound algorithm implemented in YALMIP [13]. The relaxations were solved by PENSDP [10].

Example 1.22. We consider the minimum volume problem with binary variables and the vibration constraint (1.16). The initial layout is shown in Figure 1.5, left. The dimensions are $m = 36$ and $n = 12$. The upper bound on the compliance was chosen as $\gamma = 1.0$ and the

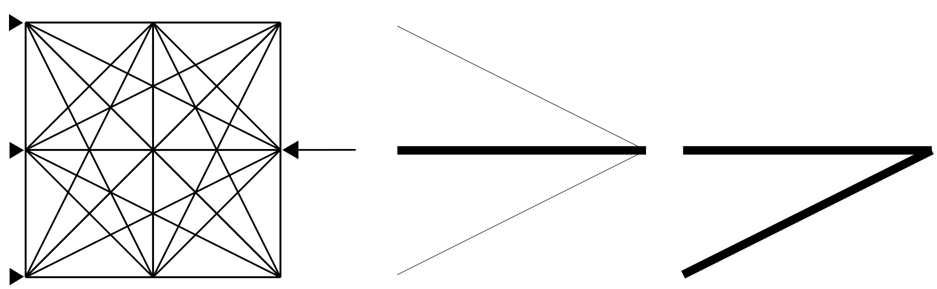

Figure 1.5. *Integer variables with vibration constraints (Ex. 1.22): initial design, relaxed solution and binary solution.*

bound on the smallest eigenfrequency $\overline{\lambda} = 0.01$. To find the optimal binary solution, the branch and bound algorithm in YALMIP only needed to visit 149 nodes. The optimal objective value of the relaxed problem was 1.0471 (with $\max_i a_i = 0.9162$), and of the binary problem 2.118. The optimal solutions for the relaxed and the binary problems are shown in Figure 1.5, middle and right. ∎

Bibliography

[1] W. ACHTZIGER, A. BEN-TAL, M. BENDSØE, AND J. ZOWE, *Equivalent displacement based formulations for maximum strength truss topology design*, IMPACT of Computing in Science and Engineering, 4 (1992), pp. 315–345. (Cited on pp. 3, 5)

[2] W. ACHTZIGER AND M. KOČVARA, *Structural topology optimization with eigenvalues*, SIAM Journal on Optimization, 18 (2007), pp. 1129–1164. (Cited on p. 7)

[3] W. ACHTZIGER AND M. STOLPE, *Global optimization of truss topology with discrete bar areas – Part II: Implementation and numerical results*, Computational Optimization and Applications, 44 (2009), pp. 315–341. (Cited on p. 11)

[4] M. ANJOS, *Conic optimization*, in Advances and Trends in Optimization with Engineering Applications, T. Terlaky, M. Anjos and S. Ahmed, eds. MOS-SIAM Series on Optimization, SIAM, 2014. (Cited on p. 8)

[5] A. BEN-TAL, M. KOČVARA, A. NEMIROVSKI, AND J. ZOWE, *Free material design via semidefinite programming: The multiload case with contact conditions*, SIAM review, 42 (2000), pp. 695–715. (Cited on p. 10)

[6] A. BEN-TAL AND A. NEMIROVSKI, *Lectures on Modern Convex Optimization*, MPS-SIAM Series on Optimization. SIAM Philadelphia, 2001. (Cited on p. 7)

[7] A. BEN-TAL, A. NEMIROVSKII, AND J. ZOWE, *Interior point polynomial time method for truss topology design*, Research report 3/92, TECHNION, Haifa, 1992. (Cited on p. 3)

[8] F. JARRE, M. KOČVARA, AND J. ZOWE, *Interior point methods for mechanical design problems*, SIAM Journal on Optimization, 8(4) (1998), pp. 1084–1107. (Cited on p. 10)

[9] Y. KANNO, M. OHSAKI, AND J. ITO, *Large-deformation and friction analysis of non-linear elastic cable networks by second-order cone programming*, International Journal for Numerical Methods in Engineering, 55 (2002), pp. 1079–1114. (Cited on p. 10)

[10] M. KOČVARA AND M. STINGL, *PENNON: Software for linear and nonlinear matrix inequalities*, in Handbook on Semidefinite, Conic and Polynomial Optimization, M. Anjos and J.B. Lasserre, eds., Springer, 2012, pp. 755–794. (Cited on p. 13)

[11] M. KOČVARA, M. STINGL, AND J. ZOWE, *Free material optimization: recent progress*, Optimization, 57 (2008), pp. 79–100. (Cited on p. 10)

[12] M. KOČVARA, *On the modelling and solving of the truss design problem with global stability constraints*, Structural and Multidisciplinary Optimization, 23(3) (2000), pp. 189–203. (Cited on p. 10)

[13] J. LÖFBERG, *YALMIP : A toolbox for modeling and optimization in MATLAB*, in Proceedings of the CACSD Conference, Taipei, Taiwan, 2004, pp. 284–289. (Cited on p. 13)

[14] R.T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970. (Cited on p. 4)

[15] M. STOLPE, *Truss topology design by linear optimization*, in Advances and Trends in Optimization with Engineering Applications, T. Terlaky, M. Anjos and S. Ahmed, eds. MOS-SIAM Series on Optimization, SIAM, 2014. (Cited on pp. 1, 2, 10)

[16] ———, *Truss topology optimization with discrete design variables by outer approximation*, Journal of Global Optimization, (2014), pp. 1–25. (Cited on pp. 11, 12)

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